On the density of wild initial data for the compressible Euler system

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Motivation: Wild data for the (incompressible) Euler system

Incompressible Euler system

Field equations

$$\mathrm{div}_x \mathbf{v} = 0,\ \partial_t \mathbf{v} + \mathrm{div}_x(\mathbf{v} \otimes \mathbf{v}) + \nabla_x P = 0,\ (t, x) \in (0, T) \times \Omega$$

$$\mathbf{v} \cdot \mathbf{n}|_{\partial\Omega} = 0,\ \mathbf{v}(0, \cdot) = \mathbf{v}_0$$

Energy inequality

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{2} \int_{R^d} |\mathbf{v}|^2 \mathrm{d}x \leq 0,\ \frac{1}{2} \int_{R^d} |\mathbf{v}|^2 \mathrm{d}x \leq \frac{1}{2} \int_{R^d} |\mathbf{v}_0|^2 \mathrm{d}x$$

Wild initial data

$$\mathbf{v}_0 \in L^2(\Omega)$$

the Euler system admits infinitely many (weak) solutions

in $(0, T)$ for any $T > 0$

Density of wild initial data

Székelyhidi, Wiedemann ARMA 2012:
Wild initial data are dense in $L^2(\Omega)$ (periodic b.c.)

Full Euler system

Field equations

$$\partial_t \varrho + \mathrm{div}_x \mathbf{m} = 0$$

$$\partial_t \mathbf{m} + \mathrm{div}_x \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right) + \nabla_x p(\varrho, \mathbf{m}, E) = 0$$

$$\partial_t E + \mathrm{div}_x \left(E \frac{\mathbf{m}}{\varrho} \right) + \mathrm{div}_x \left(p(\varrho, \mathbf{m}, E) \frac{\mathbf{m}}{\varrho} \right) = 0$$

$$\mathbf{m} \cdot \mathbf{n}|_{\partial\Omega} = 0$$

Thermodynamic stability – entropy

$$S = S(\varrho, \mathbf{m}, E) \text{ concave function of } (\varrho, \mathbf{m}, E)$$

Entropy inequality

$$\partial_t S(\varrho, \mathbf{m}, E) + \mathrm{div}_x \left(S(\varrho, \mathbf{m}, E) \frac{\mathbf{m}}{\varrho} \right) \geq 0$$

Reformulation

Helmholtz decomposition

$$\mathbf{m} = \mathbf{H}[\mathbf{m}] + \mathbf{H}^{\perp}[\mathbf{m}],\ \mathrm{div}_x \mathbf{H}[\mathbf{m}] = 0,\ \mathbf{H}^{\perp} = \nabla_x \Phi$$

Euler system revisited

$$\mathbf{v} = \mathbf{H}[\mathbf{m}],\ E = \frac{|\mathbf{v} + \nabla_x \Phi|^2}{\varrho} + \varrho e(\varrho, \vartheta),\ p = p(\varrho, \vartheta)$$

Field equations

$$\partial_t \varrho + \Delta_x \Phi = 0$$

$$\partial_t \mathbf{v} + \mathbf{H}\left[\mathrm{div}_x \left(\frac{(\mathbf{v} + \nabla_x \Phi) \otimes (\mathbf{v} + \nabla_x \Phi)}{\varrho} - \frac{1}{N} \frac{|(\mathbf{v} + \nabla_x \Phi)|^2}{\varrho} \mathbb{I} \right) \right] = 0$$

$$\partial_t (\nabla_x \Phi) + \mathbf{H}^{\perp} \left[\mathrm{div}_x \left(\frac{(\mathbf{v} + \nabla_x \Phi) \otimes (\mathbf{v} + \nabla_x \Phi)}{\varrho} - \frac{1}{N} \frac{|(\mathbf{v} + \nabla_x \Phi)|^2}{\varrho} \mathbb{I} \right) \right]$$

$$+ \nabla_x \left(\frac{1}{N} \frac{|(\mathbf{v} + \nabla_x \Phi)|^2}{\varrho} \right) + \nabla_x p(\varrho, \vartheta) = 0$$

$$\partial_t E + \mathrm{div}_x \left(E \frac{(\mathbf{v} + \nabla_x \Phi)}{\varrho} \right) + \mathrm{div}_x \left(p(\varrho, \vartheta) \frac{(\mathbf{v} + \nabla_x \Phi)}{\varrho} \right) = 0$$

