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Optimality Conditions for Time Scale Variational Problems

Komise pro obhajoby doktorských disertací v oboru
matematická analýza a příbuzné obory

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Contents

1. INTRODUCTION

The primary intention of this work is to present a thorough study of the *first* and *second order optimality conditions* for variable endpoints calculus of variations and optimal control problems on time scales. These conditions include the derivation of the time scale Euler–Lagrange equation or the weak maximum principle and the transversality condition (through the first variation), and the second variation or the accessory problem. Both necessary and sufficient optimality conditions are considered for the time scale calculus of variations problem, and necessary optimality conditions are derived for the time scale control problem.

The definiteness of the quadratic functional arising as the second variation is the key concept in the second order optimality conditions. While necessary conditions are expressed in terms of the nonnegativity of the second variation, sufficient optimality conditions are phrased in terms of its coercivity or positivity. Both the nonnegativity and positivity of the second variation are also characterized by a number of equivalent conditions, including conjugate points, properties of conjoined bases of the associated Jacobi equation, and the solutions of the corresponding Riccati matrix equation. Furthermore, we connect the Jacobi equation or system with the time scale symplectic systems, and present characterizations of the nonnegativity and positivity of the quadratic functionals corresponding to the time scale symplectic system without assuming any normality. These results are in terms of a natural conjoined basis or in terms of time scale explicit and implicit Riccati matrix equations.

Being a text on *time scales*, the presented results *unify* and *extend* the corresponding results from the continuous and discrete time theories. Moreover, some of the new time scale results are new even for the special cases of the *continuous* and *discrete time.*

The dissertation contains research results obtained by the author in the years 2003–2007, mainly together with his long term collaborator Vera Zeidan from the Michigan State University (East Lansing, Michigan, USA). More specifically, the eight papers listed on page 33 contribute to this work in a major way, although it is indispensable that they result from author's previous work on the discrete and time scale theories (see the list of publications of the author on pg. 34).

The *highlights* of the presented work are the following:

● We present o *complete study*, i.e., the necessity and sufficiency, of the *calculus of variations* problems on time scales with *general* (i.e., jointly varying)

endpoints. We present sufficiency criteria either in terms of the coercivity of the second variation (traditional approach), or in terms of the positivity of the second variation (novel approach).

• The result on the equivalence between the coercivity and positivity of the quadratic functionals in the calculus of variations is *new* even for the special case of the *continuous time* problems with *jointly varying* endpoints.

• For the first time there is a rigorous study of the *optimal control* problems *on time scales.* Moreover, we consider such optimal control problems with *equality control constraints* and with *jointly varying* endpoints.

• We present a relatively simple proof of the time scale *weak maximum principle*, even for control problems with jointly varying endpoints. For this we derive a "control generalization" of the Dubois-Reymond lemma known from the calculus of variations. To our knowledge, this proof is *new* even for the *continuous time* control problems with variable (i.e., separable or jointly varying) endpoints.

• We make a *connection* of the time scale *control problems* with the theory of *time scale symplectic systems.* More precisely, we show that the *Jacobi systems* for control problems on time scales lead naturally to time scale symplectic systems.

• We present characterizations of the nonnegativity and positivity of the quadratic functional corresponding to the time scale symplectic system *without assuming any normality.* These results are in terms of a natural conjoined basis or in terms of time scale explicit and implicit Riccati matrix equations. The results on implicit Riccati equations are *new* even for the *continuous time* linear Hamiltonian systems.

• We establish the *embedding theorem* on time scales without any restriction on the length of the time scale interval $[a, b]_{\mathbb{T}}$, which guarantees the existence and continuous dependence of solutions of a time scale dynamic equation on the initial conditions and parameters near an already existing solution on $[a, b]_{\mathbb{T}}$.

At the end of of each chapter there is a section entitled "Notes" with comments to the literature. Moreover, in each of Chapters 3–5 we include a section on open problems and perspectives outlining possible directions of future research.

2. MOTIVATION

2.1. **Motivation for optimality conditions.** In the calculus of variations and optimal control problems we are looking for a certain trajectory or control function, which gives an optimal, i.e., minimal or maximal, value of some given quantity. Such quantities are usually given in terms of an integral functional on a set of functions, which are feasible (trajectories) for the given problem and satisfy required initial and final boundary conditions.

Optimality conditions in such variational problems are of the first and second order. First order optimality conditions stem from the first variation of the problem and they are usually formulated in terms of the Euler–Lagrange equation or the adjoint equation (through the weak maximum principle), and the transversality conditions. Second order optimality conditions are connected to the definiteness (that is, the nonnegativity and coercivity) of the quadratic functional of the second variation. These conditions are necessary (nonnegativity) and sufficient (coercivity, positivity) conditions for the optimality in the original optimization problem. While the first order conditions serve as techniques to provide a list of possible candidates, the second order conditions are mechanisms used to throw out from the candidate list the nonoptimal candidates.

It is known that the above mentioned optimality conditions in terms of the second variation or quadratic functionals can be formulated in an equivalent way, namely via conjugate and coupled points, or properties of solutions of the corresponding Jacobi and Riccati equations. The focus of this work is not only deriving the optimality conditions in terms of the first and second variations, but also characterizing the definiteness of the second variation through the above equivalent concepts. Moreover, this work is focused on generalizing these results to time scale symplectic systems, which are according to latest results the most general objects, for which the qualitative theory of differential, difference, and/or dynamic equations reasonably works.

2.2. **Motivation for time scales.** Originating in the PhD thesis [44] of Stefan Hilger, the *calculus on time scales* was invented as a tool for the unification of the traditional differential and difference calculi. However, this *unification* aspect was soon supplemented by the *extension* and *generalization* features, since they provide much wider range of possible applications. On the other hand, having general results on time scales can yield new results even for the special cases of the continuous or discrete time. For example, in Chapter 5 we

obtained new results for the classical linear Hamiltonian differential systems from the general results on time scale symplectic systems.

A brief history of the time scale calculus written in a popular way together with pointing out some possible biological applications was published in [88].

From the two original classical time scales $\mathbb{R}$ and $\mathbb{Z}$ representing the continuous and purely discrete time, it was discovered that the quantum calculus or q-difference equations, whose base time set is the set $q^{\mathbb{N}}$ of powers of a given number q, is a special time scale. Such quantum calculus is frequently used in the study of physics problems, see e.g. [72].

A time scale of the form of a union of disjoint closed real intervals constitutes the appropriate background for the study of population models, see e.g. [23, pp. 15–16], [24, Section 2.4], and [21, 22].

Variational problems over time scales include a large spectrum of other problems where the time scale $\mathbb{T}$ could be, for example, a union of disjoint connected time-intervals with some discrete instances. Such problems are known also under the name "hybrid" since, as stated in [78], they are a "mixing of two fundamentally different types of problems". Here we are mixing the discrete- and the continuous-time problems. In this work we study time scale control problems, whose paradigm is particularly useful in modelling applications where high-level decision making is used to supervise process behavior. Therefore, these systems appear in many important applications stemming for instance from aerospace and power systems, where the system has to switch between various setpoints or operational modes to extend its effective operating range. Hybrid systems embrace a diverse set of applications from engineering to biology, see e.g. [77, 78] and the references therein. Over the last ten years there has been considerable activity in this area. The mathematical description of these systems can be characterized by *impulsive* differential equations. In the case where the resetting events of the equations are defined by a prescribed sequence of times that are independent of the state of the system, the system of impulsive equations is known as *time-dependent*. See for example [10, 29, 39, 40, 77]. In those references the time interval $[t_0, t_f]$ is connected and the state function x is *discontinuous* but left continuous at the resetting instances $\{t_1, t_2, \dots\}$. The impulsive differential equations have been studied there by splitting them into continuous-time and discrete-time systems. However, one can find a *time scale* of the form

$$\mathbb{T} = [t_0, t_1] \cup [t_1 + \varepsilon_1, t_2] \cup [t_2 + \varepsilon_2, t_3] \cup \cdots \subseteq [t_0, t_f + \varepsilon]$$

such that the system of impulsive differential equations is equivalent to a *time scale* system over $\mathbb{T}$. Therefore, time-dependent impulsive control systems can be viewed as special cases of the time scale control systems.

Time scale calculus has a promising potential in economic research. An application of a time scale variational problem to economics is presented in [8, 90]. Moreover, one of the current leading experts in dynamic equations on time scales, Martin Bohner, studies the applicability of time scales in the economics and finance with his undergoing NSF grant [15].

2.3. **Brief preliminaries about time scales.** The elements of the time scales theory can be found in [23]. The functions σ and ρ are respectively the forward and backward jump operators on $[a, b]_{\mathbb{T}}$ and $x^{\sigma}(t) := x(\sigma(t))$. The time scale Δ-derivative and the corresponding integral are denoted, respectively, by $x^{\Delta}(t)$ and $\int_a^b F(t)\,\Delta t$. In the special cases of the continuous and discrete times, $x^{\Delta}(t)$ reduces to the standard derivative $\dot{x}(t)$ and forward difference $\Delta x(t)$, the integral to $\int_a^b F(t)\,dt$ and $\sum_{k=0}^N F(k)$, and the jump operators to $\sigma(t) = t = \rho(t)$ and $\sigma(k) = k+1$, $\rho(k) = k-1$, respectively. The graininess function on $[a, b]_{\mathbb{T}}$ is $\mu(t) := \sigma(t) - t$. We shall use the notions of piecewise rd-continuous ($\mathrm{C}_{\mathrm{prd}}$) and piecewise rd-continuously Δ-differentiable ($\mathrm{C}^1_{\mathrm{prd}}$) functions defined in the text.

