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On Kurzweil-Stieltjes integral in a Banach space

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Abstract

In the paper we deal with the Kurzweil-Stieltjes integration of functions having value in a Banach space X. We extend results obtained by Štefan Schwabik in [4], [8] and we complete the theory so that it will be well applicable to prove results on the continuous dependence of solutions to generalized linear differential equations in a Banach space. By Schwabik, the integral $\int_a^b \mathrm{d}[F]\, g$ exists if $F\colon [a,b] \to L(X)$ has a bounded semi-variation on $[a,b]$ and $g\colon [a,b] \to X$ is regulated on $[a,b]$. We prove that this integral has a sense also if F is regulated on $[a,b]$ and g has a bounded semi-variation on $[a,b]$. Furthermore, a general form of the integration by parts theorem proposed by Š. Schwabik in [8] is presented under the assumption not covered by [8] and the substitution formula is proved.

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Key words. Kurzweil-Stieltjes integral, substitution formula

1 Introduction

It is known that integration processes based on Riemann type sums, such as Kurzweil and McShane integrals, can be extended to Banach space-valued functions. Among other contributions it is worth to highlight the monograph by Schwabik and Ye [10], which studies these type of integrals and their connections e.g. with the classicals due to Bochner and Pettis.

Concerning integrals of Stieltjes type, Hönig presented a quite complete study in [3] dealing with the interior integral. In [5] and [8] Schwabik investigated some fundamental properties of the Kurzweil-Stieltjes integral.

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The concepts of generalized (nonlinear) Kurzweil or Kurzweil-Stieltjes integrals in a Banach space have been the background of several papers related to generalized differential equations like e.g [1], [2], [6] and [7].

In this paper we are dealing with the Kurzweil-Stieltjes integral. Our aim is to supplement the existing knowledge by results needed for treating generalized linear differential equations. In particular, we prove that if $F:[a,b]\to L(X)$ and $g:[a,b]\to X$, then the integral $\displaystyle\int_a^b \mathrm{d}[F]\,g$ exists provided F is regulated on $[a,b]$ and g has a bounded semi-variation on $[a,b]$, and the integral $\displaystyle\int_a^b F\,\mathrm{d}[g]$ exist provided F has a bounded semi-variation and g is regulated. Furthermore, a general form of the integration by parts theorem proposed by Š. Schwabik in [8] will be presented under the assumptions not covered by those from [8]. Finally, the substitution formula is proved.

2 Preliminaries

Throughout these notes X is a Banach space and $L(X)$ is the Banach space of bounded linear operators on X. By $\|\cdot\|_X$ we denote the norm in X. Similarly, $\|\cdot\|_{L(X)}$ denotes the usual operator norm in $L(X)$.

Assume that $-\infty<a<b<+\infty$ and $[a,b]$ denotes the corresponding closed interval. A set $D=\{\alpha_0,\alpha_1,\dots,\alpha_m\}\subset[a,b]$ is said to be a division of $[a,b]$ if

$$a=\alpha_0<\alpha_1<\ \dots\ <\alpha_m=b\,.$$

The set of all divisions of $[a,b]$ is denoted by $\mathcal{D}[a,b]$.

A function $f:[a,b]\to X$ is called a finite step function on $[a,b]$ if there exists a division $D=\{\alpha_0,\alpha_1,\dots,\alpha_m\}$ of $[a,b]$ such that f is constant on every open interval (α_{j-1},α_j), $j=1,2,\dots,m$.

For an arbitrary function $f:[a,b]\to X$ we set

$$\|f\|_\infty=\sup_{t\in[a,b]}\|f(t)\|_X$$

and

$$\operatorname{var}_a^b f=\sup_{D\in\mathcal{D}[a,b]}\sum_{j=1}^m\|f(\alpha_j)-f(\alpha_{j-1})\|_X$$

is the variation of f over $[a,b]$. If $\operatorname{var}_a^b f<\infty$ we say that f is a function of bounded variation on $[a,b]$. $BV([a,b],X)$ denotes the Banach space of functions $f:[a,b]\to X$ of bounded variation on $[a,b]$ equipped with the norm $\|f\|_{BV}=\|f(a)\|_X+\operatorname{var}_a^b f$.

For $F\,[a,b\,]\to L(X)$ and a division $D=\{\alpha_0,\alpha_1,\dots,\alpha_m\}$ of the interval $[a,b\,]$, let

$$V_a^b(F,D)=\sup\Big\{\|\sum_{j=1}^m[F(\alpha_j)-F(\alpha_{j-1})]\,y_j\big\|_X\Big\},$$

where the supremum is taken over all possible choices of $y_j\in X$, $j=1,2,\dots,m$, with $\|y_j\|_X\le 1$. Then

$$(\mathcal{B})\,\mathrm{var}_a^b(F)=\sup\{V_a^b(x,D);D\in\mathcal{D}[a,b\,]\}\,.$$

is said to be a *semi-variation of* F on $[a,b\,]$, cf. e.g. [3]. Sometimes it is called also a $\mathcal{B}$*-variation of* F on $[a,b\,]$ (with respect to the bilinear triple $\mathcal{B}=(L(X),X,X)$, cf. e.g. [5]). Analogously, we can define the $\mathcal{B}$-variation of a function $f:[a,b\,]\to X$ using

$$V_a^b(f,D)=\sup\Big\{\|\sum_{j=1}^m F_j\,[f(\alpha_j)-f(\alpha_{j-1})]\big\|_X\Big\},$$

where the supremum is taken over all possible choices of operators $F_j\in L(X)$ with $\|F_j\|_{L(X)}\le 1$, $j=1,2,\dots,m$.

