OPTIMAL SOBOLEV EMBEDDINGS - NOTES TO THE DISSERTATION

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The principal goal of the dissertation is to study Sobolev-type inequalities involving rearrangement invariant Banach function norms. The emphasize is put on the role of Banach function spaces involved in the inequalities, in particular on their optimality with respect to a given environment.

1. Motivation and introduction

Let Ω be a domain of finite measure in $\mathbb{R}^n$ with $n \geq 2$. With no loss of generality we shall assume that $|\Omega| = 1$. We denote by $\mathfrak{M}(\Omega)$ the class of real-valued measurable functions on Ω and by $\mathfrak{M}_+(\Omega)$ the class of nonnegative functions in $\mathfrak{M}(\Omega)$.

By c, C we shall denote various positive constants independent of appropriate quantities and not necessarily the same at each occurrence. We will write $A \lesssim B$ if there is a positive constant C, independent on appropriate quantities, such that $A \leq CB$.

For two normed function spaces X, Y of functions defined on Ω, we say that X is continuously embedded into Y, and write

$$X \hookrightarrow Y, \tag{1.1}$$

when there is a positive constant C such that, for every $u \in X$, we have

$$\|u\|_Y \leq C\|u\|_X.$$

If (1.1) is satisfied, we say that the space Y is a *larger space* than X, or that $\|\cdot\|_Y$ is a *smaller norm* than $\|\cdot\|_X$. If, moreover, the spaces X and Y do not coincide in the set-theoretical sense, we say that Y is *essentially larger* than X. If none of the relations $X \hookrightarrow Y$, $Y \hookrightarrow X$ holds, then we say that the spaces X and Y are *incomparable*.

For example, we have $L^p \hookrightarrow L^q$ if and only if $0 < q \leq p \leq \infty$, since Ω is of finite measure.

By a *Sobolev inequality* we mean an estimate of a certain norm of a given function of several variables by another, possibly different, norm of its gradient. So, if X, Y is a pair of Banach spaces of functions defined on Ω, the Sobolev inequality is the estimate

$$\|u\|_Y \leq C\|\nabla^m u\|_X, \quad u \in C_0^m(\Omega). \tag{1.2}$$

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Instead of a Sobolev inequality we will also often talk about a *Sobolev embedding*, and write

$$W_0^m(X) \hookrightarrow Y. \tag{1.3}$$

Here, the subscript 0 denotes that the functions in (1.2) are of compact support on Ω and the superscript m stands for the order of the embedding (the order of the gradient involved). The spaces X and Y are respectively called the *domain space* and the *range space* of the Sobolev embedding.

Our main objective is to study *optimality* of various types of function spaces in the Sobolev embeddings. Let us explain what we mean by optimality. If the space Y in (1.3) can be replaced by a smaller space Y', say, and the Sobolev embedding

$$W_0^m(X) \hookrightarrow Y' \tag{1.4}$$

is still true, then, of course, (1.4) is an essentially stronger result than (1.3). If such an improvement is not possible (within a given category of function spaces), we say that Y is an *optimal range* (again, within its category). Analogously we treat optimality of the domain space X. If neither the range nor the domain can be improved, we talk about *optimal pair* of function spaces.

We will now illustrate the key role of various function spaces in Sobolev embeddings. We begin with the familiar Lebesgue (or L^p) spaces and then extend our thoughts to broader scales of function spaces. We will, for the sake of simplicity, restrict ourselves to the case of the first-order gradient.

Perhaps the simplest form of the classical Sobolev inequality asserts that, given $1 < p < n$, there exists $C > 0$ such that

$$\left(\int_\Omega |u(x)|^{\frac{np}{n-p}}\,dx\right)^{\frac{n-p}{np}} \le C\left(\int_\Omega |(\nabla u)(x)|^p\,dx\right)^{1/p}, \qquad u \in C_0^1(\Omega). \tag{1.5}$$

In case when $p > n$ and Ω is a Lipschitz domain, a function whose gradient belongs to $L^p(\Omega)$ is known to be Hölder continuous, namely,

$$\sup_{x,y\in\Omega} \frac{|u(x)-u(y)|}{|x-y|^{1-n/p}} \le C\left(\int_\Omega |(\nabla u)(x)|^p\,dx\right)^{1/p}, \qquad u \in C_0^1(\Omega),$$

which, in particular, implies

$$\|u\|_{L^\infty} \le C\left(\int_\Omega |(\nabla u)(x)|^p\,dx\right)^{1/p}, \qquad u \in C_0^1(\Omega). \tag{1.6}$$

The case when $p = n$ is special. We call it the *limiting case* of Sobolev embedding. It is clearly the most interesting case, and, indeed, also the most difficult one. It is known that, for every $q < \infty$,

$$\left(\int_\Omega |u(x)|^q\,dx\right)^{1/q} \le C\left(\int_\Omega |(\nabla u)(x)|^n\,dx\right)^{1/n}, \qquad u \in C_0^1(\Omega). \tag{1.7}$$

Standard examples of functions with an appropriate "logarithmic growth to infinity" near the origin, say, show that, although $np/(n-p) \to \infty$ when $p \to n-$, one cannot take the L^∞-norm on the left side of (1.7).

Let us now rewrite these inequalities in terms of embeddings. We get

$$W_0^1(L^p) \hookrightarrow L^{\frac{np}{n-p}}, \qquad 1 < p < n; \tag{1.8}$$

$$W_0^1(L^p) \hookrightarrow L^\infty, \qquad n < p < \infty; \tag{1.9}$$

$$W_0^1(L^n) \hookrightarrow L^q, \qquad \text{for every} \quad q < \infty. \tag{1.10}$$

Now, within the limited environment of L^p spaces, these results cannot be improved in any possible way. Note the interesting fact: while (1.8) contains an optimal pair of L^p norms, (1.9) does not have the optimal domain norm and (1.10) does not have the optimal range norm, since there is no largest L^p space that would render (1.9) true and, likewise, there is no smallest L^q space that would render (1.10) true.

Clearly, the embeddings in (1.9) and (1.10) are not satisfactory from the function spaces point of view, and it is natural to ask whether any reasonable improvement is at sight. The answer is in the positive, but we have to pay for the improvement by being forced to consider other, more complicated scales of function spaces than L^p spaces.

There are two basic directions in which we can extend our thoughts. The first one is the direction of Lorentz spaces, which requires us to introduce the notion of the non-increasing rearrangement. The second direction is that of Orlicz spaces, involving general convex functions in place of powers.

Given $f \in \mathfrak{M}(\Omega)$, we define its *non-increasing rearrangement*, f^*; on $(0,1)$, by

$$f^*(t) = \inf\{\lambda > 0; |\{x \in \Omega; |f(x)| > \lambda\}| \le t\}, \quad t \in (0,1).$$

Note that, when $f \in \mathfrak{M}(\Omega)$, then $f^* \in \mathfrak{M}_+(0,1)$.

For $0 < p, q \le \infty$, we define the *Lorentz space* $L^{p,q}(\Omega)$ as the collection of all functions $u \in \mathfrak{M}(\Omega)$ such that the quantity

$$\|u\|_{p,q} = \left\| u^*(t) t^{\frac{1}{p}-\frac{1}{q}} \right\|_{L^q(0,1)}$$

is finite.

We will need to compare various function spaces in the sense of a continuous embedding. In the context of Lorentz spaces, the following elementary embedding property will be useful:

$$L^{p,q} \hookrightarrow L^{p,r}, \qquad p \in (0,\infty], \ 0 < q \le r \le \infty. \tag{1.11}$$

In particular, $L^{p,p} = L^p$.

It turns out that, while (1.8) is an optimal result within the environment of Lebesgue spaces, it is not optimal in the broader category of Lorentz spaces, Indeed, we can improve it either by making the range space smaller:

$$W_0^1(L^p) \hookrightarrow L^{\frac{np}{n-p},p}, \qquad 1 < p < n, \tag{1.12}$$

or by making the domain space larger:

$$W_0^1(L^{p,\frac{np}{n-p}}) \hookrightarrow L^{\frac{np}{n-p}}, \qquad 1 < p < n. \tag{1.13}$$

These embeddings are due to Peetre [29]. Results of this sort can be also traced in the works of O'Neil and Hunt in mid 1960's.

Using (1.11), it is easy to verify that both (1.12) and (1.13) are essential improvements of (1.8), and that both these embeddings are optimal within the context of Lorentz spaces.

In order to improve (1.10), a natural appropriate step seems to be considering Orlicz spaces. We say that A is a *Young function* when A is convex and increasing on $[0,\infty)$ and

$$\lim_{t\to 0_+} t/A(t) = \lim_{t\to\infty} A(t)/t = \infty.$$

The quantity

$$\|u\|_A = \inf\left\{\lambda > 0;\ \int_\Omega A\Big(\frac{|u(x)|}{\lambda}\Big)\,dx \le 1\right\}$$

is called the *Luxemburg norm* of u and the set $L_A = L_A(\Omega)$ of all functions u such that $\|u\|_A < \infty$ is called the *Orlicz space* generated by A.

Comparison of Orlicz spaces is considerably less trivial than that of Lebesgue or Lorentz spaces. Given two Young functions A, B, we have $L_A \hookrightarrow L_B$ if and only if there is a positive constant C such that, for every $t \in (1,\infty)$,

$$B(t) \le A(Ct).$$

Moreover, L_A is essentially smaller than L_B if and only if $A \gg B$, that is, $L_A \hookrightarrow L_B$ and

$$\limsup_{t\to\infty} \frac{A(\lambda t)}{B(t)} = \infty \quad \text{for every } \lambda > 0.$$

A particular interest should be paid to a special subclass of Orlicz spaces, called *Zygmund classes*. Let A be a Young function such that $A(t) = \exp t^\alpha$ for some $\alpha > 0$ and all large values of t. We then call the corresponding Orlicz space L_A an *exponential Zygmund class* and write

$$L_A = \exp L^\alpha.$$

If $A(t) = t^p(\log t)^\alpha$ for some $p \in (0,\infty)$, $\alpha > 0$ and all large values of t, then we talk about *logarithmic Zygmund class* and write

$$L_A = L^p(\log L)^\alpha.$$

Independently of one another, Pokhozhaev [30], Trudinger [37] and Yudovich [38] have shown that there is a constant C such that

$$W_0^1(L^n) \hookrightarrow \exp L^{n'}, \tag{1.14}$$

where $\|u\|_{\exp L^{n'}}$ is the norm in the Orlicz space $\exp L^{n'}$, generated by any Young function A which is equivalent for large t to $\exp t^{n'}$, $n' = n/(n-1)$. This space is essentially smaller than any L^q-space with finite q, but, naturally, it is essentially larger than L^∞. Hence it is an essential improvement of (1.10).

Hempel, Morris and Trudinger [25] showed that the $\exp L^{n'}$-norm on the left hand side of (1.14) cannot be replaced by any essentially larger Orlicz norm. Thus, the embedding (1.14) has an optimal Orlicz range.

Though (1.14) is already quite a satisfactory result, its range space still can be further improved. Of course this is not possible by taking a smaller Orlicz range, as mentioned above. However, it can be done if we allow yet another context of function spaces than that of Orlicz spaces.

In order to get such a refinement, we will consider the so-called *Lorentz-Zygmund spaces* $L^{p,q;\alpha}(\Omega)$. These spaces were introduced and studied by Bennett and Rudnick in [14]. For $0 < p, q \leq \infty$ and $\alpha \in \mathbb{R}$, define

$$\|u\|_{p,q;\alpha} = \left\| u^*(t) t^{\frac{1}{p}-\frac{1}{q}} \left(\log\left(\frac{e}{t}\right)\right)^{\alpha} \right\|_{L^q(0,1)}.$$

Lorentz–Zygmund spaces present a generalization of Lorentz spaces as well as of Zygmund classes. Moreover, they include new non-trivial important spaces that do not fall into any of these categories. Let us recall some elementary useful relations involving these spaces: for $\alpha = 0$, we have $L^{p,q;\alpha} = L^{p,q}$. Next, for $\alpha > 0$,

$$\exp L^\alpha = L^{\infty,\infty;-\frac{1}{\alpha}} \quad \text{and} \quad L^p(\log L)^\alpha = L^{p,p;\alpha}.$$

Returning back to our Sobolev embedding, it can be shown that we can replace the $\exp L^{n'}$-norm in (1.14) by a larger norm in the following way: for $u \in C_0^1(\Omega)$, we have,

$$\left(\int_0^1 \left(\frac{u^*(t)}{\log\left(\frac{e}{t}\right)} \right)^n \frac{dt}{t} \right)^{1/n} \leq C \left(\int_\Omega |(\nabla u)(x)|^n \, dx \right)^{1/n}. \tag{1.15}$$

Using the definition of Lorentz–Zygmund spaces, we can rewrite (1.15) as

$$W_0^1(L^n) \hookrightarrow L^{\infty,n;-1}. \tag{1.16}$$

This embedding can be derived from classical capacitary estimates of Maz'ya (see [28, pages 105, 109]); it can also be traced in [29] (cf. [21] for more details), and it was stated explicitly by Hansson ([24]) and by Brézis and Wainger ([17]).