Whenever $x^{\Delta}(t)$ exists, the following formula holds:

$$x^{\sigma}(t) = x(t) + \mu(t)\,x^{\Delta}(t).$$

The product rule on $[a, \rho(b)]_{\mathbb{T}}$ is given by

$$[x(t)\,y(t)]^{\Delta} = x^{\Delta}(t)\,y(t) + x^{\sigma}(t)\,y^{\Delta}(t) = x^{\Delta}(t)\,y^{\sigma}(t) + x(t)\,y^{\Delta}(t),$$

and the integration by parts formula (corresponding to the middle expression above) is

$$\int_a^b x^{\Delta}(t)\,y(t)\,\Delta t = x(t)\,y(t)\,\big|_a^b - \int_a^b x^{\sigma}(t)\,y^{\Delta}(t)\,\Delta t,$$

3. CALCULUS OF VARIATIONS ON TIME SCALES

3.1. **Problem statement.** The time scales calculus of variations problem under consideration has the form

$$\text{minimize} \quad \mathcal{F}(y) := K\big(y(a), y(b)\big) + \int_a^b L\big(t, y^{\sigma}(t), y^{\Delta}(t)\big)\,\Delta t \qquad \text{(P)}$$

over all $y \in \mathrm{C}^1_{\mathrm{prd}}[a,b]_{\mathbb{T}}$ satisfying the general boundary condition

$$\varphi\big(y(a), y(b)\big) = 0, \tag{3.1}$$

where

$$L : [a, \rho(b)]_{\mathbb{T}} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, \qquad K : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R},$$
$$\varphi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^r, \ r \leq 2n.$$

A function $y \in \mathrm{C}^1_{\mathrm{prd}}$ is said to be *feasible* if it satisfies constraint (3.1). The *norm* in $\mathrm{C}^1_{\mathrm{prd}}$ is defined as

$$\|y\|_{\mathrm{C}^1_{\mathrm{prd}}} := \sup_{t \in [a, \rho(b)]} \big| \big(y^\sigma(t), y^\Delta(t)\big) \big|.$$

We say that $\hat{y}$ is a *weak local minimum* for (P) if there exists $\varepsilon_0 > 0$ such that $\mathcal{F}(y) \geq \mathcal{F}(\hat{y})$ for all feasible $y \in \mathrm{C}^1_{\mathrm{prd}}$ with $\|y - \hat{y}\|_{\mathrm{C}^1_{\mathrm{prd}}} < \varepsilon_0$.

The main results of this chapter are summarized in the following. In Theorem 3.1, we prove necessary conditions for the weak local optimality of a feasible $\hat{y}$. That is, under certain assumptions, the weak local optimality of $\hat{y}$ in (P) implies that $\hat{y}$ satisfies the corresponding Euler–Lagrange equation, the transversality condition, the first variation at $\hat{y}$ is zero, the second variation at $\hat{y}$ is nonnegative, and the time scale Legendre condition holds. Conversely, we show in Theorem 3.2 that if a feasible $\hat{y}$ satisfies the Euler–Lagrange equation and the transversality condition, and if the second variation at $\hat{y}$ is coercive, then $\hat{y}$ is a strict weak local minimum in (P). Alternatively, we replace in our second sufficiency result in Theorem 3.3 the assumption of the coercivity of the second variation by its positivity together with the time scale version of the strengthened Legendre condition. This allows to derive sufficient optimality conditions in terms of any equivalent condition to the positivity of the second variation, namely in terms of the time scale Riccati matrix equation.

The *second variation* at $\hat{y}$ along η is defined to be the quadratic functional

$$\mathcal{F}''(\hat{y}; \eta) := \begin{pmatrix} \eta(a) \\ \eta(b) \end{pmatrix}^T \Gamma \begin{pmatrix} \eta(a) \\ \eta(b) \end{pmatrix} + \mathcal{F}_0''(\hat{y}; \eta) \tag{3.2}$$

where

$$\mathcal{F}_0''(\hat{y}; \eta) := \int_a^b \big\{ (\eta^\sigma)^T P\, \eta^\sigma + 2\, (\eta^\sigma)^T Q\, \eta^\Delta + (\eta^\Delta)^T R\, \eta^\Delta \big\}(t)\, \Delta t,$$

and where the coefficients are

$$\Gamma := \nabla^2 K^T \big(\hat{y}(a), \hat{y}(b)\big) + \gamma^T \nabla^2 \varphi^T \big(\hat{y}(a), \hat{y}(b)\big), \tag{3.3}$$

$$P(t) := \hat{L}_{yy}(t), \quad Q(t) := \hat{L}_{yv}(t), \quad R(t) := \hat{L}_{vv}(t) \tag{3.4}$$

and γ is some vector in $\mathbb{R}^r$ specified through the transversality condition (3.7) below.

3.2. **Optimality conditions.** We derive among others the following main results. Here the assumptions (A1) and (A2) represent certain first and second order differentiability conditions on the data in problem (P). In particular, these assumptions contain the requirement that the matrix M defined in (3.5) below has full rank.

Theorem 3.1 (Necessary optimality conditions). *Let $\hat{y}$ be feasible, assume* (A1), *and let M be defined by*

$$M := \nabla\varphi\big(\hat{y}(a), \hat{y}(b)\big) \in \mathbb{R}^{r \times 2n}. \tag{3.5}$$

If $\hat{y}$ is a weak local minimum for (P), *then there exists a vector $c \in \mathbb{R}^n$ such that the following conditions hold*

   (i) *for all $t \in [a, \rho(b)]_{\mathbb{T}}$, the Euler–Lagrange equation (integral form)*

$$\hat{L}_v(t) = \int_a^t \hat{L}_y(\tau)\, \Delta\tau + c^T, \tag{3.6}$$

  (ii) *for some vector $\gamma \in \mathbb{R}^r$, the transversality condition*

$$\Big(\hat{L}_v(a), \; -\hat{L}_v(b)\Big) = \nabla K\big(\hat{y}(a), \hat{y}(b)\big) + \gamma^T M. \tag{3.7}$$

If b is left-scattered, the quantity $\hat{L}_v(b)$ in (3.7) *is defined by the formula*

$$\hat{L}_v(b) := \int_a^b \hat{L}_y(t)\, \Delta t + c^T.$$

In addition, if we assume (A2), *then the following conditions hold*

 (iii) *the second variation $\mathcal{F}''(\hat{y}; \cdot)$ is nonnegative, i.e., $\mathcal{F}''(\hat{y}; \eta) \geq 0$ for all $\eta \in \mathrm{C}_{\mathrm{prd}}^1$ with $M\begin{pmatrix} \eta(a) \\ \eta(b) \end{pmatrix} = 0$,*

 (iv) *the time scale Legendre condition*

$$R(t^{\pm}) \geq 0, \quad \text{for all dense points } t \in [\sigma(a), \rho(b)]_{\mathbb{T}}.$$

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The second variation $\mathcal{F}''(\hat{y}; \cdot)$ is said to be *coercive* if there exists $\alpha > 0$ such that

$$\mathcal{F}''(\hat{y}; \eta) \geq \alpha \left\{ |\eta(a)|^2 + |\eta(b)|^2 + \int_a^b |\eta^\Delta(t)|^2 \, \Delta t \right\} \tag{3.8}$$

for all $\eta \in \mathrm{C}^1_{\mathrm{prd}}$ with $M \begin{pmatrix} \eta(a) \\ \eta(b) \end{pmatrix} = 0$.

Theorem 3.2 (Sufficiency via coercivity)**.** *Let $\hat{y} \in \mathrm{C}^1_{\mathrm{prd}}$ be feasible and suppose that* (A2) *holds. If $\hat{y}$ satisfies the Euler–Lagrange equation* (3.6) *and, for some $\gamma \in \mathbb{R}^r$, the transversality condition* (3.7)*, and if $\mathcal{F}''(\hat{y}; \cdot)$ is coercive, i.e., there exists $\alpha > 0$ such that* (3.8) *holds, then $\hat{y}$ is a strict weak local minimum for* (P)*. More precisely, there exists $\varepsilon_0 > 0$ such that for all feasible $y \in \mathrm{C}^1_{\mathrm{prd}}$ with $\|y - \hat{y}\|_{\mathrm{C}^1_{\mathrm{prd}}} < \varepsilon_0$ we have*

$$\mathcal{F}(y) - \mathcal{F}(\hat{y}) \geq \frac{\alpha}{8} \left\{ |y(a) - \hat{y}(a)|^2 + |y(b) - \hat{y}(b)|^2 + \int_a^b |y^\Delta(t) - \hat{y}^\Delta(t)|^2 \, \Delta t \right\}.$$

Theorem 3.3 (Sufficiency via positivity)**.** *Let a function $\hat{y} \in \mathrm{C}^1_{\mathrm{prd}}$ be feasible, suppose that* (A2) *holds, and define the $n \times n$ matrices $P(t)$, $Q(t)$, $R(t)$, and M by formulas* (3.4) *and* (3.5)*. Suppose that, for some vectors $c \in \mathbb{R}^n$ and $\gamma \in \mathbb{R}^r$, $r := \mathrm{rank}\, M$, the function $\hat{y}$ satisfies the Euler–Lagrange equation* (3.6)*, the transversality condition* (3.7)*, the invertibility condition*

$R(t)$ *and* $S(t) := R(t) + \mu(t)\, Q^T(t)$ *are invertible for all $t \in [a, \rho(b)]_{\mathbb{T}}$,*

and the strengthened Legendre condition

$$R(t^\pm) \geq \alpha I \quad \textit{for all dense } t \in [\sigma(a), \rho(b)]_{\mathbb{T}},$$

for some $\alpha > 0$. Furthermore, assume that the functional $\mathcal{F}''(\hat{y}; \eta)$ defined in (3.2) *is positive definite over $\mathcal{M} \begin{pmatrix} \eta(a) \\ \eta(b) \end{pmatrix} = 0$, where the $2n \times 2n$ matrices $\mathcal{M}$ and Γ are defined by $\mathcal{M} := M^T (MM^T)^{-1} M$ and* (3.3)*. Then $\hat{y}$ is a strict weak local minimum for* (P)*.*

The results in Theorems 3.1 and 3.2 are known in the special case of continuous and discrete time calculus of variations problems. On the other hand, Theorem 3.3 is new for the continuous time problems with jointly varying endpoints.