The set of all functions $F:[a,b\,]\to L(X)$ with $(\mathcal{B})\,\mathrm{var}_a^b(F)<\infty$ is denoted by $(\mathcal{B})\,BV([a,b\,],L(X))$. Similarly as in the case of bounded variation functions, the set $(\mathcal{B})\,BV([a,b\,],L(X))$ is a Banach space with respect to the norm

$$F\in(\mathcal{B})\,BV([a,b\,],L(X))\to\|F\|_{SV}=\|F(a)\|_{L(X)}+(\mathcal{B})\,\mathrm{var}_a^b F$$

(cf. [9])

A function $f:[a,b\,]\to X$, is said to be regulated on $[a,b\,]$ if for each $t\in[a,b)$ there is $f(t+)\in X$ such that

$$\lim_{s\to t+}\|f(s)-f(t+)\|_X=0$$

and for each $t\in(a,b\,]$ there is $f(t-)\in X$ such that

$$\lim_{s\to t-}\|f(s)-f(t-)\|_X=0\,.$$

By $G([a,b\,],X)$ we denote the set of all regulated functions $f:[a,b\,]\to X$. For $t\in[a,b)$, $s\in(a,b\,]$ we put $\Delta^+f(t)=f(t+)-f(t)$ and $\Delta^-f(s)=f(s)-f(s-)$. Recall that

$$BV([a,b\,],X)\subset G([a,b\,],X)\cap(\mathcal{B})BV([a,b\,],X),$$

while $(\mathcal{B})BV([a,b\,],X)\not\subseteq G([a,b\,],X)$ (cf. e.g. [6, 1.5]). Moreover, it is known that regulated function are uniform limits of finite step functions (see [3, Theorem I.3.1]).

Now, let us recall the definition and some crucial properties of the Kurzweil-Stieltjes integral.

As usual, tagged systems $P=(D,\xi)\in\mathcal{D}[a,b\,]\times[a,b\,]^m$ where $D=\{\alpha_0,\alpha_1,\dots,\alpha_m\}$, $\xi=(\xi_1,\xi_2,\dots,\xi_m)$, are called *partitions* of $[a,b\,]$ if

$$\alpha_{j-1}\le\xi_j\le\alpha_j \quad \text{for} \quad j=1,2,\dots,m\,.$$

The set of all partitions of $[a,b\,]$ is denoted by $\mathcal{P}[a,b\,]$.

Furthermore, functions $\delta{:}[a,b\,]\to(0,\infty)$ are said to be *gauges* on $[a,b\,]$. Given a gauge δ, the partition $P=(D,\xi)$ with $D=\{\alpha_0,\alpha_1,\dots,\alpha_m\}$, $\xi=(\xi_1,\xi_2,\dots,\xi_m)$, is *$\delta$-fine* if

$$[\alpha_{j-1},\alpha_j]\subset(\xi_j-\delta(\xi_j),\xi_j+\delta(\xi_j)) \quad \text{for} \quad j=1,2,\dots,m\,.$$

We remark that for an arbitrary gauge δ on $[a,b\,]$ there always exists a δ-fine partition of $[a,b\,]$. This is stated by the Cousin lemma (see [4, Lemma 1.4]).

For given functions $F:[a,b\,]\to L(X)$ and $g:[a,b\,]\to X$ and a partition $P=(D,\xi)$ of $[a,b\,]$, where $D=\{\alpha_0,\alpha_1,\dots,\alpha_m\}$, $\xi=(\xi_1,\dots,\,\xi_m)$, we define

$$S(F,\mathrm{d}g,P)=\sum_{j=1}^m F(\xi_j)\,[g(\alpha_j)-g(\alpha_{j-1})]$$

and

$$S(\mathrm{d}F,g,P)=\sum_{j=1}^m[F(\alpha_j)-F(\alpha_{j-1})]\,g(\xi_j)\,.$$

We say that $I\in X$ is the Kurzweil-Stieltjes integral (or shortly KS-integral) of F with respect to g on $[a,b\,]$ and denote

$$I=\int_a^b F\,\mathrm{d}[g]$$

if for every $\varepsilon>0$ there exists a gauge δ on $[a,b\,]$ such that

$$\Big\|S(F,\mathrm{d}g,P)-I\Big\|_X<\varepsilon \quad \text{for all} \quad \delta-\text{fine partitions} \quad P \text{ of } \ [a,b\,]\,.$$

Similarly, $J\in X$ is the KS-integral of g with respect to F on $[a,b\,]$ if for every $\varepsilon>0$ there exists a gauge δ on $[a,b\,]$ such that

$$\Big\|S(\mathrm{d}F,g,P)-J\Big\|_X<\varepsilon \quad \text{for all} \quad \delta-\text{fine partitions} \quad P \text{ of } \ [a,b\,]\,.$$

In this case we write $J=\displaystyle\int_a^b \mathrm{d}[F]\,g$.