Moreover, it follows from the embedding theorem of Sharpley [32] that the range space in (1.16) is essentially smaller than $\exp L^{n'}$ and therefore (1.16) is an essentially better embedding than (1.14).

Lorentz-Zygmund spaces and Orlicz spaces are two independent classes of function spaces, of quite a different nature, having a nontrivial intersection (for example, Lebesgue spaces belong to both). A more interesting example is $\exp L^{n'}$, which is an Orlicz space as well as it is a Lorentz-Zygmund space. In (1.14),

$\exp L^{n'}$ is optimal as an Orlicz space, but it is not optimal as a Lorentz-Zygmund space, since it can be replaced by $L^{\infty,n;-1}$, which is essentially smaller.

We can now ask about optimality of various Sobolev embeddings within the context of each of these two categories of function spaces. We start with Lorentz–Zygmund spaces.

Problem 1.1. Is either the range space or the domain space in (1.16) optimal among Lorentz-Zygmund spaces?

An interesting particular answer to this question was given by Edmunds, Opic and the author in [6].

Theorem 1.2. *The range space $L^{\infty,n;-1}$ is the smallest Lorentz–Zygmund space such that* (1.16) *holds. There is no optimal domain Lorentz–Zygmund space in* (1.16) *since L^n can be replaced by any space from the scale*

$$\left\{L^{n,r;\frac{1}{n}-\frac{1}{r}}\right\}, \qquad 1 \le r \le n. \tag{1.17}$$

Every two spaces in (1.17) *are incomparable. The largest domain which we can obtain from this result is the sum of endpoint spaces. We thus get*

$$W_0^1(L^n + L^{n,1;-1/n'}) \hookrightarrow L^{\infty,n;-1}. \tag{1.18}$$

It is worth noticing that the answer to the optimality problem in the limiting case of embedding is not quite satisfactory when we restrict ourselves to the Lorentz–Zygmund spaces.

Turning our attention to Orlicz spaces, we first have to quote the following remarkable general result on the optimality of an Orlicz range space by A. Cianchi ([20]).

Theorem 1.3. *Let A be a Young function, satisfying*

$$\int_1^\infty \frac{\tilde{A}(s)}{s^{n'+1}}\,ds = \infty \qquad \text{and} \qquad \int_0^1 \frac{\tilde{A}(s)}{s^{n'+1}}\,ds < \infty.$$

Set

$$A_n(t) = \int_0^{t^{n'}} \left(\Phi_n^{-1}(s)\right)^{n'} ds,$$

where Φ_n^{-1} is the inverse function of

$$\Phi_n(t) = \int_0^t \frac{\tilde{A}(s)}{s^{n'+1}}\,ds.$$

Then

$$\|u\|_{L_{A_n}(\Omega)} \le \|\nabla u\|_{L_A(\Omega)}, \qquad u \in C_0^1(\Omega), \tag{1.19}$$

and (1.19) *no longer holds when A_n is replaced by a Young function B such that $B \gg A_n$.*

However, the Cianchi theorem does not give any information on the optimality of the Orlicz *domain* space, and the same goes for the above-mentioned result of Hempel, Morris and Trudinger [25].

We thus have to consider the following

Problem 1.4. In both of the embeddings

$$W_0^1(L^p) \hookrightarrow L^{\frac{np}{n-p}}, \qquad 1 < p < n, \tag{1.20}$$

$$W_0^1(L^n) \hookrightarrow \exp L^{n'}, \tag{1.21}$$

range is optimal in Orlicz spaces. Are the *domain spaces* also optimal?

Rather surprisingly, it turns out that $L^n(\Omega)$ is not optimal as an Orlicz domain space in (1.14), and, even worse, that such an optimal Orlicz domain space does not exist at all (recall that a similar situation occurring in the context of Lebesgue spaces was described by (1.7), in which case there was no optimal Lebesgue range space). This follows from

Theorem 1.5. *Let A be a Young function such that*

$$\|u\|_{\exp L^{n'}(\Omega)} \le C\|\nabla u\|_{L_A(\Omega)}. \tag{1.22}$$

Then there exists another Young function, A_1, say, such that

$$A \gg A_1 \tag{1.23}$$

and

$$\|u\|_{\exp L^{n'}(\Omega)} \le C\|\nabla u\|_{L_{A_1}(\Omega)}.$$

The proof is constructive; first, the embedding $W_0^1(L_A) \hookrightarrow \exp L^{n'}$ is shown to be equivalent to

$$\int_1^t \frac{\tilde{A}(s)}{s^{n'+1}}\,ds \le C\log t, \tag{1.24}$$

and then, given A, we construct B such that $A \gg B$ but (1.24) is still true when A is replaced by B.

The picture is different in the case of the embedding (1.20). By a completely different method involving generalized P'olya-Szegö inequality and certain thoughts on fundamental functions it is proved in [11, Chapter 9] that the domain in (1.20) is optimal in the context of Orlicz spaces.

These results show that neither the environment of Orlicz spaces, though a lot finer than that of Lebesgue spaces, does not give entirely satisfactory answers to the optimality questions.

To summarize, we obtained the best possible answers to the optimality problem within each of the categories of Orlicz spaces and Lorentz–Zygmund spaces, and observed that in neither of these classes of function spaces the answers are entirely satisfactory. We will thus turn our attention to the relatively wide class of the so-called rearrangement-invariant Banach function spaces, which provide a common roof for all the spaces that we have seen so far. In particular, an example of a questions which we intend to pursue and solve, is

Problem 1.6. Are the domain space and the range space in (1.18) optimal among rearrangement-invariant spaces?

To finish this introductory section, let us return to the embedding (1.9). If we are interested in sharp results concerning the range space, then the rearrangement techniques are not fine enough. This is a simple consequence of the fact that the range space L^∞ is itself the smallest possible rearrangement-invariant space at all. Therefore, we will also consider the following

Problem 1.7. What kind of optimality results can be obtained when the range space is one of the spaces with controlled integral or pointwise oscillation, namely a Hölder, a Campanato or a Morrey space?

All the problems mentioned above are pursued and answered in the dissertation.

2. Elementary proof of sharp Sobolev embeddings

First, in Chapter 2, we present a new elementary proof of (1.12) and (1.16). Our argument is based on a weak version of the Sobolev-Gagliardo inequality combined with the well-known Maz'ya truncation trick.

As an interesting by-product, our techniques provide us with a qualitatively new function space, giving us a further non-trivial improvement of the range space in the limiting case of Sobolev embedding. Namely, we prove

Theorem 2.1. *Assume that $p \in (0,\infty)$. Let W_p be the function class defined by the norm*

$$\|u\|_{W_p} := \left(\int_0^1 \frac{\left(u^*(\frac{t}{2}) - u^*(t)\right)^p}{t}\, dt \right)^{\frac{1}{p}} < \infty.$$

Then

$$W_0^{1,n}(\Omega) \hookrightarrow W_n(\Omega). \tag{2.1}$$

The fact that this result is really sharper than (1.16) follows from the item (v) in the following observations.

Theorem 2.2. *Assume that $|\Omega| < \infty$ and $p \in [1,\infty)$. Then*

- (i) $\|\chi_E\|_{W_p(\Omega)} = (\log 2)^{\frac{1}{p}}$ *for every measurable* $E \subset \Omega$;
- (ii) $L^\infty(\Omega) \subsetneqq W_p(\Omega)$;
- (iii) *each integer-valued* $u \in W_p(\Omega)$ *is bounded;*
- (iv) $W_p(\Omega)$ *is not a linear set;*
- (v) $W_p(\Omega) \subsetneqq BW_p(\Omega)$;
- (vi) $W_p \subsetneqq W_q$ *for every* $0 < p < q \le \infty$.

3. Optimality of Sobolev embeddings in rearrangement-invariant spaces

Our next goal is to consider optimality of function spaces in Sobolev embeddings in the context of rearrangement-invariant spaces. It is natural to consider Sobolev spaces of higher order, say, $m \in \mathbb{N}$. We define the m^{th} *order gradient*, $D^m u$, of a function u in $C_0^m(\Omega)$, by

$$D^m u = \sum_{|\alpha| \le m} \frac{\partial^\alpha u}{\partial x^\alpha},$$

where α is a multiindex of height $|\alpha|$. We also define the *reduced* m^{th} *order gradient*, $\nabla^m u$, in terms of the first order gradient $\nabla = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)$ and the Laplacian $\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$ as follows:

$$\nabla^m u = \begin{cases} \Delta^k u & \text{when } m = 2k, \\ \nabla(\Delta^k u) & \text{when } m = 2k+1, \end{cases}$$

where $\Delta^j u = \Delta(\Delta^{j-1} u)$, $j = 2, \dots, \left[\frac{m}{2}\right]$; see [13].

The reduced gradient (or 'trace' gradient) is often used in applications, one of its main advantages being perhaps the inequality

$$|u(x)| \le C(I_m(\nabla^m u))(x), \qquad u \in C_0^m(\mathbb{R}^n)$$

(cf. [27], [40], [13]), where I_m is the Riesz potential.

We shall study Sobolev embeddings for both full and reduced gradient.

Let us begin with some observations upon the non-increasing rearrangement.

Remarks 3.1. (i) The operation of non-increasing rearrangement is not subadditive. Instead, we have

$$(f+g)^*(t) \le f^*\left(\tfrac{t}{2}\right) + g^*\left(\tfrac{t}{2}\right), \quad t > 0. \tag{3.1}$$

(ii) The lack of subadditivity of non-increasing rearrangement is one of the key motivations for introducing and working with the elementary maximal function. Indeed, we have

$$(f+g)^{**}(t) \le f^{**}(t) + g^{**}(t), \quad f, g \text{ measurable}, \ 0 < t < 1.$$

(iii) Given $f, g \ge 0$ on Ω, there exists $h \ge 0$ on $(0,1)$ with

$$\int_0^1 f^*(t) h(t)\, dt = \int_\Omega f(x) g(x)\, dx, \tag{3.2}$$

where $g^* = h$.

Our basic idea is to reduce the original Sobolev inequality which involves a differential operator, to an inequality involving Hardy-type integral operator, which should be easier to handle. This reduction is quite complicated from technical point of view. For reasons caused by the technical complications it will be convenient to work with various norm-like functionals acting on functions defined on $(0,1)$. The framework of r.i. norms will be our ultimate choice but first we have to work in a broader context of quasinorms.

Definition 3.2. A *quasinorm* ϱ on $\mathfrak{M}_+(0,1)$ is defined by the following six axioms:

(A_1) $\varrho(f) \geq 0$ with $\varrho(f) = 0$ if and only if $f = 0$ a.e.;
(A_2) $\varrho(cf) = c\varrho(f), \quad c \geq 0$;
(A_3) $\varrho(f+g) \leq C[\varrho(f) + \varrho(g)]$;
(A_4) $f_n \nearrow f$ implies $\varrho(f_n) \nearrow \varrho(f)$;
(A_5) $\varrho(\chi_{(0,1)}) < \infty$;
(A_6) to each s, $0 < s < 1$, there corresponds $C = C(s) > 0$, independent of $f \in \mathfrak{M}_+(0,1)$, such that

$$\varrho(\tau_s f) \leq C\varrho(f), \quad (\tau_s f)(t) = f(st), \; 0 < t < 1.$$

If, in addition, ϱ satisfies
(A_7) $\varrho(f) = \varrho(f^*)$,
we say ϱ is a *rearrangement-invariant* (*r.i.*) *quasinorm.*

Definition 3.3. Let ϱ be an r.i. quasinorm on $\mathfrak{M}_+(0,1)$. Then, ϱ is said to be a *rearrangement-invariant* (r.i.) *norm* if we can take $C = 1$ in (A_3) and if there exists $C > 0$ such that

$$\int_0^1 f(x)\,dx \leq C\varrho(f), \quad f \in \mathfrak{M}_+(0,1). \tag{3.3}$$

The *dual* of a quasinorm ϱ is the functional

$$\varrho'(g) = \sup_{\varrho(h)=1} \int_0^1 g(t)h(t)\,dt, \quad g, h \in \mathfrak{M}_+(0,1).$$

When ϱ is rearrangement-invariant, as a consequence of (3.2) we have

$$\varrho'(g) = \varrho'_d(g^*), \tag{3.4}$$

where the *"down" dual*, ϱ'_d, is given at g by

$$\varrho'_d(g) = \sup_{\varrho(h)=1} \int_0^1 g(t)h^*(t)\,dt, \quad g, h \in \mathfrak{M}_+(0,1).$$

A functional ϱ on $\mathfrak{M}_+(0,1)$ satisfying (A_1)–(A_5) with $C = 1$ in (A_3), and (3.3), is called a *Banach function norm* on $\mathfrak{M}_+(0,1)$.