Furthermore, we develop the theory of conjugate points for the quadratic functional $\mathcal{J}(\eta) := \mathcal{F}''(\hat{y}; \eta)$ with the zero right endpoint. We present characterizations of the nonnegativity and positivity of $\mathcal{J}$ in terms of the nonexistence of such conjugate points, the "natural" conjoined basis of the Jacobi equation

$$\left[R(t)\,\eta^{\Delta} + Q^T(t)\,\eta^{\sigma}\right]^{\Delta} = P(t)\,\eta^{\sigma} + Q(t)\,\eta^{\Delta}, \tag{J}$$

and the solvability of the time scale Riccati matrix equation

$$W^{\Delta} = P(t) - [W - Q(t)]\,[R(t) + \mu(t)\,W]^{-1}\,[W - Q^T(t)]. \tag{R}$$

4. OPTIMAL CONTROL ON TIME SCALES

4.1. **Problem statement.** Consider the nonlinear time scale optimal control problem:

$$\text{minimize} \quad \mathcal{F}(x, u) := K\big(x(a), x(b)\big) + \int_a^b L\big(t, x^{\sigma}(t), u(t)\big)\,\Delta t, \tag{C^{σ}}$$

subject to $x \in \mathrm{C}^1_{\mathrm{prd}}[a, b]_{\mathbb{T}}$ and $u \in \mathrm{C}_{\mathrm{prd}}[a, \rho(b)]_{\mathbb{T}}$ (piecewise rd-continuous functions) satisfying

$$x^{\Delta}(t) = f\big(t, x^{\sigma}(t), u(t)\big), \quad t \in [a, \rho(b)]_{\mathbb{T}}, \tag{4.1}$$

$$\psi\big(t, u(t)\big) = 0, \quad t \in [a, \rho(b)]_{\mathbb{T}}, \tag{4.2}$$

$$\varphi\big(x(a), x(b)\big) = 0. \tag{4.3}$$

The data satisfy

$$\begin{aligned}
&L : [a, \rho(b)]_{\mathbb{T}} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}, \quad m \le n, &&K : \mathbb{R}^{2n} \to \mathbb{R}, \\
&f : [a, \rho(b)]_{\mathbb{T}} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n, &&\varphi : \mathbb{R}^{2n} \to \mathbb{R}^r, \quad r \le 2n, \\
&\psi : [a, \rho(b)]_{\mathbb{T}} \times \mathbb{R}^m \to \mathbb{R}^k, \quad k \le m,
\end{aligned}$$

the state $x : [a, b]_{\mathbb{T}} \to \mathbb{R}^n$, $x \in \mathrm{C}^1_{\mathrm{prd}}$, and the control $u : [a, \rho(b)]_{\mathbb{T}} \to \mathbb{R}^m$, $u \in \mathrm{C}_{\mathrm{prd}}$. The Hamiltonian corresponding to problem (C^{σ}) is

$$\mathcal{H}(t, x, u, p, \lambda, \lambda_0) := p^T f(t, x, u) + \lambda_0\,L(t, x, u) + \lambda^T \psi(t, u).$$

A pair (x, u) is said to be *feasible* if it satisfies (4.1)–(4.3). A feasible pair $(\bar{x}, \bar{u})$ is a *weak local minimum* for (C^{σ}) if there exists $\varepsilon > 0$ such that for any feasible (x, u) with $\|x - \bar{x}\|_{\mathrm{C}} < \varepsilon$ and $\|u - \bar{u}\|_{\mathrm{C}_{\mathrm{prd}}} < \varepsilon$ we have $\mathcal{F}(\bar{x}, \bar{u}) \le \mathcal{F}(x, u)$, where

$$\|u\|_{\mathrm{C}_{\mathrm{prd}}} := \sup_{t \in [a, \rho(b)]_{\mathbb{T}}} \big|u(t)\big|, \qquad \|x\|_{\mathrm{C}} := \max_{t \in [a, b]_{\mathbb{T}}} \big|x(t)\big|.$$

In this chapter we derive the first and second variations for the nonlinear time scale optimal control problem (C$^\sigma$) with control and state-endpoints equality constraints. Using the first variation, a first order necessary condition for the weak local optimality is obtained under the form of a weak maximum principle generalizing the Euler–Lagrange equation to the optimal control setting on time scales. A second order necessary condition in terms of the accessory problem is derived by using the nonnegativity of the second variation at all admissible directions. The control problem is studied under a controllability assumption, and with or without the shift in the state variable. These two forms of the problem are shown to be equivalent.

For a feasible pair $(\bar{x}, \bar{u})$ we define the $r \times 2n$ matrix M and the $k \times m$ matrices $N(t)$, $t \in [a, \rho(b)]_\mathbb{T}$, by

$$M := \nabla\varphi\big(\bar{x}(a), \bar{x}(b)\big), \quad N(t) := \nabla_u \psi\big(t, \bar{u}(t)\big). \tag{4.4}$$

Moreover, let us denote the gradients of the function f by

$$\mathcal{A}(t) := \bar{f}_x(t) := f_x\big(t, \bar{x}^\sigma(t), \bar{u}(t)\big), \quad \mathcal{B}(t) := \bar{f}_u(t) := f_u\big(t, \bar{x}^\sigma(t), \bar{u}(t)\big), \tag{4.5}$$

and define the tangent spaces $T(t)$ and the space $\mathcal{T}$ of tangent functions by

$$T(t) := \big\{v \in \mathbb{R}^m : \ N(t)\, v = 0\big\},$$

$$\mathcal{T} := \big\{v(\cdot) \in \mathrm{C}_{\mathrm{prd}}[a, \rho(b)]_\mathbb{T} : \ v(t) \in T(t) \text{ for all } t \in [a, \rho(b)]_\mathbb{T}\big\}.$$

Note that when the control constraint is not present, then $\psi : [a, \rho(b)]_\mathbb{T} \times \mathbb{R}^m \to \{0\}$ and hence, $T(t) = \mathbb{R}^m$ for all t, and in this case $\mathcal{T} = \mathrm{C}_{\mathrm{prd}}[a, \rho(b)]_\mathbb{T}$.

The linear system

$$\eta^\Delta = \mathcal{A}(t)\, \eta^\sigma + \mathcal{B}(t)\, v \tag{4.6}$$

is said to be *M-controllable over* $\mathcal{T}$ if for any vector $d \in \mathbb{R}^r$ there exist a vector $\alpha = \alpha_d \in \mathbb{R}^n$ and a function $v = v_d \in \mathcal{T}$ such that the solution $\eta(\cdot)$ of the initial value problem (4.6) with $\eta(a) = \alpha$ satisfies

$$M \begin{pmatrix} \eta(a) \\ \eta(b) \end{pmatrix} = d. \tag{4.7}$$

Problem (C$^\sigma$) is said to be *normal at* $(\bar{x}, \bar{u})$ if the matrix M has full rank and if the system

$$\left. \begin{aligned} p^\Delta = -\mathcal{A}^T(t)\, p, \quad v^T \mathcal{B}^T(t)\, p = 0, \quad \forall\, v \in T(t), \quad t \in [a, \rho(b)]_\mathbb{T}, \\ \begin{pmatrix} -p(a) \\ p(b) \end{pmatrix} = M^T \gamma, \end{aligned} \right\}$$

where $\gamma \in \mathbb{R}^r$, possesses only the trivial solution $p(\cdot) \equiv 0$ (and then also $\gamma = 0$).

4.2. **Results on controllability.** The assumptions $(A1^\sigma)$ and $(A2^\sigma)$ represent certain first and second order differentiability conditions on the data in problem (C^σ). In particular, these assumptions contain the requirement that the matrices M and $N(t)$ have full rank.

Proposition 4.1 (Controllability and normality). *Assume that the matrix M has full rank and $I - \mu(t) \mathcal{A}(t)$ is invertible on $[a, \rho(b)]_{\mathbb{T}}$. Then the linear system* (4.6) *is M-controllable over $\mathcal{T}$ if and only if the problem (C^σ) is normal at $(\bar{x}, \bar{u})$.*

Lemma 4.2 (Generalized Dubois-Reymond lemma). *Let $d \in \mathbb{R}^n$ be a vector, $\mathcal{D} \in \mathbb{R}^{r \times n}$ be a matrix, $h : [a, \rho(b)]_{\mathbb{T}} \to \mathbb{R}^l$, $h \in \mathrm{C}_{\mathrm{prd}}$, be a vector valued function, and $\mathcal{E} : [a, \rho(b)]_{\mathbb{T}} \to \mathbb{R}^{r \times l}$, $\mathcal{E} \in \mathrm{C}_{\mathrm{prd}}$, be a matrix valued function. Then*

$$d^T \alpha + \int_a^b h^T(t) \, w(t) \, \Delta t = 0, \quad \text{whenever} \quad \mathcal{D}\alpha + \int_a^b \mathcal{E}(t) \, w(t) \, \Delta t = 0$$

for $\alpha \in \mathbb{R}^n$ and $w : [a, \rho(b)]_{\mathbb{T}} \to \mathbb{R}^l$, $w \in \mathrm{C}_{\mathrm{prd}}$, if and only if there exists a vector $c \in \mathbb{R}^r$ such that

$$d = \mathcal{D}^T c \quad \text{and} \quad h(t) = \mathcal{E}^T(t) \, c \quad \text{for all } t \in [a, \rho(b)]_{\mathbb{T}}.$$

4.3. **Optimality conditions.** The following are our main results regarding necessary optimality conditions for the time scale control problem (C^σ).

