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**New guaranteed lower bounds
on eigenvalues by conforming
finite elements**

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NEW GUARANTEED LOWER BOUNDS ON EIGENVALUES BY CONFORMING FINITE ELEMENTS

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ABSTRACT. We provide two new methods for computing lower bounds of eigenvalues of symmetric elliptic second-order differential operators with mixed boundary conditions of Dirichlet, Neumann, and Robin type. The methods generalize ideas of Weinstein's and Kato's bounds and they are designed for a simple and straightforward implementation in the context of the standard finite element method. These lower bounds are obtained by a posteriori error estimators based on local flux reconstructions, which can be naturally utilized for adaptive mesh refinement. We derive these bounds, prove that they estimate the exact eigenvalues from below, and illustrate their practical performance by a numerical example.

1. INTRODUCTION

We consider the problem of finding eigenvalues $\lambda_i \in \mathbb{R}$ and nonzero eigenfunctions $u_i \in V$, $i = 1, 2, \dots$, such that

$$a(u_i, v) = \lambda_i b(u_i, v) \quad \forall v \in V, \quad (1.1)$$

where a is the inner product in the real Hilbert space V and b is a symmetric positive semidefinite bilinear form on V , see Section 2 for details. Considering arbitrary numbers $\lambda_{*,i} \in \mathbb{R}$ and functions $u_{*,i} \in V$, $b(u_{*,i}, u_{*,i}) = 1$, which are supposed to approximate eigenpairs λ_i , u_i , we introduce representatives $w_i \in V$ of residuals uniquely determined by the identity

$$a(w_i, v) = a(u_{*,i}, v) - \lambda_{*,i} b(u_{*,i}, v) \quad \forall v \in V. \quad (1.2)$$

Representatives w_i cannot be determined exactly in general, but we can compute estimates of their energy norm. In the context of the finite element method for symmetric elliptic second-order differential operators, we can efficiently reconstruct the flux $\nabla u_{*,i}$ and compute an accurate guaranteed upper bound η_i on the

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energy norm of w_i :

$$\|w_i\|_a \leq \eta_i. \quad (1.3)$$

This enables to compute the following explicit lower bounds on the eigenvalue λ_n :

$$\ell_n = \frac{1}{4} \left(-\eta_n + \sqrt{\eta_n^2 + 4\lambda_{*,n}} \right)^2, \quad (1.4)$$

$$L_n = \lambda_{*,n} \left(1 + \nu \lambda_{*,n} \sum_{i=n}^s \frac{\eta_i^2}{\lambda_{*,i}^2 (\nu - \lambda_{*,i})} \right)^{-1}, \quad (1.5)$$

where ν is assumed to satisfy $\lambda_{*,s} < \nu$ for some index $s \geq n$. Quantity ℓ_n is proved to be below λ_n if $\lambda_{n-1}\lambda_n \leq \lambda_{*,n}^2 \leq \lambda_n\lambda_{n+1}$, see Theorem 2.3 below. Similarly, L_n is a lower bound on λ_n if $\lambda_{*,s} < \nu \leq \lambda_{s+1}$ and if $\lambda_{*,i}, u_{*,i}, i = n, n+1, \dots, s$, solve a discrete eigenvalue problem, see Theorem 2.5 below.

Lower bound ℓ_n is quite general and robust, but its convergence in terms of η_n is linear, which is suboptimal. Lower bound L_n converges with the optimal quadratic speed, but it requires an additional parameter ν to be chosen, which may be complicated in some cases. The accuracy of L_n depends on the size of $\nu - \lambda_{*,s}$ and this size is limited by the spectral gap $\lambda_{s+1} - \lambda_s$. Therefore we try to choose $s \geq n$ such that the spectral gap $\lambda_{s+1} - \lambda_s$ is relatively large.

Lower bounds ℓ_n and L_n are inspired by Weinstein's [15, Corollary 6.20] and Kato's [23] bounds, respectively. However, Weinstein's and Kato's bounds are standardly formulated for an abstract operator on a Hilbert space and it is not immediately clear, how to use them in practical computations, especially if the trial functions do not possess extra regularity such that the elliptic operator can be applied pointwise. Therefore, we modify these classical results in a nontrivial way such that they can be straightforwardly used for the eigenvalue problem (1.1) and consequently in the context of the finite element method with the standard regularity of trial functions. The finite element method as a special case of the Galerkin method is well known to provide guaranteed upper bounds on eigenvalues and, thus, in combination with bounds ℓ_n and L_n we obtain two-sided bounds and full control of accuracy. For a nice survey of existing eigenvalue bounds including Weinstein's and Kato's bounds and for their generalizations we refer to [36].

The difficult problem of lower bounds of eigenvalues have already been studied for decades and several different approaches were developed. Recently, the non-conforming finite element methods were proposed [5, 21, 22, 28, 29, 31, 32, 38, 49]. These approaches provide typically an asymptotic lower bound in the sense that the lower bound is guaranteed only if the corresponding discretization mesh is sufficiently fine. Guaranteed lower bound for Laplace eigenvalues with homogeneous Dirichlet boundary conditions are obtained even on coarse meshes in [14]

by using the Crouzeix–Raviart nonconforming finite elements. Paper [30] improves this result by removing the separation condition for higher eigenvalues. A generalization of these ideas to a biharmonic operator is provided in [13].

A lower bound on the smallest eigenvalue is obtained in [39] by a nonoverlapping decomposition of the domain into subdomains, where the exact eigenvalues are known. In a sense similar method is proposed in [27]. It is based on an overlapping decomposition of the domain into geometrically simple subdomains and it yields a lower bound on the smallest eigenvalue for homogeneous Neumann or mixed Neumann–Dirichlet boundary conditions. A lower bound on the smallest eigenvalue for a triangle is obtained in [24] using a scaling. An interesting generalization of the method of eigenvalue inclusions [8, 35] for the Maxwell operator is provided in [7]. Reference [20] includes both estimators of eigenvalues and corresponding eigenfunctions.

In [42] we propose another approach based on a combination of the method of a priori-a posteriori inequalities [43, 26] and a complementarity technique [46, 47]. The method in [42] yields a linearly convergent lower bound on the principal eigenvalue and it is based on a global flux reconstructions. Bounds ℓ_n and L_n given by (1.4) and (1.5), respectively, improve this result in several aspects. They are applicable for arbitrary eigenvalues, the bound L_n is quadratically convergent, and the flux is reconstructed locally, which enables an efficient and naturally parallel implementation. Note that the used local flux reconstruction was originally proposed in [11] for source problems and we modify it for eigenvalue problems.

The rest of the paper is organized as follows. Section 2 presents Weinstein’s and Kato’s method in a general weak setting and proves the lower bounds. Section 3 introduces a particular eigenvalue problem for a symmetric second-order elliptic differential operator, proves its well posedness and briefly describes its finite element discretization. Section 4 derives a general guaranteed upper bound (1.3) on the representative of the residual based on the complementarity approach. Section 5 defines the local flux reconstruction and proves its properties. Section 6 illustrates the practical performance of the new lower bounds using the dumbbell shaped domain example. Finally, Section 7 draws the conclusions.

2. LOWER BOUNDS IN THE ABSTRACT SETTING

This section briefly describes the rigorous mathematical setting of the eigenvalue problem (1.1). The generality of this setting enables to treat the standard types of eigenvalue problems such as the Dirichlet, Neumann, Steklov, etc. in a unified manner.

Let V be a real Hilbert space with a scalar product $a(u, v)$ for $u, v \in V$. In particular the form $a(u, v)$ is continuous, bilinear, symmetric and positive definite. Further, let a form $b(u, v)$ for $u, v \in V$ be continuous, bilinear, symmetric, and positive semidefinite, i.e. $b(v, v) \geq 0$ for all $v \in V$. We use notation $\|v\|_a^2 = a(v, v)$ and $|v|_b^2 = b(v, v)$ for the norm induced by the scalar product a and the seminorm

induced by the bilinear form b , respectively. We assume that the seminorm $|\cdot|_b$ is compact with respect to the norm $\|\cdot\|_a$, i.e. from any sequence bounded in $\|\cdot\|_a$, we can extract a subsequence which is Cauchy in $|\cdot|_b$. Under these assumptions we consider the eigenvalue problem (1.1) and summarize its properties in the following theorem.

Theorem 2.1. *Under the above assumptions, problem (1.1) has the following properties.*

- (a) *There exists (at most) countable (and possibly empty) sequence of eigenvalues*

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots$$

and the corresponding eigenfunctions can be normalized as

$$b(u_i, u_j) = \delta_{ij}, \quad \forall i, j = 1, 2, \dots, N_\infty, \quad (2.1)$$

where N_∞ is the number of eigenvalues. If it is infinite, we set $N_\infty = \infty$.

- (b) *The space V can be decomposed as*

$$V = \mathcal{M} \oplus \mathcal{K}, \quad (2.2)$$

where $\mathcal{M} = \text{span}\{u_1, u_2, \dots\}$ is the linear span of all eigenfunctions, $\dim \mathcal{M} = N_\infty$, and $\mathcal{K} = \{v \in V : |v|_b = 0\}$. Consequently, any function $v \in V$ can be uniquely decomposed as

$$v = v^\mathcal{M} + v^\mathcal{K}, \quad \text{where } v^\mathcal{M} \in \mathcal{M} \text{ and } v^\mathcal{K} \in \mathcal{K}. \quad (2.3)$$

- (c) *The decomposition (2.2) satisfies*

$$a(v, u) = 0 \quad \forall v \in \mathcal{M}, \forall u \in \mathcal{K}, \quad (2.4)$$

$$b(v, u) = 0 \quad \forall v \in V, \forall u \in \mathcal{K}. \quad (2.5)$$

- (d) *Any function $v \in V$ satisfies*

$$|v|_b^2 = \sum_{i=1}^{N_\infty} |b(v, u_i)|^2, \quad (2.6)$$

$$\|v\|_a^2 = \|v^\mathcal{M}\|_a^2 + \|v^\mathcal{K}\|_a^2 \quad \text{with } \|v^\mathcal{M}\|_a^2 = \sum_{i=1}^{N_\infty} \lambda_i |b(v, u_i)|^2, \quad (2.7)$$

where $v^\mathcal{M} \in \mathcal{M}$ and $v^\mathcal{K} \in \mathcal{K}$ are given by (2.3).

Proof. To prove (a), we first mention that whenever λ_i, u_i is an eigenpair of (1.1) then $\lambda_i > 0$ and $|u_i|_b > 0$, because $0 < \|u_i\|_a^2 = \lambda_i |u_i|_b^2$. In order to use the spectral theory of compact operators, we define the solution operator $S : V \rightarrow V$ by the identity

$$a(Su, v) = b(u, v) \quad \forall v \in V. \quad (2.8)$$

It is an elementary exercise to prove that the compactness of $|\cdot|_b$ with respect to $\|\cdot\|_a$ is equivalent to the compactness of the solution operator S . Thus, the Hilbert–Schmidt spectral theorem [19, Theorem 4, Chapter II, section 3] applied

to S provides the existence of the countable sequence of eigenvalues and the corresponding orthogonal system of eigenfunctions.

Statement (b) is another consequence of the Hilbert–Schmidt spectral theorem, because it claims that $V = \mathcal{M} \oplus \ker S$, where $\ker S = \{v \in V : Sv = 0\}$ is the kernel of S . Thus, it remains to show the equality $\mathcal{K} = \ker S$. If $u \in \mathcal{K}$ then

$$|b(u, v)| \leq |u|_b |v|_b = 0 \quad \forall v \in V. \quad (2.9)$$

Thus, $a(Su, v) = b(u, v) = 0$ for all $v \in V$ and $Su = 0$. On the other hand, if $u \in \ker S$ then $0 = a(Su, u) = b(u, u) = |u|_b^2$ and $u \in \mathcal{K}$.

To prove identity (2.4) in (c), we express $v \in \mathcal{M}$ as $v = \sum_{i=1}^{N_\infty} c_i u_i$ and proceed as follows:

$$a(v, u) = \sum_{i=1}^{N_\infty} c_i a(u_i, u) = \sum_{i=1}^{N_\infty} c_i \lambda_i b(u_i, u) = \sum_{i=1}^{N_\infty} c_i \lambda_i a(u_i, Su) = 0,$$

where the last equality holds, because $u \in \mathcal{K} = \ker S$. Identity (2.5) has already been proved in (2.9).