Remarks 3.4. (i) For a Banach function norm we have the *duality principle*

$$\varrho'' = \varrho \tag{3.5}$$

(see [15, Chapter 1, Theorem 2.7]).

(ii) Axiom (A_6) automatically holds for a Banach function norm which satisfies (A_7) ([15, Chapter 3, Proposition 5.11]).

(iii) Both ϱ' and ϱ'_d obey axioms (A_1)–(A_4) and, moreover, we can take $C = 1$ in (A_3). Again, (A_5) is verified by either ϱ' or ϱ'_d if and only if (3.3) holds for ϱ. We conclude ϱ' and ϱ'_d are Banach function norms if and only if one has the L^1–imbedding (3.3) for ϱ; indeed, ϱ' will be an r.i. norm in view of (3.2) and (A_7).

Examples 3.5. (i) The Lebesgue quasinorm ϱ_p, $0 < p < \infty$, is given by

$$\varrho_p(f) = \left(\int_0^1 |f(t)|^p\,dt\right)^{\frac{1}{p}} = \left(\int_0^1 f^*(t)^p\,dt\right)^{\frac{1}{p}}, \quad f \in \mathfrak{M}_+(0,1).$$

Also,

$$\varrho_\infty(f) = \operatorname*{ess\,sup}_{0<t<1} |f(t)| = f^*(0_+)$$

is an r.i. norm on $\mathfrak{M}_+(0,1)$.

(ii) A generalization of the Lebesgue quasinorms is given by the classical Lorentz functionals $\varrho_{\phi,p}$ defined in terms of a non-negative measurable (weight) function ϕ on $(0,1)$ by

$$\varrho_{\phi,p}(f) = \varrho_p(\phi f^*). \tag{3.6}$$

We also denote

$$\gamma_{\phi,p}(f) = \varrho_p(\phi f^{**}). \tag{3.7}$$

(iii) For a Young function A, we will denote by ϱ_A the Luxemburg norm and by $\varrho_{(A)}$ the Orlicz norm on the Orlicz space $L_A(\Omega)$.

We will denote by P the Hardy average operator, defined at an integrable non-negative function g on $(0,1)$ by

$$(Pg)(t) = \frac{1}{t}\int_0^t g(s)\,ds, \quad t \in (0,1).$$

We now use (3.5) to get tractable expressions equivalent to ϱ' when $\varrho = \varrho_{\phi,p}$, $0 < p \le \infty$. In [31, Theorem 1] it was shown that if $1 < p < \infty$ (and ϕ is any locally integrable weight on $(0,1)$), then

$$\varrho'_d(g) \approx \varrho_{p'}\left(\frac{\phi^{p-1}Pg}{P\phi^p}\right) + \frac{\varrho_1(g)}{\varrho_p(\phi)}, \quad g \in \mathfrak{M}_+(0,1). \tag{3.8}$$

The corresponding expression for $0 < p \le 1$ was obtained in [35, Proposition 1]:

$$\varrho'_d(g) \approx \varrho_\infty\left(\frac{t^{1-\frac{1}{p}}(Pg)(t)}{(P\phi^p)(t)^{\frac{1}{p}}}\right), \quad g \in \mathfrak{M}_+(0,1). \tag{3.9}$$

As for $p = \infty$, it is clear that when ϕ is non-decreasing on $(0,1)$, then

$$\varrho'_d(g) = \varrho_1\left(\frac{g}{\phi}\right), \quad g \in \mathfrak{M}_+(0,1). \tag{3.10}$$

We conclude that, for $g \in \mathfrak{M}_+(0,1)$.

$$\varrho_d'(g) \approx \begin{cases} \varrho_\infty\left(\frac{t^{1-\frac{1}{p}}(Pg)(t)}{(P\phi^p)(t)^{\frac{1}{p}}}\right) & \text{if } 0<p\le 1,\\ \varrho_{p'}\left(\frac{\phi^{p-1}Pg}{P\phi^p}\right)+\frac{\varrho_1(g)}{\varrho_p(\phi)} & \text{if } 1<p<\infty,\\ \varrho_1\left(\frac{g}{\phi}\right) & \text{if } p=\infty,\ \phi \text{ non-decreasing.}\end{cases} \tag{3.11}$$

Let ϱ_R and ϱ_D be r.i. norms on $\mathfrak{M}_+(0,1)$. We shall say that ϱ_R and ϱ_D satisfy the *Sobolev inequality of degree* m, if there exists a $C>0$ such that, for every $u \in C_0^m(\Omega)$,

$$\varrho_R(u^*(t)) \le C\varrho_D(|\nabla^m u|^*(t)). \tag{3.12}$$

In order to obtain the analogs of (1.8), (1.14) and (1.16) for $\nabla^m u$ we have to replace n by n/m throughout (for the resulting inequalities see [28], [29], [33], [36], [24] and [17]).

Here ϱ_D, ϱ_R stand for *domain* and *range* norms in the Sobolev imbedding, respectively.

We are interested in when the norms involved in the Sobolev inequalities are optimal in the sense that ϱ_R cannot be replaced by an essentially larger r.i. norm and ϱ_D cannot be replaced by an essentially smaller r.i. norm.

Remark 3.6. Let ϱ_R and ϱ_D be quasinorms. Then, as defined above, the first-order Sobolev inequality holds if there is a constant $C>0$ such that for every $u \in C_0^1(\Omega)$

$$\varrho_R\left(u^*(t)\right) \le C\varrho_D\left(|\nabla u|^*(t)\right). \tag{3.13}$$

We will however also work with an inequality slightly different from (3.13), namely

$$\varrho_R\left(u^*(t)\right) \le C\varrho_D\left(\frac{d}{dt}\int_{\{x\in\Omega:\,|u(x)|>u^*(t)\}}|(\nabla u)(x)|\,dx\right), \quad u \in C_0^1(\Omega). \tag{3.14}$$

We shall point out that (3.13) always implies (3.14) and that they are equivalent when ϱ_R and ϱ_D are *norms.*

Reduction to Hardy operators. The first step in our argument is a reduction of the Sobolev embedding to the boundedness of certain one-dimensional Hardy operators.

To see how the Hardy operators arise, consider a smooth radial function $u(x) = u(|x|)$ supported in the ball $B = \left\{x \in \mathbb{R}^n;\ |x| < \omega_n^{-\frac{1}{n}}\right\}$ of unit measure centered at the origin. Setting $r=|x|$, one has

$$|(\nabla u)(r)| = |u'(r)|,$$

with $u(r) = \int_r^{\omega_n^{-\frac{1}{n}}} u'(s)\,ds$ or

$$u(\omega_n^{-\frac{1}{n}} r^{\frac{1}{n}}) = n^{-1}\omega_n^{-\frac{1}{n}} \int_r^1 f(t)t^{\frac{1}{n}-1}\,dt, \quad f(t) = u'(s),\ s = \omega_n^{-\frac{1}{n}} t^{\frac{1}{n}}.$$

Again,

$$(\nabla^2 u)(r) = (\Delta u)(r) = u''(r) + \tfrac{n-1}{r}u'(r), \quad u(\omega_n^{-\frac{1}{n}}) = u'(\omega_n^{-\frac{1}{n}}) = 0,$$

so that

$$u(r) = \tfrac{1}{2-n}\left(r^{2-n}\int_0^r (\nabla^2 u)(s)s^{n-1}\,ds + \int_r^{\omega_n^{-\frac{1}{n}}} (\nabla^2 u)(s)s\,ds \right)$$

or

$$u(\omega_n^{-\frac{1}{n}} r^{\frac{1}{n}}) = \frac{1}{n(2-n)\omega_n^{\frac{2}{n}}}\left(r^{\frac{2}{n}-1}\int_0^r f(t)\,dt + \int_r^1 f(t)t^{\frac{2}{n}-1}\,dt \right), \tag{3.15}$$

with $f(t) = (\nabla^2 u)(s)$, $s = \omega_n^{-\frac{1}{n}} t^{\frac{1}{n}}$.

For general u, the connection with Hardy operators is made by a version of the Pólya–Szegö inequality when $m = 1$ and by a convolution inequality of O'Neil when $m > 1$. This connection is sharp when u is radially decreasing. It is clear that the higher-order case will be different from the first-order case, involving, as it does, two Hardy operators rather than just one.

Now we can state the general reduction theorem.

Theorem 3.7. *Fix $m, n \in \mathbb{Z}$, satisfying $n \geq 2$ and $1 \leq m \leq n-1$. Let ϱ_R be an r.i. quasinorm on $\mathfrak{M}_+(0,1)$. Then, when $m = 1$, a necessary and sufficient condition that* (3.14) *hold with ϱ_R and a quasinorm ϱ_D on $\mathfrak{M}_+(0,1)$ is the existence of $C > 0$ for which*

$$\varrho_R\left(\int_t^1 f(s)s^{\frac{1}{n}-1}\,ds \right) \leq C\varrho_D(f), \quad f \in \mathfrak{M}_+(0,1). \tag{3.16}$$

When $n \geq 3$ and $2 \leq m \leq n-1$, a necessary and sufficient condition that (3.12) *hold for ϱ_R and another r.i. quasinorm ϱ_D on $\mathfrak{M}_+(0,1)$ is the existence of $C > 0$ for which*

$$\varrho_R\left(\int_t^1 f^{**}(s)s^{\frac{m}{n}-1}\,ds \right) \leq C\varrho_D(f), \quad f \in \mathfrak{M}_+(0,1). \tag{3.17}$$

Note that when $m = 1$, ϱ_D does not have to be necessarily rearrangement-invariant.

In the case when $m = 1$ and ϱ_R and ϱ_D are *r.i. norms* rather than just quasinorms, we can formulate a refined version of Theorem 3.7.

Theorem 3.8. *Let ϱ_R and ϱ_D be r.i. norms on $\mathfrak{M}_+(0,1)$. Then, in order that there is a constant $C>0$ such that* (3.13) *holds, it is necessary and sufficient that there exist a $K>0$ for which*

$$\varrho_R\left(\int_t^1 f(s)s^{\frac{1}{n}-1}\,ds\right)\le K\varrho_D(f),\quad f\in\mathfrak{M}_+(0,1). \tag{3.18}$$

In view of the above result and Theorem 3.7, we have

Corollary 3.9. *Let ϱ_D and ϱ_R be r.i. quasinorms on $\mathfrak{M}_+(0,1)$. Then,* (3.13) *implies* (3.14)*. Moreover, the two inequalities are equivalent when ϱ_R and ϱ_D are norms.*

Corollary 3.10. *Let ϱ_D and ϱ_R be r.i. norms on $\mathfrak{M}_+(0,1)$.* (i) *Assume that*

$$\varrho_R\left(\int_t^1 f^{**}(s)s^{\frac{1}{n}-1}\,ds\right)\le C\varrho_D(f),\quad f\in\mathfrak{M}_+(0,1). \tag{3.19}$$

Then the Sobolev inequality (3.13) *holds.*

(ii) *Assume that the estimate*

$$\varrho_R\left(\int_t^1 f^{**}(s)s^{\frac{1}{n}-1}\,ds\right)\le C\varrho_R\left(\int_t^1 f^{*}(s)s^{\frac{1}{n}-1}\,ds\right) \tag{3.20}$$

is satisfied for every $f\in\mathfrak{M}_+(0,1)$. Then (3.19) *is equivalent* (3.13).

Remark 3.11. A short argument involving Fubini's theorem and (A_6) yields (3.19) equivalent to

$$\varrho_R\left(t^{\frac{1}{n}-1}\int_0^t f(s)\,ds+\int_t^1 f(s)s^{\frac{1}{n}-1}\,ds\right)\le C\varrho_D(f),\quad f\in\mathfrak{M}_+(0,1). \tag{3.21}$$

Theorem 3.12. *Assume that ϱ_R is an r.i. norm such that* (3.20) *is satisfied. Then*

$$\varrho_R\left(\int_t^1 f(s)s^{\frac{1}{n}-1}\,ds\right)\le C\varrho_R\left(\int_t^1 f^{*}(s)s^{\frac{1}{n}-1}\,ds\right),\quad f\in\mathfrak{M}_+(0,1). \tag{3.22}$$

In particular, ϱ_D defined by

$$\varrho_D(f)=\varrho_R\left(\int_t^1 f^{*}(s)s^{\frac{1}{n}-1}\,ds\right),\quad f\in\mathfrak{M}_+(0,1), \tag{3.23}$$

is equivalent to the smallest r.i. norm which renders (3.13) *true.*

It would be of interest to know when a given r.i. norm ϱ_R satisfies (3.20). A sufficient condition can be expressed by means of the lower Boyd index.