Theorem 4.3 (Weak maximum principle on time scales). *Assume that $(\bar{x}, \bar{u})$ is a weak local minimum for (C^σ) such that the assumption $(A1^\sigma)$ holds. Then there exist a constant $\lambda_0 \geq 0$, a vector $\bar{\gamma} \in \mathbb{R}^r$, a function $\bar{\lambda} : [a, \rho(b)]_{\mathbb{T}} \to \mathbb{R}^k$, $\bar{\lambda} \in \mathrm{C}_{\mathrm{prd}}$, and a function $\bar{p} : [a, b]_{\mathbb{T}} \to \mathbb{R}^n$, $\bar{p} \in \mathrm{C}^1_{\mathrm{prd}}$, such that $\lambda_0 + \|\bar{p}\|_C \neq 0$ and satisfying the following conditions:*

(i) *the adjoint equation: for all $t \in [a, \rho(b)]_{\mathbb{T}}$*

$$-\bar{p}^\Delta(t) = \mathcal{A}^T(t) \, \bar{p}(t) + \lambda_0 \, \bar{L}_x^T(t), \qquad (4.8)$$

(ii) *the stationarity condition: for all $t \in [a, \rho(b)]_{\mathbb{T}}$*

$$\mathcal{B}^T(t) \, \bar{p}(t) + \lambda_0 \, \bar{L}_u^T(t) + N^T(t) \, \bar{\lambda}(t) = 0, \qquad (4.9)$$

(iii) *the transversality condition:*

$$\begin{pmatrix} -\bar{p}(a) \\ \bar{p}(b) \end{pmatrix} = \lambda_0 \, \nabla K^T \big(\bar{x}(a), \bar{x}(b) \big) + M^T \bar{\gamma}, \qquad (4.10)$$

where $\mathcal{A}(t)$ *and* $\mathcal{B}(t)$ *are defined by* (4.5). *Moreover, if the system* (4.6) *is* M*-controllable over* $\mathcal{T}$*, then we may take* $\lambda_0 = 1$ *and in this case* $\bar{\gamma}$*,* $\bar{\lambda}(\cdot)$*, and* $\bar{p}(\cdot)$ *are unique.*

The time scale weak maximum principle in Theorem 4.3 unifies the corresponding continuous time weak maximum principle, see e.g. [41, Theorem 2.1, Chapter 6], and the discrete weak maximum principle e.g. from [57, Theorem 1] or [59, Theorem 1]. Traditionally, the weak maximum principle for the continuous time control problem is derived through the *strong* Pontryagin principle, see e.g. [41], while for the discrete time setting the mathematical programming approach is commonly used, see e.g. [25, 57, 59, 79]. In the calculus of variations, whether continuous, discrete, or time scale, the Euler–Lagrange equation is usually derived through the combination of the first variation and the Dubois-Reymond Lemma, see e.g. [14, 37, 74]. For optimal control problems over the continuous time, Hestenes in [43, Theorem 4.1] derived what is equivalent to a generalized Dubois-Reymond Lemma only for the case of fixed endpoint(s). This result could be used in conjunction with [41, Section 6.7] to deduce the weak maximum principle for the fixed endpoint(s) continuous time optimal control problems. On the other hand, a Hilbert space method is used in [42, Section 8] to establish the weak maximum principle for linear-quadratic control problems with fixed endpoints. However, to our knowledge, there is no direct method to derive these results (that is, the generalized Dubois-Reymond Lemma for the control setting and the weak maximum principle) for the *variable endpoints* case.

The second variation of the functional $\mathcal{F}$ in problem (C^σ), or the *accessory problem*, at $(\bar{x}, \bar{u})$ in the direction (η, v) is defined as

$$\mathcal{F}''(\bar{x}, \bar{u}; \eta, v) := \begin{pmatrix} \eta(a) \\ \eta(b) \end{pmatrix}^T \Gamma \begin{pmatrix} \eta(a) \\ \eta(b) \end{pmatrix} + \int_a^b \begin{pmatrix} \eta^\sigma(t) \\ v(t) \end{pmatrix}^T \nabla^2_{(x,v)} \bar{\mathcal{H}}(t) \begin{pmatrix} \eta^\sigma(t) \\ v(t) \end{pmatrix} \Delta t,$$

where $\bar{\mathcal{H}}(t) := \mathcal{H}\big(t, \bar{x}^\sigma(t), \bar{u}(t), \bar{p}(t), \bar{\lambda}(t), 1\big)$ and

$$\Gamma := \nabla^2 K \big(\bar{x}(a), \bar{x}(b) \big) + \bar{\gamma}^T \nabla^2 \varphi \big(\bar{x}(a), \bar{x}(b) \big). \qquad (4.11)$$

Next we obtain a second order necessary optimality condition for (C^σ) that the second variation of $\mathcal{F}$ is nonnegative at $(\bar{x}, \bar{u})$ in the direction of all admissible pairs (η, v).

Theorem 4.4 (Second variation). *Assume that $(\bar{x}, \bar{u})$ is a weak local minimum for (C^σ), assumption $(A2^\sigma)$ holds, and the system (4.6) is M-controllable over $\mathcal{T}$. Then $\mathcal{F}''(\bar{x}, \bar{u}; \eta, v) \geq 0$ for any admissible pair (η, v), where $\bar{\gamma}$, $\bar{\lambda}(\cdot)$, and $\bar{p}(\cdot)$ satisfy the weak maximum principle (Theorem 4.3).*

We also consider the isoperimetric control problem on time scales and develop the corresponding weak maximum principle and the accessory problem.

4.4. **Control problem without shift.** There are two formulations of dynamic and variational problems over time scales. The difference basically lies in the presence of either x^σ or x in the data of the problem. The presence of x is traditionally used in the classical discrete optimal control setting, see e.g. [25, 79]. On the other hand, the case when the data depend on x^σ is now commonly used in modern discrete theory [7, 59, 62, 74] and more recently, in time scale dynamic equations, see e.g. [3, 4, 32, 35, 36, 46, 61]. The reason for this latter is that it produces qualitative properties (such as, for example, the oscillation) of the same form as those known in the continuous time setting.

Consider the optimal control problem *without* a shift on x, namely,

$$\text{minimize} \quad F(x, u) := K\big(x(a), x(b)\big) + \int_a^b g\big(t, x(t), u(t)\big)\,\Delta t, \qquad \text{(C)}$$

subject to $x \in \mathrm{C}^1_{\mathrm{prd}}[a, b]_\mathbb{T}$ and $u \in \mathrm{C}_{\mathrm{prd}}[a, \rho(b)]_\mathbb{T}$ satisfying (4.2), (4.3), and

$$x^\Delta(t) = h\big(t, x(t), u(t)\big), \quad t \in [a, \rho(b)]_\mathbb{T},$$

with the corresponding Hamiltonian

$$H(t, x, u, p, \lambda, \lambda_0) := p^T h(t, x, u) + \lambda_0\, g(t, x, u) + \lambda^T \psi(t, u). \qquad (4.12)$$

The definitions of the feasibility and weak local minimum for (C) are similar to those for (C^σ) using the data of (C), see Subsection 4.1.

Despite that problems (C) and (C^σ) appear to be visually different, we shall prove that they are in fact equivalent near a feasible pair $(\bar{x}, \bar{u})$, and thus any result pertaining one form can be translated into the other form via the transformation displayed below. This transformation, as well as any similar ones, is based on the implicit function theorem.

The first and second order regularity of the data in problem (C) is given by assumptions called (A1) and (A2), which are the same as (A1$^\sigma$) and (A2$^\sigma$), in which the quantities (g, h, ψ), $\bar{x}$, and $I + \mu(t)\, h_x(t, \bar{x}(t), \bar{u}(t))$ replace (L, f, ψ), $\bar{x}^\sigma$, and $I - \mu(t)\, f_x(t, \bar{x}^\sigma(t), \bar{u}(t))$, respectively.

Let us denote the gradients of the function h by

$$A(t) := \bar{h}_x(t) := h_x\big(t, \bar{x}(t), \bar{u}(t)\big), \quad B(t) := \bar{h}_u(t) := h_u\big(t, \bar{x}(t), \bar{u}(t)\big), \quad (4.13)$$

and recall the definition of the matrix M in (4.4).

The linear system

$$\eta^\Delta = A(t)\, \eta + B(t)\, v \tag{4.14}$$

is said to be *M-controllable over* $\mathcal{T}$ if for any vector $d \in \mathbb{R}^r$ there exist a vector $\alpha = \alpha_d \in \mathbb{R}^n$ and a function $v = v_d \in \mathcal{T}$ such that the solution of the initial value problem (4.14) with $\eta(a) = \alpha$ satisfies (4.7).

Problem (C) is said to be *normal at* $(\bar{x}, \bar{u})$ if the matrix M has full rank and if the system

$$\left. \begin{array}{c} p^\Delta = -A^T(t)\, p^\sigma, \quad v^T B^T(t)\, p^\sigma = 0, \quad \forall\, v \in T(t), \quad t \in [a, \rho(b)]_{\mathbb{T}}, \\[2mm] \begin{pmatrix} -p(a) \\ p(b) \end{pmatrix} = M^T \gamma, \end{array} \right\}$$

for some vector $\gamma \in \mathbb{R}^r$, possesses only the trivial solution $p(\cdot) \equiv 0$ (and then also $\gamma = 0$).

Similarly as in Proposition 4.1 we have the following.

Proposition 4.5 (Controllability and normality)**.** *Assume that the matrix M has full rank and $I + \mu(t)\, A(t)$ is invertible on $[a, \rho(b)]_{\mathbb{T}}$. Then the linear system* (4.14) *is M-controllable over* $\mathcal{T}$ *if and only if the problem* (C) *is normal at* $(\bar{x}, \bar{u})$.

The weak maximum principle for problem (C) then reads as follows.