Finally, the equalities in (d) follow from the splitting (2.3). Since $b(v, u^\mathcal{K}) = a(v, Su^\mathcal{K}) = 0$ for all $u^\mathcal{K} \in \mathcal{K} = \ker S$ and all $v \in V$, we obtain $|v|_b = |v^\mathcal{M}|_b$. The expansion $v^\mathcal{M} = \sum_{i=1}^{N_\infty} c_i u_i$ with $c_i = b(v, u_i) = a(v, u_i)/\lambda_i$ and the Parseval's identity $|v^\mathcal{M}|_b^2 = \sum_{i=1}^{N_\infty} c_i^2$ yields (2.6). Similarly, by (2.4) we obtain $\|v\|_a^2 = \|v^\mathcal{M}\|_a^2 + \|v^\mathcal{K}\|_a^2$ and easy computation gives $\|v^\mathcal{M}\|_a^2 = \sum_{i=1}^{N_\infty} c_i^2 \lambda_i$, which shows (2.7). \square

Properties listed in Theorem 2.1 enable to prove an estimate of a certain distance between an arbitrary number $\lambda_{*,n} \in \mathbb{R}$ and the exact spectrum.

Theorem 2.2. *Let λ_i , $i = 1, 2, \dots$, be eigenvalues of (1.1). Let $u_{*,n} \in V \setminus \{0\}$ and $\lambda_{*,n} \in \mathbb{R}$ be arbitrary. Consider $w_n \in V$ given by (1.2). Then*

$$\min_i \frac{|\lambda_i - \lambda_{*,n}|^2}{\lambda_i} \leq \frac{\|w_n\|_a^2}{|u_{*,n}|_b^2}. \quad (2.10)$$

Proof. By (2.7) and (1.2) we express

$$\begin{aligned} \|w_n\|_a^2 - \|w_n^\mathcal{K}\|_a^2 &= \sum_{i=1}^{N_\infty} \lambda_i |b(w_n, u_i)|^2 = \sum_{i=1}^{N_\infty} \frac{|a(w_n, u_i)|^2}{\lambda_i} \\ &= \sum_{i=1}^{N_\infty} \frac{|a(u_{*,n}, u_i) - \lambda_{*,n} b(u_{*,n}, u_i)|^2}{\lambda_i} = \sum_{i=1}^{N_\infty} \frac{|\lambda_i - \lambda_{*,n}|^2}{\lambda_i} |b(u_{*,n}, u_i)|^2. \end{aligned} \quad (2.11)$$

Thus,

$$\|w_n\|_a^2 \geq \min_i \frac{|\lambda_i - \lambda_{*,n}|^2}{\lambda_i} \sum_{i=1}^{N_\infty} |b(u_{*,n}, u_i)|^2$$

and Parseval's identity (2.6) immediately yields the statement (2.10). \square

Next we prove that Theorem 2.2 yields a lower bound for the eigenvalue λ_n , provided an approximation $\lambda_{*,n}$ is not far away from λ_n in the sense that

$$\sqrt{\lambda_{n-1}\lambda_n} \leq \lambda_{*,n} \leq \sqrt{\lambda_n\lambda_{n+1}}. \quad (2.12)$$

Theorem 2.3. *Let $u_{*,n} \in V$, $|u_{*,n}|_b = 1$, be arbitrary and let $\lambda_{*,n} \in \mathbb{R}$ satisfies the closeness condition (2.12). Further, let there be $\eta_n \in \mathbb{R}$ such that $w_n \in V$ given by (1.2) satisfies (1.3). Then ℓ_n defined in (1.4) satisfies*

$$\ell_n \leq \lambda_n. \quad (2.13)$$

Proof. For brevity, let us put $\lambda_* = \lambda_{*,n}$ and $u_* = u_{*,n}$. Let us notice that condition (2.12) implies that $\lambda_i\lambda_n \leq \lambda_*^2$ for all $i = 1, 2, \dots, n-1$ and that $\lambda_*^2 \leq \lambda_n\lambda_i$ for all $i = n+1, n+2, \dots, N_\infty$. Consequently, inequality $\lambda_*^2(\lambda_i - \lambda_n) \leq \lambda_n\lambda_i(\lambda_i - \lambda_n)$ holds true for all $i = 1, 2, \dots, N_\infty$. This inequality is however equivalent to

$$\frac{(\lambda_n - \lambda_*)^2}{\lambda_n} \leq \frac{(\lambda_i - \lambda_*)^2}{\lambda_i} \quad \forall i = 1, 2, \dots, N_\infty.$$

Hence, using this estimate in (2.10), bound (1.3), and assumption $|u_{*,n}|_b = 1$, we immediately obtain

$$\frac{(\lambda_n - \lambda_*)^2}{\lambda_n} \leq \eta_n^2.$$

This can be rewritten as the quadratic inequality $\lambda_n^2 - (2\lambda_* + \eta_n^2)\lambda_n + \lambda_*^2 \leq 0$ and this inequality can only be satisfied if (2.13) holds true. \square

Lower bound ℓ_n is quite universal and robust, but its convergence with respect to η_n is linear, which is suboptimal. However, inspired by Kato's bound [23] we are able to derive a quadratically convergent lower bound. Before we do it, we introduce an auxiliary lemma.

Lemma 2.4. *Let $u_* \in V$ and $\lambda_* \in \mathbb{R}$ be arbitrary and let $w \in V$ be given by*

$$a(w, v) = a(u_*, v) - \lambda_* b(u_*, v) \quad \forall v \in V, \quad (2.14)$$

see (1.2). Then components $u_^{\mathcal{K}}$ and $w^{\mathcal{K}}$ defined in (2.3) satisfy*

$$\|u_*^{\mathcal{K}}\|_a = \|w^{\mathcal{K}}\|_a. \quad (2.15)$$

Moreover,

$$\sum_{i=1}^{N_\infty} \frac{|b(u_*, u_i)|^2}{\lambda_i} = \frac{1}{\lambda_*^2} (\|w\|_a^2 + 2\lambda_*|u_*|_b^2 - \|u_*\|_a^2). \quad (2.16)$$

Proof. Using $v = w^{\mathcal{K}}$ in (2.14) and orthogonalities (2.4) and (2.5), we obtain $\|w^{\mathcal{K}}\|_a^2 = a(u_*^{\mathcal{K}}, w^{\mathcal{K}})$. Similarly, by using $v = u_*^{\mathcal{K}}$ in (2.14), we have $a(w^{\mathcal{K}}, u_*^{\mathcal{K}}) = \|u_*^{\mathcal{K}}\|_a^2$ and equality (2.15) is proved.

Using (2.11), (2.7), and (2.6), we obtain equality

$$\begin{aligned} \|w\|_a^2 - \|w^{\mathcal{K}}\|_a^2 &= \sum_{i=1}^{N_\infty} \left(\lambda_i - 2\lambda_* + \frac{\lambda_*^2}{\lambda_i} \right) |b(u_*, u_i)|^2 \\ &= \|u_*\|_a^2 - \|u_*^{\mathcal{K}}\|_a^2 - 2\lambda_* |u_*|_b^2 + \lambda_*^2 \sum_{i=1}^{N_\infty} \frac{|b(u_*, u_i)|^2}{\lambda_i}. \end{aligned}$$

Identity (2.16) now follows by employing (2.15) and a simple rearrangement. \square

Theorem 2.5. *Let $\tilde{u}_{*,i} \in V$ for $i = r, r+1, \dots, s$ be arbitrary. Let $\tilde{V}_* = \text{span}\{\tilde{u}_{*,r}, \tilde{u}_{*,r+1}, \dots, \tilde{u}_{*,s}\}$. Let $\lambda_{*,i} > 0$ and $u_{*,i} \in \tilde{V}_*$, $|u_{*,i}|_b = 1$, $i = r, r+1, \dots, s$, be the eigenvalues sorted in ascending order and the corresponding eigenfunctions of problem*

$$a(u_{*,i}, v_*) = \lambda_{*,i} b(u_{*,i}, v_*) \quad \forall v_* \in \tilde{V}_*. \quad (2.17)$$

Let there exist $\nu > 0$ satisfying

$$\lambda_{s-1} \leq \lambda_{*,s} < \nu \leq \lambda_{s+1}, \quad (2.18)$$

where λ_{s+1} is the eigenvalue of (1.1). Further, let $w_i \in V$ be given by (1.2) and let $\eta_i > 0$ bounds $\|w_i\|_a$ for all $i = r, r+1, \dots, s$ as in (1.3). Then L_n defined in (1.5) satisfies

$$L_n \leq \lambda_n \quad (2.19)$$

for all $n = r, r+1, \dots, s$.

Proof. Since (2.17) corresponds to a generalized matrix eigenvalue problem with a symmetric and positive definite matrix, eigenfunctions $u_{*,r}, u_{*,r+1}, \dots, u_{*,s}$ form an orthogonal system:

$$a(u_{*,i}, u_{*,j}) = \lambda_{*,i} \delta_{ij} \quad \text{and} \quad b(u_{*,i}, u_{*,j}) = \delta_{ij} \quad \forall i, j = r, r+1, \dots, s. \quad (2.20)$$

Let $n \in \{r, r+1, \dots, s\}$ be arbitrary. Let us consider a function $z_* = \sum_{i=n}^s \gamma_i u_{*,i}$, where coefficients γ_i are uniquely determined by requirements $b(z_*, u_i) = 0$ for $i = n+1, n+2, \dots, s$ and $\sum_{i=n}^s \gamma_i^2 = 1$. Using (2.20), we easily derive expressions

$$\|z_*\|_a^2 = \sum_{i=n}^s \lambda_{*,i} \gamma_i^2 \quad \text{and} \quad |z_*|_b^2 = 1. \quad (2.21)$$

Now, we consider $w^z \in V$ given by

$$a(w^z, v) = a(z_*, v) - \lambda_{*,n} b(z_*, v) \quad \forall v \in V \quad (2.22)$$

and work out an expression for $\|w^z\|_a$. Using (2.22), definition of z_* , and (1.2), we have

$$\begin{aligned} \|w^z\|_a^2 &= a(z_*, w^z) - \lambda_{*,n} b(z_*, w^z) = \sum_{i=n}^s \gamma_i [a(u_{*,i}, w^z) - \lambda_{*,n} b(u_{*,i}, w^z)] \\ &= \sum_{i=n}^s \gamma_i [a(w_i, w^z) + (\lambda_{*,i} - \lambda_{*,n}) b(u_{*,i}, w^z)]. \end{aligned} \quad (2.23)$$

Considering $i \in \{n, n+1, \dots, s\}$, we notice that identities (1.2) and (2.20) easily imply $a(u_{*,j}, w_i) = 0$ and $a(w_j, w_i) = -\lambda_{*,j} b(u_{*,j}, w_i)$ for all $j = n, n+1, \dots, s$. Consequently, using (2.22) with $v = w_i$ and definition of z_* , we derive equality

$$a(w_i, w^z) = \sum_{j=n}^s \gamma_j [a(u_{*,j}, w_i) - \lambda_{*,n} b(u_{*,j}, w_i)] = \sum_{j=n}^s \gamma_j \frac{\lambda_{*,n}}{\lambda_{*,j}} a(w_j, w_i). \quad (2.24)$$

Setting $v = u_{*,i}$ in (2.22), using definition of z_* and orthogonality (2.20), we have

$$a(u_{*,i}, w^z) = a(z_*, u_{*,i}) - \lambda_{*,n} b(z_*, u_{*,i}) = \gamma_i (\lambda_{*,i} - \lambda_{*,n}). \quad (2.25)$$

The last auxiliary step is to take $v = w^z$ in (1.2) and utilize (2.24) and (2.25) to obtain

$$\lambda_{*,i} b(u_{*,i}, w^z) = a(u_{*,i}, w^z) - a(w_i, w^z) = \gamma_i (\lambda_{*,i} - \lambda_{*,n}) - \sum_{j=n}^s \gamma_j \frac{\lambda_{*,n}}{\lambda_{*,j}} a(w_j, w_i). \quad (2.26)$$

Finally, we substitute (2.24) and (2.26) into (2.23) and after straightforward manipulations we arrive at

$$\|w^z\|_a^2 = \sum_{i=n}^s \gamma_i^2 \frac{(\lambda_{*,i} - \lambda_{*,n})^2}{\lambda_{*,i}} + \lambda_{*,n}^2 \sum_{i=n}^s \sum_{j=n}^s \frac{\gamma_i \gamma_j}{\lambda_{*,i} \lambda_{*,j}} a(w_i, w_j). \quad (2.27)$$

The following chain of inequalities leads to the lower bound (2.19). Due to the (2.18) we easily verify that $(\lambda_i - \lambda_n)(\lambda_i - \nu) \geq 0$ for all $i \leq n$ and all $i \geq s+1$. Combining this with the definition of z_* , we have

$$0 \leq \sum_{i=1}^{N_\infty} \frac{1}{\lambda_i} (\lambda_i - \lambda_n)(\lambda_i - \nu) |b(z_*, u_i)|^2 = \sum_{i=1}^{N_\infty} \left(\lambda_i - (\lambda_n + \nu) + \frac{\lambda_n \nu}{\lambda_i} \right) |b(z_*, u_i)|^2.$$