Convex integration ansatz

Acoustic system

$$\partial_t \mathbf{v} + \mathbf{H}\left[\mathrm{div}_x \left(\frac{(\mathbf{v} + \nabla_x \Phi) \otimes (\mathbf{v} + \nabla_x \Phi)}{\varrho} - \frac{1}{N} \frac{|(\mathbf{v} + \nabla_x \Phi)|^2}{\varrho} \mathbb{I} \right)\right] = 0$$

$$\partial_t (\nabla_x \Phi) + \mathbf{H}^{\perp}\left[\mathrm{div}_x \left(\frac{(\mathbf{v} + \nabla_x \Phi) \otimes (\mathbf{v} + \nabla_x \Phi)}{\varrho} - \frac{1}{N} \frac{|(\mathbf{v} + \nabla_x \Phi)|^2}{\varrho} \mathbb{I} \right)\right]$$

$$+ \nabla_x \left(\frac{1}{N} \frac{|(\mathbf{v} + \nabla_x \Phi)|^2}{\varrho} \right) + \nabla_x p(\varrho, \vartheta) = 0$$

Convex integration ansatz (Extended Euler system)

$$\partial_t \mathbf{v} + \mathrm{div}_x \left(\frac{(\mathbf{v} + \nabla_x \Phi) \otimes (\mathbf{v} + \nabla_x \Phi)}{\varrho} - \frac{1}{N} \frac{|(\mathbf{v} + \nabla_x \Phi)|^2}{\varrho} \mathbb{I} \right) = 0$$

$$\mathrm{div}_x \mathbf{v} = 0$$

$$\partial_t (\nabla_x \Phi) + \nabla_x \left(\frac{1}{N} \frac{|(\mathbf{v} + \nabla_x \Phi)|^2}{\varrho} \right) + \nabla_x p(\varrho, \vartheta) = 0$$

$$\partial_t \varrho + \Delta_x \Phi = 0$$

Infinitely many solutions – wild initial data

Wild solutions by convex integration (EF, Klingenberg, Kreml, Markfelder JDE 2019

Theorem. For any given piecewise constant initial density ϱ_0 and ϑ_0, there exist (infinitely many) initial momenta $\mathbf{m}_0 = \mathbf{v}_0$ such that the Extended Euler system admits infinitely many weak solutions satisfying the entropy balance as equality. In particular, these are admissible weak solutions to the Euler system

Wild initial data

Any (weak) solution of the Euler system satisfying Extended Euler system is called *wild solution*. The corresponding initial data are *wild data*

W (wild) convergence

Data space

$$
\begin{aligned}
&\mathrm{L}^1_{+,s_0}(\Omega; R^{N+2}) \\
&= \left\{ [\varrho, \mathbf{m}, E] \in L^1(\Omega; R^{N+2}) \ \middle| \ \varrho \geq 0,\ E \geq 0,\ s(\varrho, \mathbf{m}, E) \geq s_0 > -\infty \right\}.
\end{aligned}
$$

W–convergence $[\varrho_{0,n}, \mathbf{m}_{0,n}, E_{0,n}] \to [\varrho_0, \mathbf{m}_0, E_0]$