Analogously, if $H:[a,b\,]\to L(X)$, we define the integral $\displaystyle\int_a^b H\,\mathrm{d}[F]\,g$ using sums of the form

$$S(H,\mathrm{d}F,g,P)=\sum_{j=1}^m H(\xi_j)[F(\alpha_j)-F(\alpha_{j-1})]\,g(\xi_j).$$

The KS-integral is linear and additive with respect to intervals. Basic results concerning KS-integral can be found in [5] and [12]. Obviously, if the Riemann-Stieltjes integral (RS)$\int_a^b F\,\mathrm{d}[g]$ exists, then the KS-integral $\int_a^b F\,\mathrm{d}[g]$ also exists and

$$\int_a^b F\,\mathrm{d}[g] = (\mathrm{RS})\int_{\mathrm{a}}^{\mathrm{b}} \mathrm{F}\,\mathrm{d}[\mathrm{g}].$$

Some of the further results needed later are summarized in the following assertions:

2.1 . Proposition. *Let $F:[a,b]\to L(X)$ and $g:[a,b]\to X$.*

(i) [5, Proposition 10] *Let $F\in(\mathcal{B})\,BV([a,b],L(X))$ and $g:[a,b]\to X$ be such that $\int_a^b \mathrm{d}[F]\,g$ exists. Then*

$$\Big\|\int_a^b \mathrm{d}[F]\,g\Big\|_X \le (\mathcal{B})\,(\mathrm{var}_a^b F)\,\|g\|_\infty\,.$$

(ii) [5, Proposition 11] *Let $F\in(\mathcal{B})\,BV([a,b],L(X))$ and $g_n:[a,b]\to X$ be such that $\int_a^b \mathrm{d}[F]\,g_n$ exists for all $n\in\mathbb{N}$ and $\lim_{n\to\infty}\|g_n-g\|_\infty=0$. Then*

$$\int_a^b \mathrm{d}[F]\,g \quad \textit{exists and} \quad \int_a^b \mathrm{d}[F]\,g = \lim_{n\to\infty}\int_a^b \mathrm{d}[F]\,g_n\,.$$

(iii) [5, Proposition 15]*If $F\in(\mathcal{B})BV([a,b],L(X))$ and $g\in G([a,b],X)$ then $\int_a^b \mathrm{d}[F]\,g$ exists.*

(iv) [8, Theorem 13]*If $F\in G([a,b],L(X))\cap(\mathcal{B})BV([a,b],L(X))$ and $g\in BV([a,b],X)$ then both the integrals $\int_a^b F\,\mathrm{d}[g]$ and $\int_a^b \mathrm{d}[F]\,g$ exist, the sum*

$$\sum_{a\le\tau<b}\Delta^+F(\tau)\,\Delta^+g(\tau) - \sum_{a<\tau\le b}\Delta^-F(\tau)\,\Delta^-g(\tau)$$

converges in X and the equality

$$\int_a^b F\,\mathrm{d}[g] + \int_a^b \mathrm{d}[F]\,g$$
$$= F(b)\,g(b) - F(a)\,g(a) - \sum_{a\le t<b}\Delta^+F(t)\,\Delta^+g(t) + \sum_{a<t\le b}\Delta^-F(t)\,\Delta^-g(t)$$

is true.

3 Main results

In this section we will present our main results. First, we will prove two auxiliary properties of the KS-integral which, in the case that $X \neq \mathbb{R}^n$, are not available in the literature.

3.1 . Lemma. (i) *Let* $F \in (\mathcal{B})BV([a,b],L(X))$, $g \in G([a,b],X)$ *be such that* $\int_a^b F\,\mathrm{d}[g]$ *exists. Then*

$$\|S(F,\mathrm{d}g,P)\|_X \le 2\,\|F\|_{SV}\,\|g\|_\infty\,, \quad \textit{for each } P \in \mathcal{P}[a,b] \tag{3.1}$$

and

$$\Big\|\int_a^b F\,\mathrm{d}[g]\Big\|_X \le 2\,\|F\|_{SV}\,\|g\|_\infty. \tag{3.2}$$

(ii) *Let* $F \in G([a,b],L(X))$, $g \in (\mathcal{B})\,BV([a,b],X)$ *be such that* $\int_a^b \mathrm{d}[F]\,g$ *exists. Then*

$$\|S(\mathrm{d}F,g,P)\|_X \le 2\,\|F\|_\infty\,\|g\|_{SV} \quad \textit{for each} \;\; P \in \mathcal{P}[a,b]. \tag{3.3}$$

and

$$\Big\|\int_a^b \mathrm{d}[F]\,g\Big\|_X \le 2\,\|F\|_\infty\,\|g\|_{SV}. \tag{3.4}$$

Proof. It is easy to check that, for an arbitrary partition $P=(D,\xi)$ of $[a,b]$ with $D=\{\alpha_0,\alpha_1,\dots,\alpha_m\}$ and $\xi=(\xi_1,\xi_2,\dots,\xi_m)$, we have

$$\begin{aligned}
&S(F,\mathrm{d}g,P)\\
&\quad = F(\xi_1)\,[g(\alpha_1)-g(a)] + F(\xi_2)\,[g(\alpha_2)-g(\alpha_1)] + \dots + F(\xi_m)\,[g(b)-g(\alpha_{m-1})]\\
&\quad = F(b)\,g(b) - F(a)g\,(a)\\
&\qquad - \,[F(\xi_1)-F(a)]\,g(a) - [F(\xi_2)-g(\xi_1)] - \dots - [F(b)-F(\xi_m)]\,g(b)\\
&\quad = F(b)\,g(b) - F(a)\,g(a) - \sum_{j=0}^{m}[F(\xi_{j+1})-F(\xi_j)]\,g(\alpha_j)\,,
\end{aligned}$$

where $\xi_0 = a$ and $\xi_{m+1} = b$. Consequently

$$\begin{aligned}
\|S(F,\mathrm{d}g,P)\|_X &\le \big(\|F(a)\|_{L(X)} + \|F(b)\|_{L(X)}\big)\,\|g\|_\infty\\
&\qquad + \Big\|\sum_{j=0}^{m}[F(\xi_{j+1})-F(\xi_j)]\,\frac{g(\alpha_i)}{\|g(\alpha_i)\|_X}\,\|g(\alpha_i)\|_X\Big\|_X\\
&\le \left(\|F(a)\|_{L(X)} + \|F(b)\|_{L(X)} + \Big\|\sum_{j=0}^{m}[F(\xi_{j+1})-F(\xi_j)]\,\frac{g(\alpha_i)}{\|g(\alpha_i)\|_X}\Big\|_X\right)\|g\|_\infty\\
&\le \big(\|F(a)\|_{L(X)} + \|F(b)\|_{L(X)} + (\mathcal{B})\,\mathrm{var}_a^b F\big)\,\|g\|_\infty \le 2\,\|F\|_{SV}\,\|g\|_\infty\,,
\end{aligned}$$

i.e. (3.1) is true.

