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SCHOTTKY VERTEX OPERATOR CLUSTER ALGEBRAS

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Abstract. Using recursion formulas for vertex operator algebra higher genus characters with formal parameters identified with local coordinates around marked points on a Riemann surface of arbitrary genus, we introduce the notion of a vertex operator cluster algebra structure. Cluster elements and mutation rules are explicitly defined, and the simplest example of a vertex operator cluster algebra is presented.

1. Introduction

The theory of cluster algebras is connected to many different areas of mathematics, e.g., the representation theory of finite dimensional algebras, Lie theory, Poisson geometry and Teichmüller theory [30, 31, 33]. Among these topics are dilogarithm identities for conformal field theories [51, 52], quantum algebras [34, 35], quivers [37, 38, 53]. Cluster algebras have numerous applications [22, 23, 24, 25, 30, 31, 18, 39, 51, 52, 11]. Cluster algebras appear in several applications in Conformal Field Theory [11, 51, 52]. It is natural to find an analogue of a cluster algebra structure in the language of vertex operator algebras. In particular, a structure that incorporates non-commutative nature of vertex operator algebra relations.

In this paper we make use of the higher genus reduction formulas for vertex algebra characters. The recursion properties of vertex operator algebra characters allow us to introduce an algebraic structure which we call a vertex operator cluster algebra, simultaneously incorporating properties of cluster algebras, non-commutative nature, and analytical and geometrical features of character function theory of vertex operator algebras on Riemann surfaces. Vertex operator cluster algebra seeds are defined over non-commutative variables (elements of vertex algebra), coordinates around marked points on a Riemann surface, and functions depending on a number of vertex operators. the origin of ordinary cluster algebras arising [18] on Riemann surfaces.

In Section 5 we recall basic definitions related to cluster algebras. In Section 2 the construction of n-point characters on the Schottky reparameterization of a genus g Riemann surface is reminded. A short introduction to vertex operator algebras is given in Appendix 4. Appendix 3 contains the formulation of a vertex operator cluster algebra structure and Proposition 2 describing the involutivity property of a vertex algebra setup. Appendix 6 contains a description of the Schottky parameterization

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of a genus g Riemann surface. Appendix 7 describes auxiliary objects and matrices needed for the Schottky genus g Zhu reduction formula.

2. Vertex algebra characters and Zhu reduction formula in Schottky parameterization

In this section we recall [32] the construction and Proposition 1 for vertex operator algebra character functions on a genus g Riemann surface. In particular, the formal partition and n-point correlation functions for a vertex operator algebra associated to a genus g Riemann surface $\mathcal{S}_g$ are introduced in the Schottky scheme with sewing relation (6.5). All expressions here are functions of formal variables $w_{\pm a}$, ρ_a and vertex operator parameters. Then we recall the genus g Zhu recursion formula with universal coefficients that have a geometrical meaning and are meromorphic on a Riemann surface $\mathcal{S}_g$ for all $(w_{\pm a}, \rho_a) \in \mathfrak{C}_g$ (for notations see Appendix 6). These coefficients are generalizations of the elliptic Weierstrass functions [57].

For a $2g$ vertex algebra V states

$$\boldsymbol{b} = (b_{-1}, b_1; \ldots; b_{-g}; b_g),$$

and corresponding local coordinates

$$\boldsymbol{w} = (w_{-1}, w_1; \ldots; w_{-g}, w_g),$$

of points $2g$ $(p_{-1}, p_1; \ldots; p_{-g}, p_g)$ on the sphere $\mathcal{S}_0$, consider the genus zero $2g$-point correlation function

$$\begin{aligned} Z^{(0)}(\boldsymbol{b}, \boldsymbol{w}) =& Z^{(0)}(b_{-1}, w_{-1}; b_1, w_1; \ldots; b_{-g}, w_{-g}; b_g, w_g) \\ =& \prod_{a \in \mathcal{I}_+} \rho_a^{\mathrm{wt}(b_a)} Z^{(0)}(\bar{b}_1, w_{-1}; b_1, w_1; \ldots; \bar{b}_g, w_{-g}; b_g, w_g). \end{aligned}$$

where $\mathcal{I}_+ = \{1, 2, \ldots, g\}$. Let

$$\boldsymbol{b}_+ = (b_1, \ldots, b_g),$$

denote an element of a V-tensor product $V^{\otimes g}$-basis with dual basis

$$\boldsymbol{b}_- = (b_{-1}, \ldots, b_{-g}),$$

with respect to the bilinear form $\langle \cdot, \cdot \rangle_{\rho_a}$ (cf. Appendix 4).

Let w_a for $a \in \mathcal{I}$ be $2g$ formal variables. One identify them with the canonical Schottky parameters (see Appendix 6). We can define the genus g partition function as

$$Z_V^{(g)} = Z_V^{(g)}(\boldsymbol{w}, \boldsymbol{\rho}) = \sum_{\boldsymbol{b}_+} Z^{(0)}(\boldsymbol{b}, \boldsymbol{w}), \tag{2.1}$$

for

$$(\boldsymbol{w}, \boldsymbol{\rho}) = (w_{\pm 1}, \rho_1; \ldots; w_{\pm g}, \rho_g).$$

This definition is motivated by the sewing relation (6.5).