Given an r.i. norm ϱ on $\mathfrak{M}_+(0,1)$, the *lower Boyd index* i_ϱ is given by

$$i_\varrho=\lim_{t\to 0_+}\frac{\log(\frac{1}{t})}{\log h_\varrho(t)},$$

where

$$h_\varrho(t) = \sup_{f \not\equiv 0} \frac{\varrho(\tau_t f)}{\varrho(f)}, \quad \tau_t f(s) = f(st), \quad f \in \mathfrak{M}_+(0,1), \quad 0 < s, t < 1.$$

Theorem 3.13. *Let $n \geq 2$ and $1 \leq m \leq n-1$. Let ϱ_R be an r.i. norm on $\mathfrak{M}_+(0,1)$. Then*

$$\varrho_R \left(\int_t^1 f^{**}(s) s^{\frac{m}{n}-1} \, ds \right) \leq C \varrho_R \left(\int_t^1 f^{*}(s) s^{\frac{m}{n}-1} \, ds \right), \tag{3.24}$$

holds whenever the lower index i_R of ϱ_R satisfies

$$i_R > \frac{n}{n-m}. \tag{3.25}$$

The optimal domain and the optimal range. Now we are in a position when we can apply the reduction theorem (Theorem 3.7) to associate to a given range r.i. quasinorm ϱ_R the essentially smallest quasinorm ϱ_D such that (3.14) holds when $m = 1$ and (3.12) holds when $n \geq 3$ and $2 \leq m \leq n-1$. in the latter case, ϱ_D is rearrangement invariant.

Theorem 3.14. *Suppose $m, n \in \mathbb{Z}_+$, with $n \geq 2$ and $1 \leq m \leq n-1$. Let ϱ_R be an r.i. quasinorm on $\mathfrak{M}_+(0,1)$. For $f \in \mathfrak{M}_+(0,1)$ define*

$$\varrho_D(f) = \varrho_R \left(\int_t^1 f(s) s^{\frac{1}{n}-1} \, ds \right) \qquad \text{if } m = 1 \tag{3.26}$$

and

$$\varrho_D(f) = \varrho_R \left(\int_t^1 f^{**}(s) s^{\frac{m}{n}-1} \, ds \right) \qquad \text{if } n \geq 3, \ 2 \leq m \leq n-1. \tag{3.27}$$

Then, ϱ_D is a quasinorm on $\mathfrak{M}_+(0,1)$ such that (3.14) holds when $m = 1$ and (3.12) holds when $n \geq 3$ and $2 \leq m \leq n-1$. Moreover, it is the smallest such quasinorm when $m = 1$ and the smallest such r.i. quasinorm when $n \geq 3$ and $2 \leq m \leq n-1$ in the sense that if ϱ is another, then there exists $C > 0$ for which

$$\varrho_D(f) \leq C \varrho(f), \ f \in \mathfrak{M}_+(0,1).$$

The following observation is immediately seen.

Theorem 3.15. *Suppose $m, n \in \mathbb{Z}_+$, with $n \geq 2$ and $1 \leq m \leq n-1$. Let ϱ_R be an r.i. quasinorm on $\mathfrak{M}_+(0,1)$. If ϱ_R satisfies the subadditivity property*

$$\varrho_R(f+g) \leq \varrho_R(f) + \varrho_R(g), \quad f, g \in \mathfrak{M}_+(0,1), \tag{3.28}$$

then so will the ϱ_D defined by (3.26) and (3.27). In any case, ϱ_D satisfies the L^1-embedding property (3.3) when $n \geq 3$ and $2 \leq m \leq n-1$.

Remark 3.16. The example $\varrho_R(f) = \varrho_1(f)$, $f \in \mathfrak{M}_+(0,1)$, for which (3.26) becomes $\varrho_D(f) = \varrho_1(t^{\frac{1}{n}} f(t))$, shows ϱ_D need not satisfy (3.3).

Now we turn our attention to the construction of the optimal range norm when the domain quasinorm is given.

Theorem 3.17. *Suppose $n \in \mathbb{Z}_+$, with $n \geq 2$. Suppose ϱ_D is a quasinorm on $\mathfrak{M}_+(0,1)$ such that*

$$\int_0^1 f(t) t^{\frac{1}{n}}\, dt \leq C \varrho_D(f), \quad f \in \mathfrak{M}_+(0,1). \tag{3.29}$$

Then, the functional σ defined by

$$\sigma(g) = \varrho_D'(t^{\frac{1}{n}} g^{**}(t))$$

is an r.i. norm on $\mathfrak{M}_+(0,1)$. Moreover; (3.14) holds for $\varrho_R = \sigma'$, σ' being the largest such r.i. range norm.

For the higher-order embedding, we have

Theorem 3.18. *Suppose $m, n \in \mathbb{Z}_+$, with $n \geq 3$ and $2 \leq m \leq n-1$. Suppose ϱ is a quasinorm on $\mathfrak{M}_+(0,1)$ for which the L^1 imbedding (3.3) holds. Then, the functional σ defined at $g \in \mathfrak{M}_+(0,1)$ by*

$$\sigma(g) = \varrho'\left(\int_t^1 g^{**}(s) s^{\frac{m}{n}-1}\, ds\right)$$

is an r.i. norm on $\mathfrak{M}_+(0,1)$. Moreover, if ϱ is rearrangement-invariant, then (3.12) holds for $\varrho_R = \sigma'$ and $\varrho_D = \varrho$, σ' being the largest such r.i. range norm.

Remark 3.19. The results concerning the optimal domain norm when the range norm is given are quite satisfactory, because they provide an explicit construction of the ϱ_D. On the other hand, the results concerning the optimal range when the domain is given, are rather implicit as they require two duality steps. In particular examples, the characterization of the required dual norms can be quite difficult. In some cases however a precise characterization is available. We shall first demonstrate the results with some examples.

Examples.

Theorem 3.20 (Optimal range)**.** (i) *Let $1 \leq p < n$. Then, the Lorentz space $L^{\frac{np}{n-p},p}$ is the smallest r.i. space which renders the embedding*

$$W_0^1(L^p) \hookrightarrow L^{\frac{np}{n-p},p}$$

true.

(ii) *The Lorentz–Zygmund space $L^{\infty,n;-1}$ is the smallest r.i. space which renders the embedding*

$$W_0^1(L^n) \hookrightarrow L^{\infty,n;-1}$$

true.

Let us note that Theorem 3.20 does not contradict Theorem 2.1, as $W_n(\Omega)$ is not an r.i. space. (Of course it is obviously a rearrangement-invariant structure but it is not linear as pointed out in Theorem 2.2 (iv).)

Theorem 3.21 (Optimal domain). (i) *Let $n' < p < \infty$. Then, the Lorentz space $L^{\frac{np}{n+p},p}$ is the largest r.i. space which renders the embedding*

$$W_0^1(L^{\frac{np}{n+p},p}) \hookrightarrow L^p$$

true.

(ii) *The space L^1 is the largest r.i. space which renders the embedding*

$$W_0^1(L^1) \hookrightarrow L^{n'}$$

true.

(iii) *The space $L^{n,1}$ is the largest r.i. space which renders the embedding*

$$W_0^1(L^{n,1}) \hookrightarrow L^\infty$$

true.

4. Optimality of Sobolev embeddings in the context of Orlicz spaces

Having the material of Chapter 3, we are in a position to prove the results concerning Orlicz spaces, and, thereby, to answer Problem 1.4. In particular, we give a detailed constructive proof of Theorem 1.5, mentioned in the introduction. This gives an answer to the second question in Problem 1.4.

Next, in order to answer the first one, we turn our attention to the sub-limiting case. Our main results in this direction are the following two theorems.

Theorem 4.1. *Let A be a Young function. Let ϱ_A be the corresponding Orlicz norm on $\mathfrak{M}_+(0,1)$. Let ϱ_R be an r.i. norm on $\mathfrak{M}_+(0,1)$ such that*

$$\varrho_R(|\nabla u|^*) \lesssim \varrho_A(u^*) \quad \text{for every } u \in L_A(\Omega). \tag{4.1}$$

Assume further that

$$\frac{1}{A^{-1}(\frac{1}{t})} \approx t^{\frac{1}{n}} \varrho_R\left(\chi_{(0,t)}\right). \tag{4.2}$$

Then $\varrho_A(\Omega)$ is the smallest Orlicz norm such that (4.1) holds.

Theorem 4.2. *Let A be a Young function. Let ϱ_A be the corresponding Orlicz norm on $\mathfrak{M}_+(0,1)$. Let ϱ_R be an r.i. norm on $\mathfrak{M}_+(0,1)$ such that*

$$\varrho_R(|\nabla^m u|^*) \lesssim \varrho_A(u^*) \quad \text{for every } u \in L_A(\Omega). \tag{4.3}$$

Assume that

$$\frac{1}{A^{-1}(\frac{1}{t})} \approx t \varrho_R\left(\chi_{(t,1)}(s) s^{\frac{m}{n}-1}\right), \qquad t \in (0, \frac{1}{2}). \tag{4.4}$$

Then ϱ_A is the smallest Orlicz norm such that (4.3) holds.

Let us now overview the situation of optimality of domain and range norms of Sobolev inequality in the context of Orlicz spaces. Theorem 1.3 of A. Cianchi shows that, given a fixed Orlicz *domain* space, there always exists the optimal Orlicz range space. On the other hand, the situation described in Theorem 1.5 shows that for a given Orlicz *range* space, the optimal Orlicz domain space need not necessarily exist. Still, this situation is not universal: consider the simplest possible example of Orlicz range space, i.e. a Lebesgue space $L^q(\Omega)$, $n' < q < \infty$. Then, as we have shown, the optimal Orlicz domain space is the Lebesgue space $L^r(\Omega)$ with

$$r = \frac{qn}{q+n} < q. \tag{4.5}$$

A natural question now occurs: what governs the difference between the case represented by the range space $\exp L^{n'}(\Omega)$ (for which there is no optimal Orlicz domain space) and the case represented by the range space $L^q(\Omega)$, $q \in (n', \infty)$ (for which the optimal Orlicz domain space is readily found)?

Certain insight into this problem is achieved when the optimal fundamental function is calculated and the corresponding Orlicz space is considered. Let us outline the principal ideas of our method:

(i) we start with a given Orlicz norm ϱ_A;
(ii) we find the corresponding optimal rearrangement-invariant domain norm ϱ_D;
(iii) we calculate its fundamental function $\varphi = \varphi_\varrho$;
(iv) we find the (unique) Orlicz norm whose fundamental function is equivalent to φ_ϱ, and denote this norm by ϱ_B;
(v) we find out whether or not ϱ_B satisfies, together with ϱ_A, the Sobolev inequality;
(vi) if so, then ϱ_B is the optimal Orlicz domain norm for the given range norm ϱ_A in the Sobolev inequality.

5. Optimal Pairs of Lorentz–Karamata Quasinorms

In order to obtain a wide variety of examples interesting from the practical point of view we shall now turn our attention to a fairly large class of function spaces. To this end we shall introduce a new class of function spaces generated by functionals which we call Lorentz–Karamata quasinorms (these are classical Lorentz quasinorms involving the slowly-varying functions of Karamata). Let us note that all the function norms which have occurred so far in particular examples fall into this category.

Definition 5.1. A positive function b is said to be *slowly varying* (s.v.) on $(1, \infty)$, in the sense of Karamata, if for each $\varepsilon > 0$, $t^\varepsilon b(t)$ is equivalent to an increasing function and $t^{-\varepsilon} b(t)$ is equivalent to a decreasing function.

Examples 5.2. The following functions are slowly varying on $(1, \infty)$:
$b(t) = (e + \log t)^\alpha \left(\log(e + \log t)\right)^\beta$, α, $\beta \in \mathbb{R}$;

$b(t) = \exp(\sqrt{\log t})$.