Theorem 4.6 (Weak maximum principle on time scales)**.** *Assume that $(\bar{x}, \bar{u})$ is a weak local minimum for* (C) *such that the assumption* (A1) *holds. Then there exist a constant $\lambda_0 \geq 0$, a vector $\bar{\gamma} \in \mathbb{R}^r$, a function $\bar{\lambda} : [a, \rho(b)]_{\mathbb{T}} \to \mathbb{R}^k$, $\bar{\lambda} \in \mathrm{C}_{\mathrm{prd}}$, and a function $\bar{p} : [a, b]_{\mathbb{T}} \to \mathbb{R}^n$, $\bar{p} \in \mathrm{C}^1_{\mathrm{prd}}$, such that $\lambda_0 + \|\bar{p}\|_C \neq 0$ and satisfying the following conditions:*

(i) *the adjoint equation: for all $t \in [a, \rho(b)]_{\mathbb{T}}$*

$$-\bar{p}^\Delta(t) = A^T(t)\, \bar{p}^\sigma(t) + \lambda_0\, \bar{g}_x^T(t), \tag{4.15}$$

(ii) *the stationarity condition: for all $t \in [a, \rho(b)]_{\mathbb{T}}$*

$$B^T(t)\,\bar{p}^\sigma(t) + \lambda_0\,\bar{g}_u^T(t) + N^T(t)\,\bar{\lambda}(t) = 0, \tag{4.16}$$

(iii) *the transversality condition* (4.10),

where $A(t)$ and $B(t)$ are defined by (4.13), *and $\bar{g}_x(t)$ and $\bar{g}_u(t)$ are evaluated at $\big(t, \bar{x}(t), \bar{u}(t)\big)$. Moreover, if the system* (4.14) *is M-controllable over $\mathcal{T}$, then we may take $\lambda_0 = 1$ and in this case $\bar{\gamma}$, $\bar{\lambda}(\cdot)$, and $\bar{p}(\cdot)$ are unique.*

Comparing the weak maximum principles as well as the normality notions and the first and second variations for problems (C^σ) and (C), we can see that starting *with the shift* in x in (C^σ) leads to the adjoint and stationarity equations (4.8) and (4.9) *without the shift* in $\bar{p}$. On the other hand, starting *with no shift* in x in (C) leads to the adjoint and stationarity equations (4.15) and (4.16) *with the shift* in $\bar{p}$. This reflects the natural duality between the state variable x and the adjoint variable p.

The second variation of the functional F in problem (C) at $(\bar{x}, \bar{u})$ in the direction (η, v) is defined as

$$F''(\bar{x}, \bar{u}; \eta, v) := \begin{pmatrix} \eta(a) \\ \eta(b) \end{pmatrix}^T \Gamma \begin{pmatrix} \eta(a) \\ \eta(b) \end{pmatrix} + \int_a^b \begin{pmatrix} \eta(t) \\ v(t) \end{pmatrix}^T \nabla_{(x,v)}^2 \bar{H}(t) \begin{pmatrix} \eta(t) \\ v(t) \end{pmatrix} \Delta t,$$

where $\bar{H}(t) := H\big(t, \bar{x}(t), \bar{u}(t), \bar{p}^\sigma(t), \bar{\lambda}(t), 1\big)$, the Hamiltonian H is defined by (4.12), and Γ is from (4.11).

Theorem 4.7 (Second variation)**.** *Assume that $(\bar{x}, \bar{u})$ is a weak local minimum for* (C), *assumption* (A2) *holds, and the system* (4.14) *is M-controllable over $\mathcal{T}$. Then $F''(\bar{x}, \bar{u}; \eta, v) \geq 0$ for any admissible pair (η, v), where $\bar{\gamma}$, $\bar{\lambda}(\cdot)$, and $\bar{p}(\cdot)$ satisfy the weak maximum principle (Theorem 4.6).*

5. Time scale symplectic systems

Linear Hamiltonian differential systems

$$X' = A(t)\,X + B(t)\,U, \quad U' = C(t)\,X - A^T(t)\,U, \quad t \in [a, b], \tag{H_c}$$

and discrete symplectic systems

$$X_{k+1} = \mathcal{A}_k X_k + \mathcal{B}_k U_k, \quad U_{k+1} = \mathcal{C}_k X_k + \mathcal{D}_k U_k, \quad k \in [0, N]_{\mathbb{N}}, \tag{S_d}$$

arise as Jacobi systems for nonlinear calculus of variations and optimal control problems. The properties of their special solutions, the principal solution

or more generally the so-called natural conjoined bases, determine the non-negativity and positivity of the corresponding quadratic functional arising as the second variation for such nonlinear variational problems. Therefore, such conditions can be used as second order necessary and sufficient optimality criteria.

In this chapter we will show that the continuous time linear Hamiltonian systems (H_c) and discrete symplectic systems (S_d) and their corresponding quadratic functionals can be unified and extended into one theory – the theory of time scale symplectic systems. In particular, we study the definiteness of the associated time scale quadratic functional in terms of a natural conjoined basis and in terms of the explicit and implicit Riccati matrix equations.

5.1. **Problem statement.** Let Γ_a, Γ_b, R_a, and R_b be given $n \times n$-matrices with Γ_a and Γ_b symmetric. Consider the quadratic functional

$$\mathcal{I}(x,u) := x^T(a)\,\Gamma_a\,x(a) + x^T(b)\,\Gamma_b\,x(b) + \mathcal{I}_0(x,u), \qquad (5.1)$$

where

$$\mathcal{I}_0(x,u) := \int_a^b \big\{ x^T \mathcal{C}^T (I + \mu \mathcal{A})\,x + 2\mu\,x^T \mathcal{C}^T \mathcal{B} u + u^T (I + \mu \mathcal{D})^T \mathcal{B} u \big\}(t)\,\Delta t,$$

over all *admissible* (x,u), i.e.,

$$x^\Delta(t) = \mathcal{A}(t)\,x(t) + \mathcal{B}(t)\,u(t), \quad t \in [a, \rho(b)]_{\mathbb{T}}, \qquad (5.2)$$

and $x(t)$ satisfies

$$x(a) \in \operatorname{Im} R_a, \quad x(b) \in \operatorname{Im} R_b. \qquad (5.3)$$

We assume that $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D} : [a, \rho(b)]_{\mathbb{T}} \to \mathbb{R}^{n \times n}$, $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D} \in \mathrm{C}_{\mathrm{prd}}$, are such that the $2n \times 2n$ matrix $\mathcal{S}(t) := \left(\begin{smallmatrix} \mathcal{A}(t) & \mathcal{B}(t) \\ \mathcal{C}(t) & \mathcal{D}(t) \end{smallmatrix} \right)$ satisfies the identity

$$\mathcal{S}^T(t)\,\mathcal{J} + \mathcal{J}\mathcal{S}(t) + \mu(t)\,\mathcal{S}^T(t)\,\mathcal{J}\mathcal{S}(t) = 0 \quad \text{on } [a, \rho(b)]_{\mathbb{T}}. \qquad (5.4)$$

Equation (5.4) implies that $I + \mu(t)\,\mathcal{S}(t)$ is a symplectic matrix. To the functional $\mathcal{I}$ we associate the so-called *time scale symplectic* (or *Hamiltonian*) *system*

$$X^\Delta = \mathcal{A}(t)\,X + \mathcal{B}(t)\,U, \quad U^\Delta = \mathcal{C}(t)\,X + \mathcal{D}(t)\,U. \qquad (\mathcal{S})$$

When $[a,b]_{\mathbb{T}}$ is a real connected interval $[a,b]$, the identity (5.4) defining the time scale symplectic system $(\mathcal{S})$ becomes

$$\mathcal{S}^T(t)\,\mathcal{J} + \mathcal{J}\mathcal{S}(t) = 0 \quad \text{on } [a,b],$$

which means that $(\mathcal{S})$ reduces to the Hamiltonian system (H_c), where $A(t) := \mathcal{A}(t)$, $B(t) := \mathcal{B}(t)$, $C(t) := \mathcal{C}(t)$, and $\mathcal{D}(t) = -\mathcal{A}^T(t)$. When $[a,b]_{\mathbb{T}}$ is the

discrete interval $[0, N+1]_{\mathbb{N}}$, the identity (5.4) is now equivalent to saying that $I + \mathcal{S}(k)$ is a symplectic matrix. Thus, with $A_k := I + \mathcal{A}(k)$, $B_k := \mathcal{B}(k)$, $C_k := \mathcal{C}(k)$, and $D_k := I + \mathcal{D}(k)$, the system $(\mathcal{S})$ reduces to (S_d).

Time scale symplectic systems we first mentioned in [49, Remark 4.3] as a possible natural extension of therein studied time scale linear Hamiltonian systems. This idea was then developed in the paper [34] by the author and Došlý, and in [48] by the author, where the Reid roundabout theorem for the quadratic functional $\mathcal{I}$ with zero endpoints was established (under a normality assumption). An alternative terminology to the "time scale symplectic system" is the "time scale Hamiltonian system", which is used in [4]. Since then, several other papers deal with the theory of time scale symplectic systems, such as [17, 33, 50, 81, 82, 83], yet excluding the work on general time scale symplectic systems without normality assumption.

A solution (X, U) of $(\mathcal{S})$ is a *conjoined basis* if $X^T(t)\, U(t)$ is symmetric and rank $\begin{pmatrix} X^T(t) & U^T(t) \end{pmatrix} = n$ at some (and hence at any) point $t \in [a, b]_{\mathbb{T}}$. A *natural conjoined basis* of $(\mathcal{S})$, denoted by (X_a, U_a), is any conjoined basis satisfying the initial conditions

$$X_a(a) = R_a, \quad X_a^T(a)\, U_a(a) = X_a^T(a)\, \Gamma_a\, X_a(a).$$

The conjoined basis definition then implies that we also have $X_a^T(a)\, U_a(a)$ symmetric and rank $\begin{pmatrix} X_a^T(a) & U_a^T(a) \end{pmatrix} = n$. Given matrices R_a and Γ_a, by [75, Corollary 3.1.3], there always exists an $n \times n$ matrix U (not necessarily unique) such that $R_a^T U$ is symmetric, rank $\begin{pmatrix} R_a^T & U^T \end{pmatrix} = n$, and $R_a^T U = R_a^T \Gamma_a R_a$. The conjoined basis (X_a, U_a) of $(\mathcal{S})$ satisfying $X_a(a) = R_a$ and $U_a(a) = U$ is then a natural conjoined basis. Hence, the set of natural conjoined bases is nonempty.