Remark 1. Note that $Z_V^{(g)}$ depends on ρ_a via the dual vectors $\boldsymbol{b}_-$ as in (4.17). The genus g partition function for the tensor product $V_1 \otimes V_2$ of two vertex operator algebras V_1 and V_2 is

$$Z_{V_1 \otimes V_2}^{(g)} = Z_{V_1}^{(g)} \, Z_{V_2}^{(g)}.$$

Now we recall a formal Zhu reduction expression for all genus g Schottky n-point character functions. One defines the genus g formal n-point function for n vectors $(v_1, \ldots, v_n) \in V$ inserted $(y_1, \ldots, y_n)$ by

$$Z_V^{(g)}(\boldsymbol{v}, \boldsymbol{y}) = Z_V^{(g)}(\boldsymbol{v}, \boldsymbol{y}; \boldsymbol{w}, \boldsymbol{\rho}) = \sum_{\boldsymbol{b}_+} Z^{(0)}(\boldsymbol{v}, \boldsymbol{y}; \boldsymbol{b}, \boldsymbol{w}), \tag{2.2}$$

where

$$Z^{(0)}(\boldsymbol{v}, \boldsymbol{y}; \boldsymbol{b}, \boldsymbol{w}) = Z^{(0)}(v_1, y_1; \ldots; v_n, y_n; b_{-1}, w_{-1}; \ldots; b_g, w_g).$$

Let U be a vertex operator subalgebra of V where V has a U-module decomposition

$$V = \bigoplus_{\alpha \in A} W_\alpha,$$

for U-modules W_α and some indexing set A. Let

$$W_{\boldsymbol{\alpha}} = \bigotimes_{a=1}^{g} W_{\alpha_a},$$

denote a tensor product of g modules

$$Z_{W_{\boldsymbol{\alpha}}}^{(g)}(\boldsymbol{v}, \boldsymbol{y}) = \sum_{\boldsymbol{b}_+ \in W_{\boldsymbol{\alpha}}} Z^{(0)}(\boldsymbol{v}, \boldsymbol{y}; \boldsymbol{b}, \boldsymbol{w}), \tag{2.3}$$

where here the sum is over a basis $\{\boldsymbol{b}_+\}$ for $W_{\boldsymbol{\alpha}}$. It follows that

$$Z_V^{(g)}(\boldsymbol{v}, \boldsymbol{y}) = \sum_{\boldsymbol{\alpha} \in \boldsymbol{A}} Z_{W_{\boldsymbol{\alpha}}}^{(g)}(\boldsymbol{v}, \boldsymbol{y}), \tag{2.4}$$

where the sum ranges over $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_g) \in \boldsymbol{A}$, for $\boldsymbol{A} = A^{\otimes g}$. Finally, it is useful to define corresponding formal n-point correlation differential forms

$$\begin{aligned} \mathcal{F}_V^{(g)}(\boldsymbol{v}, \boldsymbol{y}) &= Z^{(g)}(\boldsymbol{v}, \boldsymbol{y}) \ \boldsymbol{dy}^{\mathrm{wt}(\boldsymbol{v})}, \\ \mathcal{F}_{W_{\boldsymbol{\alpha}}}^{(g)}(\boldsymbol{v}, \boldsymbol{y}) &= Z_{W_{\boldsymbol{\alpha}}}^{(g)}(\boldsymbol{v}, \boldsymbol{y}) \ \boldsymbol{dy}^{\mathrm{wt}(\boldsymbol{v})}, \end{aligned} \tag{2.5}$$

where

$$\boldsymbol{dy}^{\mathrm{wt}(\boldsymbol{v})} = \prod_{k=1}^{n} dy_k^{\mathrm{wt}(v_k)}.$$

Recall notations and identifications given in Appendix 7. Then one has:

Proposition 1. *The genus g $(n+1)$-point formal differential $\mathcal{F}_{W_{\boldsymbol{\alpha}}}^{(g)}(u, x; \boldsymbol{v}, \boldsymbol{y})$ for a quasiprimary vector $u \in U$ of weight $\mathrm{wt}(u) = p$ inserted at a point p_0, with the coordinate x, and general vectors $(v_1, \ldots, v_n)$ inserted at points $p_1, \ldots, p_n$ with coordinates $(y_1, \ldots, y_n)$ correspondingly, respectively, satisfies the recursive identity*

$$\begin{aligned} &\mathcal{F}_{W_{\boldsymbol{\alpha}}}^{(g)}(u, x; \boldsymbol{v}, \boldsymbol{y}) \\ &= \sum_{k=1}^{n} \sum_{j \geqslant 0} \partial^{(0,j)} \ \Psi_p(x, y_k) \ \mathcal{F}_{W_{\boldsymbol{\alpha}}}^{(g)}(v_1, y_1; \ldots; u(j)v_k, y_k; \ldots; v_n, y_n) \ dy_k^j \\ &\quad + \sum_{a=1}^{g} \Theta_a(x) \ O_a^{W_{\boldsymbol{\alpha}}}(u; \boldsymbol{v}, \boldsymbol{y}). \end{aligned} \tag{2.6}$$

Here $\partial^{(0,j)}$ is given by

$$\partial^{(i,j)} f(x,y) = \partial_x^{(i)} \partial_y^{(j)} f(x,y),$$

for a function $f(x,y)$, and $\partial^{(0,j)}$ denotes partial derivatives with respect to x and y_j. The forms $\Psi_p(x, y_k)\ dy_k^j$ given by (7.14), $\Theta_a(x)$ is of (7.16), and $O_a^{W_{\boldsymbol{\alpha}}}(u; \boldsymbol{v}, \boldsymbol{y})$ of (7.17).