Now, let an arbitrary $\varepsilon > 0$ be given. By our assumptions there is a gauge δ on $[a,b]$ such that

$$\Big\| S(F, \mathrm{d}g, P) - \int_a^b F \, \mathrm{d}[g] \Big\|_X < \varepsilon \quad \text{whenever } P \text{ is } \delta - \text{fine.}$$

Let P_0 be an arbitrary δ-fine partition of $[a,b]$. Then by (3.1) we have

$$\begin{aligned} \Big\| \int_a^b F \, \mathrm{d}[g] \Big\|_X &\le \Big\| S(F, \mathrm{d}g, P_0) - \int_a^b F \, \mathrm{d}[g] \Big\|_X + \| S(F, \mathrm{d}g, P_0) \|_X \\ &< \varepsilon + 2 \, \|F\|_{SV} \, \|g\|_\infty \end{aligned}$$

Since $\varepsilon > 0$ can be arbitrary, it follows that inequality (3.2) is true.

The proof of (3.3) and (3.4) can be obtained in a similar way. □

3.2. Lemma. *Let $g : [a,b] \to X$ be a finite step function. Then for any $F : [a,b] \to L(X)$ the integral $\int_a^b F \, \mathrm{d}[g]$ exists.*

Proof. One can check that $g : [a,b] \to L(X)$ is a finite step function if and only if it is a finite linear combination of the functions of the form

$$\chi_{[a,\tau]}(t) \, \widetilde{x}, \ \chi_{[\sigma,b]}(t) \, \widetilde{y}, \ \chi_{[a]}(t) \, \widetilde{z}, \ \chi_{[b]}(t) \, \widetilde{w},$$

where τ, σ are some points from (a,b) and $\widetilde{x}, \widetilde{y}, \widetilde{z}, \widetilde{w}$ may be arbitrary elements of X. Hence, by the linearity of the integral, it is sufficient to prove the formula (3.8) for functions g of the form:

$$\chi_{[a,\tau]} \widetilde{x}, \quad \chi_{[\tau,b]} \widetilde{x}, \quad \chi_{[a]} \, \widetilde{x}, \quad \chi_{[b]} \, \widetilde{x},$$

where $\tau \in (a,b)$ and $\widetilde{x} \in X$.

Let $\tau \in (a,b)$, $\widetilde{x} \in X$ and $g = \widetilde{x} \, \chi_{[a,\tau]}$. Given $\varepsilon > 0$ define

$$\delta(t) = \begin{cases} \varepsilon & \text{if } t = \tau. \\ \frac{1}{2} |\tau - t| & \text{if } t \ne \tau \end{cases}$$

Then, for any δ-fine partition P of $[a,b]$, τ is the tag and $S(F, \mathrm{d}g, P) = -F(\tau) \, \widetilde{x}$. Hence

$$\int_a^b F \, \mathrm{d}[g] = -F(\tau) \, \widetilde{x} \, .$$

The proofs of the cases $g = \chi_{[\tau,b]} \, \widetilde{x}$, $g = \chi_{[a]} \, \widetilde{x}$ and $g = \chi_{[b]} \, \widetilde{x}$ are analogous. □

Next theorem is the first main result of this paper. It supplements the Schwabik's existence result stated in Proposition 2.1 (iii).

3.3. Theorem. (i) *If $F \in G([a,b], L(X))$, $g \in (\mathcal{B})BV([a,b], X)$, then the integral $\int_a^b \mathrm{d}[F] \, g$ exists.*

(ii) *If* $F \in (\mathcal{B})\,BV([a,b],L(X))$, $g \in G([a,b],X)$, *then the integral* $\displaystyle\int_a^b F\,\mathrm{d}[g]$ *exists.*

Proof. (i) Let $F_n : [a,b] \to L(X)$, $n \in \mathbb{N}$, be a sequence of finite step functions such that

$$\lim_{n\to\infty} \|F_n - F\|_\infty = 0.$$

Since $F_n \in BV([a,b],L(X))$ for each $n \in \mathbb{N}$, it follows from Proposition 2.1 (iii) that for each $n \in \mathbb{N}$ the integral $\displaystyle\int_a^b \mathrm{d}[F_n]\,g$ exists. Moreover, these integrals define a Cauchy sequence in the Banach space X. Indeed, given $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that $\|F_n - F\|_\infty < \varepsilon$, for $n \ge n_0$. Thus, using Lemma 3.1, we obtain

$$\Big\| \int_a^b \mathrm{d}[F_n - F_m]\,g \Big\|_X \le 2\,\|F_n - F_m\|_\infty\,\|g\|_{SV} \le 4\,\varepsilon\,\|g\|_{SV} \quad \text{for all} \quad m,n \ge n_0.$$