Employing identities (2.7), (2.6), and (2.16) with $w = w^z$ defined by (2.22), $\lambda_* = \lambda_{*,n}$, and $u_* = z_*$, we obtain

$$0 \leq \|z_*\|_a^2 - \|z_*^{\mathcal{K}}\|_a^2 - (\lambda_n + \nu) |z_*|_b^2 + \frac{\lambda_n \nu}{\lambda_{*,n}^2} (\|w^z\|_a^2 + 2\lambda_{*,n} |z_*|_b^2 - \|z_*\|_a^2).$$

Now, we substitute relations (2.21) and (2.27) into this inequality, use that fact that $\|z_*^{\mathcal{K}}\|_a^2 \geq 0$ and derive

$$0 \leq \sum_{i=n}^s \lambda_{*,i} \gamma_i^2 - \lambda_n - \nu + \frac{\lambda_n \nu}{\lambda_{*,n}^2} \left(\sum_{i=n}^s \gamma_i^2 \frac{(\lambda_{*,i} - \lambda_{*,n})^2}{\lambda_{*,i}} + \lambda_{*,n}^2 \sum_{i=n}^s \sum_{j=n}^s \frac{\gamma_i \gamma_j}{\lambda_{*,i} \lambda_{*,j}} a(w_i, w_j) + 2\lambda_{*,n} - \sum_{i=n}^s \lambda_{*,i} \gamma_i^2 \right).$$

Using identity $\sum_{i=n}^s \gamma_i^2 = 1$, we rearrange this inequality as

$$\sum_{i=n}^s (\nu - \lambda_{*,i}) \gamma_i^2 \leq \frac{\lambda_n}{\lambda_{*,n}} \left[\sum_{i=n}^s \frac{\lambda_{*,n}}{\lambda_{*,i}} (\nu - \lambda_{*,i}) \gamma_i^2 + \nu \lambda_{*,n} \sum_{i=n}^s \sum_{j=n}^s \frac{\gamma_i \gamma_j}{\lambda_{*,i} \lambda_{*,j}} a(w_i, w_j) \right]. \quad (2.28)$$

The first sum on the right hand side can be estimated as

$$\sum_{i=n}^s \frac{\lambda_{*,n}}{\lambda_{*,i}} (\nu - \lambda_{*,i}) \gamma_i^2 \leq \sum_{i=n}^s (\nu - \lambda_{*,i}) \gamma_i^2, \quad (2.29)$$

because $\lambda_{*,n} \leq \lambda_{*,i} < \nu$ for all $i = n, n+1, \dots, s$. The double sum on the right hand side of (2.28) can be bounded using $a(w_i, w_j) \leq \|w_i\|_a \|w_j\|_a \leq \eta_i \eta_j$ and the Cauchy–Schwarz inequality as

$$\begin{aligned} \sum_{i=n}^s \sum_{j=n}^s \frac{\gamma_i \gamma_j}{\lambda_{*,i} \lambda_{*,j}} a(w_i, w_j) &\leq \left(\sum_{i=n}^s |\gamma_i| \frac{\eta_i}{\lambda_{*,i}} \right)^2 \\ &\leq \left(\sum_{i=n}^s (\nu - \lambda_{*,i}) \gamma_i^2 \right) \left(\sum_{i=n}^s \frac{\eta_i^2}{\lambda_{*,i}^2 (\nu - \lambda_{*,i})} \right). \end{aligned} \quad (2.30)$$

Estimating the right-hand side of (2.28) by (2.29) and (2.30) and dividing by $\sum_{i=n}^s (\nu - \lambda_{*,i}) \gamma_i^2$, we end up with inequality

$$1 \leq \frac{\lambda_n}{\lambda_{*,n}} \left(1 + \nu \lambda_{*,n} \sum_{i=n}^s \frac{\eta_i^2}{\lambda_{*,i}^2 (\nu - \lambda_{*,i})} \right),$$

which is equivalent to (2.19). \square

Let us make a few remarks about the result presented in Theorem 2.5. First, choosing $n = s$, the bound (2.19) simplifies to

$$\lambda_{*,n} \left(1 + \frac{\nu}{\lambda_{*,n} (\nu - \lambda_{*,n})} \eta_n^2 \right)^{-1} \leq \lambda_n.$$

Second, accurate estimates η_i of the representatives of residuals $\|w_i\|_a$ are computed by the flux reconstruction approach, which we describe in details below in Section 5.

Third, lower bound (2.19) depends on ν , which is required to satisfy (2.18). This condition cannot be guaranteed unless a lower bound on λ_{s+1} is known. If an analytic (and perhaps rough) lower bound on λ_{s+1} is known, we can use (2.19) to compute more accurate lower bounds on λ_s and smaller eigenvalues. However, a priori known lower bounds on eigenvalues are rare in practice. In Section 6 we illustrate how to compute the needed lower bound by a homotopy method described in [34, 35].

A practical approach in applications, where the lower bounds need not to be guaranteed, is to set $\nu = \ell_{s+1}$ computed by (1.4). The bound ℓ_{s+1} is guaranteed to be below λ_{s+1} under the closeness condition (2.12), but this condition cannot be verified unless lower bounds on the corresponding eigenvalues are already known. However, resolving the eigenvalue problem with sufficient accuracy provides a good confidence that the closeness condition holds true. In Section 6 we illustrate how to use the computed lower bounds to verify a posteriori if the closeness condition is likely to hold. In addition, we mention that the closeness condition (2.12) is just a sufficient condition in Theorem 2.3 and the bound (1.4) can be below the exact eigenvalue even if the closeness condition is not satisfied. Numerical experiments we performed indicate that this is actually very common situation in practical computations, see Section 6 below.

The fourth remark concerns the optimal use of Theorem 2.5. Estimate (2.19) provides $s - r + 1$ lower bounds on eigenvalues $\lambda_r, \lambda_{r+1}, \dots, \lambda_s$, respectively. In particular, we can use the lower bound on λ_s as a new value of ν and use Theorem 2.5 again to obtain new lower bounds on $\lambda_r, \lambda_{r+1}, \dots, \lambda_{s-1}$. This process can be repeated $s - r$ times using for ν the best lower bound computed so far, see [35]. As the final lower bounds we naturally choose the largest one computed. In Section 6 below, we combine this recursive process with the lower bound (1.4) to obtain sharper bounds on rough meshes.

Finally, we remark that Theorem 2.5 requires the exact equality in (2.17). If we compute approximate eigenfunctions $\tilde{u}_{*,i}$ using standard approaches based on the finite element discretization, we can usually take $u_{*,i} = \tilde{u}_{*,i}$ and the error in identity (2.17) is on the level of machine precision. If not, then we suggest to compute $u_{*,i}$ by solving the (small) eigenvalue problem (2.17) by a direct method. Of course, the exact equality in (2.17) cannot be reached due to round-off errors, but this issue can be solved by the interval arithmetics as suggested in [34, 35], for example.

3. LOWER BOUNDS FOR SYMMETRIC ELLIPTIC OPERATORS

This section introduces a symmetric elliptic eigenvalue problem for second-order partial differential operators with mixed boundary conditions and all necessary assumptions. It also briefly describes its finite element discretization.

We consider the following eigenvalue problem: find $\lambda_n > 0$ and $u_n \neq 0$ such that

$$\begin{aligned} -\operatorname{div}(\mathcal{A}\nabla u_n) + cu_n &= \lambda_n\beta_1 u_n && \text{in } \Omega, \\ (\mathcal{A}\nabla u_n) \cdot \mathbf{n}_\Omega + \alpha u_n &= \lambda_n\beta_2 u_n && \text{on } \Gamma_N, \\ u_n &= 0 && \text{on } \Gamma_D. \end{aligned} \quad (3.1)$$

In order to formulate this problem in a weak sense, we consider $\Omega \subset \mathbb{R}^2$ to be a Lipschitz domain with boundary $\partial\Omega$ split into two relatively open disjoint parts Γ_N and Γ_D . Symbol \mathbf{n}_Ω stands for the unit outward normal vector. Diffusion matrix $\mathcal{A} \in [L^\infty(\Omega)]^{2 \times 2}$, reaction coefficient $c \in L^\infty(\Omega)$ and coefficients $\beta_1 \in L^\infty(\Omega)$, $\alpha, \beta_2 \in L^\infty(\Gamma_N)$ are assumed to be piecewise constant. Further, we assume $\beta_1 \geq 0$ and $\beta_2 \geq 0$. Note that the assumption of piecewise constant coefficients and of the two-dimensionality of the domain are not essential. They are needed due to technical reasons connected with the flux reconstruction. In order to guarantee the symmetry and ellipticity, we assume $c \geq 0$ and $\alpha \geq 0$ and the matrix \mathcal{A} to be symmetric and uniformly positive definite, i.e. we assume existence of a constant $C > 0$ such that

$$\boldsymbol{\xi}^T \mathcal{A}(x) \boldsymbol{\xi} \geq C |\boldsymbol{\xi}|^2 \quad \forall \boldsymbol{\xi} \in \mathbb{R}^2 \text{ and for almost all } x \in \Omega,$$

where $|\cdot|$ stands for the Euclidean norm.

Defining the usual space

$$V = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_D\}, \quad (3.2)$$

we introduce the following weak formulation of (3.1): find $\lambda_n > 0$ and $u_n \in V \setminus \{0\}$ such that

$$a(u_n, v) = \lambda_n b(u_n, v) \quad \forall v \in V, \quad (3.3)$$

where

$$a(u, v) = (\mathcal{A}\nabla u, \nabla v) + (cu, v) + (\alpha u, v)_{\Gamma_N}, \quad (3.4)$$

$$b(u, v) = (\beta_1 u, v) + (\beta_2 u, v)_{\Gamma_N}, \quad (3.5)$$

(\cdot, \cdot) stands for the $L^2(\Omega)$, and $(\cdot, \cdot)_{\Gamma_N}$ for the $L^2(\Gamma_N)$ inner products.

We assume the form $a(u, v)$ to be a scalar product in V . This is the case if at least one of the following conditions is satisfied: (a) $c > 0$ on a subset of Ω of positive measure, (b) $\alpha > 0$ on a subset of Γ_N of positive measure, (c) measure of Γ_D is positive. In agreement with the notation introduced above, we denote by $\|\cdot\|_a$ and $|\cdot|_b$ the norm induced by $a(\cdot, \cdot)$ and the seminorm induced by $b(\cdot, \cdot)$, respectively.

The following theorem shows the validity of the crucial compactness assumption. Consequently, eigenproblem (3.3) is well defined and possesses all properties listed in Theorem 2.1.

Theorem 3.1. *Let bilinear forms $a(\cdot, \cdot)$ and $b(\cdot, \cdot)$ be defined by (3.4) and (3.5) with the above listed requirements on the coefficients. Let $a(\cdot, \cdot)$ be a scalar product in V . Then the seminorm $|\cdot|_b$ is compact with respect to the norm $\|\cdot\|_a$.*

Proof. Notice that the definition (3.5) of the form $b(u, v)$ for $u, v \in V$ is understood as follows

$$b(u, v) = (\beta_1 Iu, Iv)_{L^2(\Omega)} + (\beta_2 \gamma u, \gamma v)_{L^2(\Gamma_N)},$$

where $I : V \rightarrow L^2(\Omega)$ is the identity mapping and $\gamma : V \rightarrow L^2(\Gamma_N)$ the trace operator. The identity I is compact due to Rellich theorem [1, Theorem 6.3] and the compactness of the trace operator γ is proved in [25, Theorem 6.10.5]; see also [9].