3. Schottky vertex operator cluster algebras

In this section we formulate Proposition 2 concerning a cluster vertex algebra associated to a vertex operator algebra in the case of Schottky parameterization of a genus g Riemann surface. That Proposition clarifies a cluster-like algebra structure for a vertex operator algebra. Let us fix a strong-type (cf. Appendix 4) vertex operator algebra V. Chose $n+1$-marked points p_0 and p_i, $i=1,\ldots,n$ on a genus g Riemann surface formed by the Schottky parameterization (cf. Appendix 6). In the vicinity of each marked point p_0, p_i define local coordinates x, y_i, with zero at points p_0, p_i correspondingly.

Consider n-tuples of arbitrary states $v_i \in V$, and corresponding vertex operators

$$\mathbf{Y}(\boldsymbol{v}, \boldsymbol{y}) = \left(Y(v_1, y_1), \ldots, Y(v_n, y_n)\right),$$

with coordinates $(y_1, \ldots, y_n)$, around points p_i, $i=1,\ldots,n$.

Definition 1. We define a vertex operator cluster algebra seed

$$\left(\boldsymbol{v}, \mathbf{Y}(\boldsymbol{v}, \boldsymbol{y}), \mathcal{F}_n^{(g)}(\boldsymbol{v}, \boldsymbol{y})\right), \tag{3.1}$$

where

$$\mathcal{F}_n^{(g)}(\boldsymbol{v}, \boldsymbol{y}) = \mathcal{F}_{W_{\boldsymbol{\alpha}}}^{(g)}(\boldsymbol{v}, \boldsymbol{y}),$$

(and in particular $\mathcal{F}_V^{(g)}(\boldsymbol{v}, \boldsymbol{y})$) is a genus g n-point character function $\mathcal{F}_n^{(g)}(\boldsymbol{v}, \boldsymbol{y})$ (2.5).

The mutation is defined as follows:

Definition 2. For $\boldsymbol{v}$, we define the mutation $\boldsymbol{v}'$ of $\boldsymbol{v}$ in the direction $k \in 1, \ldots, n$ as

$$\boldsymbol{v}' = \mu_k(u,m)\boldsymbol{v} = \left(v_1, \ldots, F_k(u(m)).v_k, \ldots, v_n\right), \tag{3.2}$$

for some $m \geqslant 0$, and V-valued functions $F_k(v(m))$. Note that due to the property (4.3) we get a finite number of terms as a result of the action of $v(m)$ on v_k, $1 \leqslant k \leqslant n$. For the n-tuple of vertex operators we define

$$\begin{aligned} \mathbf{Y}\left(\boldsymbol{v}', \boldsymbol{y}\right) &= \mu_k(u,m)\ \mathbf{Y}\left(\boldsymbol{v}, \boldsymbol{y}\right) \\ &= \left(Y(v_1, y_1), \ldots, Y(G_k(u(m)).v_k, y_k), \ldots, Y(v_n, y_n)\right), \end{aligned} \tag{3.3}$$

where $G_k(u(m))$ are other V-valued functions. For $u \in V$, $w \in \mathbb{C}$, the mutation $\mu(u, x, \boldsymbol{y})$ of $\mathcal{F}_n^{(g)}(\boldsymbol{v}, \boldsymbol{y})$,

$$\mathcal{F}_n^{(g)\prime}(\boldsymbol{v}, \boldsymbol{y}) = \mu(u, x, \boldsymbol{y})\ \mathcal{F}_n^{(g)}(\boldsymbol{v}, \boldsymbol{y}), \tag{3.4}$$

is defined by summation over mutations in all possible directions k, $1 \leqslant k \leqslant n$, with auxiliary functions $f(m,k,x,\boldsymbol{y})$, $k \in 1,\ldots,n$:

$$\begin{aligned}\mathcal{F}_n^{(g)\prime}(\boldsymbol{v},\boldsymbol{y}) \\ &= \sum_{k=1}^{n}\sum_{m\geqslant 0} f(m,k,x,\boldsymbol{y})\; \mathcal{F}_n^{(g)}(v_1,y_1;\ldots;H_k(u(m)).v_k,y_k;\ldots;v_n,y_n), \\ &\quad + \widetilde{\mathcal{F}}_n^{(g)}(u,x;\boldsymbol{v},\boldsymbol{y}).\end{aligned} \tag{3.5}$$

where $\widetilde{\mathcal{F}}_n^{(g)}(u,x;\boldsymbol{v},\boldsymbol{y})$, denotes the higher terms in the genus g Zhu reduction formulas (2.6), and $H_k(u(m))$ are V-valued functions. Then (3.2), (3.3), (3.5) define the mutation of the seed (3.1).

Definition 3. Definitions 1–2, the genus g Zhu reduction procedure, and involutivity condition for mutation determine the structure of a genus g vertex operator cluster algebra $\mathcal{CG}_n^{(g)}$ of dimension n. We call the full vertex operator cluster algebra the union $\bigcup_{n\geqslant 0} \mathcal{CG}_n^{(g)}$.