Therefore there is $I \in X$ such that $I = \displaystyle\lim_{n\to\infty}\int_a^b \mathrm{d}[F_n]\,g$. This implies that there exists $N \in \mathbb{N}$ such that $N \ge n_0$ and

$$\Big\| \int_a^b \mathrm{d}[F_N]\,g - I \Big\|_X < \varepsilon.$$

Let δ be a gauge on $[a,b]$ such that

$$\Big\| S(\mathrm{d}F_N, g, P) - \int_a^b \mathrm{d}[F_N]\,g \Big\|_X < \varepsilon \quad \text{whenever} \quad P \text{ is } \delta-\text{fine}.$$

Having this in mind and using (3.3), for an arbitrary δ-fine partition P of $[a,b]$, we get

$$\begin{aligned}
&\Big\| S(\mathrm{d}F, g, P) - I \Big\|_X \\
&\quad \le \Big\| S(\mathrm{d}F, g, P) - S(\mathrm{d}F_N, g, P) \Big\|_X + \Big\| S(\mathrm{d}F_N, g, P) - \int_a^b \mathrm{d}[F_N]\,g \Big\|_X \\
&\qquad + \Big\| \int_a^b \mathrm{d}[F_N]\,g - I \Big\|_X \\
&\quad < 2\,\|F - F_N\|_\infty\,\|g\|_{SV} + 2\varepsilon < 2\,\varepsilon\,(\|g\|_{SV} + 1),
\end{aligned}$$

which concludes the proof of the assertion (i).

The assertion (ii) can be proved by the same arguments using Lemma 3.2 instead of Proposition 2.1. □

A direct consequence of Lemma 3.1 and Theorem 3.3 is the following assertion.

3.4. Corollary. (i) *Let* $g, g_n \in G([a,b],X)$, $n \in \mathbb{N}$ *be such that* $\lim_{n\to\infty}\|g_n - g\|_\infty = 0$. *Then for any* $F \in (\mathcal{B})\,BV([a,b],L(X))$, *the integrals*

$$\int_a^b F\,\mathrm{d}[g] \quad \text{and} \quad \int_a^b F\,\mathrm{d}[g_n],\ n \in \mathbb{N},$$

exist and

$$\lim_{n\to\infty}\int_a^b F\,\mathrm{d}g_n=\int_a^b F\,\mathrm{d}g.$$

(ii) *Let $F,\ F_n\in G([a,b],L(X))$, $n\in\mathbb{N}$ be such that* $\lim_{n\to\infty}\|F_n-F\|_\infty=0$. *Then for any $g\in(\mathcal{B})\,BV([a,b],X)$, the integrals*

$$\int_a^b \mathrm{d}[F]\,g \quad \textit{and} \quad \int_a^b \mathrm{d}[F_n]\,g,\ n\in\mathbb{N},$$

exist and

$$\lim_{n\to\infty}\int_a^b \mathrm{d}[F_n]\,g=\int_a^b \mathrm{d}[F]\,g.$$

Thanks to Theorem 3.3, we are now also able to extend the integration by parts theorem by Schwabik (cf. Proposition 2.1 (iv) or [8, Theorem 10]) and the Substitution Theorem (cf. e.g. [13, Theorem 2.3.19] for $X=\mathbb{R}^n$) to the form more suitable for applications to generalized differential equations. This will be the content of the rest of these notes. Whenever we treat functions of bounded variation, we believe to be able to extend in a close future the corresponding results to regulated functions having a bounded semi-variation.

3.5. Lemma. (i) *If $F\in G([a,b],L(X))$ and $g\in BV([a,b],X)$, then*

$$\sum_{t\in[a,b)}\|\Delta^+F(t)\Delta^+g(t)\|_X+\sum_{t\in(a,b]}\|\Delta^-F(t)\Delta^-g(t)\|_X\le 2\,\|F\|_\infty\,\mathrm{var}_a^b g\,. \tag{3.5}$$

(ii) *If $F\in BV([a,b],L(X))$ and $g\in G([a,b],X)$, then*

$$\sum_{t\in[a,b)}\|\Delta^+F(t)\Delta^+g(t)\|_X+\sum_{t\in(a,b]}\|\Delta^-F(t)\Delta^-g(t)\|_X\le 2\,(\mathrm{var}_a^b F)\,\|g\|_\infty\,.$$

Proof. (i) Let $F\in G([a,b],L(X))$ and $g\in BV([a,b],X)$. It is known that the points of discontinuities of a regulated function are at most countable (see [3, Corollary I.3.2.b]). Let $\{s_k\}$ be the set of common points of discontinuity of the functions F and g in (a,b), so we can write

$$\begin{aligned}&\sum_{t\in[a,b)}\|\Delta^+F(t)\,\Delta^+g(t)\|_X+\sum_{t\in(a,b]}\|\Delta^-F(t)\,\Delta^-g(t)\|_X\\&\quad=\|\Delta^+F(a)\,\Delta^+g(a)\|_X+\|\Delta^-F(b)\,\Delta^-g(b)\|_X\\&\qquad+\sum_{k=1}^{\infty}\Big[\|\Delta^+F(s_k)\,\Delta^+g(s_k)\|_X+\|\Delta^-F(s_k)\,\Delta^-g(s_k)\|_X\Big]\end{aligned}$$

For $n\in\mathbb{N}$, define

$$\begin{aligned}S_n&=\|\Delta^+F(a)\,\Delta^+g(a)\|_X+\|\Delta^-F(b)\,\Delta^-g(b)\|_X\\&\quad+\sum_{k=1}^{n}\big[\|\Delta^-F(s_k)\,\Delta^-g(s_k)\|_X+\|\Delta^+F(s_k)\,\Delta^+g(s_k)\|_X\big]\,.\end{aligned}$$

Let $\varepsilon>0$ and $n\in\mathbb{N}$ be given and let $\{t_1,t_2,\dots,t_n\}\subset(a,b)$ be such that

$$\{t_1,t_2,\dots,t_n\}=\{s_1,s_2,\dots,s_n\} \quad\text{and}\quad a<t_1<t_2<\dots<t_n<b\,.$$