Remarks 5.3. Suppose that b is slowly varying on $(1, \infty)$. Then

(i) b^r is slowly varying on $(1, \infty)$ for all $r \in \mathbb{R}$;

(ii) $\int_{t^{-1}}^1 s^{-1} b(s^{-1})\, ds$ is slowly varying on $(1, \infty)$ and (see [40, Chapter 2, p. 186])

$$\lim_{t\to\infty} \frac{b(t)}{\int_{t^{-1}}^1 s^{-1} b(s^{-1})\, ds} = 0; \tag{5.1}$$

(iii) $\lim\limits_{t\to\infty} \frac{b(ct)}{b(t)} = 1$ for all $c > 0$.

(iv) In particular, both $b(t^{-1})^{-1}$ and $\frac{b(t)^{q-1}}{\int_0^{\frac{1}{t}} s^{-1} b(s^{-1})^q\, ds}$ (when $q < \infty$) are slowly varying on $(1, \infty)$ whenever b is.

Definition 5.4. Let $p, q \in (0, \infty]$ and let b be a slowly-varying function on $(1, \infty)$. Assume that $\|t^{-\frac{1}{q}} b(t^{-1})\|_{L^q(0,1)} < \infty$ when $p = \infty$. Then the *Lorentz–Karamata* (L–K) *space* $L^{p,q;b} = L^{p,q;b}$ is the collection of all measurable f such that $\|f\|_{L^{p,q;b}} < \infty$, where

$$\|f\|_{L^{p,q;b}} = \|t^{\frac{1}{p}-\frac{1}{q}} b(\tfrac{1}{t}) f^*(t)\|_{L^q(0,1)}. \tag{5.2}$$

Remark 5.5. (i) The functional $\|f\|_{L^{p,q;b}}$ is a norm if $p > q$ or if $p = q$ and b is non-decreasing on $(1, \infty)$. Moreover, it is equivalent to the r.i. norm

$$\|f\|_{L^{(p,q;b)}} = \|t^{\frac{1}{p}-\frac{1}{q}} b(\tfrac{1}{t}) f^{**}(t)\|_{L^q(0,1)}$$

if and only if $p > 1$.

We shall now present the main results on optimality of pairs of Lorentz–Karamata spaces in the Sobolev embeddings ([5, Section 5]).

Theorem 5.6. *Fix p and q with $1 \le p, q \le \infty$. Suppose b is a slowly varying function on $(1, \infty)$, which is such that $\phi(t) = t^{\frac{1}{p}-\frac{1}{q}} b(t^{-1})$ satisfies $\varrho_q(\phi) < \infty$. Let*

$$\varrho_R(f) = \begin{cases} \varrho_q(\phi f^*) & \text{when} \quad p > q, \\ \varrho_q(\phi f^{**}) & \text{when} \quad p \le q, \end{cases}$$

and

$$\varrho_D(f) = \varrho_R\left(\int_t^1 f(s) s^{\frac{1}{n}-1}\, ds\right), \quad n \in \mathbb{Z}_+,\ n \ge 2.$$

Then, ϱ_R and ϱ_D are optimal in (3.14) *as an r.i. norm and a Banach function norm, respectively.*

We remark that ϱ_R is equivalent to an r.i. norm by Remark 5.5 and ϱ_D is a Banach function norm by Theorems 3.14 and 3.15.

In view of Theorem 3.14, only the optimality of ϱ_R needs to be shown. To this end, we prove

$$\varrho'_R(g) \approx \varrho'_D(t^{\frac{1}{n}} g^{**}(t)),$$

and then invoke Theorem 3.17 with $\sigma = \varrho_R'$ and $\varrho = \varrho_D$.

For the higher-order case, we have

Theorem 5.7. *Let p, q, ϕ and ϱ_R be as in* Theorem 5.6. *Given $f \in \mathfrak{M}_+(0,1)$, define*

$$\varrho_D(f) = \varrho_R\left(\int_t^1 f^{**}(s)s^{\frac{m}{n}-1}\,ds\right).$$

Then, ϱ_R and ϱ_D are r.i. norms which are optimal in (3.12) *if $1 \le p' < \frac{n}{m}$; otherwise, ϱ_D is optimal but ϱ_R is not except, possibly, when $p' = \frac{n}{m}$, $q = \infty$ and b is bounded away from zero.*

The proof is based on the following idea: to prove that ϱ_R is optimal for its optimal ϱ_D, it is enough to show that it is optimal for *some* norm. Such a norm is constructed. To this end, a rather long and deep argument is used, involving weighted inequalities, known duality relations, calculations on fundamental functions, and last but not least the following new duality theorem for special function norms, which is of independent interest.

Theorem 5.8. *Let $1 < p < \infty$ and suppose the weight ϕ on $(0,1)$ satisfies $\int_0^1 \phi(t)^p\,dt < \infty$. Assume, further, that $\int_0^1 \frac{\phi(t)^p}{t^p}\,dt = \infty$ and*

$$\int_0^r \phi(t)^p\,dt \le Cr^p\left(1 + \int_r^1 \frac{\phi(t)^p}{t^p}\,dt\right), \quad 0 < r < 1. \tag{5.3}$$

Then, the r.i. norm $\gamma_{\phi,p}$ has dual norm

$$\varrho'(g) \approx \varrho_{p'}(\psi g^*), \quad g \in \mathfrak{M}_+(0,1),$$

where

$$\psi(s)^{p'} = \frac{d}{ds}\left[\left(1 + \int_s^1 \frac{\phi(y)^p}{y^p}\,dy\right)^{1-p'}\right], \quad 0 < s < 1. \tag{5.4}$$

Remark 5.9. The simplest expression equivalent to $\varrho_D(f)$ in Theorem 5.6 is

(i) $\varrho_q\left(t^{-\frac{1}{q}}b(t^{-1})\int_t^1 f(s)s^{\frac{1}{n}-1}\,ds\right)$ when $p' = 1$;

(ii) $\varrho_q\left(t^{\frac{1}{n}+\frac{1}{p}-\frac{1}{q}}b(t^{-1})f(t)\right)$ when $p' > 1$.

Likewise, the simplest expression equivalent to $\varrho_D(f)$ in Theorem 5.7 is

(i) $\varrho_q\left(t^{-\frac{1}{q}}b(t^{-1})\int_t^1 f^*(s)s^{\frac{m}{n}-1}\,ds\right)$ when $p' = 1$;

(ii) $\varrho_q\left(t^{\frac{m}{n}+\frac{1}{p}-\frac{1}{q}}b(t^{-1})f^*(t)\right)$ when $1 < p' < \frac{n}{m}$;

(iii) $\varrho_q\left(t^{1-\frac{1}{q}}b(t^{-1})(Pf^*)(t)\right)$ when $p' = \frac{n}{m}$, $\varrho_q\left(t^{-\frac{1}{q}}b(t^{-1})\right) = \infty$, and $\frac{d}{dt}b(t^{-1})^{-1}$ is non-increasing;

(iv) $\varrho_1(f)$ when $p' > \frac{n}{m}$ or $p' = \frac{n}{m}$ and $\varrho_q\left(t^{-\frac{1}{q}}b(t^{-1})\right) < \infty$.

We observe that in either case the expression in (i) cannot be replaced by the one in (ii) when $p' = 1$, since

$$\varrho_q\left(t^{-\frac{1}{q}}b(t^{-1})\int_t^1 \chi_{(0,a)}(s)s^{\frac{m}{n}-1}\,ds\right) \le C\varrho_q\left(t^{\frac{m}{n}-\frac{1}{q}}b(t^{-1})\chi_{(0,a)}(t)\right),$$

$0 < a < 1$, implies

$$\int_0^a t^{-1}b(t^{-1})^q\,dt \le Cb(a^{-1})^q, \quad 0 < a < 1,$$

which contradicts (5.1).

Again, the expression in (iii) cannot be replaced by

$$\varrho_q\left(t^{1-\frac{1}{q}}b(t^{-1})f^*(t)\right).$$

Examples. We present here examples of norms, ϱ_D and ϱ_R, which, in view of Theorems 5.6, 5.7 and Theorem 3.12 are optimal in (3.13) or (3.14). Once again, $m, n \in \mathbb{Z}_+$, $n \ge 2$, $1 \le m \le n-1$, and b is a slowly varying function on $(1, \infty)$.

1. Let $1 < q < \frac{n}{m}$,

$$\varrho_D(f) = \varrho_q\left(t^{-\frac{m}{n}}\int_t^1 f^{**}(s)s^{\frac{m}{n}-1}\,ds\right) \approx \varrho_q(f^{**}) \approx \varrho_q(f)$$

and

$$\varrho_R(f) = \varrho_q\left(t^{-\frac{m}{n}}f^*(t)\right).$$

Then, ϱ_R and ϱ_D are an optimal pair in (3.12).

2. Suppose that $\varrho_D(f) = \varrho_{\frac{n}{m}}(f)$ and that $\varrho_R = \varrho_{\infty,\frac{m}{n};-1}$ is the norm of Hansson and Brézis–Wainger:

$$\varrho_R(f) = \varrho_{\frac{n}{m}}\left(t^{-\frac{m}{n}}\left(\log\frac{e}{t}\right)^{-1}f^*(t)\right). \tag{5.5}$$

Then, ϱ_R and ϱ_D are r.i. norms satisfying (3.12); moreover, by Theorems 3.17, 3.17 and 5.8, ϱ_R is optimal, though ϱ_D is not, as it can be replaced by the essentially smaller norm (cf. [6, Theorem 9.2])

$$\varrho(f) = \inf\left\{\varrho_{\frac{n}{m}}(f_0) + \varrho_1\left(t^{\frac{m}{n}-1}\left(\log\frac{e}{t}\right)^{\frac{m}{n}-1}f_1^*(t)\right) : f = f_0 + f_1\right\}.$$

When $m > 1$, the ϱ_R in (5.5) and ϱ_D, defined by

$$\begin{aligned}\varrho_D(f) &= \varrho_{\frac{n}{m}}\left(t^{-\frac{m}{n}}\left(\log\frac{1}{t}\right)^{-1}\int_t^1 f^{**}(s)s^{\frac{m}{n}-1}\,ds\right)\\ &\approx \varrho_{\frac{n}{m}}\left(t^{-\frac{m}{n}}\left(\log\frac{1}{t}\right)^{-1}\int_t^1 f^{*}(s)s^{\frac{m}{n}-1}\,ds\right)\end{aligned}$$

are optimal in (3.12). As for the case $m = 1$, the r.i. norm ϱ_R and the norm

$$\varrho_D(f) = \varrho_n\left(t^{-\frac{1}{n}}\left(\log\frac{e}{t}\right)^{-1}\int_t^1 f(s)s^{\frac{1}{n}-1}\,ds\right)$$

are optimal in (3.14) and also in (3.13) .

More generally, the r.i. norms

$$\begin{aligned}\varrho_D(f) &= \varrho_{\frac{n}{m}}\left(t^{-\frac{m}{n}}b(t^{-1})\int_t^1 f^{**}(s)s^{\frac{m}{n}-1}\,ds\right)\\ &\approx \varrho_{\frac{n}{m}}\left(t^{-\frac{m}{n}}b(t^{-1})\int_t^1 f^{*}(s)s^{\frac{m}{n}-1}\,ds\right)\end{aligned}$$

and

$$\varrho_R(f) = \varrho_{\frac{n}{m}}\left(t^{-\frac{m}{n}}b(t^{-1})f^*(t)\right)$$

optimal in (3.12) if $n \geq 3$ and $2 \leq m \leq n-1$, while this ϱ_R (with $m = 1$) and the norm

$$\varrho_D(f) = \varrho_n\left(t^{-\frac{1}{n}}b(t^{-1})\int_t^1 f(s)s^{\frac{1}{n}-1}\,ds\right)$$

are optimal in (3.14) and also in (3.13). When $b(t) = [1+\log(1+\log t)]^{-1}$, these results extend and give the best possible refinement of double-exponential type results of [22]; when $b(t) = (\log(et))^{-\alpha}$, $\alpha > 0$, they do the same for the inequality in [23].

3. Take

$$\varrho_D(f) = \varrho_\infty\left(b(t^{-1})\int_t^1 f^{**}(s)s^{\frac{m}{n}-1}\,ds\right) \approx \varrho_\infty\left(b(t^{-1})\int_t^1 f^{*}(s)s^{\frac{m}{n}-1}\,ds\right)$$

and

$$\varrho_R(f) = \varrho_\infty\left(b(t^{-1})f^{**}(t)\right) \approx \varrho_\infty\left(b(t^{-1})f^{*}(t)\right)$$

to get an optimal pair of r.i. norms in (3.12). Given $b(t) \equiv 1$ and $m = 1$, this yields the pair $\varrho_D(f) = \varrho_1\left(t^{\frac{1}{n}-1}f^*(t)\right)$ and $\varrho_R(f) = \varrho_\infty(f)$, obtained in [3, Theorem 5.3] by different means.