- $$\varrho_{0,n} > 0, \quad s(\varrho_{0,n}, \mathbf{m}_{0,n}, E_{0,n}) \geq s_0 > -\infty$$
- $$[\varrho_{0,n}, \mathbf{m}_{0,n}, E_{0,n}] \to [\varrho_0, \mathbf{m}_0, E_0] \quad \text{in } L^1(\Omega; R^{N+2})$$
- the initial data $[\varrho_{0,n}, \mathbf{m}_{0,n}, E_{0,n}]$ give rise to a sequence of admissible weak solutions $[\varrho_n, \mathbf{m}_n, E_n]$ satisfying

$$\int_0^T \int_\Omega \left(\frac{\mathbf{m}_n \otimes \mathbf{m}_n}{\varrho_n} - \frac{1}{N} \frac{|\mathbf{m}_n|^2}{\varrho_n} \mathbb{I} \right) : \nabla_x^2 \varphi \ \mathrm{d}x\mathrm{d}t \to 0 \quad \text{as } n \to \infty$$

$$\text{for any } \varphi \in C_c^\infty((0, T) \times \Omega)$$

The last condition is automatically satisfied for wild initial data!

Main result

Reachable set

We say that a trio $[\varrho_0, \mathbf{m}_0, E_0]$ is *reachable* if there exists a sequence of initial data $\{\varrho_{0,n}, \mathbf{m}_{0,n}, E_{0,n}\}_{n=1}^{\infty}$ such that

$$[\varrho_{0,n}, \mathbf{m}_{0,n}, E_{0,n}] \overset{(W)}{\rightarrow} [\varrho_0, \mathbf{m}_0, E_0].$$

Theorem (EF, Klingenberg, Markfelder Calc. Variations PDE 2020)

Let $s_0 \in R$ be given. Let $\Omega \subset R^N$, $N = 2, 3$ be a bounded smooth domain. Then the complement of the set of reachable data is an open dense set in $L^1_{+,s_0}(\Omega; R^{N+2})$.

Corollary

The complement of the set of wild initial data (the data that give rise to solution of Extended Euler system) contains an open dense set in $L^1_{+,s_0}(\Omega; R^{N+2})$.

Localization

Domain partition

$$Q \text{ open } Q \subset (0,T) \times \Omega,\ \overline{Q} = [0,T] \times \overline{\Omega}$$

Closed partition
Family of partitions $\mathcal{Q}$ closed with respect to Hausdorff complementary topology

W - convergence $[\varrho_{0,n}, \mathbf{m}_{0,n}, E_{0,n}] \stackrel{(W[\mathcal{Q}])}{\to} [\varrho_0, \mathbf{m}_0, E_0]$

- $$\varrho_{0,n} > 0, \quad s(\varrho_{0,n}, \mathbf{m}_{0,n}, E_{0,n}) \geq s_0 > -\infty;$$
- $$[\varrho_{0,n}, \mathbf{m}_{0,n}, E_{0,n}] \to [\varrho_0, \mathbf{m}_0, E_0] \quad \text{in } L^1(\Omega; R^{N+2});$$
- the initial data $[\varrho_{0,n}, \mathbf{m}_{0,n}, E_{0,n}]$ give rise to a sequence of admissible weak solutions $[\varrho_n, \mathbf{m}_n, E_n]$ satisfying

$$\int_0^T \int_\Omega \left(\frac{\mathbf{m}_n \otimes \mathbf{m}_n}{\varrho_n} - \frac{1}{N} \frac{|\mathbf{m}_n|^2}{\varrho_n} \mathbb{I} \right) : \nabla_x^2 \varphi \ \mathrm{d}x\mathrm{d}t \to 0 \quad \text{as } n \to \infty$$

$$\text{for any } \varphi \in C_c^\infty(Q)$$

for some $Q \in \mathcal{Q}$

Localization – main result

Theorem (EF, Klingenberg, Markfelder Calc. Variations PDE 2020)

Let $s_0 \in R$ be given and $\Omega \subset R^N$, $N = 2, 3$ be a bounded smooth domain.
Let $\mathcal{Q}$ be a closed partition set in $(0, T) \times \Omega$.
Then the complement of the set of $\mathcal{Q}-$reachable data is an open dense set in $\mathrm{L}^1_{+,s_0}(\Omega; R^{N+2})$.