A pair (x, u) is called *admissible* (on the interval $[a, b]_{\mathbb{T}}$) if $x \in \mathrm{C}^1_{\mathrm{prd}}$, $\mathcal{B}u \in \mathrm{C}_{\mathrm{prd}}$, and it satisfies equation (5.2), i.e., the first equation of system $(\mathcal{S})$. We remark that in most cases it is sufficient to require that $u \in \mathrm{C}_{\mathrm{prd}}$, since then the product $\mathcal{B}u$ is also in $\mathrm{C}_{\mathrm{prd}}$. However, it could happen that $\mathcal{B}u$ is in $\mathrm{C}_{\mathrm{prd}}$ when u is not.

The quadratic functional $\mathcal{I}$ is *nonnegative* (or *nonnegative definite*), we write $\mathcal{I} \geq 0$, if $\mathcal{I}(x, u) \geq 0$ for all admissible pairs (x, u) satisfying the boundary conditions (5.3). The quadratic functional $\mathcal{I}$ is *positive* (or *positive definite*), we write $\mathcal{I} > 0$, if $\mathcal{I}(x, u) > 0$ for all admissible (x, u) satisfying (5.3), and $x \not\equiv 0$ on $[a, b]_{\mathbb{T}}$.

We shall also work with the following $n \times n$-matrices (suppressing the argument t in these definitions), defined via a given conjoined basis (X, U) of

$(\mathcal{S})$,

$$M := [I - X^{\sigma}(X^{\sigma})^{\dagger}]\mathcal{B}, \quad T := I - M^{\dagger}M, \quad P := X(X^{\sigma})^{\dagger}\mathcal{B}. \qquad (5.5)$$

Note that in this chapter the matrix function $M(t)$ has nothing to do with the constant matrix M from Sections 3 and 4.

We say that a matrix valued function $X(t)$ has *piecewise constant kernel on* $[a,b]_{\mathbb{T}}$ if there exist points $\{t_k\}_{k=0}^{m} \subseteq [a,b]_{\mathbb{T}}$ with $a = t_0 < t_1 < \cdots < t_{m-1} < t_m = b$ such that

$$\text{Ker}\, X(t) \text{ is constant for all } t \in (t_{k-1}, t_k)_{\mathbb{T}}, \ k = 1, \ldots m. \qquad (5.6)$$

Condition (5.6) is void on intervals $(t_{k-1}, t_k)_{\mathbb{T}}$ where $t_k = \sigma(t_{k-1})$.

5.2. **Jacobi systems for time scale control problems.** The motivation for the study of time scale symplectic systems originated in the unification of the continuous time linear Hamiltonian systems (H_c) and the discrete symplectic systems (S_d). However, it is known in these two special cases that these systems are Jacobi systems for nonlinear calculus of variations and control problems, see e.g. [28,41,56,57,89,94,96]. In particular, the systems (H_c) and (S_d) are the Euler–Lagrange systems for the quadratic functionals $\mathcal{I}_c$ and $\mathcal{I}_d$, respectively.

This fact is also known for the time scale *calculus of variations problems*. On the other hand, this question remained open for the *time scale control setting*. In this section we fill this gap and show that time scale symplectic systems $(\mathcal{S})$ play the same role for general control problems on time scales. In particular, by using the time scale weak maximum principle in Theorems 4.3 and 4.6, we prove that nonlinear time scale control problems lead to time scale linear Hamiltonian systems (in fact, two kinds of linear Hamiltonian systems, one with the shift in η and one with the shift in q, depending on whether the original control problem has or does not have the shift in x), which naturally possess a symplectic structure and hence, can be embedded into time scale symplectic systems. This fact highlights the theory of time scale symplectic systems as an ultimate field for second order optimality conditions for such variational problems. Furthermore, we also prove that the system $(\mathcal{S})$ is indeed the Euler–Lagrange system for the functional $\mathcal{I}$.

5.3. **Definiteness of quadratic functionals.** The first main result of this section is the following characterization of the positivity of $\mathcal{I}$ with separable endpoints via the nonexistence of the generalized focal points for a natural conjoined basis of $(\mathcal{S})$.

Theorem 5.1 ($\mathcal{I} > 0$, separated endpoints). *The quadratic functional $\mathcal{I}$ in* (5.1) *is positive definite if and only if a natural conjoined basis (X_a, U_a) of $(\mathcal{S})$ has no generalized focal points in $(a, b]_\mathbb{T}$, that is, the following two conditions*

(i) *the kernel condition*

$$\operatorname{Ker} X_a(t) \subseteq \operatorname{Ker} X_a(\tau) \quad \text{for all } t, \tau \in [a, b]_\mathbb{T}, \ \tau \leq t,$$

(ii) *the P–condition*

$$P_a(t) := X_a(t) \, [X_a^\sigma(t)]^\dagger \, \mathcal{B}(t) \geq 0 \quad \text{for all } t \in [a, \rho(b)]_\mathbb{T}, \tag{5.7}$$

hold, and satisfies

(iii) *the final endpoint inequality*

$$U_a(b) \, X_a^\dagger(b) + \Gamma_b > 0 \quad \text{on } \operatorname{Im} R_b \cap \operatorname{Im} X_a(b).$$

The characterization of the positivity of $\mathcal{I}$ in Theorem 5.1 directly generalizes [48, Theorem 6] to abnormal systems. This theorem also generalizes and unifies the corresponding continuous-time results in [75, Theorem 2.4.2] and [93, Theorem 5.5] and the discrete-time results in [19, Theorem 3.2] and [59, Theorem 5].

The second main result of this paper characterizes the nonnegativity of $\mathcal{I}$ in terms of a modification of the P–condition (5.7).

Theorem 5.2 ($\mathcal{I} \geq 0$, separated endpoints). *The quadratic functional $\mathcal{I}$ in* (5.1) *is nonnegative if and only if a natural conjoined basis (X_a, U_a) of $(\mathcal{S})$ satisfies*

(i) $X_a(t)$ *has piecewise constant kernel on $[a, b]_\mathbb{T}$,*

(ii) *the image condition, for all admissible (x, u) satisfying* (5.3),

$$x(t) \in \operatorname{Im} X_a(t) \quad \text{for all } t \in (a, b]_\mathbb{T}, \tag{5.8}$$

(iii) *the P–condition*

$$T(t) \, P_a(t) \, T(t) \geq 0 \quad \text{for all } t \in [a, \rho(b)]_\mathbb{T},$$

where the matrix $P_a(t)$ is defined by (5.7) *and $T(t)$ is given in* (5.5) *through (X_a, U_a),*

(iv) *the final endpoint inequality*

$$U_a(b) \, X_a^\dagger(b) + \Gamma_b \geq 0 \quad \text{on } \operatorname{Im} R_b \cap \operatorname{Im} X_a(b).$$

Note that the image condition in (5.8) is satisfied trivially at $t = a$, because the admissible pairs (x, u) satisfy the initial boundary condition $x(a) \in \operatorname{Im} R_a$.

The characterization of the nonnegativity of $\mathcal{I}$ in Theorem 5.2 generalizes and unifies the corresponding continuous-time result in [76, Theorem 2] and the discrete-time result in [18, Theorem 2].

Next we present a theorem on the piecewise constant kernel of conjoined bases of $(\mathcal{S})$. Although the general idea of the proof is similar to the continuous-time case, the proof itself is much more involved. This result is a time scale generalization of [76, Theorem 3].

Theorem 5.3 (Piecewise constant kernel). *Assume that the quadratic functional $\mathcal{I}_0$ is nonnegative over the zero endpoints. Then for any conjoined basis (X, U) of $(\mathcal{S})$, the matrix $X(t)$ has piecewise constant kernel on $[a, b]_\mathbb{T}$, that is, there exist points $\{t_k\}_{k=0}^m \subseteq [a, b]_\mathbb{T}$ with $a = t_0 < t_1 < \cdots < t_{m-1} < t_m = b$ such that condition (5.6) holds. Moreover, we have*

$$\operatorname{Ker} X(t) \subseteq \operatorname{Ker} X(t_{k-1}) \cap \operatorname{Ker} X(t_k) \ \text{for all } t \in (t_{k-1}, t_k)_\mathbb{T}, \ k = 1, \dots, m. \tag{5.9}$$

Conditions (5.6) and (5.9) are void on intervals $(t_{k-1}, t_k)_\mathbb{T}$ where $t_k = \sigma(t_{k-1})$.

The absence of any *normality* assumption in the main results of this chapter has opened the door for the utility of new techniques that are particularly useful when extending results from fixed to varying endpoints. For instance, when extending results from fixed to varying endpoints, one known approach is based on adding an isolated point to the original time scale, thus creating a *new time scale* to which the results on fixed endpoints can be applied. The application of this technique was limited by the fact that a *normality* assumption does not carry through to the *newly* constructed time scale, see an application to the Riccati equation in [65, Section 6]. Moreover, such a technique was not admissible in the continuous-time setting, since adding an isolated point to a connected interval would not produce a connected interval, but it is admissible in the *time scale* setting. On the other hand, there is a known method for extending results from the case of separable endpoints to jointly varying endpoints which is based on applying the separable endpoints results to an equivalent augmented problem having *separated* endpoints. Unfortunately, the augmented problem turns out to be abnormal, see the applications in [65, Section 4], and thus the presence of a normality condition in the results for separable endpoints renders this method inapplicable. Hence, the results of this chapter revive all those techniques.