Then,

$$\begin{aligned}S_n &= \|\Delta^+F(a)\,\Delta^+g(a)\|_X+\|\Delta^-F(b)\,\Delta^-g(b)\|_X\\ &\quad+\sum_{k=1}^n\big[\|\Delta^-F(t_k)\,\Delta^-g(t_k)\|_X+\|\Delta^+F(t_k)\,\Delta^+g(t_k)\|_X\big]\,.\end{aligned}$$

Furthermore, for each $k=1,2,\dots,n$, choose $\delta_k>0$ in such a way that

$$\|g(t_k+\delta_k)-g(t_k+)\|_X<\frac{\varepsilon}{8\,(n+1)\,\|F\|_\infty},$$

$$\|g(t_k-\delta_k)-g(t_k-)\|_X<\frac{\varepsilon}{8\,(n+1)\,\|F\|_\infty}$$

and

$$[t_k-\delta_k,t_k+\delta_k]\cap\{t_1,t_2,\dots,t_n\}=\{t_k\}\,.$$

Analogously, let $\delta_0>0$ be such that

$$a+\delta_0<t_1 \quad\text{and}\quad \|g(a+\delta_0)-g(a+)\|_X<\frac{\varepsilon}{8\,\|F\|_\infty}$$

and

$$b-\delta_0>t_n \quad\text{and}\quad \|g(b-)-g(b-\delta_0)\|_X<\frac{\varepsilon}{8\,\|F\|_\infty}\,.$$

Now, noting that

$$\|\Delta^+F(t)\|_{L(X)}\le 2\,\|F\|_\infty \quad\text{for } t\in[a,b)$$

and

$$\|\Delta^-F(t)\|_{L(X)}\le 2\,\|F\|_\infty \quad\text{for } t\in(a,b\,],$$

we can see that

$$\begin{aligned}S_n \le\ & 2\,\|F\|_\infty\Big(\|g(a+)-g(a+\delta_0)\|_X+\|g(a+\delta_0)-g(a)\|_X\Big)\\ &+2\,\|F\|_\infty\sum_{k=1}^n\|g(t_k+)-g(t_k+\delta_k)\|_X\\ &+2\,\|F\|_\infty\sum_{k=1}^n\|g(t_k+\delta_k)-g(t_k)\|_X\\ &+2\,\|F\|_\infty\sum_{k=1}^n\|g(t_k-)-g(t_k-\delta_k)\|_X\\ &+2\,\|F\|_\infty\sum_{k=1}^n\|g(t_k-\delta_k)-g(t_k)\|_X\end{aligned}$$

$$
\begin{aligned}
&\quad + 2\,\|F\|_\infty \Big(\|g(b) - g(b-\delta_0)\|_X + g(b-\delta_0) - g(b-)\|_X \Big) \\
&< \frac{\varepsilon}{4} + 2\,\|F\|_\infty\,\|g(a+\delta_0) - g(a)\|_X \\
&\quad + \frac{n\,\varepsilon}{4\,(n{+}1)} + 2\,\|F\|_\infty \sum_{k=1}^{n} \|g(t_k+\delta_k) - g(t_k)\|_X \\
&\quad + \frac{n\,\varepsilon}{4\,(n{+}1)} + 2\,\|F\|_\infty \sum_{k=1}^{n} \|g(t_k) - g(t_k-\delta_k)\|_X \\
&\quad + 2\,\|F\|_\infty\,\|g(b) - g(b-\delta_0)\|_X + \frac{\varepsilon}{4}\,.
\end{aligned}
$$

To summarize, we have

$$
\begin{aligned}
S_n < \varepsilon + 2\,\|F\|_\infty &\Big(\|g(a+\delta_0) - g(a)\|_X + \sum_{k=1}^{n} \|g(t_k+\delta_k) - g(t_k)\|_X \Big) \\
+ 2\,\|F\|_\infty &\Big(\sum_{k=1}^{n} \|g(t_k) - g(t_k-\delta_k)\|_X + \|g(b) - g(b-\delta_0)\|_X \Big).
\end{aligned}
$$

This implies that $S_n \le \varepsilon + 2\,\|F\|_\infty\,(\mathrm{var}_a^b g)$ holds for any $n \in \mathbb{N}$. Moreover, as $\varepsilon > 0$ can be arbitrarily small, we finally deduce that

$$
S_n \le 2\,\|F\|_\infty\,(\mathrm{var}_a^b g) \quad \text{for any} \quad n \in \mathbb{N}\,,
$$

wherefrom the desired estimate (3.5) follows.

(ii) Similarly, we could proceed if $F \in BV([a,b], L(X))$ and $g \in G([a,b], X)$. □

3.6 . Corollary. (Integration by parts.) *Let $F \in BV([a,b], L(X))$ and $g \in G([a,b], X)$ (or $F \in G([a,b], L(X))$ and $g \in BV([a,b], X)$). Then both the integrals*

$$
\int_a^b F\,\mathrm{d}[g] \quad \textit{and} \quad \int_a^b \mathrm{d}[F]\,g
$$

exist and

$$
\left.
\begin{aligned}
&\int_a^b F\,\mathrm{d}[g] + \int_a^b \mathrm{d}[F]\,g \\
&\quad = F(b)\,g(b) - F(a)\,g(a) - \sum_{a\le t<b} \Delta^+ F(t)\,\Delta^+ g(t) + \sum_{a<t\le b} \Delta^- F(t)\,\Delta^- g(t)
\end{aligned}
\right\} \tag{3.6}
$$

holds.