Remark 2. Exchange matrix of ordinary cluster algebras is replaced in this construction with genus g Schottky characters for a vertex operator algebra. These are higher genus generalizations of matrix elements at genus zero [26], and traces of vertex operator algebra modules at genus one [57].

Using (2.6), we obtain in (3.2), (3.3), and (3.5):

$$\begin{aligned} f(m,k,x,\boldsymbol{y}) &= \partial^{(0,j)}\Psi_p(x,y_k)\, dy_k^j, \\ \widetilde{\mathcal{F}}_n^{(g)}(u,x;\boldsymbol{v},\boldsymbol{y}) &= \sum_{a=1}^{g}\Theta_a(x)\; O_a^{W_{\boldsymbol{\alpha}}}(u;\boldsymbol{v},\boldsymbol{y}). \end{aligned} \tag{3.6}$$

Next we provide an example of the Schottky vertex operator cluster algebra. We formulate

Proposition 2. *For a vertex operator algebra V such that $\dim V_k = 1$, $k \in \mathbb{Z}$, with $u = \mathbf{1}_V$, $w \in \mathbb{C}$, and*

$$F_k(u(m)).v = G_k(u(m)).v = \xi_{u,v}u(-1).v,$$

$$H_k(u(m)) = u(m),$$

for $m \geqslant 0$, and $\xi_{u,v} \in \mathbb{C}$, $\xi_{u,v}^2 = 1$, depending on u and v, in (3.2), (3.3), and (3.5), the mutation

$$\mu = \left(\mu_k(\mathbf{1}_V,-1),\mu_k(\mathbf{1}_V,-1),\mu(\mathbf{1}_V,x,\boldsymbol{y})\right),$$

$$\left(\boldsymbol{v}',\mathbf{Y}(\boldsymbol{v}',\boldsymbol{y}),\mathcal{F}_n^{(g)\prime}(\boldsymbol{v}',\boldsymbol{y})\right) = \mu\;\left(\boldsymbol{v},\mathbf{Y}(\boldsymbol{v},\boldsymbol{y}),\mathcal{F}_n^{(g)}(\boldsymbol{v},\boldsymbol{y})\right), \tag{3.7}$$

defined by (3.2), (3.3), (3.5) is an involution, i.e.,

$$\mu\,\mu = \mathrm{Id}.$$

Proof. According to (4.11) for $u = \mathbf{1}_V \in V_0$, and $v_k \in V_l$, $1 \leqslant k \leqslant n$, $l \in \mathbb{Z}$,

$$u(-1)u(-1).v_k : V_l \to V_l.$$

Due to the genus g Zhu reduction formula and (2.6), we have

$$\begin{aligned}
\mathcal{F}_n^{(g)\prime}(\boldsymbol{v}, \boldsymbol{y}) &= \mu(\mathbf{1}_V, x, \boldsymbol{y})\ \mathcal{F}_n^{(g)}(\boldsymbol{v}, \boldsymbol{y}) \\
&= \sum_{k=1}^{n} \sum_{m \geqslant 0} f(m, k, x, \boldsymbol{y})\ \mathcal{F}_n^{(g)}(v_1, z_1; \ldots; \mathbf{1}_V[m].v_k, z_k; \ldots; v_n, z_n; \tau_1, \tau_2, \epsilon) \\
&\qquad\qquad + \widetilde{\mathcal{F}}_n^{(g)}(\mathbf{1}_V, x; \boldsymbol{v}, \boldsymbol{y}) \\
&= \mathcal{F}_{n+1}^{(g)}(\mathbf{1}_V, x; \boldsymbol{v}, \boldsymbol{y}) = \mathcal{F}_n^{(g)}(\boldsymbol{v}, \boldsymbol{y}). \qquad (3.8)
\end{aligned}$$

Thus, in this case, the mutation μ is an involution. □

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4. Appendix: Vertex Operator Algebras

A vertex operator algebra [8, 13, 26, 27, 36, 41, 46] is determined by a quadruple $(V, Y, \mathbf{1}_V, \omega)$, where is a linear space endowed with a $\mathbb{Z}$-grading with

$$V = \bigoplus_{r \in \mathbb{Z}} V_r,$$

with $\dim V_r < \infty$. The state $\mathbf{1}_V \in V_0$, $\mathbf{1}_V \neq 0$, is the vacuum vector and $\omega \in V_2$ is the conformal vector with properties described below. The vertex operator Y is a linear map

$$Y : V \to \mathrm{End}(V)\left[\left[z, z^{-1}\right]\right],$$

for formal variable z so that for any vector $u \in V$ we have a vertex operator

$$Y(u, z) = \sum_{n \in \mathbb{Z}} u(n) z^{-n-1}. \qquad (4.1)$$

The linear operators (modes) $u(n) : V \to V$ satisfy creativity

$$Y(u, z)\mathbf{1}_V = u + O(z), \qquad (4.2)$$

and lower truncation

$$u(n)v = 0, \qquad (4.3)$$

conditions for each u, $v \in V$ and $n \gg 0$. For the conformal vector ω one has

$$Y(\omega, z) = \sum_{n \in \mathbb{Z}} L(n) z^{-n-2}, \qquad (4.4)$$

where $L(n)$ satisfies the Virasoro algebra for some central charge C

$$[\, L(m), L(n)\,] = (m-n)L(m+n) + \frac{C}{12}(m^3 - m)\delta_{m,-n}\mathrm{Id}_V, \qquad (4.5)$$