Compactness of I and γ implies that from any sequence $\{v_n\} \subset V$ bounded in $\|\cdot\|_a$ we can extract a subsequence $\{v_{n_k}\}$ such that $\{Iv_{n_k}\}$ is Cauchy in $L^2(\Omega)$ and $\{\gamma v_{n_k}\}$ is Cauchy in $L^2(\Gamma_N)$. Since

$$|v|_b^2 \leq \max \{ \|\beta_1\|_{L^\infty(\Omega)}, \|\beta_2\|_{L^\infty(\Gamma_N)} \} \left(\|Iv\|_{L^2(\Omega)}^2 + \|\gamma v\|_{L^2(\Gamma_N)}^2 \right) \quad \forall v \in V,$$

we immediately see that the subsequence $\{v_{n_k}\}$ is Cauchy in $|\cdot|_b$ as well. \square

We discretize the eigenvalue problem (3.3) by the standard conforming finite element method. To avoid technicalities with curved elements, we assume the domain Ω to be polygonal and consider a conforming (face-to-face) triangular mesh \mathcal{T}_h consisting of closed triangles called elements. Further, we define the finite element space

$$V_h = \{v_h \in V : v_h|_K \in P_p(K), \forall K \in \mathcal{T}_h\}, \quad (3.6)$$

where $P_p(K)$ stands for the space of polynomials of degree at most p on the triangle $K \in \mathcal{T}_h$. With this notation, the finite element approximation of the eigenvalue problem (3.3) reads: Find $\lambda_{h,n} > 0$ and $u_{h,n} \in V_h \setminus \{0\}$ such that

$$a(u_{h,n}, v_h) = \lambda_{h,n} b(u_{h,n}, v_h) \quad \forall v_h \in V_h. \quad (3.7)$$

4. GUARANTEED BOUNDS BASED ON THE COMPLEMENTARY ENERGY

Lower bounds ℓ_n and L_n given by (1.4) and (1.5), respectively, require a computable guaranteed upper bounds η_n on the energy norm $\|w_n\|_a$ of the representative of the residual defined in (1.2). In this section, we show how to compute this η_n . The technique described here is based on the complementary energy (or two energy principle), which can be traced back to the hypercircle method [44], see also [2, 10, 40]. In this section, we provide an estimate that is close to optimal with respect to the data computed solely on individual elements. We achieve this by combining several possibilities how to estimate various terms involved.

Let V be given by (3.2) and let $\lambda_{*,n} > 0$ and $u_{*,n} \in V \setminus \{0\}$ be an arbitrary approximation of an eigenpair of (3.1). Further, let $\mathbf{q} \in \mathbf{H}(\text{div}, \Omega)$ be an arbitrary vector field. Let us note that the quality of the resulting error bound depends

heavily on an appropriate choice of \mathbf{q} . A particular way how to compute suitable \mathbf{q} efficiently is described below in Section 5. However, the result presented in this section is valid for $\mathbf{q} \in \mathbf{H}(\operatorname{div}, \Omega)$ independently on the way it is constructed.

First, we introduce certain notation. Based on $\lambda_{*,n}$, $u_{*,n}$, \mathbf{q} , we define flux F , residual r in Ω , and boundary residual g on Γ_N as

$$\begin{aligned} F &= \mathcal{A}\nabla u_{*,n} - \mathbf{q}, \\ r &= cu_{*,n} - \lambda_{*,n}\beta_1 u_{*,n} - \operatorname{div} \mathbf{q}, \\ g &= \alpha u_{*,n} - \lambda_{*,n}\beta_2 u_{*,n} + \mathbf{q} \cdot \mathbf{n}_\Omega. \end{aligned} \quad (4.1)$$

Then, we compute their various norms for every element $K \in \mathcal{T}_h$:

$$F_K = \|\mathcal{A}_K^{-1/2} F\|_K, \quad (4.2)$$

$$r_{1,K} = \begin{cases} \|c_K^{-1/2} r\|_K & \text{if } c_K > 0, \\ \infty & \text{if } c_K = 0, \end{cases} \quad (4.3)$$

$$r_{2,K} = \begin{cases} h_K \pi^{-1} (\lambda_{\mathcal{A}_K}^{\min})^{-1/2} \|r\|_K & \text{if } \int_K r \, dx = 0, \\ \infty & \text{otherwise,} \end{cases} \quad (4.4)$$

$$r_{3,K} = \begin{cases} \|\beta_{1,K}^{-1/2} r\|_K & \text{if } \beta_{1,K} > 0, \\ \infty & \text{if } \beta_{1,K} = 0. \end{cases} \quad (4.5)$$

Here, quantities \mathcal{A}_K , c_K , and $\beta_{1,K}$ are constant values of coefficients \mathcal{A} , c , and β_1 restricted to the element K , respectively. Symbol $\|\cdot\|_K$ stands for the usual norm in $L^2(K)$. In general, we adopt the notation $\|\cdot\|_Q$ and $(\cdot, \cdot)_Q$ for the norm and the inner product in $L^2(Q)$, where Q is a domain. We also denote by $\|\cdot\|_{a,K}$ and $|\cdot|_{b,K}$ the local energy norm and the local b -seminorm, i.e. $\|v\|_{a,K}^2 = (\mathcal{A}_K \nabla v, \nabla v)_K + (c_K v, v)_K + (\alpha v, v)_{\partial K \cap \Gamma_N}$ and $|v|_{b,K}^2 = (\beta_{1,K} v, v)_K + (\beta_2 v, v)_{\partial K \cap \Gamma_N}$, cf. (3.4) and (3.5). We recall that $\lambda_{\mathcal{A}_K}^{\min}$ stands for the smallest eigenvalue of the local coefficient matrix \mathcal{A}_K and h_K for the diameter of the element K .

Further, we define similar quantities on those edges of the element $K \in \mathcal{T}_h$ that lie on Γ_N . We put $\mathcal{E}_K^N = \{\gamma : \gamma \text{ is an edge of } K \text{ and } \gamma \subset \partial K \cap \Gamma_N\}$ and for all elements $K \in \mathcal{T}_K$ and all edges $\gamma \in \mathcal{E}_K^N$ we define

$$g_{1,\gamma} = \begin{cases} \|\alpha_\gamma^{-1/2} g\|_\gamma & \text{if } \alpha_\gamma > 0, \\ \infty & \text{if } \alpha_\gamma = 0, \end{cases} \quad (4.6)$$

$$g_{2,\gamma} = \begin{cases} \min \left\{ C_K^{\mathcal{A}_K}, \overline{C}_K^{\mathcal{A}_K} \right\} \|g\|_\gamma, & \text{if } c_K > 0 \text{ or } \int_\gamma g \, dx = 0, \\ \infty & \text{otherwise,} \end{cases} \quad (4.7)$$

$$g_{3,\gamma} = \begin{cases} \|\beta_{2,\gamma}^{-1/2} g\|_\gamma & \text{if } \beta_{2,\gamma} > 0, \\ \infty & \text{if } \beta_{2,\gamma} = 0. \end{cases} \quad (4.8)$$

Similarly as above, α_γ and $\beta_{2,\gamma}$ are the constant values of α and β_2 on edges $\gamma \in \mathcal{E}_K^N$. Constants $C_K^{\mathcal{A}_K}$ and $\overline{C}_K^{\mathcal{A}_K}$ are given by simple modifications of [4, 3,

Lemma 1] as

$$\begin{aligned} (C_K^{\mathcal{A}_K})^2 &= \frac{|\gamma|}{d|K|c_K^{1/2}} \left(\frac{4h_K^2}{\lambda_{\mathcal{A}_K}^{\min}} + \frac{d^2}{c_K} \right)^{1/2}, \\ (\overline{C}_K^{\mathcal{A}_K})^2 &= \frac{|\gamma|}{d|K|} \overline{M}_K^{\mathcal{A}_K} \left(\frac{2h_K}{(\lambda_{\mathcal{A}_K}^{\min})^{1/2}} + d \overline{M}_K^{\mathcal{A}_K} \right), \end{aligned}$$

where $\overline{M}_K^{\mathcal{A}_K} = \min \left\{ h_K \pi^{-1} (\lambda_{\mathcal{A}_K}^{\min})^{-1/2}, c_K^{-1/2} \right\}$ and $d = 2$ stands for the dimension. Note that we formally consider the possibility $C_K^{\mathcal{A}_K} = \infty$ in the case of $c_K = 0$ and $\overline{C}_K^{\mathcal{A}_K} = \infty$ if $\int_{\gamma} g \, dx \neq 0$.

To proceed, we introduce sets $\mathcal{E}_K^{\text{N}0} = \{\gamma \in \mathcal{E}_K^{\text{N}} : \beta_{2,\gamma} = 0\}$ and $\mathcal{E}_K^{\text{N}+} = \{\gamma \in \mathcal{E}_K^{\text{N}} : \beta_{2,\gamma} > 0\}$. For all elements $K \in \mathcal{T}_h$ we define

$$M_K = \min \left\{ \left(F_K^2 + r_{1,K}^2 + \sum_{\gamma \in \mathcal{E}_K^{\text{N}}} g_{1,\gamma}^2 \right)^{1/2}, \left(F_K^2 + r_{1,K}^2 \right)^{1/2} + \sum_{\gamma \in \mathcal{E}_K^{\text{N}}} g_{2,\gamma}, \right. \\ \left. \left(F_K^2 + \sum_{\gamma \in \mathcal{E}_K^{\text{N}}} g_{1,\gamma}^2 \right)^{1/2} + r_{2,K}, F_K + r_{2,K} + \sum_{\gamma \in \mathcal{E}_K^{\text{N}}} g_{2,\gamma} \right\} \quad (4.9)$$

and also quantity M_K^0 , which is given by (4.9) with \mathcal{E}_K^{N} replaced by $\mathcal{E}_K^{\text{N}0}$. It is also useful to put

$$Y_K = \min \left\{ \left(F_K^2 + \sum_{\gamma \in \mathcal{E}_K^{\text{N}0}} g_{1,\gamma}^2 \right)^{1/2}, F_K + \sum_{\gamma \in \mathcal{E}_K^{\text{N}0}} g_{2,\gamma} \right\}, \quad (4.10)$$

$$R_K = \left(r_{3,K}^2 + \sum_{\gamma \in \mathcal{E}_K^{\text{N}+}} g_{3,\gamma}^2 \right)^{1/2}, \quad (4.11)$$

$$G_K = \left(\sum_{\gamma \in \mathcal{E}_K^{\text{N}+}} g_{3,\gamma}^2 \right)^{1/2}. \quad (4.12)$$

Finally, if a quantity $\underline{\lambda}_1$ such that $0 < \underline{\lambda}_1 \leq \lambda_1$ is available, we put

$$\begin{aligned} \mathcal{T}_h^{++} &= \{K \in \mathcal{T}_h : \beta_{1,K} > 0 \text{ and } Y_K + \underline{\lambda}_1^{-1/2} R_K \leq M_K\}, \\ \mathcal{T}_h^{+0} &= \{K \in \mathcal{T}_h : \beta_{1,K} = 0, \mathcal{E}_K^{\text{N}+} \neq \emptyset, \text{ and } \underline{\lambda}_1^{-1/2} G_K + M_K^0 \leq M_K\}, \end{aligned}$$

$\mathcal{T}_h^+ = \mathcal{T}_h^{++} \cup \mathcal{T}_h^{+0}$ and $\mathcal{T}_h^0 = \mathcal{T}_h \setminus \mathcal{T}_h^+$. Note that if lower bound $\underline{\lambda}_1$ is not available, we formally set $\mathcal{T}_h^{++} = \mathcal{T}_h^{+0} = \emptyset$. Using this notation, we formulate the following theorem.

Theorem 4.1. *Let $\lambda_{*,n} \in \mathbb{R}$ and $u_{*,n} \in V \setminus \{0\}$ be arbitrary and let $w_n \in V$ be given by (1.2). Using the above notation, we have*

$$\|w_n\|_a \leq \eta_n = \min\{\eta_n^{(a)}, \eta_n^{(b)}\} \quad (4.13)$$

where

$$\begin{aligned} \eta_n^{(a)} &= \left(\sum_{K \in \mathcal{T}_h} M_K^2 \right)^{1/2}, \quad \eta_n^{(b)} = \left[\sum_{K \in \mathcal{T}_h^{++}} Y_K^2 + \sum_{K \in \mathcal{T}_h^{+0}} (M_K^0)^2 + \sum_{K \in \mathcal{T}_h^0} M_K^2 \right]^{1/2} \\ &\quad + \underline{\lambda}_1^{-1/2} \left[\sum_{K \in \mathcal{T}_h^{++}} R_K^2 + \sum_{K \in \mathcal{T}_h^{+0}} G_K^2 \right]^{1/2}. \end{aligned}$$