Furthermore, the fact that no normality is assumed in Theorems 5.1 and 5.2 is important for many applications. For example, except of the results on Riccati equations in the next subsection, one can now derive without any normality Sturmian comparison theorems, extensions of Theorems 5.1 and 5.2 to jointly varying endpoints, perturbation results for positive and nonnegative definite quadratic functionals, and the characterization of the positivity of $\mathcal{I}$ in terms of a time scale Riccati inequality. These mentioned results were obtained by the author and Zeidan in [65] and by the author and Růžičková in [53, 54].

5.4. **Riccati matrix equation – explicit.** In this section we will construct a solution of the time scale *Riccati equation*

$$R[Q](t) = 0, \quad t \in [a, \rho(b)]_{\mathbb{T}}, \tag{$\mathcal{R}$}$$

involving the Riccati operator

$$R[Q](t) := Q^{\Delta} - [\mathcal{C}(t) + \mathcal{D}(t)\,Q] + Q^{\sigma}[\mathcal{A}(t) + \mathcal{B}(t)\,Q].$$

It is known that equation $(\mathcal{R})$ has a symmetric solution $Q(t)$ on $[a, b]_{\mathbb{T}}$ if and only if the symplectic system $(\mathcal{S})$ has a conjoined basis (X, U) such that $X(t)$ is *invertible* for all $t \in [a, b]_{\mathbb{T}}$, see e.g. [34, Theorem 3] or [33, Theorem 10.9]. In this case, the solution $Q(t)$ of $(\mathcal{R})$ is given by $Q(t) = U(t)\,X^{-1}(t)$ on $[a, b]_{\mathbb{T}}$.

Note that, given boundary conditions (5.3) in terms of the matrices R_a and R_b, we can set $\mathcal{M}_a := I - R_a R_a^{\dagger}$ and $\mathcal{M}_b := I - R_b R_b^{\dagger}$ to obtain equivalent boundary conditions

$$\mathcal{M}_a\,x(a) = 0, \quad \mathcal{M}_b\,x(b) = 0. \tag{5.10}$$

Conversely, given the boundary conditions (5.10) in terms of the projections $\mathcal{M}_a$ and $\mathcal{M}_b$, the obvious choice $R_a := I - \mathcal{M}_a$ and $R_b := I - \mathcal{M}_b$ yields the boundary conditions (5.3). Furthermore, we may assume without loss of generality that the matrices Γ_a and Γ_b satisfy $\Gamma_a = (I - \mathcal{M}_a)\,\Gamma_a\,(I - \mathcal{M}_a)$ and $\Gamma_b = (I - \mathcal{M}_b)\,\Gamma_b\,(I - \mathcal{M}_b)$.

In view of the above remark, we can choose a specific natural conjoined basis (X_a, U_a), called in this context "the" natural conjoined basis, by using the initial conditions

$$X_a(a) = I - \mathcal{M}_a, \quad U_a(a) = \Gamma_a + \mathcal{M}_a.$$

The main result on the solvability of the explicit Riccati equation $(\mathcal{R})$ is then the following.

Theorem 5.4 ($\mathcal{I} > 0$, separated endpoints). *The following statements are equivalent.*

   (i) *The quadratic functional $\mathcal{I}$ in (5.1) is positive over (5.10).*

  (ii) *There exists a conjoined basis (X, U) of $(\mathcal{S})$ with no generalized focal points in $(a, b]_{\mathbb{T}}$ such that $X(t)$ is invertible for all $t \in [a, b]_{\mathbb{T}}$ and satisfying*

$$X^T(a)\, [\, \Gamma_a\, X(a) - U(a)\,] > 0 \quad on\ \operatorname{Ker} \mathcal{M}_a\, X(a),$$
$$X^T(b)\, [\, \Gamma_b\, X(b) + U(b)\,] > 0 \quad on\ \operatorname{Ker} \mathcal{M}_b\, X(b).$$

 (iii) *There exists a symmetric solution $Q(t)$ on $[a, b]_{\mathbb{T}}$ of the time scale Riccati matrix equation $(\mathcal{R})$ such that for all $t \in [a, \rho(b)]_{\mathbb{T}}$*

$$I + \mu(t)\, [\mathcal{A}(t) + \mathcal{B}(t)\, Q(t)\,]\ is\ invertible,$$
$$\big\{ I + \mu(t)\, [\mathcal{A}(t) + \mathcal{B}(t)\, Q(t)\,] \big\}^{-1} \mathcal{B}(t) \geq 0,$$

and satisfying the initial and final endpoint inequalities

$$\Gamma_a - Q(a) > 0 \quad on\ \operatorname{Ker} \mathcal{M}_a,$$
$$\Gamma_b + Q(b) > 0 \quad on\ \operatorname{Ker} \mathcal{M}_b.$$

In Theorem 5.4 we are able to remove a certain "dense-normality" assumption used in [48, Theorem 1] or in [33, Theorem 10.52] and, furthermore, extend this result to *variable endpoints*. The corresponding statements to Theorem 5.4 in the special cases of the continuous and discrete times can be found respectively in [85, Theorem 7.4] and [59, Theorem 7]. Furthermore, it is shown in [65, Theorem 7.1] that the Riccati matrix equation $(\mathcal{R})$ can be replaced by the *Riccati inequality*

$$R[Q](t)\, \big\{ I + \mu(t)\, [\mathcal{A}(t) + \mathcal{B}(t)\, Q)] \big\}^{-1} \leq 0, \quad t \in [a, \rho(b)]_{\mathbb{T}}.$$

This constitutes a generalization and unification of the continuous time Riccati inequality in [85, Theorems 7.2–7.4], see also [84, Theorem VII.5.3], the discrete time Riccati inequality in [51, Theorem 1], and the time scale Riccati inequality in [55, Theorem 3.1]. Note the latter reference deals with a linear *Hamiltonian* system under a normality assumption and zero endpoints.

5.5. **Riccati matrix equations – implicit.** The result of Theorem 5.4 on the "explicit" Riccati equation $(\mathcal{R})$ is based on the existence of *some* conjoined basis (X, U) of $(\mathcal{S})$ which has $X(t)$ invertible on $[a, b]_{\mathbb{T}}$. On the other hand, if we require to use a *specific* conjoined basis of $(\mathcal{S})$, such as the natural

conjoined basis (X_a, U_a), then the "explicit" Riccati equation $(\mathcal{R})$ has to be replaced by an implicit Riccati equation. The word "implicit" means that the Riccati operator $R[Q](t)$ is projected over the image of the matrix $X_a(t)$.

The main feature of the results in this section resides in the absence of the invertibility assumption of $X_a(t)$ and thus permitting the inclusion of abnormal systems. Instead we are contented with the assumption that the Moore–Penrose generalized inverse $X_a^\dagger$ is *continuous* on the time scale interval $(a, b)_{\mathbb{T}}$ or $(a, b]_{\mathbb{T}}$. In the continuous time case this means that $\operatorname{Ker} X_a$ is constant on the connected interval (a, b) or $(a, b]$, while in the discrete case this assumption is vacuous. However, there are time scales, such as the union of disjoint closed intervals, for which $X_a^\dagger$ is continuous but $\operatorname{Ker} X_a$ is not constant. The image condition (5.8) is known to play a key role for the nonnegativity of $\mathcal{I}$, see Theorem 5.2. One of our Riccati type results (Theorem 5.5) involves this image condition, while in the other one (Theorem 5.6) this condition is replaced by a certain Riccati type inequality.

Given a symmetric $n \times n$ matrix function $Q(t)$ on $[a, b]_{\mathbb{T}}$, we define the symmetric matrix

$$\mathcal{P}(t) := \left\{ \mathcal{B} + \mu \left(\mathcal{D}^T - \mathcal{B}^T Q^\sigma \right) \mathcal{B} \right\}(t).$$

The symmetry of $\mathcal{P}(t)$ follows from the properties of the coefficients of $(\mathcal{S})$ obtained from the defining identity (5.4).

Theorem 5.5 ($\mathcal{I} \geq 0$, separable endpoints)**.** *Assume that $X_a^\dagger$ is continuous on $(a, b)_{\mathbb{T}}$. The quadratic functional $\mathcal{I}$ in (5.1) is nonnegative over (5.10) if and only if X_a has piecewise constant kernel on $[a, b]_{\mathbb{T}}$ and there exists a symmetric $n \times n$ matrix function $Q(t)$ on $[a, b]_{\mathbb{T}}$ such that $Q \in \mathrm{C}^1_{\mathrm{prd}}(a, b)_{\mathbb{T}}$ and satisfying*

(i) *the time scale implicit Riccati equation:*

$$[X_a^\sigma(t)]^T R[Q](t) X_a(t) = 0 \qquad on \ (a, \rho(b))_{\mathbb{T}}, \tag{5.11}$$

and the equation in (5.11) holds also at $t = a$ if a is right-scattered, and at $t = \rho(b)$ if b is left-scattered,

(ii) *the initial condition:*

$$Q(a) = \Gamma_a, \quad if \ a \ is \ right\text{-}scattered, \tag{5.12}$$

$$(I - \mathcal{M}_a) \lim_{t \to a^+} Q(t) X_a(t) = \Gamma_a, \quad if \ a \ is \ right\text{-}dense, \tag{5.13}$$

(iii) *the final endpoint inequality:*

$$Q(b) + \Gamma_b \geq 0 \quad on \ \mathrm{Ker}\,\mathcal{M}_b \cap \mathrm{Im}\,X_a(b), \quad if \ b \ is \ left\text{-}scattered, \qquad (5.14)$$

$$X_a^T(b) \lim_{t \to b^-} [\Gamma_b + Q(t)]\, X_a(t) \geq 0 \quad on \ \mathrm{Ker}\,\mathcal{M}_b\, X_a(b), \quad if \ b \ is \ left\text{-}dense,$$
$$\qquad (5.15)$$

(iv) *the $\mathcal{P}$–condition:*

$$T(t)\,\mathcal{P}(t)\,T(t) \geq 0 \qquad on \ [a, \rho(b)]_{\mathbb{T}}, \qquad (5.16)$$

where the matrix $T(t)$ is defined in (5.5) through (X_a, U_a),

(v) *for any admissible (x, u) satisfying (5.10) we have the image condition (5.8) and*

$$\mu(t)\,[I - T(t)]\,[u(t) - Q(t)\,x(t)] = 0 \qquad on \ [a, \rho(b)]_{\mathbb{T}}.$$

The characterization of the nonnegativity of $\mathcal{I}$ in Theorem 5.5 uses the image condition (5.8). In our next result we are able to remove this image condition, except at $t = b$ if b is a left-dense point, but the price for this improvement is assuming the condition

$$Q(t)\,X(t) = U(t)\,X^\dagger(t)\,X(t) \qquad\qquad (5.17)$$

that connects together the solutions Q and X_a at all scattered points, and a Riccati type equation for all admissible x instead of the one for the solution X_a.