where Id_V is identity operator on V. Each vertex operator satisfies the translation property

$$\partial_z Y(u,z) = Y\left(L(-1)u, z\right). \tag{4.6}$$

The Virasoro operator $L(0)$ provides the $\mathbb{Z}$-grading with

$$L(0)u = ru,$$

for $u \in V_r$, $r \in \mathbb{Z}$. Finally, the vertex operators satisfy the Jacobi identity

$$\begin{aligned} z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right) Y(u,z_1)Y(v,z_2) - z_0^{-1}\delta\left(\frac{z_2-z_1}{-z_0}\right) Y(v,z_2)Y(u,z_1) \\ = z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right) Y\left(Y(u,z_0)v, z_2\right). \end{aligned} \tag{4.7}$$

These axioms imply locality, skew-symmetry, associativity and commutativity conditions:

$$(z_1-z_2)^N Y(u,z_1)Y(v,z_2) = (z_1-z_2)^N Y(v,z_2)Y(u,z_1), \tag{4.8}$$

$$Y(u,z)v = e^{zL(-1)}Y(v,-z)u,$$

$$(z_0+z_2)^N Y(u,z_0+z_2)Y(v,z_2)w = (z_0+z_2)^N Y(Y(u,z_0)v,z_2)w,$$

$$u(k)Y(v,z) - Y(v,z)u(k) = \sum_{j\geqslant 0} \binom{k}{j} Y(u(j)v,z) z^{k-j}, \tag{4.9}$$

for u, v, $w \in V$ and integers $N \gg 0$. For $v = \mathbf{1}_V$ one has

$$Y(\mathbf{1}_V, z) = \mathrm{Id}_V. \tag{4.10}$$

Note also that modes of homogeneous states are graded operators on V, i.e., for $v \in V_k$,

$$v(n) : V_m \to V_{m+k-n-1}. \tag{4.11}$$

In particular, let us define the zero mode $o(v)$ of a state of weight $wt(v) = k$, i.e., $v \in V_k$, as

$$o(v) = v(wt(v)-1), \tag{4.12}$$

extending to V additively.

Definition 4. Given a vertex operator algebra V, one defines the adjoint vertex operator with respect to $\alpha \in \mathbb{C}$, by

$$\begin{aligned} Y^\dagger(u,z) &= \sum_{n\in\mathbb{Z}} u^\dagger(n) z^{-n-1} \\ &= Y\left(\exp\left(\frac{z}{\alpha}L(1)\right)\left(-\frac{\alpha}{z^2}\right)^{L(0)} u, \frac{\alpha}{z}\right), \end{aligned} \tag{4.13}$$

associated with the formal Möbius map [26]

$$z \mapsto \frac{\alpha}{z}.$$

Definition 5. An element $u \in V$ is called quasiprimary if

$$L(1)u = 0.$$

For quasiprimary u of weight $\mathrm{wt}(u)$ one has

$$u^\dagger(n) = (-1)^{\mathrm{wt}(u)} \alpha^{n+1-\mathrm{wt}(u)} u(2\mathrm{wt}(u) - n - 2).$$

Definition 6. A bilinear form

$$\langle ., . \rangle : V \times V \to \mathbb{C},$$

is called invariant if [26, 44]

$$\langle Y(u,z)a, b\rangle = \langle a, Y^\dagger(u,z)b\rangle, \tag{4.14}$$

for all a, b, $u \in V$.

Notice that the adjoint vertex operator $Y^\dagger(.,.)$ as well as the bilinear form $\langle ., . \rangle$, depend on α. In terms of modes, we have

$$\langle u(n)a, b\rangle = \langle a, u^\dagger(n)b\rangle. \tag{4.15}$$

Choosing $u = \omega$, and for $n = 1$ implies

$$\langle L(0)a, b\rangle = \langle a, L(0)b\rangle.$$

Thus,

$$\langle a, b\rangle = 0,$$

when $\mathrm{wt}(a) \neq \mathrm{wt}(b)$.

Definition 7. A vertex operator algebra is called of strong-type if

$$V_0 = \mathbb{C}\mathbf{1}_V,$$

and V is simple and self-dual, i.e., V is isomorphic to the dual module V' as a V-module.

It is proven in [44] that a strong-type vertex operator algebra V has a unique invariant non-degenerate bilinear form up to normalization. This motivates

Definition 8. The form $\langle ., . \rangle$ on a strong-type vertex operator algebra V is the unique invariant bilinear form $\langle ., . \rangle$ normalized by

$$\langle \mathbf{1}_V, \mathbf{1}_V \rangle = 1.$$

Given a vertex operator algebra $(V, Y(.,.), \mathbf{1}, \omega)$, one can find an isomorphic vertex operator algebra $(V, Y[.,.], \mathbf{1}, \tilde{\omega})$ called [57] the square-bracket vertex operator algebra. Both algebras have the same underlying vector space V, vacuum vector $\mathbf{1}_V$, and central charge. The vertex operator $Y[.,.]$ is determined by

$$Y[v, z] = \sum_{n \in \mathbb{Z}} v[n] z^{-n-1} = Y\left(q_z^{L(0)} v, q_z - 1\right).$$

The new square-bracket conformal vector is

$$\tilde{\omega} = \omega - \frac{c}{24}\mathbf{1},$$

with the vertex operator

$$Y[\tilde{\omega}, z] = \sum_{n \in \mathbb{Z}} L[n] z^{-n-2}.$$

The square-bracket Virasoro operator mode $L[0]$ provides an alternative $\mathbb{Z}$–grading on V, i.e., $\mathrm{wt}[v] = k$ if

$$L[0]v = kv,$$

where $\mathrm{wt}[v] = \mathrm{wt}(v)$ for primary v, and $L(n)v = 0$ for all $n > 0$. We can similarly define a square-bracket bilinear form $\langle ., . \rangle_{\mathrm{sq}}$.