4. Finally, set

$$\varrho_D(f) = \varrho_1\left(t^{-\frac{m}{n}}b(t^{-1})\int_t^1 f(s)s^{\frac{m}{n}-1}\,ds\right) \approx \varrho_1(b(t^{-1})f(t))$$

and

$$\varrho_R(f) = \varrho_1\left(t^{-\frac{m}{n}}b(t^{-1})f^*(t)\right). \tag{5.6}$$

When $m = 1$, these norms are optimal in (3.14), but only ϱ_R is rearrangement-invariant, unless $b(t) \approx 1$, in which case $\varrho_D(f) \approx \varrho_1(f)$ and the pair is optimal in (3.13). When $n \geq 3$ and $2 \leq m \leq n-1$, with ϱ_R as in (5.6) and

$$\begin{aligned}\varrho_D(f) &= \varrho_1\left(t^{-\frac{m}{n}} b(t^{-1}) \int_t^1 f^{**}(s) s^{\frac{m}{n}-1}\, ds\right)\\ &\approx \varrho_1\left(b(t^{-1})^{**}(t)\right),\end{aligned}$$

we have ϱ_D, but not ϱ_R, optimal in (3.12). To obtain optimal pairs, in which the domain norms, ϱ_D, have indices equal to 1 we must require that $b(t) \equiv 1$ or that b increases to infinity, with $\frac{d}{dt} b(t^{-1})^{-1}$ non-increasing, then set

$$\varrho_D(f) = \varrho_\infty\left(t^{1-\frac{m}{n}} b(t^{-1}) \int_t^1 f^{**}(s) s^{\frac{m}{n}-1}\, ds\right) \approx \varrho_\infty\left(b(t^{-1}) \int_0^t f^*(s)\, ds\right)$$

and

$$\varrho_R(f) = \varrho_\infty\left(t^{1-\frac{m}{n}} b(t^{-1}) f^*(t)\right).$$

The Optimal Domain in the Limiting Case. A particular case of the norm in (5.5) finishes the analysis of the optimality of the limiting case of Sobolev inequality initiated by the domain norm $\varrho = \varrho_n$. We can reformulate the result in terms of function spaces in the following way: For the domain space $L^n(\Omega)$, the smallest possible rearrangement-invariant range space is the Lorentz-Zygmund space $L^{\infty,n;-1}(\Omega)$, that is, the space of Brézis and Wainger. However, then, $L^n(\Omega)$ is *not optimal* (the largest possible) rearrangement-invariant domain space for the Sobolev embedding into $L^{\infty,n;-1}(\Omega)$. This is already known to us from Theorem 1.2. By Theorem 3.14, the optimal rearrangement-invariant domain space, denoted by $X = X(\Omega)$, say, is normed by

$$\|f\|_X = \varrho_{\infty,n;-1}\left(\int_t^1 s^{-\frac{1}{n'}} f^{**}(s)\, ds\right). \tag{5.7}$$

It was shown in [11, Section 8] that the space X is still essentially larger than $\left(L^n + L^{n,1;-\frac{1}{n'}}\right)$, obtained in [6] (Theorem 1.2). It turns out that X is a new type of a very important function space. In [11, Section 8], a detailed study of X was carried out. The following theorem describes its relations to familiar function spaces.

Theorem 5.10. *Let the space X be defined by* (5.7). *Then X is is an r.i. space. Moreover,*

(i) *the fundamental function φ_X of X satisfies*

$$\varphi_X(t) \approx t^{\frac{1}{n}}\left(\log\left(\frac{e}{t}\right)\right)^{-\frac{1}{n'}}, \quad t \in (0,1); \tag{5.8}$$

(ii) *the following relations hold:*

$$\left(L^n + L^{n,1;-\frac{1}{n'}}\right) \hookrightarrow X, \tag{5.9}$$

$$X \subset \left(L^{n,n;-\frac{1}{n'}} \cap \bigcap_{\alpha>1} L^{n,1;-\frac{\alpha}{n'}} \right); \tag{5.10}$$

and both the embedding (5.9) *and the inclusion* (5.10) *are strict;*

(iii) X *is incomparable to every space from the scale of Lorentz-Zygmund spaces*

$$\left\{ L^{n,r;-\frac{1}{n'}} \right\}, \quad r \in (1,n);$$

(iv) X *is incomparable to every space from the scale of Orlicz spaces*

$$\left\{ L^{n,n;-\frac{\alpha}{n'}} \right\}, \quad \alpha \in (0,1).$$

The proof of this result is based on rather fine calculations involving usual tools for rearrangements such as the Hardy's lemma or the inequality of Hardy, Littlewood and Pólya. Also some recent results on embeddings of various weighted function spaces are useful.

6. Supremum operators, duality and optimality

Let us summarize some of the steps we have taken so far. For the sake of simplicity, we restrict ourselves to the case $m = 1$. First, we have reduced a Sobolev embedding to the boundedness of a weighted integral operator of Hardy type. This result enabled us to construct the optimal domain norm when the range norm is given. Our key concern about this construction is the rather unpleasant fact that, in general, ϱ_D from (3.26) does not have to be necessarily an r.i. norm. First, it does not have to be rearrangement-invariant, and second, (A_6) (with $\varrho = \varrho_D$) might not be true. To illustrate this situation, we consider $\varrho_R(f) = \int_0^1 f(t)\,dt$. Then, of course, ϱ_R is an r.i. norm but $\varrho_D(f) = \int_0^1 f(t)t^{\frac{1}{n}}\,dt$, which is not rearrangement-invariant and, moreover, (A_6) (with $\varrho = \varrho_D$) clearly does not hold.

In view of our reduction theorem, a natural candidate for the optimal *r.i. norm* (for $m = 1$) is the one given at $f \in \mathfrak{M}_+(0,1)$ by

$$\varrho_D(f) = \varrho_R\left(\int_t^1 f^*(s)s^{\frac{1}{n}-1}\,ds \right). \tag{6.1}$$

This functional immediately removes one of the hurdles: it is obviously rearrangement-invariant. However, it creates new dangers instead. First, it does not necessarily have to be a norm, and, second, even worse, the Sobolev inequality itself might no longer be true. Indeed, as we know from the reduction theorem, for ϱ_D defined by (6.1), the Sobolev inequality is equivalent to the estimate

$$\varrho_R\left(\int_t^1 f(s)s^{\frac{1}{n}-1}\,ds \right) \le C\varrho_R\left(\int_t^1 f^*(s)s^{\frac{1}{n}-1}\,ds \right) \quad \text{for all } f \ge 0. \tag{6.2}$$

This however might be false, as can be demonstrated with the example $\varrho_R(f) = \int_0^1 f^*(t)\,dt$. Then, (6.2) turns into

$$\int_0^1 f(t)t^{\frac{1}{n}}\,dt \le C\int_0^1 f^*(t)t^{\frac{1}{n}}\,dt,$$

and this is certainly not true for all $f \in \mathfrak{M}_+(0,1)$ (take for example $f(t) = (1-t)^{-1}$). Moreover, $\varrho_D(f) = \int_0^1 f^*(t)t^{\frac{1}{n}}\,dt$, which is not equivalent to a norm.

We are thus motivated to seek conditions which imply that ϱ_D is equivalent to a norm and that (6.2) is satisfied. One such a sufficient condition is the estimate

(6.3)
$$\varrho_R\left(\int_t^1 f^{**}(s)s^{\frac{1}{n}-1}\,ds\right) \le C\varrho_R\left(\int_t^1 f^*(s)s^{\frac{1}{n}-1}\,ds\right) \quad \text{for all } f \in \mathfrak{M}_+(0,1).$$

When (6.3) is satisfied, then (Theorem 3.12) (6.2) holds and, moreover, since f^{**} is sub-additive in $f \in \mathfrak{M}_+(0,1)$, ϱ_D is a norm. Condition (6.3) is however rather strong and rules out important cases such as the one obtained from the Lorentz norm

$$\varrho_R(f) = \varrho_{\frac{n}{n-1},1}(f) = \int_0^1 f^*(t)t^{-\frac{1}{n}}\,dt, \quad f \in \mathfrak{M}_+(0,1),$$

for which

$$\varrho_R\left(\int_t^1 f^*(s)s^{\frac{1}{n}-1}\,ds\right) \approx \int_0^1 f^*(t)\,dt,$$

while

$$\varrho_R\left(\int_t^1 f^{**}(s)s^{\frac{1}{n}-1}\,ds\right) \approx \int_0^1 f^*(t)\log\frac{1}{t}\,dt.$$

Last, we obtained in Theorems 3.17 and 3.18 formulae solving the symmetric problem to find an optimal ϱ_R when ϱ_D is given. Our main concern about these results is that the optimal range norm ϱ_R is obtained only in an implicit form, via its dual norm. It is not always easy to work out the norm ϱ_R from the formula. But, for example, for $\varrho_D(f) = \|f\|_n$, the optimal range r.i. norm satisfies $\varrho_R'(g) = \varrho_D'(t^{\frac{1}{n}}(g^{**})(t))$. The ϱ_R can be now worked out on using [31, Remark on p. 147]. We get

$$\varrho_R(f) \approx \left(\int_0^1 \left(\frac{f^*(t)}{\log(\frac{e}{t})}\right)^n \frac{dt}{t}\right)^{\frac{1}{n}} \quad \text{for all} \quad f \in \mathfrak{M}_+(0,1). \tag{6.4}$$

The optimal domain norm (we shall call it $\tilde{\varrho}_D$), corresponding to this ϱ_R, is

$$\tilde{\varrho}_D(g) = \left(\int_0^1 \left(\int_t^1 s^{\frac{1}{n}-1}g^*(s)\,ds\right)^n \left(\log\left(\frac{e}{t}\right)\right)^{-n}\frac{dt}{t}\right)^{\frac{1}{n}}, \quad g \in \mathfrak{M}_+(0,1). \tag{6.5}$$

This is a new function norm which naturally occurs in the optimality problem, therefore it is worth of studying. One of the most important challenges is the characterization of its dual norm.

In [12], we established mild sufficient conditions under which explicit description of the corresponding optimal domain r.i. norm is available. For instance, we obtain considerably better conditions for the validity of (6.2) and for ϱ_D be equivalent to an r.i. norm than (6.3). We also give a very reasonable sufficient conditions in order that an explicit formula for optimal range r.i. norm is available.

The key part is played by the operator T given at $g \in \mathfrak{M}_+(0,1)$ and $t \in (0,1)$ by

$$(Tg)(t) = t^{-\frac{m}{n}} \sup_{t<s<1} s^{\frac{m}{n}} g^*(s).$$

This is a "weighted modification" of a particular case of the *Hardy-type operator involving suprema*

$$(R_\gamma g)(t) = \sup_{t<s<1} s^{\frac{\gamma}{n}} g^*(s),$$

which was for $\gamma \in (0,n)$ introduced in [2].

In [12], we introduced the operator J. For a nonnegative function g on $(0,1)$, let Jg be the nonnegative non-increasing derivative of the least concave majorant of the quasi-concave function $\sup_{0<s\le t} s^{1-\frac{1}{n}} g^*(s)$. Then it is a simple exercise to verify that

$$\sup_{0<s\le t} s^{1-\frac{1}{n}} g^*(s) \le \int_0^t Jg(s)\,ds \le 2 \sup_{0<s\le t} s^{1-\frac{1}{n}} g^*(s) \quad \text{for all measurable } g.$$

With the help of the operators T and J, we obtain a desired duality formula.

Theorem 6.1. *Let ϱ_R be an r.i. norm whose dual, ϱ_R', satisfies, for some $C>0$,*

$$\varrho_R'(Tg) \le C\varrho_R'(g) \quad \text{for all } g \ge 0. \tag{6.6}$$

(i) *Then* (6.2) *holds.*

(ii) *Define ϱ_D by* (6.1)*. Set*

$$\sigma(g) = \varrho_R'(Jg).$$

Then,

$$\varrho_D(f) \approx \sigma'(f) \quad \text{for all } f \ge 0.$$

(iii) *The functional ϱ_D defined by* (6.1) *is an r.i. norm; moreover, it is the smallest r.i. norm for which the Sobolev inequality holds.*

This theorem shows the importance of the supremum operator. In particular, it is very important to be able to decide whether the condition (6.6) holds, that is, when the operator T is bounded on the dual of ϱ_R. To this end we first characterize the interpolation properties of the operator T.