Proof. a) Let $F \in BV([a,b], L(X))$, $g \in G([a,b], X)$ and let $\{g_n\}$ be a sequence of finite step functions on $[a,b]$ which tends uniformly to g on $[a,b]$. Then by Proposition 2.1 (iv) we have

$$
\left.
\begin{aligned}
&\int_a^b F\,\mathrm{d}[g_n] + \int_a^b \mathrm{d}[F]\,g_n - F(b)\,g_n(b) + F(a)\,g_n(a) \\
&\quad = \sum_{a\le t<b} \Delta^- F(t)\,\Delta^- g_n(t) - \sum_{a<t\le b} \Delta^- F(t)\,\Delta^- g_n(t)
\end{aligned}
\right\} \tag{3.7}
$$

for any $n\in\mathbb{N}$. Thus, by Corollary 3.4, the relation

$$\lim_{n\to\infty}\left(\int_a^b F\,\mathrm{d}[g_n]+\int_a^b \mathrm{d}[F]\,g_n-F(b)\,g_n(b)+F(a)\,g_n(a)\right)$$
$$=\int_a^b F\,\mathrm{d}[g]+\int_a^b \mathrm{d}[F]\,g-F(b)\,g(b)+F(a)\,g(a)$$

holds. Further, by Lemma 3.5 (ii) the estimate

$$\sum_{a\le t<b}\|\Delta^+F(t)\,\Delta^+(g(t)-g_n(t))\|_X+\sum_{a<t\le b}\|\Delta^-F(t)\,\Delta^-(g(t)-g_n(t))\|_X$$
$$\le 2\,(\mathrm{var}_a^bF)\,\|g-g_n\|_\infty$$

is true. Consequently,

$$\lim_{n\to\infty}\left(\sum_{a\le t<b}\Delta^+F(t)\,\Delta^+g_n(t)-\sum_{a<t\le b}\Delta^-F(t)\,\Delta^-g_n(t)\right)$$
$$=\sum_{a\le t<b}\Delta^+F(t)\,\Delta^+g(t)-\sum_{a<t\le b}\Delta^-F(t)\,\Delta^-g(t).$$

To summarize, letting $n\to\infty$ in (3.7), we obtain (3.6).

b) Similarly, we could proceed if $F\in G([a,b],L(X))$ and $g\in BV([a,b],X)$. □

Now, notice that using arguments analogous to those from the proofs of the assertions (i) and (ii) in Proposition 2.1 we can justify the following proposition.

3.7. Proposition. *Let F, $H:[a,b]\to L(X)$ and $g:[a,b]\to X$.*

(i) *If $F\in BV([a,b],L(X))$, $H:[a,b]\to L(X)$ and $g:[a,b]\to X$ are such that $\int_a^b H\,\mathrm{d}[F]\,g$ exists, then*

$$\left\|\int_a^b H\,\mathrm{d}[F(s)]\,g(s)\right\|_X\le\|H\|_\infty\,(\mathrm{var}_a^bF)\,\|g\|_\infty\,.$$

(ii) *Let $F\in BV([a,b],L(X))$, $H_n:[a,b]\to L(X)$, $n\in\mathbb{N}$ and let $g\colon[a,b]\to X$ be bounded and such that $\int_a^b H_n\,\mathrm{d}[F]\,g$ exists. If $\lim_{n\to\infty}\|H_n-H\|_\infty=0$, then the integral $\int_a^b H\,\mathrm{d}[F]\,g$ exists and*

$$\lim_{n\to\infty}\left\|\int_a^b H_n\,\mathrm{d}[F]\,g-\int_a^b H\,\mathrm{d}[F]\,g\right\|_X=0\,.$$

3.8. Theorem. (Substitution Theorem.) *Let $F \in BV([a,b], L(X))$ and let $g:[a,b] \to X$ be bounded and such that the integral $\int_a^b \mathrm{d}[F]\,g$ exists. Then both the integrals*

$$\int_a^b H(s)\,\mathrm{d}_s\Big[\int_a^s \mathrm{d}[F]\,g\Big] \quad \textit{and} \quad \int_a^b H\,\mathrm{d}[F]\,g$$

exist and the equality

$$\int_a^b H(s)\,\mathrm{d}\Big[\int_a^t \mathrm{d}[F]\,g\Big] = \int_a^b H\,\mathrm{d}[F]\,g \tag{3.8}$$

holds for each $H \in G([a,b], L(X))$.

Proof. *Step* 1. First, we show that (3.8) holds for each finite step function $H:[a,b]\to L(X)$.

By the linearity of the integral and since a finite step function $H\colon [a,b] \to L(X)$ is a finite linear combination of the functions of the form

$$\chi_{[a,\tau]}(t)\,\widetilde{H}_1,\ \chi_{[\sigma,b]}(t)\,\widetilde{H}_2,\ \chi_{[a]}(t)\,\widetilde{H}_3,\ \chi_{[b]}(t)\,\widetilde{H}_4,$$

where τ, $\sigma \in (a,b)$ and $\widetilde{H}_i \in L(X)$ $i = 1,2,3,4$, it is enough to justify (3.8) for functions H of such a form.

Let $\tau \in (a,b)$, $\widetilde{H} \in L(X)$, $H = \chi_{[a,\tau]}(t)\,\widetilde{H}$ and $K(t) = \int_a^t \mathrm{d}[F]\,g$ for $t \in [a,b]$.