Next we recall a lemma from [32]. The bilinear form $\langle \cdot, \cdot \rangle$ of is invertible and that

$$\langle u, v \rangle = 0,$$

Let $\{b\}$ be a homogeneous basis for V with the dual basis $\{\bar{b}\}$.

Lemma 1. *For u quasiprimary of weight p we have*

$$\sum_{b \in V_n} (u(m)b) \otimes \bar{b} = \sum_{b \in V_{n+p-m-1}} b \otimes \left(u^{\dagger}(m) \bar{b} \right). \tag{4.16}$$

Remark 3. Suppose that U is a vertex operator subalgebra of V and $W \subset V$ is a U-module. For $u \in U$ and homogeneous W-basis $\{w\}$ we may then extend (4.16) to obtain

$$\sum_{w \in W_n} (u(m)w) \otimes \overline{w} = \sum_{w \in W_{n+p-m-1}} w \otimes \left(u^{\dagger}(m) \overline{w} \right).$$

For the Schottky setup we have the following properties associated to the ρ-sewing. For each $a \in \mathcal{I}_+$, let $\{b_a\}$ denote a homogeneous V-basis and let $\{\bar{b}_a\}$ be the dual basis with $\langle \cdot, \cdot \rangle_1$, i.e., with $\rho = 1$. Define

$$b_{-a} = \rho_a^{\mathrm{wt}(b_a)} \bar{b}_a, \quad a \in \mathcal{I}_+, \tag{4.17}$$

for a formal ρ_a. We then identify ρ_a with a Schottky sewing parameter. Then $\{b_{-a}\}$ is a dual basis for the bilinear form $\langle \cdot, \cdot \rangle_{\rho_a}$ with adjoint modes

$$u^{\dagger}_{\rho_a}(m) = (-1)^p \rho_a^{m-p+1} u(2p-2-m), \tag{4.18}$$

for u quasiprimary of weight p.

5. Appendix: Definition of a cluster algebra

Let us first recall the notion of a cluster algebra [19, 20, 21] following of [55]. We consider commutative cluster algebras of rank n. The set of all cluster variables is constructed recursively from an initial set of n cluster variables using mutations. Every mutation defines a new cluster variable as a rational function of the cluster variables constructed previously. Thus, recursively, every cluster variable is a certain rational function in the initial n cluster variables. These rational functions are Laurent polynomials [19].

A cluster algebra is determined by its initial seed which consists of a cluster

$$\mathbf{x} = (x_1, \ldots, x_n),$$

of algebraically independent set of generators, a coefficient tuple

$$\mathbf{y} = (y_1, \ldots, y_n),$$

and a skew-symmetrizable $n \times n$ integer exchange matrix

$$B = (b_{ij}),$$

i.e., $b_{i,j} = -b_{j,i}$. The coefficients $\{y_1, \ldots, y_n\}$ are taken in a torsion free abelian group $\mathbb{P}$. The mutation in direction k defines a new cluster

$$x'_k x_k = y^+ \prod_{b_{k,i}>0} x_i^{b_{k,i}} + y^- \prod_{b_{k,i}<0} x_i^{-b_{k,i}}, \tag{5.1}$$

where $y^\pm$ are certain monomials in $(y_1, \ldots, y_n)$. Mutations also transform the coefficient tuple y and the matrix B.

If ζ is any cluster variable, then u is obtained from the initial cluster $\boldsymbol{x}$ by a sequence of mutations, then [19] ζ can be written as a Laurent polynomial in variables $(x_1, \ldots, x_n)$, that is,

$$f(\boldsymbol{x}) = \zeta \prod_{i=1}^n x_i^{d_i}, \tag{5.2}$$

for some d_i, where $f(\boldsymbol{x})$ is a polynomial with coefficients in the group ring $\mathbb{ZP}$ of the coefficient group $\mathbb{P}$. A cluster algebra is of finite type if it has only a finite number of seeds. In [20] it was shown that cluster algebras of finite type can be classified in terms of the Dynkin diagrams of finite-dimensional simple Lie algebras.

5.1. **Formal definition.** Let $\mathbb{P}$ be an abelian group with binary operation $\oplus$, $\mathbb{ZP}$ be the group ring of $\mathbb{P}$, and let $\mathbb{QP}(\boldsymbol{x})$ be the field of rational functions in n variables with coefficients in $\mathbb{QP}$.

Definition 9. A seed is a triple $(\mathbf{x}, \mathbf{y}, B)$, where $\mathbf{x} = \{x_1, \ldots, x_n\}$ is a basis of $\mathbb{QP}(x_1, \ldots, x_n)$, $\mathbf{y} = \{y_1, \ldots, y_n\}$, is an n-tuple of elements $y_i \in \mathbb{P}$, and B is a skew-symmetrizable matrix.