Proof. For brevity, we will write $w = w_n$ within this proof. First, we prove the estimate $\|w\|_a \leq \eta_n^{(a)}$. Using (2.14) and the divergence theorem, we easily obtain identity

$$\|w\|_a^2 = (F, \nabla w) + (r, w) + (g, w)_{\Gamma_N} = \sum_{K \in \mathcal{T}_h} \mathcal{L}_K, \quad (4.14)$$

where $\mathcal{L}_K = (F, \nabla w)_K + (r, w)_K + \sum_{\gamma \in \mathcal{E}_K^N} (g, w)_\gamma$. The terms in \mathcal{L}_K can be estimated by the Cauchy–Schwarz inequality. We bound the first one as

$$(F, \nabla w)_K = (\mathcal{A}^{-1/2} F, \mathcal{A}^{1/2} \nabla w) \leq \|\mathcal{A}^{-1/2} F\|_K \|\mathcal{A}^{1/2} \nabla w\|_K = F_K \|\nabla w\|_{\mathcal{A}, K}, \quad (4.15)$$

where we use the notation $\|\nabla w\|_{\mathcal{A}, K}^2 = (\mathcal{A} \nabla w, \nabla w)_K$. The second term can be bounded in three ways. First, in those elements $K \in \mathcal{T}_h$ where $c_K > 0$, we have $(r, w)_K \leq c_K^{-1/2} \|r\|_K \|c_K^{1/2} w\|_K$. Second, in elements $K \in \mathcal{T}_h$ where $\int_K r \, dx = 0$, we can consider $\bar{w}_K = |K|^{-1} \int_K w \, dx$, Poincaré inequality $\|w - \bar{w}_K\|_K \leq h_K \pi^{-1} \|\nabla w\|_K$ [33], the smallest eigenvalue $\lambda_{\mathcal{A}_K}^{\min}$ of \mathcal{A}_K , and derive estimate $(r, w)_K = (r, w - \bar{w}_K)_K \leq \|r\|_K \|w - \bar{w}_K\|_K \leq h_K \pi^{-1} (\lambda_{\mathcal{A}_K}^{\min})^{-1/2} \|r\|_K \|\nabla w\|_{\mathcal{A}, K}$. Third, in elements $K \in \mathcal{T}_h$, where $\beta_{1,K} > 0$, we have $(r, w)_K \leq \beta_{1,K}^{-1/2} \|r\|_K \|\beta_{1,K}^{1/2} w\|_K$. Thus, using definitions (4.3)–(4.5), we obtain estimates

$$(r, w)_K \leq r_{1,K} \|c_K^{1/2} w\|_K, \quad (4.16)$$

$$(r, w)_K \leq r_{2,K} \|\nabla w\|_{\mathcal{A}, K}, \quad (4.17)$$

$$(r, w)_K \leq r_{3,K} \|\beta_{1,K}^{1/2} w\|_K \quad (4.18)$$

for all elements $K \in \mathcal{T}_h$.

In a similar way, we estimate the third term. First, on edges $\gamma \in \mathcal{E}_K^N$ where $\alpha_\gamma > 0$, we have $(g, w)_\gamma \leq \alpha_\gamma^{-1/2} \|g\|_\gamma \|\alpha_\gamma^{1/2} w\|_\gamma$. Second, on those edges $\gamma \in \mathcal{E}_K^N$ where

$\int_{\gamma} g \, dx = 0$, we can use the trace inequality [4, Lemma 1] and obtain $(g, w)_{\gamma} = (g, w - \bar{w}_{\gamma})_{\gamma} \leq \|g\|_{\gamma} \|w - \bar{w}_{\gamma}\|_{\gamma} \leq \bar{C}_K^{A_K} \|g\|_{\gamma} \|w\|_{a,K}$, where $\bar{w}_{\gamma} = |\gamma|^{-1} \int_{\gamma} w \, dx$. Third, if $c_K > 0$ and $\gamma \in \mathcal{E}_K^N$ then the trace inequality [4, Lemma 1] yields $(g, w)_{\gamma} = \|g\|_{\gamma} \|w\|_{\gamma} \leq C_K^{A_K} \|g\|_{\gamma} \|w\|_{a,K}$. Fourth, on edges $\gamma \in \mathcal{E}_K^N$ where $\beta_{2,\gamma} > 0$, we have $(g, w)_{\gamma} \leq \beta_{2,\gamma}^{-1/2} \|g\|_{\gamma} \|\beta_{2,\gamma}^{1/2} w\|_{\gamma}$. Thus, using definitions (4.6)–(4.8) we obtain the following bounds

$$(g, w)_{\gamma} \leq g_{1,\gamma} \|\alpha_{\gamma}^{1/2} w\|_{\gamma}, \quad (4.19)$$

$$(g, w)_{\gamma} \leq g_{2,\gamma} \|w\|_{a,K}, \quad (4.20)$$

$$(g, w)_{\gamma} \leq g_{3,\gamma} \|\beta_{2,\gamma}^{1/2} w\|_{\gamma}. \quad (4.21)$$

Now, we bound \mathcal{L}_K in four ways using various combinations of bounds (4.15), (4.16)–(4.17), and (4.19)–(4.20):

$$\mathcal{L}_K \leq F_K \|\nabla w\|_{\mathcal{A},K} + r_{1,K} \|c_K^{1/2} w\|_K + \sum_{\gamma \in \mathcal{E}_K^N} g_{1,\gamma} \|\alpha_{\gamma}^{1/2} w\|_{\gamma},$$

$$\mathcal{L}_K \leq F_K \|\nabla w\|_{\mathcal{A},K} + r_{1,K} \|c_K^{1/2} w\|_K + \sum_{\gamma \in \mathcal{E}_K^N} g_{2,\gamma} \|w\|_{a,K},$$

$$\mathcal{L}_K \leq F_K \|\nabla w\|_{\mathcal{A},K} + r_{2,K} \|\nabla w\|_{\mathcal{A},K} + \sum_{\gamma \in \mathcal{E}_K^N} g_{1,\gamma} \|\alpha_{\gamma}^{1/2} w\|_{\gamma},$$

$$\mathcal{L}_K \leq F_K \|\nabla w\|_{\mathcal{A},K} + r_{2,K} \|\nabla w\|_{\mathcal{A},K} + \sum_{\gamma \in \mathcal{E}_K^N} g_{2,\gamma} \|w\|_{a,K}.$$

Using the Cauchy-Schwarz inequality in these estimates, we conclude that

$$\mathcal{L}_K \leq M_K \|w\|_{a,K} \quad (4.22)$$

and consequently equality (4.14) yields

$$\|w\|_a^2 \leq \sum_{K \in \mathcal{T}_h} M_K \|w\|_{a,K} \leq \left(\sum_{K \in \mathcal{T}_h} M_K^2 \right)^{1/2} \|w\|_a,$$

which readily provides the desired bound $\|w\|_a \leq \eta_m^{(a)}$.

To prove $\|w\|_a \leq \eta_m^{(b)}$, we first derive several auxiliary estimates using the same technique as above. First, using (4.2), (4.6), and (4.7), we obtain

$$(F, \nabla w)_K + \sum_{\gamma \in \mathcal{E}_K^{N_0}} (g, w)_{\gamma} \leq Y_K \|w\|_{a,K}, \quad (4.23)$$

where Y_K is given by (4.10). Similarly, by (4.5) and (4.8), we have

$$(r, w)_K + \sum_{\gamma \in \mathcal{E}_K^{N_+}} (g, w)_{\gamma} \leq R_K |w|_{b,K}, \quad (4.24)$$

where R_K is provided in (4.11). Next, using the same approach as for (4.22), we derive the bound

$$(F, \nabla w)_K + (r, w)_K + \sum_{\gamma \in \mathcal{E}_K^{\text{NO}}} (g, w)_\gamma \leq M_K^0 \|w\|_{a,K}, \quad (4.25)$$

where M_K^0 was introduced below (4.9). Finally, we use (4.8) and estimate

$$\sum_{\gamma \in \mathcal{E}_K^{\text{N}^+}} (g, w)_\gamma \leq G_K |w|_{b,K}, \quad (4.26)$$

where G_K is defined in (4.12).

Thus, using estimates (4.23), (4.24), (4.25), and (4.26) in (4.14), we obtain

$$\begin{aligned} \|w\|_a^2 &= \sum_{K \in \mathcal{T}_h^{++}} \mathcal{L}_K + \sum_{K \in \mathcal{T}_h^{+0}} \mathcal{L}_K + \sum_{K \in \mathcal{T}_h^0} \mathcal{L}_K \leq \sum_{K \in \mathcal{T}_h^{++}} [Y_K \|w\|_{a,K} \\ &\quad + R_K |w|_{b,K}] + \sum_{K \in \mathcal{T}_h^{+0}} [M_K^0 \|w\|_{a,K} + G_K |w|_{b,K}] + \sum_{K \in \mathcal{T}_h^0} M_K \|w\|_{a,K}. \end{aligned}$$

This can be further estimated using the Cauchy–Schwarz inequality as

$$\begin{aligned} \|w\|_a^2 &\leq \left[\sum_{K \in \mathcal{T}_h^{++}} Y_K^2 + \sum_{K \in \mathcal{T}_h^{+0}} (M_K^0)^2 + \sum_{K \in \mathcal{T}_h^0} M_K^2 \right]^{1/2} \|w\|_a \\ &\quad + \left[\sum_{K \in \mathcal{T}_h^{++}} R_K^2 + \sum_{K \in \mathcal{T}_h^{+0}} G_K^2 \right]^{1/2} |w|_b \end{aligned}$$

Since $|w|_b \leq \lambda_1^{-1/2} \|w\|_a \leq \underline{\lambda}_1^{-1/2} \|w\|_a$, the proof is finished. \square

5. LOCAL FLUX RECONSTRUCTION

In this section, we describe a local procedure how to construct a suitable flux reconstruction $\mathbf{q} \in \mathbf{H}(\text{div}, \Omega)$ needed to evaluate η_n in Theorem 4.1. Specifically, we use the flux reconstruction proposed in [11] and generalize it to the eigenvalue problem (3.3). The description of the local flux reconstruction is technical and it is inspired mainly by works [18] and [16]. We will denote the computable flux reconstruction by \mathbf{q}_h in order to distinguish it from an arbitrary element $\mathbf{q} \in \mathbf{H}(\text{div}, \Omega)$. The flux reconstruction \mathbf{q}_h is computed as a reconstruction of the already computed approximate gradient $\nabla u_{h,n}$, where $u_{h,n} \in V \setminus \{0\}$ is given by (3.7).

The flux reconstruction \mathbf{q}_h is naturally defined in the Raviart–Thomas finite element spaces, see e.g. [12] and [37]. The Raviart–Thomas space of order p is defined on the mesh \mathcal{T}_h as

$$\mathbf{W}_h = \{\mathbf{w}_h \in \mathbf{H}(\text{div}, \Omega) : \mathbf{w}_h|_K \in \mathbf{RT}_p(K) \quad \forall K \in \mathcal{T}_h\}, \quad (5.1)$$

where $\mathbf{RT}_p(K) = [P_p(K)]^2 \oplus \mathbf{x}P_p(K)$ and $\mathbf{x} = (x_1, x_2)$ is the vector of coordinates.

We introduce the notation for vertices (nodes) of the mesh \mathcal{T}_h . Let \mathcal{N}_h denote the set of all vertices in \mathcal{T}_h . The subsets of those lying on $\bar{\Gamma}_D$, on Γ_N , and in the interior of Ω are denoted by \mathcal{N}_h^D , \mathcal{N}_h^N , and \mathcal{N}_h^I , respectively. Notice that if a vertex is located at the interface between the Dirichlet and Neumann boundary, it is not in \mathcal{N}_h^N , but only in \mathcal{N}_h^D . We also denote by \mathcal{N}_h^K and \mathcal{E}_h^K the sets of three vertices and three edges of the element K , respectively.

We construct the flux reconstruction $\mathbf{q}_h \in \mathbf{W}_h$ by solving local Neumann and Neumann/Dirichlet mixed finite element problems defined on patches of elements sharing a given vertex. Let $\mathbf{a} \in \mathcal{N}_h$ be an arbitrary vertex, we denote by $\psi_{\mathbf{a}}$ the standard piecewise linear and continuous hat function associated with \mathbf{a} . This function vanishes at all vertices of \mathcal{T}_h except of \mathbf{a} , where it has value 1. Note that $\psi_{\mathbf{a}} \in V_h$ for vertices $\mathbf{a} \in \mathcal{N}_h^I \cup \mathcal{N}_h^N$, but $\psi_{\mathbf{a}} \notin V_h$ for $\mathbf{a} \in \mathcal{N}_h^D$. Further, let $\mathcal{T}_{\mathbf{a}} = \{K \in \mathcal{T}_h : \mathbf{a} \in K\}$ be the set of elements sharing the vertex \mathbf{a} and $\omega_{\mathbf{a}} = \text{int} \bigcup \{K : K \in \mathcal{T}_{\mathbf{a}}\}$ the patch of elements sharing the vertex \mathbf{a} . We denote by $\mathcal{E}_{\mathbf{a}}^I$ the set of interior edges in the patch $\omega_{\mathbf{a}}$, by $\mathcal{E}_{\mathbf{a}}^{B,E}$ the set of those edges on the boundary $\partial\omega_{\mathbf{a}}$ that do not contain \mathbf{a} , and by $\mathcal{E}_{\mathbf{a}}^{B,D}$ and $\mathcal{E}_{\mathbf{a}}^{B,N}$ the sets of edges on the boundary $\partial\omega_{\mathbf{a}}$ with an end point at \mathbf{a} lying either on Γ_D or on Γ_N , respectively. Note that sets $\mathcal{E}_{\mathbf{a}}^{B,D}$ and $\mathcal{E}_{\mathbf{a}}^{B,N}$ can be nonempty only if $\mathbf{a} \in \mathcal{N}_h^D \cup \mathcal{N}_h^N$, i.e. for boundary patches.