Obviously,

$$\int_a^\tau H\,\mathrm{d}[F]\,g = \int_a^\tau H\,\mathrm{d}[K] = \widetilde{H}\int_a^\tau \mathrm{d}[F]\,g\,. \tag{3.9}$$

Let $\varepsilon > 0$ be given and let

$$\delta(t) = \begin{cases} \varepsilon & \text{if } t = \tau. \\ \frac{1}{2}|\tau - t| & \text{if } \tau < t \le b\,. \end{cases}$$

Then, for any δ-fine partition P of $[\tau,b]$ with $D = \{\alpha_0, \alpha_1, \dots, \alpha_m\}$ and $\xi = (\xi_1, \xi_2 \dots, \xi_m)$ we have $\xi_1 = \alpha_0 = \tau$, $\alpha_1 < \tau + \varepsilon$ and

$$S(H, \mathrm{d}F, g, P) = \widetilde{H}\,[F(\alpha_1) - F(\tau)]\,g(\tau) \quad \text{and} \quad S(H, \mathrm{d}K, P) = \widetilde{H}\,[K(\alpha_1) - K(\tau)].$$

As a result and as a consequence of the Hake theorem for KS-integrals (cf.e.g. [Corollary 24][5]) we get

$$\int_\tau^b H\,\mathrm{d}[F]\,g = \widetilde{H}\,\Delta^+F(\tau)\,g(\tau) \quad \text{and} \quad \int_\tau^b H\,\mathrm{d}[K] = \widetilde{H}\,\Delta^+K(\tau) = \widetilde{H}\,\Delta^+F(\tau)\,g(\tau),$$

i.e.

$$\int_\tau^b H\,\mathrm{d}[F]\,g = \int_\tau^b H\,\mathrm{d}[K] = \widetilde{H}\,\Delta^+F(\tau)\,g(\tau)\,.$$

This, together with (3.9) yields (3.8).

The proofs of the remaining cases $H=\chi_{[\tau,b]}\,\widetilde{H}$, $H=\chi_{[a]}\,\widetilde{H}$ and $H=\chi_{[b]}\,\widetilde{H}$ can be done in a similar way.

Step 2. Let $H\in G([a,b],L(X))$. Denote again $K(t)=\displaystyle\int_a^t \mathrm{d}[F]\,g$ for $t\in[a,b]$ and consider the sequence $H_n:[a,b]\to L(X)$, $n\in\mathbb{N}$, of finite step functions such that $\lim_{n\to\infty}\|H_n-H\|_\infty=0$. By Proposition 3.7 (ii) and *Step* 1 we have

$$\lim_{n\to\infty}\int_a^b H_n\,\mathrm{d}[K]=\lim_{n\to\infty}\int_a^b H_n\,\mathrm{d}[F]\,g=\int_a^b H\,\mathrm{d}[F]\,g\,.$$

Let $\varepsilon>0$ be given. Choose $n_0\in\mathbb{N}$ and a gauge δ on $[a,b]$ in such a way that

$$\|H_n-H\|_\infty<\varepsilon,\quad \left\|\int_a^b H_n\,\mathrm{d}[K]-\int_a^b H\,\mathrm{d}[F]\,g\right\|_X<\varepsilon\quad\text{hold for}\quad n\ge n_0$$

and

$$\left\|S(H_{n_0},\mathrm{d}K,P)-\int_a^b H_{n_0}\,\mathrm{d}[K]\right\|_X<\varepsilon\quad\text{for all }\delta-\text{fine partitions}\quad P\text{ on }[a,b]\,.$$

Then, for an arbitrary δ-fine partition $P=(D,\xi)$ of $[a,b]$ with $D=\{\alpha_0,\alpha_1,\dots,\alpha_m\}$ and $\xi=(\xi_1,x_2,\dots,\xi_m)$ we have

$$\begin{aligned}
\|S(H,\mathrm{d}K,P)-S(H_{n_0},\mathrm{d}K,P)\|_X&=\left\|\sum_{j=1}^m\big(H(\xi_j)-H_{n_0}(\xi_j)\big)\Big[\int_{\alpha_{j-1}}^{\alpha_j}\mathrm{d}[F]\,g\Big]\right\|_X\\
\le\|H-H_{n_0}\|_\infty\sum_{j=1}^m\left\|\int_{\alpha_{j-1}}^{\alpha_j}\mathrm{d}[F]\,g\right\|_X&\le\|H-H_{n_0}\|_\infty\sum_{j=1}^m\Big[(\mathrm{var}_{\alpha_{j-1}}^{\alpha_j}F)\,\|g\|_\infty\Big]\\
=\|H-H_{n_0}\|_\infty\,(\mathrm{var}_a^bF)\|g\|_\infty&<\varepsilon\,(\mathrm{var}_a^bF)\,\|g\|_\infty.
\end{aligned}$$

To summarize, we have

$$\begin{aligned}
\left\|S(H,\mathrm{d}K,P)-\int_a^b H\,\mathrm{d}[F]\,g\right\|_X&\le\|S(H,\mathrm{d}K,P)-S(H_{n_0},\mathrm{d}K,P)\|_X\\
&\quad+\left\|S(H_{n_0},\mathrm{d}K,P)-\int_a^b H_{n_0}\,\mathrm{d}[K]\right\|_X+\left\|\int_a^b H\,\mathrm{d}[K]-\int_a^b H\,\mathrm{d}[F]\,g\right\|_X\\
&<\varepsilon\,\big(2+(\mathrm{var}_a^bF)\,\|g\|_\infty\big)
\end{aligned}$$

for each δ-fine partition P of $[a,b]$, i.e., (3.8) is true. □

3.9 . Remark. Notice that, on the contrary to the finite dimensional case, in a case of a general Banach space X the Substitution Theorem can not be obtained as a corollary of the Saks-Henstock Lemma.

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