Definition 10. Given a seed

$$(\mathbf{x}, \mathbf{y}, B),$$

its mutation $\mu_k(\mathbf{x}, \mathbf{y}, B)$ in direction k is a new seed $(\mathbf{x}', \mathbf{y}', B')$ defined as follows. Let $[x]_+ = max(x, 0)$. Then we have $B' = (b'_{ij})$ with

$$b'_{ij} = \left[\begin{array}{l} b_{ij} \text{ for } i = k \text{ or } j = k, \\ b_{ij} + [-b_{ik}]_+ b_{kj} + b_{ik}[b_{kj}]_+, \ \ \text{otherwise.} \end{array} \right. \tag{5.3}$$

For new coefficients $\mathbf{y}' = (y'_1, \ldots, y'_n)$, with

$$y'_j = \left[\begin{array}{l} y_k^{-1} \text{ if } j = k, \\ y_j y_k^{[b_{kj}]_+} (y_k \oplus 1)^{-b_{kj}} \text{ if } j \neq k, \end{array} \right. \tag{5.4}$$

and $\mathbf{x} =(\mathrm{x}_1, \ldots, x_n)$, where

$$(y_k \oplus 1)\ x_k\ x'_k = y_k \prod_{i=1}^n x_i^{[b_{ik}]_+} + \prod_{i=1}^n x_i^{[-b_{ik}]_+}. \tag{5.5}$$

Mutations are involutions, i.e.,

$$\mu_k\ \mu_k\ (\mathbf{x}, \mathbf{y}, B) = (\mathbf{x}, \mathbf{y}, B).$$

6. Appendix: The Schottky uniformization of Riemann surfaces

In this appendix we recall the Schottky uniformization of Riemann surfaces [32]. Consider a compact marked Riemann surface $\mathcal{S}_g$ of genus g, e.g., [17, 48, 15, 7], with canonical homology basis α_a, β_a for $a \in \mathcal{I}_+ = \{1, 2, \ldots, g\}$. We recall the construction of a genus g Riemann surface $\mathcal{S}_g$ using the Schottky uniformization where we sew g handles to the Riemann sphere

$$\mathcal{S}_0 \cong \widehat{\mathbb{C}} = \mathbb{C} \bigcup \{\infty\},$$

e.g., [29, 7]. Every Riemann surface can be non-uniquely Schottky uniformized [6]. For $a \in \mathcal{I} = \{\pm 1, \pm 2, \ldots, \pm g\}$, let $\mathcal{C}_a \subset \mathcal{S}_0$ be $2g$ non-intersecting Jordan curves. For $z \in \mathcal{C}_a$, $z' \in \mathcal{C}_{-a}$, $W_{\pm a} \in \widehat{\mathbb{C}}$, $a \in \mathcal{I}_+$, and q_a with

$$0 < |q_a| < 1,$$

let curves be identified by the sewing relation

$$\frac{z' - W_{-a}}{z' - W_a} \cdot \frac{z - W_a}{z - W_{-a}} = q_a. \tag{6.1}$$

For $a \in \mathcal{I}_+$, Introduce

$$\sigma_a = (W_{-a} - W_a)^{-1/2} \begin{pmatrix} 1 & -W_{-a} \\ 1 & -W_a \end{pmatrix}, \tag{6.2}$$

and

$$\gamma_a = \sigma_a^{-1} \begin{pmatrix} q_a^{1/2} & 0 \\ 0 & q_a^{-1/2} \end{pmatrix} \sigma_a, \tag{6.3}$$

Thus

$$z' = \gamma_a z.$$

Note that

$$\sigma_a(W_{-a}) = 0,$$

and

$$\sigma_a(W_a) = \infty,$$

are, respectively, attractive and repelling fixed points of the map

$$Z \to Z' = q_a Z,$$

for

$$Z = \sigma_a z,$$

and

$$Z' = \sigma_a z'.$$

Here W_{-a} and W_a are the corresponding fixed points for γ_a. One identifies the standard homology cycles α_a with $\mathcal{C}_{-a}$ and β_a with a path connecting $z \in \mathcal{C}_a$ to

$$z' = \gamma_a z, \in \mathcal{C}_{-a}.$$

and $z' \in \mathcal{C}_{-a}$.

Definition 11. The genus g Schottky group Γ is the free group with generators γ_a. Define

$$\gamma_{-a} = \gamma_a^{-1}.$$

The independent elements of Γ are reduced words of length k of the form

$$\gamma = \gamma_{a_1} \ldots \gamma_{a_k},$$

where $a_i \neq -a_{i+1}$ for each $i = 1, \ldots, k-1$.