We introduce auxiliary quantities

$$\tilde{r}_h^{\mathbf{a}} = c\psi_{\mathbf{a}}u_{h,n} - \lambda_{h,n}\beta_1\psi_{\mathbf{a}}u_{h,n} + (\mathcal{A}\nabla\psi_{\mathbf{a}}) \cdot \nabla u_{h,n}, \quad (5.2)$$

$$\tilde{g}_h^{\mathbf{a}} = \alpha\psi_{\mathbf{a}}u_{h,n} - \lambda_{h,n}\beta_2\psi_{\mathbf{a}}u_{h,n}. \quad (5.3)$$

Note that these quantities are defined in such a way that

$$a(u_{h,n}, \psi_{\mathbf{a}}) - \lambda_{h,n}b(u_{h,n}, \psi_{\mathbf{a}}) = \int_{\omega_{\mathbf{a}}} \tilde{r}_h^{\mathbf{a}} dx + \sum_{\gamma \in \mathcal{E}_{\mathbf{a}}^{B,N}} \int_{\gamma} \tilde{g}_h^{\mathbf{a}} ds \quad \forall \mathbf{a} \in \mathcal{N}_h. \quad (5.4)$$

The local flux reconstruction is defined in the Raviart–Thomas spaces on patches $\omega_{\mathbf{a}}$ with suitable boundary conditions. We introduce the space

$$\mathbf{W}_{\mathbf{a}}^0 = \{\mathbf{w}_h \in \mathbf{H}(\text{div}, \omega_{\mathbf{a}}) : \mathbf{w}_h|_K \in \mathbf{RT}_p(K) \quad \forall K \in \mathcal{T}_{\mathbf{a}} \\ \text{and} \quad \mathbf{w}_h \cdot \mathbf{n}_{\gamma} = 0 \text{ on edges } \gamma \in \mathcal{E}_{\mathbf{a}}^{B,E} \cup \mathcal{E}_{\mathbf{a}}^{B,N}\} \quad (5.5)$$

and the affine set

$$\mathbf{W}_{\mathbf{a}} = \{\mathbf{w}_h \in \mathbf{H}(\text{div}, \omega_{\mathbf{a}}) : \mathbf{w}_h|_K \in \mathbf{RT}_p(K) \quad \forall K \in \mathcal{T}_{\mathbf{a}}, \quad \mathbf{w}_h \cdot \mathbf{n}_{\gamma} = 0 \\ \text{on edges } \gamma \in \mathcal{E}_{\mathbf{a}}^{B,E} \text{ and } \mathbf{w}_h \cdot \mathbf{n}_{\gamma} = -\Pi_{\gamma}(\tilde{g}_h^{\mathbf{a}}) \text{ on edges } \gamma \in \mathcal{E}_{\mathbf{a}}^{B,N}\}. \quad (5.6)$$

The symbol Π_{γ} stands for the $L^2(\gamma)$ -orthogonal projection onto the space $P_p(\gamma)$ of polynomials of degree at most p on the edge γ . We also define the space $P_p(\mathcal{T}_{\mathbf{a}}) = \{v_h \in L^2(\omega_{\mathbf{a}}) : v_h|_K \in P_p(K) \quad \forall K \in \mathcal{T}_{\mathbf{a}}\}$ of piecewise polynomial and in

general discontinuous functions. Further, we introduce the space

$$P_p^*(\mathcal{T}_a) = \begin{cases} \{v_h \in P_p(\mathcal{T}_a) : \int_{\omega_a} v_h \, dx = 0\}, & \text{for } \mathbf{a} \in \mathcal{N}_h^I \cup \mathcal{N}_h^N, \\ P_p(\mathcal{T}_a), & \text{for } \mathbf{a} \in \mathcal{N}_h^D. \end{cases} \quad (5.7)$$

Using these spaces, we define the flux reconstruction $\mathbf{q}_h \in \mathbf{W}_h$ as the sum

$$\mathbf{q}_h = \sum_{\mathbf{a} \in \mathcal{N}_h} \mathbf{q}_h^a, \quad (5.8)$$

where $\mathbf{q}_h^a \in \mathbf{W}_a$ together with $d_h^a \in P_p^*(\mathcal{T}_a)$ solves the mixed finite element problem

$$(\mathcal{A}^{-1} \mathbf{q}_h^a, \mathbf{w}_h)_{\omega_a} - (d_h^a, \operatorname{div} \mathbf{w}_h)_{\omega_a} = (\psi_a \nabla u_{h,n}, \mathbf{w}_h)_{\omega_a} \quad \forall \mathbf{w}_h \in \mathbf{W}_a^0, \quad (5.9)$$

$$(\operatorname{div} \mathbf{q}_h^a, v_h)_{\omega_a} = (\tilde{r}_h^a, v_h)_{\omega_a} \quad \forall v_h \in P_p^*(\mathcal{T}_a). \quad (5.10)$$

Let us note that this mixed finite element problem is equivalent to the minimization of $\left\| \psi_a \mathcal{A}^{\frac{1}{2}} \nabla u_{h,n} - \mathcal{A}^{-\frac{1}{2}} \mathbf{s}_h^a \right\|_{\omega_a}$ over all $\mathbf{s}_h^a \in \mathbf{W}_a$ satisfying the constraint $\operatorname{div} \mathbf{s}_h^a = \Pi_p(\tilde{r}_h^a)$ in ω_a , where Π_p denotes the $L^2(\omega_a)$ -orthogonal projection onto $P_p(\mathcal{T}_a)$.

The flux reconstruction (5.8) is defined in such a way that quantities r and g vanish. We prove this fact below in Lemma 5.2. However, first, we need to prove that identity (5.10) can be actually tested by any polynomial.

Lemma 5.1. *Let $\mathbf{q}_h^a \in \mathbf{W}_a$ and $d_h^a \in P_p^*(\mathcal{T}_a)$ be a solution of problem (5.9)–(5.10). Then*

$$(\operatorname{div} \mathbf{q}_h^a, v_h)_{\omega_a} = (\tilde{r}_h^a, v_h)_{\omega_a} \quad \forall v_h \in P_p(\mathcal{T}_a). \quad (5.11)$$

Proof. Notice that if $\mathbf{a} \in \mathcal{N}_h^D$ then there is nothing to prove due to definition (5.7). If $\mathbf{a} \in \mathcal{N}_h^I \cup \mathcal{N}_h^N$ then we can use ψ_a as a test function in (3.7). Consequently, identity (5.4) and definition (5.6) imply

$$(\tilde{r}_h^a, 1)_{\omega_a} = - \sum_{\gamma \in \mathcal{E}_a^{\text{B,N}}} (\tilde{g}_h^a, 1)_\gamma = (\mathbf{q}_h^a \cdot \mathbf{n}_{\partial\omega_a}, 1)_{\partial\omega_a} = (\operatorname{div} \mathbf{q}_h^a, 1)_{\omega_a}. \quad (5.12)$$

□

Let us note that for vertices $\mathbf{a} \in \mathcal{N}_h^I \cup \mathcal{N}_h^N$ the problem (5.9)–(5.10) corresponds to a pure Neumann problem for d_h^a . This problem is solvable, because the corresponding equilibrium condition is exactly (5.12). In addition, its solution is unique thanks to the fact that the space $P_p^*(\mathcal{T}_a)$ does not contain constant functions. For $\mathbf{a} \in \mathcal{N}_h^D$, the problem (5.9)–(5.10) corresponds to a well posed Dirichlet–Neumann problem and the space $P_p^*(\mathcal{T}_a)$ contains constant functions. The existence and uniqueness of problem (5.9)–(5.10) can be rigorously proved in the same way as in [48].

Finally, we present the result that quantities r and g , see (4.1), vanish for the described flux reconstruction \mathbf{q}_h .

Lemma 5.2. *Let $\mathbf{q}_h \in \mathbf{W}_h$ be given by (5.8) and the problem (5.9)–(5.10). Then*

$$cu_{h,n} - \lambda_{h,n}\beta_1 u_{h,n} - \operatorname{div} \mathbf{q}_h = 0 \quad \text{a.e. in } \Omega, \quad (5.13)$$

$$\alpha u_{h,n} - \lambda_{h,n}\beta_2 u_{h,n} + \mathbf{q}_h \cdot \mathbf{n}_\Omega = 0 \quad \text{a.e. on } \Gamma_N. \quad (5.14)$$

Proof. Let us set $r_h = cu_{h,n} - \lambda_{h,n}\beta_1 u_{h,n} - \operatorname{div} \mathbf{q}_h$. Clearly, $r_h|_K \in P_p(K)$ for all $K \in \mathcal{T}_h$. Using the decomposition of unity $\sum_{\mathbf{a} \in \mathcal{N}_h^K} \psi_{\mathbf{a}} = 1$, notation (5.2), and equality (5.11), we obtain

$$\begin{aligned} \|r_h\|^2 &= \sum_{\mathbf{a} \in \mathcal{N}_h^K} (c\psi_{\mathbf{a}}u_{h,n} - \lambda_{h,n}\beta_1\psi_{\mathbf{a}}u_{h,n} - \operatorname{div} \mathbf{q}_h^{\mathbf{a}}, r_h)_{\omega_{\mathbf{a}}} \\ &= \sum_{\mathbf{a} \in \mathcal{N}_h^K} (\tilde{r}_h^{\mathbf{a}} - (\mathcal{A}\nabla\psi_{\mathbf{a}}) \cdot \nabla u_{h,n} - \operatorname{div} \mathbf{q}_h^{\mathbf{a}}, r_h)_{\omega_{\mathbf{a}}} \\ &= - \sum_{\mathbf{a} \in \mathcal{N}_h^K} ((\mathcal{A}\nabla\psi_{\mathbf{a}}) \cdot \nabla u_{h,n}, r_h)_{\omega_{\mathbf{a}}} = 0. \end{aligned}$$

Thus, r_h vanishes almost everywhere in Ω .

To prove the second statement, we set $g_h = \alpha u_{h,n} - \lambda_{h,n}\beta_2 u_{h,n} + \mathbf{q}_h \cdot \mathbf{n}_\Omega$. Let γ be an arbitrary edge on the Neumann boundary. Clearly, $g_h|_\gamma \in P_p(\gamma)$. The decomposition of unity $\sum_{\mathbf{a} \in \mathcal{N}_\gamma^T} \psi_{\mathbf{a}} = 1$ with \mathcal{N}_γ^T being the set of the two end-points of the edge γ and definitions (5.6) and (5.3) then give

$$\begin{aligned} \|g_h\|_\gamma^2 &= \sum_{\mathbf{a} \in \mathcal{N}_\gamma^T} (\alpha\psi_{\mathbf{a}}u_{h,n} - \lambda_{h,n}\beta_2\psi_{\mathbf{a}}u_{h,n} + \mathbf{q}_h^{\mathbf{a}} \cdot \mathbf{n}_\gamma, g_h)_\gamma \\ &= \sum_{\mathbf{a} \in \mathcal{N}_\gamma^T} (\alpha\psi_{\mathbf{a}}u_{h,n} - \lambda_{h,n}\beta_2\psi_{\mathbf{a}}u_{h,n} - \Pi_\gamma(\tilde{g}_h^{\mathbf{a}}), g_h)_\gamma = 0. \end{aligned} \quad (5.15)$$

Thus, g_h vanishes almost everywhere on γ and, hence, on Γ_N . \square

The following corollary summarizes the fact that properties (5.13) and (5.14) of the flux reconstruction \mathbf{q}_h simplify the error estimator presented in Theorem 4.1 considerably.