Example of $\mathcal{Q}$

$$\mathcal{Q} = \Big\{ Q \subset (0, T) \times \Omega \ \Big| \ Q = ((0, T) \times \Omega) \setminus (\cup_{i=1}^{M} H_i), \ H_i - \text{a hyperplane in } R^{N+1} \Big\},$$

Method of proof, I

- **Step 1**

$$[\varrho_0, \mathbf{m}_0, E_0] \ \boxed{\text{regular reachable data}}$$

$$[\varrho_{0,n}, \mathbf{m}_{0,n}, E_{0,n}] \stackrel{(W)}{\to} [\varrho_0, \mathbf{m}_0, E_0]$$

- **Step 2**
  $[\varrho_n, \mathbf{m}_n, E_n]$ generates a dissipative measure valued solution of the Euler system (Březina, EF 2018)
- **Step 3**

$$[\varrho_0, \mathbf{m}_0, E_0] \text{ regular } \Rightarrow \text{ existence of local smooth solution } [\varrho, \mathbf{m}, E]$$

- **Step 4**
  Weak strong uniqueness principle (Březina, EF 2018)

$$[\varrho_n, \mathbf{m}_n, E_n] \to [\varrho, \mathbf{m}, E] \text{ strongly on } [0, T_{\max})$$

Method of proof, II

- **Step 5**

$$\int_0^T \int_\Omega \left(\frac{\mathbf{m}_n \otimes \mathbf{m}_n}{\varrho_n} - \frac{1}{N} \frac{|\mathbf{m}_n|^2}{\varrho_n} \mathbb{I} \right) : \nabla_x^2 \varphi \ \mathrm{d}x\mathrm{d}t \to 0 \quad \text{as } n \to \infty$$

$$\Rightarrow$$

$$\int_0^T \int_\Omega \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} - \frac{1}{N} \frac{|\mathbf{m}|^2}{\varrho} \mathbb{I} \right) : \nabla_x^2 \varphi \ \mathrm{d}x\mathrm{d}t = 0 \quad \text{as } n \to \infty$$

for any $\varphi \in C_c((0, T_{\max}) \times \Omega)$

- **Step 6**

continuity of strong solution up to $t = 0$ $\Rightarrow$

$$\Rightarrow$$

$$\mathrm{div}_x \mathrm{div}_x \left(\frac{\mathbf{m}_0 \otimes \mathbf{m}_0}{\varrho_0} \right) - \frac{1}{N} \Delta_x \left(\frac{|\mathbf{m}_0|^2}{\varrho_0} \right) = 0 \quad \text{in } \Omega$$

- **Step 7**
  Identify the class of functions satisfying the limit elliptic PDE

Elliptic problem

$$\mathrm{div}_x \mathrm{div}_x \left(\frac{\mathbf{m}_0 \otimes \mathbf{m}_0}{\varrho_0} \right) - \frac{1}{N} \Delta_x \left(\frac{|\mathbf{m}_0|^2}{\varrho_0} \right) = 0$$

Weak formulation

$$\int_\Omega \left(\frac{\mathbf{m}_0 \otimes \mathbf{m}_0}{\varrho_0} \right) : \nabla_x^2 \varphi - \frac{1}{N} \frac{|\mathbf{m}_0|^2}{\varrho_0} \Delta_x \varphi \ \mathrm{d}x = 0 \qquad (1)$$

Solution set

- $\mathbf{w} = \frac{\mathbf{m}_0}{\varrho_0}$
- The set

$$\mathcal{S} = \left\{ \mathbf{w} \in L^2(\Omega; R^N) \ \middle| \ \mathbf{w} \text{ solves } (1) \right\}$$

-

$\mathcal{S}$ is closed in L^2

$\mathcal{S}$ is nowhere dense