Theorem 6.2. (i) *T is bounded on L^1 and also on $L^{n,\infty}$;*

(ii) *there is a $C>0$ such that*

$$(Tg)^{**}(t) \le CT(g^{**})(t) \quad \text{for all measurable } g \text{ and } t \in (0,1).$$

Remarks 6.3. (i) In fact, it is not difficult to show that T is bounded on every space L^p, $0 < p < n$ (in this sense, L^1 is not really an "endpoint"). Consequently, T is also bounded on every space $L^{p,q}$, $p \in (0, n)$, $q \in [1, \infty]$.

(ii) Since $\left(T\chi_{(0,1)}\right)(t) = t^{-\frac{1}{n}}$, and $t^{-\frac{1}{n}}$ is the extremal function in the unit ball of $L^{n,\infty}$, T cannot map any rearrangement-invariant space into a space smaller that $L^{n,\infty}$. Hence, $L^{n,\infty}$ is a natural interpolation endpoint for the operator T. Similarly, since Tg majorizes g^* for every g, T cannot map any r.i. space X into a smaller space than X. We thus always have $T(X) \supset (X \cup L^{n,\infty})$ for every r.i. space X.

(iii) There is one more reason why $L^{n,\infty}$ is the natural endpoint of the scale of spaces on which T is bounded: when $\varrho_R' = \varrho_{n,\infty}$, then $\varrho_R = \varrho_{\frac{n}{n-1},1}$, and ϱ_D is thus the L^1–norm, which is known to be smallest of all r.i. norms.

As an application of our results we shall now characterize the dual of the norm from (6.5).

Example 6.4. Let $\tilde{\varrho_D}$ be defined by (6.5). Then

$$\tilde{\varrho_D}'(g) \approx \left(\int_0^1 \left[\sup_{0<s\le t} s^{1-\frac{1}{n}} g^*(s)\right]^{\frac{n}{n-1}} \frac{dt}{t}\right)^{\frac{n-1}{n}} \qquad \text{for all } g \ge 0.$$

In particular, the boundedness of T on $L^{(1,\frac{n}{n-1})}$ implies (by Theorem 6.1 (iii)) that $\tilde{\varrho_D}$ is an r.i. norm.

The boundedness of the operator R_γ on fairly general classical Lorentz spaces was characterized in [2].

Theorem 6.5. *Let $n \in \mathbb{N}$, $\gamma \in [0, n)$, $1 < p \le q < \infty$, and let w, v be nonnegative measurable functions on $(0, \infty)$ with v satisfying $\int_0^x v(t)\,dt < \infty$ for every $x \in (0, \infty)$. Then there is a positive constant C such that the inequality*

$$\left(\int_0^\infty \left[(R_\gamma\phi)(t)\right]^q w(t)\,dt\right)^{1/q} \le C\left(\int_0^\infty \phi^p(t)\,v(t)\,dt\right)^{1/p}$$

holds for all ϕ non-increasing if and only if

$$r^{\frac{\gamma}{n}}\left(\int_0^r w(t)\,dt\right)^{1/q} \le C\left(\int_0^r v(t)\,dt\right)^{1/p} \tag{6.7}$$

holds for all $r \in (0, \infty)$.

This characterization can be used to give a sharp pointwise estimate for the fractional maximal operator

$$(M_\gamma f)(x) = \sup_{Q\ni x} |Q|^{\frac{\gamma}{n}-1} \int_Q |f(y)|\,dy, \quad x \in \mathbb{R}^n,$$

where the supremum is extended over all cubes $Q \subset \mathbb{R}^n$ with sides parallel to the coordinate axes.

Theorem 6.6. *Let $n \in \mathbb{N}$ and $\gamma \in [0,n)$. Then there exists a positive constant C depending only on n and γ, such that*

$$(6.8)\qquad (M_\gamma f)^*(t) \le C \sup_{t<\tau<\infty} \tau^{\frac{\gamma}{n}} f^{**}(\tau), \qquad t \in (0,\infty),$$

for every $f \in L^1_{\rm loc}(\mathbb{R}^n)$. Inequality (6.8) is sharp in the sense that for every ϕ non-increasing there exists a function f on $\mathbb{R}^n$ such that $f^* = \phi$ a.e. on $(0,\infty)$ and

$$(6.9)\qquad (M_\gamma f)^*(t) \ge c \sup_{t<\tau<\infty} \tau^{\frac{\gamma}{n}} f^{**}(\tau), \qquad t \in (0,\infty),$$

where, again, c is a positive constant which depends only on n and γ. Moreover, the expression $\sup_{t<\tau<\infty} \tau^{\frac{\gamma}{n}} f^{**}(\tau)$ can be replaced by $\big(t^{\frac{\gamma}{n}} f^{**}(t) + \sup_{t<\tau<\infty} \tau^{\frac{\gamma}{n}} f^*(\tau)\big)$ in both (6.8) and (6.9).

Let us recall that such sharp estimates had been known before for other important integral operators of harmonic analysis such as the Hardy–Littlewood maximal operator, the Hilbert transform and the Riesz potential, and to obtain such an estimate for the fractional maximal operator had been an open problem for long time.

The results of the preceding two theorems were used in [2] to characterize when the fractional maximal operator is bounded on a classical Lorentz space.

Theorem 6.7. *Let $n \in \mathbb{N}$, $\gamma \in [0,n)$, $1 < p \le q < \infty$, and let w, v be non-negative and measurable functions on $(0,\infty)$ with v satisfying $\int_0^x v(t)\,dt < \infty$ for every $x \in (0,\infty)$. Then*

$$\left(\int_0^\infty \Big[(M_\gamma f)^*(t)\Big]^q w(t)\,dt\right)^{1/q} \le C\left(\int_0^\infty f^*(t)^p\, v(t)\,dt\right)^{1/p}$$

if and only if (6.7) and

$$\left(\int_r^\infty t^{q(\frac{\gamma}{n}-1)} w(t)\,dt\right)^{1/q} \left(\int_0^r \left(t^{-1}\int_0^t v(y)\,dy\right)^{-p'} v(t)\,dt\right)^{1/p'} \le C$$

hold for all $r \in (0,\infty)$.

With the help of the techniques described in this section, we can state the main results concerning full (non-reduced) Sobolev embeddings. The first of them is the general reduction theorem.

Theorem 6.8. *Suppose $m, n \in \mathbb{Z}_+$, where $n \ge 2$ and $1 \le m \le n-1$. Let ϱ_R and ϱ_D be r.i. norms on $\mathfrak{M}_+(0,1)$. Then there exists a constant C) such that*

$$(6.10)\qquad \varrho_R\left(u^*(t)\right) \le C \varrho_D\left(|D^m u|^*(t)\right), \qquad u \in C_0^m(\Omega),$$

if and only if there exists a $K > 0$ for which

$$(6.11)\qquad \varrho_R\left(\int_t^1 f(s) s^{\frac{m}{n}-1}\,ds\right) \le K \varrho_D(f), \qquad f \in \mathfrak{M}_+(0,1).$$

The key ingredients of the proof involve interpolation theory and the properties of the supremum operators. Next, we characterize the optimal range and the optimal domain.

Theorem 6.9. *Let ϱ_D be an r.i. norm on $\mathfrak{M}_+(0,1)$. Then, the optimal r.i. norm ϱ_R such that* (6.10) *holds for ϱ_D satisfies*

$$\varrho_R(f) \approx \int_0^1 f^*(t)\,dt + \sup_{\varrho_D'(S_{\frac{n}{m}} g)\le 1} \int_0^1 t^{-\frac{m}{n}} \left[f^{**}(t) - f^*(t)\right] g^*(t)\,dt, \tag{6.12}$$

$f \in \mathfrak{M}_+(0,1)$, where the operator $S_{\frac{n}{m}}$ defined at $f \in \mathfrak{M}_+(0,1)$ by

$$\left(S_{\frac{n}{m}} f\right)(t) = t^{\frac{m}{n}-1} \sup_{0<s<t} s^{1-\frac{m}{n}} f^*(s), \qquad t \in (0,1). \tag{6.13}$$

Theorem 6.10. *Let ϱ_R be an r.i. norm on $\mathfrak{M}_+(0,1)$ such that $L_{\varrho_R} \hookrightarrow L^{\frac{n}{n-m},1}$. Then,*

$$\varrho_D(f) := \sup_{h\sim f} \varrho_R\left(\int_t^1 h(s) s^{\frac{m}{n}-1}\,ds\right), \qquad f \in \mathfrak{M}_+(0,1), \tag{6.14}$$

is an r.i. norm on $\mathfrak{M}_+(0,1)$ for which

$$\varrho_R\left(\int_t^1 f(s) s^{\frac{m}{n}-1}\,ds\right) \le \varrho_D(f), \qquad f \in \mathfrak{M}_+(0,1);$$

further, ϱ_D is essentially the smallest such r.i. norm.

The results are demonstrated with plenty of examples, to name just one kind, let us formulate the results for Orlicz spaces.

Theorem 6.11. *Assume that A is a Young function, let $\varrho_D = \varrho_A$, and assume that*

$$S_{\frac{n}{m}} : L_{\tilde{A}} \to L_{\tilde{A}}. \tag{6.15}$$

Then, the optimal r.i. norm ϱ_R such that (6.10) *holds, satisfies, for every $f \in \mathfrak{M}_+(0,1)$,*

$$\varrho_R(f) \approx \int_0^1 f^*(t)\,dt + \varrho_A\left(t^{-1} \int_t^{2t} s^{-\frac{m}{n}} \left[f^{**}(s) - f^*(s)\right]\,ds\right). \tag{6.16}$$

Theorem 6.12. *Assume that A is a Young function and let $\varrho_R = \varrho_A$. Assume that there exists a $k > 0$ such that the complementary function $\tilde{A}$ of A satisfies*

$$\int_{kt}^\infty \frac{\tilde{A}(s)}{s^{\frac{n}{m}+1}}\,ds \le \frac{\tilde{A}(t)}{t^{\frac{n}{m}}}, \quad t \in (1,\infty). \tag{6.17}$$

Then, the optimal r.i. norm ϱ_D such that (6.10) *holds, satisfies, for every $f \in \mathfrak{M}_+(0,1)$,*

$$\varrho_D(f) \approx \varrho_A\left(\int_t^1 f^*(s) s^{-\frac{m}{n}}\,ds\right). \tag{6.18}$$

7. Sobolev embeddings into BMO and VMO, Hölder, Campanato and Morrey spaces

Finally we shall present results on Sobolev embeddings into spaces with controlled pointwise and integral oscillation.

We start with the space BMO of functions having *bounded mean oscillation*, and the space VMO of functions having *vanishing mean oscillation.*

These spaces proved to be particularly useful in various areas of analysis, especially harmonic analysis and interpolation theory and the Sobolev embeddings with these spaces as ranges are quite useful also in the theory of partial differential equations (see e.g. [19], [18] or [16].

The space BMO has been successfully used as an interpolation substitute for L^∞ where L^∞ does not work (for example, in the interpolation of Hilbert transform and related singular integrals).

For the sake of simplicity, we restrict our results to the case when the underlying domain is a cube in $\mathbb{R}^n$.

Definition 7.1. Let Q be a cube in $\mathbb{R}^n$. The space BMO(Q) is the class of integrable functions f on Q such that

$$||f||_{\star,Q} = \sup_{Q'\subset Q} \frac{1}{|Q'|} \int_{Q'} |f(x) - f_{Q'}|\, dx < \infty,$$

where $f_{Q'} = |Q'|^{-1} \int_{Q'} f$, and the supremum is extended over all subcubes Q' of Q.

We say that f belongs to VMO(Q), if

$$\lim_{s\to 0_+} \rho_f(s) = 0, \tag{7.1}$$

where

$$\rho_f(s) = \sup_{|Q'|\leq s} \frac{1}{|Q'|} \int_{Q'} |f(x) - f_{Q'}|\, dx. \tag{7.2}$$

Remark 7.2. Obviously,

$$L^\infty \subset \mathrm{BMO}, \quad \mathrm{VMO} \subset \mathrm{BMO}. \tag{7.3}$$

Considering the function $\log|x|$ near the origin, we can show that $L^\infty \neq$ BMO and VMO $\neq$ BMO. Moreover, L^∞ and VMO are incomparable, as is demonstrated with the functions $\sin(\log|x|) \in L^\infty\setminus$ VMO and $\sqrt{\log|x|} \in$ VMO $\setminus L^\infty$.

The next three theorems give a complete answer to the question of optimal r.i. domain for L^∞, BMO and VMO (recall that for the latter we already know the answer) plus another characterizing condition.