Corollary 5.3. *Let the flux reconstruction $\mathbf{q}_h \in \mathbf{W}_h$ be given by (5.8) and problem (5.9)–(5.10). Let $\lambda_{h,n} \in \mathbb{R}$ and $u_{h,n} \in V_h \setminus \{0\}$ satisfy (3.7). Further, let $w_n \in V$ be given by (1.2) with bilinear forms defined in (3.4)–(3.5) and with $\lambda_{*,n} = \lambda_{h,n}$ and $u_{*,n} = u_{h,n}$. Then*

$$\|w_n\|_a \leq \eta_n, \quad \text{where} \quad \eta_n^2 = \sum_{K \in \mathcal{T}_h} F_K^2 = \|\mathcal{A}^{1/2} (\nabla u_{h,n} - \mathcal{A}^{-1} \mathbf{q}_h)\|_{L^2(\Omega)}^2. \quad (5.16)$$

Proof. The statement follows immediately from Theorem 4.1 and properties (5.13) and (5.14). \square

Remark 5.4. In floating-point arithmetics, we cannot solve problem (5.9)–(5.10) exactly due to round-off errors. Consequently, hypotheses of Corollary 5.3 are

not satisfied, and the bound (5.16) is not guaranteed. The point is that quantities r and g do not vanish exactly in this case. In order to overcome this issue, we recommend to use Theorem 4.1 with $\mathbf{q} = \mathbf{q}_h$. For example, if $r \neq 0$, $c = 0$, and $\beta_1 > 0$ in Ω then $r_{1,K} = \infty$, $r_{2,K} = \infty$, and $r_{3,K} < \infty$ for all $K \in \mathcal{T}_h$. Consequently, $M_K = \infty$, $\mathcal{T}_h^{++} = \mathcal{T}_h$, and $\mathcal{T}_h^{+0} = \mathcal{T}_h^0 = \emptyset$. If we consider for simplicity $\Gamma_N = \emptyset$ then $Y_K = F_K$, $R_K = r_{3,K}$, and estimator (4.13) reduces to

$$\eta_n^2 = \sum_{K \in \mathcal{T}_h} F_K^2 + \underline{\lambda}_1^{-1/2} \sum_{K \in \mathcal{T}_h} r_{3,K}^2. \quad (5.17)$$

This estimator provides a guaranteed upper bound on $\|w_n\|_a$ even if r does not equal to zero exactly. However, it has to be said that in practical computations the quantity r and, thus, all quantities $r_{3,K}$ are typically on the level of machine precision. Therefore, the difference of estimators given by (5.16) and (5.17) is often negligible.

6. NUMERICAL EXAMPLE IN DUMBBELL SHAPE DOMAIN

This section illustrates the numerical performance of lower bounds ℓ_n and L_n given by (1.4) and (1.5), respectively. These bounds depend on quantities η_i , which are guaranteed bounds on representatives of residuals, see (1.3). We compute these quantities by the local flux reconstruction procedure described in Section 5.

We will compute two-sided bounds on the first ten eigenvalues of the Laplacian in a dumbbell shaped domain [45] with homogeneous Dirichlet boundary conditions. Thus, we consider problem (3.1) in the domain Ω showed in Figure 1 (left) with $\Gamma_N = \emptyset$, \mathcal{A} being the identity matrix, and with constant coefficients $c = 0$, $\alpha = 0$, $\beta_1 = 1$, $\beta_2 = 0$. We solve this problem by the standard conforming finite element method with piecewise linear and continuous trial and test functions, i.e., we consider the finite element space (3.6) with $p = 1$ and the finite element approximation given by (3.7). The computed eigenvalues $\lambda_{h,n}$ are upper bounds on the corresponding exact eigenvalues. Lower bounds are computed in two ways. First, we use lower bound L_n only and the quantity ν satisfying (2.18) is obtained by the homotopy method. Second, we combine both lower bounds ℓ_n and L_n .

Due to singularities of eigenfunctions in reentrant corners, we utilize a mesh adaptive algorithm. This algorithm follows the standard loop:

- **SOLVE.** Given a mesh \mathcal{T}_h , we compute finite element approximate eigenpairs $\lambda_{h,n}$, $u_{h,n}$, for $n = r, \dots, s$, see (3.7).
- **ESTIMATE.** We solve local problems (5.9)–(5.10), reconstruct the flux (5.8), and compute error estimator (4.13) for $n = r, \dots, s$. We check if the left-hand side inequality of condition (2.18) is satisfied. If so, we compute the lower bound (1.5) for $n = r, \dots, s$. In order to obtain as accurate estimates

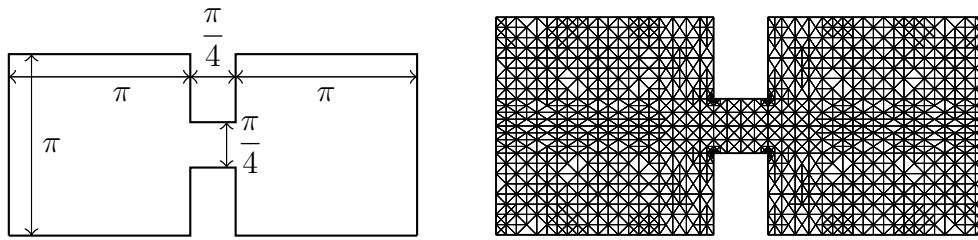


FIGURE 1. The left panel shows the dimensions of the dumbbell shaped domain $\Omega = \Omega^{(4)}$. The right panel presents the adaptively refined mesh after 20 adaptive steps.

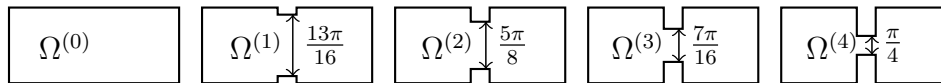


FIGURE 2. The five domains we use for the homotopy transition between the rectangle $\Omega^{(0)}$ and the dumbbell shaped domain $\Omega^{(4)}$.

as possible, we evaluate these lower bounds recursively, as described at the end of Section 2.

- **MARK.** We use quantities F_K , see (4.2) and (5.16), to mark elements for refinement. For each element we have $s-r+1$ quantities F_K corresponding to eigenvalues λ_n , $n = r, \dots, s$. We use their maximum as the error indicator for the mesh refinement. We mark elements by the bulk criterion [17].
- **REFINE.** We refine the marked elements by the newest vertex bisection algorithm, see e.g. [41].

In order to compare the accuracy of computed eigenvalue bounds, we stop this adaptive algorithm as soon as the number of degrees of freedom exceeds 750 000.

Bound L_n with the homotopy method. The quantity ν satisfying (2.18) is computed by the homotopy method described in detail in [34]. In particular, we use the specific procedure from [35, Sec. 6.1, par. b)]. We consider a sequence of five domains $\Omega^{(m)}$, $m = 0, 1, \dots, 4$. These domains are interiors of the union of squares $(0, \pi)^2$, $(5\pi/4, 9\pi/4) \times (0, \pi)$, and the rectangle $[\pi, 5\pi/4] \times (y_1, \pi - y_1)$, where $y_1 = 3m\pi/32$ and $m = 0, 1, \dots, 4$, see Figure 2. Notice that $\Omega^{(0)} = (0, 9\pi/4) \times (0, \pi)$ is a rectangle and $\Omega^{(4)} = \Omega$ is the targeted dumbbell shaped domain. Moreover, these domains are nested $\Omega^{(4)} \subset \Omega^{(3)} \subset \dots \subset \Omega^{(0)}$ and thus the eigenvalues on the larger domain are below the corresponding eigenvalues on the smaller domain. More accurately, if $\lambda_n^{(m)}$ denotes the n -th eigenvalue of the Laplace eigenvalue problem in the domain $\Omega^{(m)}$ then the Courant minimax principle [6] implies that

$$\lambda_n^{(m-1)} \leq \lambda_n^{(m)}, \quad \text{for } m = 1, 2, 3, 4 \text{ and } n = 1, 2, \dots \quad (6.1)$$

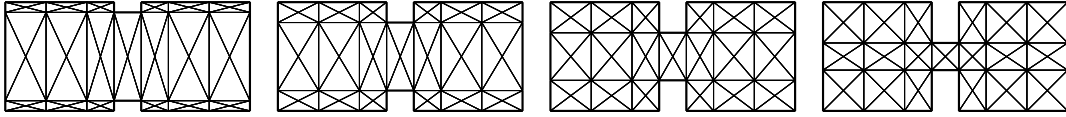


FIGURE 3. Initial meshes on domains $\Omega^{(1)}, \dots, \Omega^{(4)}$, respectively.

Eigenvalues on the rectangle $\Omega^{(0)}$ are known analytically:

$$\lambda_{i,j}^{(0)} = 16i^2/81 + j^2, \quad i, j = 1, 2, \dots$$

One of these values can be chosen for ν to compute lower bounds on the domain $\Omega^{(1)}$. Due to the large spectral gap $\lambda_{21}^{(0)} - \lambda_{20}^{(0)}$, see Table 1, we choose $\nu = 16.1111 < 145/9 = \lambda_{6,3}^{(0)} = \lambda_{21}^{(0)}$. By (6.1) we have $\nu < \lambda_{21}^{(1)}$. Hence, the right-hand side inequality of condition (2.18) is satisfied for $s = 20$ and we can use (1.5), see Theorem 2.5, to compute lower bounds on $\lambda_1^{(1)}, \dots, \lambda_{20}^{(1)}$.

We compute these lower bounds by the adaptive algorithm described above using initial meshes depicted in Figure 3. Note that in order to evaluate $\eta_n^{(b)}$ in (4.13) we use $\underline{\lambda}_1 = \lambda_1^{(0)} = 97/81$ computed analytically for the rectangle $\Omega^{(0)}$. This value is guaranteed to be below $\lambda_1^{(m)}$ for all $m = 1, 2, 3, 4$, see (6.1).

To continue the homotopy method, we set $\nu = L_{20}^{(1)}$, where $L_{20}^{(1)}$ is the lower bound on $\lambda_{20}^{(1)}$ for the domain $\Omega^{(1)}$. Using this value, we repeat the adaptive process for $\Omega^{(2)}$ and compute lower bounds $L_1^{(2)}, \dots, L_{19}^{(2)}$ on the first nineteen eigenvalues for the domain $\Omega^{(2)}$. Then we choose $\nu = L_{18}^{(2)}$ and compute by the same procedure lower bounds $L_1^{(3)}, \dots, L_{17}^{(3)}$ for the domain $\Omega^{(3)}$. Note that we did not choose $L_{19}^{(2)}$ for ν , because the spectral gap $\lambda_{19}^{(2)} - \lambda_{18}^{(2)}$ is too small, see Table 1. Finally, we take $\nu = L_{17}^{(3)}$ and compute lower bounds on the first sixteen eigenvalues for the targeted domain $\Omega^{(4)} = \Omega$.

Table 1 presents the analytically computed eigenvalues for $\Omega^{(0)}$ and two-sided bounds on eigenvalues obtained in the last adaptive step for domains $\Omega^{(1)}, \dots, \Omega^{(4)}$. We observe that the smaller eigenvalues are approximated with higher accuracy than the larger eigenvalues. This is caused by the recursive evaluation of the lower bound (1.5), see the description at the end of Section 2, and also by the fact that the higher eigenfunctions are more oscillatory and hence more difficult to approximate.

We note that the original goal was to compute two-sided bounds of the first ten eigenvalues, but it is advantageous to compute slightly more of them, because the performance of the lower bound L_n depends on the size of the corresponding spectral gaps. For example in the case of $\Omega^{(4)}$, we obtain ν in the reasonable large spectral gap between λ_{16} and λ_{17} , which enables to find accurate lower bounds for the first ten eigenvalues.

