Isomorphic Kottman constant of a Banach space

Tomasz Kania

Matematický ústav AV ČR, Praha

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joint work with J.M.F. Castillo, M. Gonzalez, and P. Papini

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The Elton–Odell theorem (1981). There exists $\varepsilon = \varepsilon(X)$ such that S_X contains a $(1+\varepsilon)$-separated sequence.

Symmetric separation in the separable case

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Theorem (Russo, '19)

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$\nu(X) = \sup_{i \neq k} |\langle f_i, x\rangle| + |\langle f_k, x\rangle|$, $\|x\|' = \max\{\|x\|, \nu(x)\}$.

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- $K(c_0) = 2 = \tilde{K}_s(c_0)$,
- $K(\ell_p) = 2^{1/p} = \tilde{K}_s(\ell_p)$;
- Kryczka–Prus: $K(X) \geqslant \sqrt[5]{4}$ for any non-reflexive X.

Preliminary observations

- For a countably incomplete ultrafilter $\mathscr{U}$ and a space X, we have

$$1 < K(X) \leqslant K_f(X) = K(X^{\mathscr{U}}) \leqslant 2,$$

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- There exists a space Z for which

$$K(Z) < K(Z^{**}),$$

  and it is easy to check that this space also satisfies $K_s(Z) < K_s(Z^{**})$. The said space is a J-sum of ℓ_1^n ($n \in \mathbb{N}$) in the sense of Bellenot; it has the property that $K(Z) < 2$, yet Z^{**} admits a quotient map onto ℓ_1 so that $K_s(Z^{**}) = 2$.

Castillo–González–K.–Papini

For every space X, $2 \leqslant K(X) \cdot K(X^*)$.

Based on a simple application of Ramsey's theorem:

Lemma

Let (x_n) be a bounded sequence in a Banach space. Then there exists an infinite subset M of $\mathbb{N}$ such that $\|x_i - x_j\|$ converges as $i, j \in M$, $i, j \to \infty$.

Proof.

X contains a basic seq. with basis constant at most $1+\varepsilon$: $(x_n)_{n=1}^\infty$ in X and $(x_n^*)_{n=1}^\infty$ in X^* with $\|x_n\| = 1$ and $\|x_n^*\| \leqslant 1+\varepsilon$ ($n \in \mathbb{N}$) s.t. $\langle x_i^*, x_j \rangle = \delta_{ij}$. For $i \neq j$,

$$2 = \langle x_i^* - x_j^*, x_i - x_j \rangle \leqslant \|x_i^* - x_j^*\| \cdot \|x_i - x_j\|.$$

Let us set $y_n^* = (1+\varepsilon)^{-1} x_n^*$. (Passing to a subsequence) $\|y_i^* - y_j^*\|$ and $\|x_i - x_j\|$ converge to k^* and to k, resp. in the sense of the Lemma. Then

$$2(1+\varepsilon)^{-1} \leqslant k^* \cdot k \leqslant K(X^*) \cdot K(X),$$

hence $2 \leqslant K(X) \cdot K(X^*)$.

Twisted sums

Castillo–González–K.–Papini

For a short exact sequence of Banach spaces

$$0 \to Y \to X \to Z \to 0,$$

we have

$$\tilde{K}(X) = \max\{\tilde{K}(Y), \tilde{K}(Z)\}.$$

Main idea: the constant is cts w.r.t. to the Kadets metric

$$d_K(M, N) = \inf \max \Big\{ \sup_{x \in iB_M} \operatorname{dist}(x, jB_N), \sup_{y \in jB_N} \operatorname{dist}(y, iB_M) \Big\},$$

where the inf is taken w.r.t all isometric embeddings i, j of M, N into common spaces.

- J.M.F. Castillo and P.L. Papini, On the Kottman constants in Banach spaces, *Banach Center Publ*. 92 (2011), 75–84.
- P. Hájek, T. Kania, and T. Russo, Symmetrically separated sequences in the unit sphere of a Banach space, *J. Funct. Anal.* 275 (2018), 3148–3168.
- C. A. Kottman, Subsets of the unit ball that are separated by more than one. *Studia Math.* 53 (1975), 15–27.

Preprints:

- P. Hájek, T. Kania, and T. Russo, Separated sets and Auerbach systems in Banach spaces, `arXiv:1803.11501`.
- J. M. F. Castillo, M. González, T. Kania, and P. Papini, The isomorphic Kottman constant of a Banach space, `arXiv:1910.01626`.