Theorem 7.3. *The following are equivalent:*

(i) $$W_0^1(X) \hookrightarrow L^\infty;$$

(ii)
$$\|r^{-1/n'}\|_{X'} < \infty;$$

(iii)
$$X \hookrightarrow L^{n,1}.$$

Theorem 7.4. *The following are equivalent:*

(i)
$$W_0^1(X) \hookrightarrow \mathrm{BMO};$$

(ii)
$$\sup_{0<s<1} \frac{1}{s}\|r^{1/n}\chi_{(0,s)}(r)\|_{X'} < \infty;$$

(iii)
$$X \hookrightarrow L^{n,\infty}.$$

Theorem 7.5. *The following are equivalent:*

(i)
$$\lim_{s\to 0_+} \sup_{\|\nabla u\|_X \le 1} \varrho_u(s) = 0;$$

(ii)
$$\lim_{s\to 0_+} \frac{1}{s}\|r^{1/n}\chi_{(0,s)}(r)\|_{X'} = 0;$$

(iii)
$$X \subset (L^{n,\infty})_a,$$

where for a Banach function space X we denote by X_a its subspace of those functions that have in X absolutely continuous norm.

In particular, the optimal range for embedding into BMO is the Lorentz space $L^{n,\infty}$. It is of interest to compare this to the fact that the optimal range for embedding into L^∞ is the Lorentz space $L^{n,1}$

Let us finish with the result on optimality of Orlicz domain in the Sobolev embeddings into such spaces.

Theorem 7.6. *Let A be a Young function. Then*

$$W_0^1(L_A) \hookrightarrow \mathrm{BMO} \qquad \text{iff} \qquad \int_0^t \widetilde{A}(s)\,ds \le Ct^{n'+1}$$

and

$$W_0^1(L_A) \hookrightarrow L^\infty \qquad \text{iff} \qquad \int_1^\infty \widetilde{A}(s)s^{-n'-1}\,ds < \infty.$$

Theorem 7.7. (i) *The space L^n is the largest Orlicz space L_A such that $W_0^1(L_A) \hookrightarrow L^\infty$.*

(ii) *There does not exist any largest Orlicz space L_A such that $W_0^1(L_A) \hookrightarrow$ BMO.*

Now let us turn to general Hölder, Campanato and Morrey spaces.

A classical result due to Morrey states that if $p > n$, then any function from $W^{1,p}(Q)$ equals a.e. a Hölder continuous function with exponent $1 - \frac{p}{n}$. Precisely, the embedding

$$W^{1,p} \hookrightarrow C^{0,\alpha}, \quad \alpha = 1 - \tfrac{p}{n}, \tag{7.4}$$

holds, where $C^{0,\alpha}(Q)$ denotes the space of Hölder continuous functions with exponent $\alpha \in (0,1]$ endowed with the seminorm

$$\|f\|_{C^{0,\alpha}} = \sup_{\substack{x,y\in Q\\ x\neq y}} \frac{|f(x)-f(y)|}{|x-y|^\alpha}.$$

More generally, given a continuous function $\varphi : (0,\infty) \to (0,\infty)$, which will be referred to as an *admissible function*, one can consider the space $C^{0,\varphi}(Q)$ equipped with the seminorm

$$\|f\|_{C^{0,\varphi}} = \sup_{\substack{x,y\in Q\\ x\neq y}} \frac{|f(x)-f(y)|}{\varphi(|x-y|)}. \tag{7.5}$$

Obviously, $C^{0,\varphi}(Q)$ contains uniformly continuous functions if and only if

$$\lim_{t\to 0+} \varphi(t) = 0.$$

Hölder type spaces are a basic tool in various areas of analysis, including, for instance, theory of regularity in the calculus of variations and in partial differential equations. In these and other applications, however, one is often forced to to work with related function spaces defined in terms of integral, rather than pointwise, oscillation. These are the spaces of Campanato and Morrey type. In analogy with (7.5) (see [34]), given an admissible function φ, the Campanato space $L^C_\varphi(Q)$ is defined as the space of all real-valued measurable functions f on Q for which the seminorm

$$\|f\|_{L^C_\varphi} = \sup_{Q'\subset Q} \frac{1}{|Q'|\varphi(|Q'|^{\frac{1}{n}})} \int_{Q'} |f(x) - f_{Q'}|\, dx$$

is finite, where the supremum is taken over all subcubes Q' of Q with sides parallel to those of Q and $f_{Q'} = |Q'|^{-1} \int_{Q'} f(y)\, dy$, the mean value of f over Q'. Similarly, the Morrey space $L^M_\varphi(Q)$ is defined as the space of all functions f as above such that the norm

$$\|f\|_{L^M_\varphi} = \sup_{Q'\subset Q} \frac{1}{|Q'|\varphi(|Q'|^{\frac{1}{n}})} \int_{Q'} |f(x)|\, dx$$

is finite. Here, $|E|$ denotes the Lebesgue measure of a measurable set $E \subset \mathbb{R}^n$.

In the case when $\varphi(t) = t^\alpha$ with appropriate $\alpha \in \mathbb{R}$, the spaces $L^C_\varphi(Q)$ and $L^M_\varphi(Q)$ coincide with the classical Campanato and Morrey spaces and will be denoted simply by $L^C_\alpha(Q)$ and $L^M_\alpha(Q)$, respectively. The importance of Campanato spaces in the theory of regularity stems from the fact that they overlap with Hölder spaces. Indeed, a basic result (see for example [26]), tells us that $L^C_\alpha = C^{0,\alpha}$ if $0 < \alpha \le 1$. The overlapping of Campanato and Morrey spaces is also nonempty, since $L^C_\alpha = L^M_\alpha$ if $-n \le \alpha < 0$. In the borderline case when $\alpha = 0$, $L^M_0 = L^\infty$, whereas $L^C_0 = \mathrm{BMO}$, the space of functions of bounded mean oscillation over Q (see e.g. [26] for a comprehensive exposition of these spaces).

AUTHOR'S PUBLICATIONS RELATED TO THE DISSERTATION

[1] M. Carro, L. Pick, J. Soria and V. Stepanov, *On embeddings between classical Lorentz spaces*, Math. Ineq. Appl **4** (2001), 397–428.

[2] A. Cianchi, R. Kerman, B. Opic and L. Pick, *A sharp rearrangement inequality for fractional maximal operator*, Studia Math **138** (2000), 277–284.

[3] A. Cianchi and L. Pick, *Sobolev embeddings into BMO, VMO, and L^∞*, Ark. Mat **36** (1998), 317–340.

[4] A. Cianchi and L. Pick, *Sobolev embeddings into spaces of Campanato, Morrey and Hölder type*, J. Math. Anal. Appl. **282** (2003), 128–150.

[5] D.E. Edmunds, R. Kerman and L. Pick, *Optimal Sobolev embeddings involving rearrangement-invariant quasinorms*, J. Funct. Anal. **170** (2000), 307–355.

[6] W.D. Evans, B. Opic and L. Pick, *Interpolation of operators on scales of generalized Lorentz-Zygmund spaces*, Math. Nachr. **182** (1996), 127–181.

[7] A. Gogatishvili and L. Pick, *Duality principles and reduction theorems*, Math. Ineq. Appl **3** (2000), 539–558.

[8] A. Gogatishvili and L. Pick, *Discretization and anti-discretization of rearrangement-invariant norms*, Publ. Mat. **47** (2003), 311–358.

[9] J. Malý and L. Pick, *An elementary proof of sharp Sobolev embeddings*, accepted for publication to Proc. Amer. Math. Soc.

[10] B. Opic and L. Pick, *On generalized Lorentz-Zygmund spaces*, Math. Ineq. Appl **2** (1999), 391–467.

[11] L. Pick, *Optimal Sobolev embeddings*, Nonlinear Analysis, Function Spaces and Applications 6, Proceedings of the 6th International Spring School held in Prague, Czech Republic, May-June 1998, M. Krbec and A. Kufner (eds.), Olympia Press, Prague, 1999, 156–199.

[12] L. Pick, *Supremum operators and optimal Sobolev inequalities*, Function Spaces, Differential Operators and Nonlinear Analysis. Proceedings of the Spring School held in Syöte Centre, Pudasjärvi (Northern Finland), June 1999, V. Mustonen and J. Rákosník (eds.), Mathematical Institute AS CR, Prague, 2000, 207–219.

OTHER REFERENCES

[13] D.R. Adams, *A sharp inequality of J. Moser for higher order derivatives*, Annals of Math. **128** (1988), 385–398.

[14] C. Bennett and K. Rudnick, *On Lorentz-Zygmund spaces*, Dissert. Math **175** (1980), 1–72.

[15] C. Bennett and R. Sharpley, *Interpolation of Operators*, Pure and Applied Mathematics Vol. 129, Academic Press, Boston 1988.

[16] M. Bramanti and C. Cerutti, *$W^{1,2}_p$ solvability for the Cauchy–Dirichlet problem for parabolic equations with VMO coefficients*, Comm. Part. Diff. Eq **18** (1993), 1735–1763.

[17] H. Brézis and S. Wainger, *A note on limiting cases of Sobolev embeddings and convolution inequalities*, Comm. Partial Diff. Eq. **5** (1980), 773–789.

[18] F. Chiarenza , *L^p-regularity for systems of PDE's with coefficients in VMO*, Nonlinear Analysis, Function Spaces and Applications Vol. **5**, Prometheus, Prague 1995, 1–32.

[19] F. Chiarenza, M. Frasca and P. Longo, *$W^{2,p}$-solvability of the Dirichlet problem for non divergence elliptic equations with VMO coefficients*, Trans. Amer. Math. Soc **330** (1993), 841–853.
[20] A. Cianchi, *A sharp embedding theorem for Orlicz–Sobolev spaces*, Indiana Univ. Math. J **45** (1996), 39–65.
[21] M. Cwikel and E. Pustylnik, *Sobolev type embeddings in the limiting case*, J. Fourier Anal. Appl **4** (1998), 433–446.
[22] D.E. Edmunds, P. Gurka and B. Opic, *Double exponential integrability of convolution operators in generalized Lorentz–Zygmund spaces*, Indiana Univ. Math. J. **44** (1995), 19–43.
[23] N. Fusco, P.L. Lions and C. Sbordone, *Sobolev imbedding theorems in borderline cases*, Proc. Amer. Math. Soc **124** (1996), 561–565.
[24] K. Hansson, *Imbedding theorems of Sobolev type in potential theory*, Math. Scand. **45** (1979), 77–102.
[25] J.A. Hempel,G.R. Morris and N.S. Trudinger, *On the sharpness of a limiting case of the Sobolev imbedding theorem*, Bull. Australian Math. Soc **3** (1970), 369–373.
[26] A. Kufner, O. John and S. Fučík, *Function spaces*, Noordhoff, Leyden, Academia, Praha, 1977.
[27] V.G. Maz'ya, *A theorem on the multidimensional Schrödinger operator* (Russian), Izv. Akad. Nauk. **28** (1964), 1145-1172.
[28] V.G. Maz'ya, *Sobolev Spaces*, Springer, Berlin 1985.
[29] J. Peetre, *Espaces d'interpolation et théorème de Soboleff*, Ann. Inst. Fourier **16** (1966), 279–317.
[30] S.I. Pokhozhaev, *On eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$*, Dokl. Akad. Nauk SSR **165** (1965), 36–39.
[31] E. Sawyer, *Boundedness of classical operators on classical Lorentz spaces*, Studia Math. **96** (1990), 145–158.
[32] R. Sharpley, *Counterexamples for classical operators in Lorentz–Zygmund spaces*, Studia Math. **68** (1980), 141–158.
[33] S.L. Sobolev, *Applications of Functional Analysis in Mathematical Physics*, Transl. of Mathem. Monographs, American Math. Soc., Providence, R.I. **7**, 1963.
[34] S. Spanne, *Some function spaces defined using the mean oscillation over cubes*, Ann. Scu. Norm. Sup. Pisa **19** (1965), 593-608.
[35] V.D. Stepanov, *The weighted Hardy's inequality for nonincreasing functions*, Trans. Amer. Math. Soc. **338** (1993), 173–186.
[36] R.S. Strichartz, *A note on Trudinger's extension of Sobolev's inequality*, Indiana Univ. Math. J **21** (1972), 841–842.
[37] N.S. Trudinger, *On imbeddings into Orlicz spaces and some applications*, J. Math. Mech **17** (1967), 473–483.
[38] V.I. Yudovich, *Some estimates connected with integral operators and with solutions of elliptic equations*, Soviet Math. Doklady **2** (1961), 746–749.
[39] W.P. Ziemer, *Weakly Differentiable Functions*, Springer, New York etc. 1989.
[40] A. Zygmund, *Trigonometric Series*, Cambridge University Press, Cambridge.

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