	Bound L_n with the homotopy method					Combination of ℓ_n and L_n
	$\Omega^{(0)}$	$\Omega^{(1)}$	$\Omega^{(2)}$	$\Omega^{(3)}$	$\Omega^{(4)}$	
λ_1	1.19753	1.351 ³⁵ ₄₈	1.5974 ⁰ ₉	1.8301 ³ ₉	1.955 ⁷⁶ ₈₁	1.955 ⁷⁶ ₈₁
λ_2	1.79012	1.804 ¹⁸ ₂₇	1.8414 ³ ₈	1.8988 ⁰ ₃	1.960 ⁶⁷ ₇₀	1.960 ⁶⁷ ₇₀
λ_3	2.77778	2.926 ⁰² ₂₈	3.30 ¹⁸¹ ₂₁₁	4.11 ⁰⁵⁸ ₁₃₀	4. ⁷⁹⁹⁹⁸ ₈₀₀₈₅	4. ⁷⁹⁹⁹⁸ ₈₀₀₈₅
λ_4	4.16049	4.20 ⁷⁸⁷ ₉₁₆	4.33 ⁶⁷¹ ₇₄₃	4.558 ²² ₈₆	4.829 ³⁶ ₉₉	4.829 ³⁶ ₉₉
λ_5	4.19753	4.62 ⁰³⁰ ₃₂₃	4.8 ⁷⁸⁸² ₈₀₁₈	4.971 ²⁷ ₇₇	4.996 ⁵⁰ ₉₁	4.996 ⁵⁰ ₉₁
λ_6	4.79012	4.83 ⁵⁵⁸ ₇₂₀	4.91 ⁴⁷³ ₅₄₈	4.97 ³⁸⁴ ₄₀₉	4.996 ⁷² ₉₂	4.996 ⁷² ₉₂
λ_7	5.77778	6.02 ⁴⁹⁰ ₈₁₄	6.234 ¹⁴ ₈₄	7.08 ²⁴³ ₄₀₆	7.98 ⁴⁶⁰ ₇₁₄	7.98 ⁴⁶⁰ ₇₁₄
λ_8	5.93827	6.4 ⁰⁸⁷⁵ ₁₃₃₀	7.40 ⁴⁰⁶ ₇₈₄	7.87 ¹⁵² ₅₀₂	7.98 ⁵⁹⁴ ₇₂₁	7.98 ⁵⁹⁴ ₇₂₁
λ_9	7.16049	7.32 ⁴²⁰ ₉₁₉	7.63 ²⁹⁷ ₅₅₃	7.88 ⁶⁶⁹ ₈₆₅	9.35 ²⁷² ₇₅₂	9.35 ²⁷² ₇₅₂
λ_{10}	8.11111	8.1 ⁸⁷²³ ₉₀₁₀	8.37 ⁶⁰⁵ ₇₄₃	8.78 ²⁴¹ ₄₄₀	9.5 ⁰⁷⁵³ ₁₁₀₈	9.5 ⁰⁷⁵³ ₁₁₀₈
λ_{11}	8.93827	9. ³⁸⁹³⁹ ₄₁₀₇₄	9.9 ³⁸⁰⁸ ₅₀₄₅	9.9 ⁸⁸⁵⁹ ₉₂₉₆	9.99 ⁷⁵⁵ ₉₉₃	9.99 ⁷⁵⁵ ₉₉₃
λ_{12}	9.19753	9.7 ⁶³⁷⁵ ₈₆₇₈	9.95 ¹⁵⁰ ₇₇₂	9.99 ⁰⁸³ ₃₂₉	9.99 ⁸⁷⁴ ₉₉₃	9.99 ⁸⁷⁵ ₉₉₃
λ_{13}	9.79012	9.8 ⁵⁸⁵⁴ ₇₂₉₅	10.7 ⁷¹⁷ ₈₁₄	11.12 ⁴¹ ₆₅	12. ⁶²⁷¹ ₇₃₁₉	12. ⁶⁵⁴⁰ ₇₃₁₉
λ_{14}	10.6790	10. ⁶⁹⁴⁰ ₇₀₂₃	10.9 ²⁵² ₃₂₂	12.6 ¹⁹⁴ ₄₀₈	12. ⁷⁴⁰² ₈₃₀₆	12. ⁷⁵³¹ ₈₃₀₆
λ_{15}	10.7778	11. ³⁹⁶¹ ₄₁₅₈	12.0 ³⁰⁸ ₄₅₉	12.6 ⁶⁶⁹ ₈₃₂	12. ⁸⁹⁷⁴ ₉₆₉₃	12. ⁸⁹³⁴ ₉₆₉₃
λ_{16}	11.1111	12.0 ⁵³⁸ ₈₅₅	12.6 ³³⁶ ₅₀₅	12. ⁶⁹⁵³ ₇₀₇₃	12.9 ³³⁵ ₆₉₅	12. ⁸⁹³⁷ ₉₆₉₅
λ_{17}	12.1605	12.4 ²³⁸ ₄₁₇	12.8 ⁷³⁵ ₈₃₄	13.13 ¹⁶ ₈₆		
λ_{18}	13.6420	13.7 ³⁵¹ ₅₇₆	14.0 ⁴³⁷ ₅₇₀			
λ_{19}	13.6790	13. ⁸⁹¹⁵ ₉₀₆₈	14.6 ⁴⁴³ ₆₇₅			
λ_{20}	13.9383	15. ¹⁹⁸⁹ ₂₄₂₂				
λ_{21}	16.1111					

TABLE 1. Two-sided bounds computed in the last adaptive step. Digits in the upper/lower index indicate the lower/upper bound, respectively. Eigenvalues on the rectangle $\Omega^{(0)}$ are known analytically. Columns $\Omega^{(1)}, \dots, \Omega^{(4)}$ correspond to homotopy steps for lower bound L_n . The last column presents the combination of lower bounds ℓ_n and L_n for the dumbbell shaped domain $\Omega^{(4)} = \Omega$.

Combination of ℓ_n and L_n . The rough idea is to compute the lower bound on λ_n as $\max\{\ell_n, L_n\}$ and to use $\nu = \ell_{n+1}$ to evaluate L_n . Quantities ℓ_n and L_n are given by (1.4) and (1.5), respectively. This approach is computationally less demanding, because the expensive homotopy is not performed. On the other hand, this approach does not provide guaranteed lower bounds on eigenvalues due to the closeness condition (2.12), which cannot be verified in practice. However, numerical experiments we performed indicate that the computed bounds are very reliable. In our numerical tests, we never found the computed lower bound to be above the best available upper bound. This applies not only to the situation when the problem is well resolved and the closeness condition (2.12) and the lower

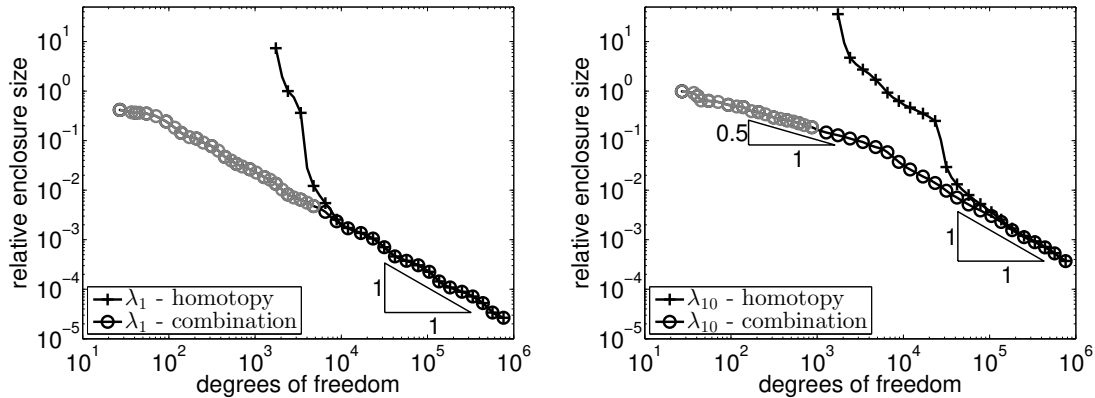


FIGURE 4. Convergence curves for relative enclosure sizes $(\lambda_{h,n} - L_n)/L_n$ (homotopy) and $(\lambda_{h,n} - \underline{\lambda}_{h,n})/\underline{\lambda}_{h,n}$ (combination) for $n = 1$ (left) and $n = 10$ (right) with respect to the number of degrees of freedom. Crosses correspond to the last homotopy step, circles to the combination of ℓ_n and L_n , see (1.4) and (1.5). Data points, where the best bounds indicate invalidity of closeness condition (2.12) are coloured in grey.

bound condition (2.18) are likely to hold, but even to cases when the accuracy is low and these conditions are probably not satisfied. The reason is that these conditions are sufficient and if they are not satisfied then the computed lower bounds can still be below the true eigenvalues. Numerical results show that this is a very common situation.

Let us describe the algorithm in more details. It is based on the above specified adaptive algorithm with two changes. First, step **SOLVE** computes eigenpairs for $n = r, \dots, s + 1$. Second, step **ESTIMATE** is changed as follows:

- Solve local problems (5.9)–(5.10), reconstruct the flux (5.8), and compute error estimator η_n by (4.13) for all eigenpairs $\lambda_{h,n}$, $u_{h,n}$, $n = r, \dots, s + 1$.
- Compute lower bounds $\ell_r, \dots, \ell_{s+1}$ by (1.4).
- Set $\nu = \ell_{s+1}$, check the left-hand side inequality of condition (2.18), and compute lower bounds L_r, \dots, L_s by the recursive application of (1.5).
- The final lower bound is given by $\underline{\lambda}_{h,n} = \max\{\ell_n, L_n\}$ for all $n = r, \dots, s$.

The results of this procedure for the Laplace eigenvalue problem in the dumb-bell shaped domain with $r = 1$ and $s = 15$ are provided in the last column of Table 1. For illustration, Figure 1 (right) shows the adaptively refined mesh after 20 adaptive steps. In Figure 4, we also present convergence curves for relative enclosure sizes $E_{\text{rel},n}^{\text{est}} = (\lambda_{h,n} - \underline{\lambda}_{h,n})/\underline{\lambda}_{h,n}$. Note that $E_{\text{rel},n}^{\text{est}}$ bounds the true relative error: $(\lambda_{h,n} - \lambda_n)/\lambda_n \leq E_{\text{rel},n}^{\text{est}}$ and, thus, provides a reliable information about the true error. Figure 4 shows $E_{\text{rel},n}^{\text{est}}$ for the first (left) and the tenth (right) eigenvalue. We observe that the combination of the adaptive algorithm and the

quadratically convergent bound L_n yields the expected order of convergence on sufficiently fine meshes. We point out that suboptimal lower bound ℓ_n is useful on relatively coarse meshes, where it provides more accurate results than L_n . Indeed, convergence curves provided in Figure 4 typically start with a suboptimal slope corresponding to ℓ_n and as soon as the mesh is sufficiently fine, the optimal bound L_n overcomes. We also note that L_n is not evaluated on those rough meshes where the left-hand side inequality in (2.18) is not satisfied, see for example the curves for the homotopy method, where data points for small numbers of degrees of freedom are missing. Finally, we used the best available lower bounds $\underline{\lambda}_{h,n}^{\text{best}}$ in the last adaptive step and tested if $\lambda_{h,n} \leq \sqrt{\underline{\lambda}_{h,n}^{\text{best}} \underline{\lambda}_{h,n+1}^{\text{best}}}$. Those points, where this test passed and closeness condition (2.12) is satisfied, are indicated as black circles in Figure 4. Notice that even in grey data points, where closeness condition (2.12) is probably not valid, lower bound ℓ_n always produced a value below the best available lower bound and hence below the true eigenvalue.

Comparing results in Table 1, we observe that values in the last column are almost the same as values in the previous column. Thus, the combination of bounds ℓ_n and L_n yielded almost the same lower bounds on eigenvalues although they are not guaranteed by the theory. Thus, if the bounds are not required to be guaranteed, we recommend to use the simple combination of bounds ℓ_n and L_n rather than the homotopy method.

7. CONCLUSIONS

In this paper we generalized classical Weinstein's and Kato's bounds to the weak setting suitable for direct application of the finite element method. Needed guaranteed bounds on the representative of the residual are computed by the complementarity technique using the local flux reconstruction. Formulas (1.4) and (1.5) for lower bounds ℓ_n and L_n are simple and explicit. The bound L_n is quadratically convergent and provides guaranteed lower bounds on all eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_s$, provided a guaranteed lower bound ν on λ_{s+1} is available. The computational efficiency of the bound L_n depends on the size of the spectral gap $\lambda_{s+1} - \lambda_s$. Therefore, we recommend to first identify the index s such that the spectral gap $\lambda_{s+1} - \lambda_s$ is relatively large and then use a guaranteed lower bound ν on λ_{s+1} to estimate the smaller eigenvalues from below. For the optimal usage of this bound, we recommend the recursive algorithm described at the end of Section 2.

The guaranteed lower bound on λ_{s+1} can be computed, for example, by the homotopy method as we illustrated in Section 6. Numerical experiments, however, show that L_n need not be sufficiently accurate on rough meshes. Therefore, we recommend to compute both ℓ_n and L_n and use the larger of them. We note that ℓ_n can be evaluated very cheaply as soon as η_n is available. The bound ℓ_n is guaranteed to be below the corresponding exact eigenvalue λ_n if the (upper) finite element approximation $\lambda_{h,n}$ is in a sense closer to λ_n than to λ_{n+1} , see

(2.12). This condition can be verified if guaranteed lower bounds on λ_n and λ_{n+1} are available, for example using L_n and L_{n+1} .

If the particular application does not require the computed lower bounds to be guaranteed then the homotopy method is not necessary. We can use ℓ_n as the first lower bounds and then employ them to compute more accurate bounds L_n . We would like to emphasize that the conditions guaranteeing that ℓ_n and L_n are below the corresponding exact eigenvalues are sufficient conditions only. Thus, if they are not satisfied then the computed bounds can still be below the exact values. Performed numerical experiments confirm that this is actually a very common case in practice.

In our future research we will focus on the proof of the local efficiency of the estimator given by Corollary 5.3 with the local flux reconstruction satisfying (5.8) and (5.9)–(5.10). We then plan to use this local efficiency result to prove the convergence of the corresponding adaptive algorithm.

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