Theorem 5.6 ($\mathcal{I} \geq 0$, separable endpoints)**.** *Assume that $X_a^\dagger$ is continuous on $(a, b)_{\mathbb{T}}$. Then $\mathcal{I} \geq 0$ over (5.10) if and only if X_a has piecewise constant kernel on $[a, b]_{\mathbb{T}}$ and there exists a symmetric $n \times n$ matrix function $Q(t)$ on $[a, b]_{\mathbb{T}}$ such that $Q \in \mathrm{C}^1_{\mathrm{prd}}(a, b)_{\mathbb{T}}$ and satisfying*

  (i) *the initial condition (5.12) and (5.13),*

 (ii) *the final endpoint condition (5.14) and (5.15),*

(iii) *for all points $t \in [a, b]_{\mathbb{T}}$ which are right-scattered or left-scattered, Q and (X_a, U_a) satisfy equation (5.17),*

(iv) *the $\mathcal{P}$–condition (5.16), where the matrix $T(t)$ is defined in formulas (5.5) through (X_a, U_a),*

 (v) *for any admissible (x, u) with (5.10) we have*
    (a) *the Riccati type identity*

$$[x^\sigma(t)]^T R[Q](t)\,x(t) = 0 \qquad on \ (a, \rho(b))_{\mathbb{T}}, \qquad (5.18)$$

and the equation in (5.18) holds also at $t = a$ if a is right-scattered and at $t = \rho(b)$ if b is left-scattered,

(b) *the image condition:*

$$x(b) \in \operatorname{Im} X_a(b) \qquad \text{if } b \text{ is left-dense.}$$

The strengthening of the conditions on $Q(t)$ at $t = \rho(b)$ or $t = b$ in Theorem 5.5 yields the result on the positivity of the quadratic functional $\mathcal{I}$ defined in (5.1).

Theorem 5.7 ($\mathcal{I} > 0$, separable endpoints)**.** *Assume that $X_a^\dagger$ is continuous on $(a, b]_\mathbb{T}$. Then $\mathcal{I} > 0$ over (5.10) if and only if X_a has piecewise constant kernel on $[a, b]_\mathbb{T}$ and there exists a symmetric $n \times n$ matrix function $Q(t)$ on $[a, b]_\mathbb{T}$ such that $Q \in \mathrm{C}_{\mathrm{prd}}^1(a, b]_\mathbb{T}$ and satisfying*

(i) *the time scale implicit Riccati equation*

$$R[Q](t)\, X_a(t) = 0 \qquad \text{on } (a, \rho(b)]_\mathbb{T}, \tag{5.19}$$

and the equation in (5.19) holds also at $t = a$ if a is right-scattered,

(ii) *the initial condition (5.12) and (5.13),*

(iii) *the final endpoint inequality*

$$Q(b) + \Gamma_b > 0 \qquad \text{on } \operatorname{Ker} \mathcal{M}_b \cap \operatorname{Im} X_a(b),$$

(iv) *the $\mathcal{P}$–condition*

$$\mathcal{P}(t) \geq 0 \qquad \text{on } [a, \rho(b)]_\mathbb{T}.$$

These kind of results on implicit Riccati equations that we obtain in this subsection do not exits in the literature for the special case of *continuous time* even if X_a is assumed to be *invertible* on (a, b) or $(a, b]$. Thereby we complete the study of quadratic functionals in the abnormal case that was initiated for the continuous time by Kratz in his pioneering work [76]. On the other hand, one of the nonnegativity results is new for the special case of *discrete time*, as it does not impose any extra restriction on Q as in [52], while the second nonnegativity result reduces to its discrete counterpart in [52].

In view of Theorems 4.4 and 4.7 and Subsection 5.2, the characterizations of the nonnegativity of the quadratic functional $\mathcal{I}$ in (5.1) in Theorems 5.2, 5.5 and 5.6 constitute second order necessary optimality conditions for the time scale control problems (C^σ) and (C) from Section 4 with *separated endpoints*.

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Resumé

Předkládaná disertace představuje podrobnou studii podmínek optimality prvního a druhého řádu pro problémy variačního počtu a optimálního řízení na časových škálách (time scales).

Ve variačním počtu a optimálním řízení hledáme určitou trajektorii či kontrolní funkci, která dává optimální, tj. minimální či maximální, hodnotu nějaké předem určené veličiny. Tyto veličiny jsou obvykle zadány pomocí integrálního funkcionálu na nějaké množině funkcí, které jsou přípustné (trajektorie) pro daný problém a splňují požadované počáteční a koncové okrajové podmínky.

Podmínky optimality v takových variačních úlohách jsou prvního a druhého řádu. Podmínky optimality prvního řádu pocházejí z první variace problému a jsou obvykle formulovány ve tvaru Euler–Lagrangeovy rovnice nebo adjungované rovnice (pomocí slabého Pontryaginova principu maxima) a podmínek transversality. Podmínky optimality druhého řádu jsou spjaty s definitností (tj. s nezáporností a koercivitou) kvadratického funkcionálu druhé variace problému. Tyto podmínky jsou pak nutnými (nezápornost) a postačujícími (koercivita, pozitivita) podmínkami optimality pro původní optimalizační úlohu. Zatímco podmínky optimality prvního řádu slouží k nalezení seznamu možných optimálních kandidátů, aplikace podmínek optimality druhého řádu pak vyloučí ty kandidáty, kteří optimální nejsou.

Je známo, že výše uvedené podmínky optimality druhého řádu lze formulovat ekvivalentním způsobem, zejména pomocí konjugovaných a sdružených bodů či pomocí vlastností řešení přidružené Jacobiho nebo Riccatiho rovnice. Záměrem této práce je tedy nejen odvození takových podmínek optimality prvního a druhého řádu, ale také jejich charakterizace pomocí uvedených ekvivaletních pojmů. Tato práce se také zaměřuje na zobecnění těchto výsledků na symplektické systémy na časových škálách, které jsou podle posledních výzkumů nejobecnějšími objekty, pro něž rozumně funguje kvalitativní teorie diferenciálních a diferenčních rovnic a rovnic na časových škálách.

Pod pojmem "časová škála" rozumíme libovolnou neprázdnou uzavřenou podmnožinu reálných čísel. Studium dynamických rovnic na časových škálách umožňuje sjednotit a zobecnit příslušné výsledky pro klasické diferenciální rovnice a diferenční rovnice a současně zdůvodnit rozdíly mezi nimi. Na druhé straně se často stane (jakožto i v této práci), že odvozené nové výsledky pro úlohu na časových škálách jsou dokonce novými výsledky i pro diferenciální či diferenční rovnice.

Přínos této práce spočívá v následujícím:

- Předkládáme *úplnou studii*, tj. nutné i postačující podmínky optimality, pro úlohu variačního počtu na časových škálách s *obecnými* okrajovými podmínkami. Postačující podmínky optimality jsou formulovány buď pomocí koercivity druhé variace (tradiční přístup) nebo pomocí pozitivity druhé variace (nový přístup).

- Dokázaná ekvivalence mezi koercivitou a pozitivitou kvadratických funkcionálů ve variačním počtu je novým výsledkem pro *spojité problémy s obecnými konci.*

- V literatuře se poprvé objevuje text o *optimálním řízení na časových škálách.* Studované problémy optimálního řízení uvažujeme navíc s *omezeními kontrolních funkcí* ve formě rovnic a pro *obecné okrajové podmínky.*

- Uvádíme relativně jednoduchý důkaz *slabého Pontryaginova principu maxima* na časových škálách pro problémy optimálního řízení s obecnými okrajovými podmínkami. Pro tento účel jsme odvodili zobecnění Dubois-Reymondova lemmatu, které je známo ve variačním počtu. Podle našich informací je takový důkaz nový dokonce pro spojité problémy optimálního řízení s proměnnými (tj. separovanými či obecnými) okrajovými podmínkami.

- V práci odvozujeme přímou *souvislost* mezi problémy *optimálního řízení na časových škálách* a *symplektickými systémy na časových škálách.* Přesněji, ukazujeme, že Jacobiho systém pro úlohu optimální řízení na časových škálách přirozeně vede na symplektický systém na časových škálách.

- Uvádíme charakterizace nezápornosti a pozitivity kvadratických funkcionálů příslušejících k symplektickým systémům na časových škálách *bez předpokladu normality.* Tyto výsledky jsou formulovány pomocí přirozené izotropické báze nebo pomocí explicitní či implicitní Riccatiho rovnice. Výsledky pro implicitní Riccatiho rovnici jsou *nové* i pro *spojité lineární Hamiltonovské systémy.*

- Dokázali jsme *větu o vnoření na časových škálách* (či větu o spojité závislosti řešení nelineární dynamické rovnice na časových škálách na počátečních podmínkách a parametrech), která nepředpokládá žádné omezení na délku uvažovaného (kompaktního) intervalu $[a, b]_{\mathbb{T}}$, v blízkosti již existujícího řešení na $[a, b]_{\mathbb{T}}$.