Let $\Lambda(\Gamma)$ denote the limit set of Γ, i.e., the set of limit points of the action of Γ on $\widehat{\mathbb{C}}$. Then

$$\mathcal{S}_g \simeq \Omega_0/\Gamma$$

where

$$\Omega_0 = \widehat{\mathbb{C}} - \Lambda(\Gamma).$$

We let $\mathcal{D} \subset \widehat{\mathbb{C}}$ denote the standard connected fundamental region with oriented boundary curves $\mathcal{C}_a$. Define

$$w_a = \gamma_{-a}.\infty.$$

Using (6.1) we find

$$w_a = \frac{W_a - q_a W_{-a}}{1 - q_a}, \tag{6.4}$$

for $a \in \mathcal{I}$. where we define $q_{-a} = q_a$. Then (6.1) is equivalent to

$$(z' - w_{-a})(z - w_a) = \rho_a, \tag{6.5}$$

with

$$\rho_{\pm a} = -\frac{q_a (W_a - W_{-a})^2}{(1 - q_a)^2}. \tag{6.6}$$

(6.5) implies

$$\gamma_a z = w_{-a} + \frac{\rho_a}{z - w_a}. \tag{6.7}$$

Let Δ_a be the disc with centre w_a and radius $|\rho_a|^{\frac{1}{2}}$. One chooses the Jordan curve $\mathcal{C}_a$ to be the boundary of Δ_a. Then γ_a maps the exterior (interior) of Δ_a to the interior (exterior) of Δ_{-a} since

$$|\gamma_a z - w_{-a}||z - w_a| = |\rho_a|.$$

The discs Δ_a, Δ_b are non-intersecting if and only if

$$|w_a - w_b| > |\rho_a|^{\frac{1}{2}} + |\rho_b|^{\frac{1}{2}}, \tag{6.8}$$

for all $a \neq b$. One defines $\mathfrak{C}_g$ to be the set

$$\{(w_a, w_{-a}, \rho_a) | a \in \mathcal{I}_+\} \subset \mathbb{C}^{3g},$$

satisfying (6.8). We refer to $\mathfrak{C}_g$ as the Schottky parameter space.

The relation (6.1) is Möbius invariant for

$$\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C}),$$

with

$$(z, z', W_a, q_a) \to (\gamma z, \gamma z', \gamma W_a, q_a),$$

giving an $\mathrm{SL}_2(\mathbb{C})$ action on $\mathfrak{C}_g$ as follows

$$\gamma : (w_a, \rho_a) \mapsto \left(\frac{(Aw_a + B)(Cw_{-a} + D) - \rho_a AC}{(Cw_a + D)(Cw_{-a} + D) - \rho_a C^2}, \frac{\rho_a}{((Cw_a + D)(Cw_{-a} + D) - \rho_a C^2)^2} \right). \tag{6.9}$$

Definition 12. One defines the Schottky space as

$$\mathfrak{S}_g = \mathfrak{C}_g / \mathrm{SL}_2(\mathbb{C}),$$

This provides a natural covering space for the moduli space of genus g Riemann surfaces (of dimension 1 for $g = 1$ and $3g - 3$ for $g \geqslant 2$).

7. Appendix: coefficient functions in the Zhu reduction formula

For purposes of the formula (2.6) we recall here certain definitions [32]. Define a column vector

$$X = (X_a(m)),$$

indexed by $m \geqslant 0$ and $a \in \mathcal{I}$ with components

$$X_a(m) = \rho_a^{-\frac{m}{2}} \sum_{\boldsymbol{b}_+} Z^{(0)}(\ldots; u(m)b_a, w_a; \ldots), \tag{7.1}$$

and a row vector

$$p(x) = (p_a(x, m)),$$

for $m \geqslant 0, a \in \mathcal{I}$ with components

$$p_a(x, m) = \rho_a^{\frac{m}{2}} \partial^{(0,m)} \psi_p^{(0)}(x, w_a). \tag{7.2}$$

Introduce the column vector

$$G = (G_a(m)),$$

for $m \geqslant 0, a \in \mathcal{I}$, given by

$$G = \sum_{k=1}^{n} \sum_{j \geqslant 0} \partial_k^{(j)} \; q(y_k) \; Z_V^{(g)}(v_1, y_1; \ldots; u(j)v_k, y_k; \ldots; v_n, y_n),$$

where $q(y) = (q_a(y; m))$, for $m \geqslant 0$, $a \in \mathcal{I}$, is a column vector with components

$$q_a(y; m) = (-1)^p \rho_a^{\frac{m+1}{2}} \partial^{(m,0)} \psi_p^{(0)}(w_{-a}, y), \tag{7.3}$$

and

$$R = (R_{ab}(m, n)),$$

for m, $n \geqslant 0$ and a, $b \in \mathcal{I}$ is a doubly indexed matrix with components

$$R_{ab}(m,n) = \begin{cases} (-1)^p \rho_a^{\frac{m+1}{2}} \rho_b^{\frac{n}{2}} \partial^{(m,n)} \psi_p^{(0)}(w_{-a}, w_b), & a \neq -b, \\ (-1)^p \rho_a^{\frac{m+n+1}{2}} \mathcal{E}_m^n(w_{-a}), & a = -b, \end{cases} \tag{7.4}$$

where

$$\mathcal{E}_m^n(y) = \sum_{\ell=0}^{2p-2} \partial^{(m)} f_\ell(y)\, \partial^{(n)} y^\ell, \tag{7.5}$$

$$\psi_p^{(0)}(x,y) = \frac{1}{x-y} + \sum_{\ell=0}^{2p-2} f_\ell(x) y^\ell, \tag{7.6}$$

for any Laurent series $f_\ell(x)$ for $\ell = 0, \ldots, 2p-2$. Define the doubly indexed matrix $\Delta = (\Delta_{ab}(m,n))$ by

$$\Delta_{ab}(m,n) = \delta_{m,n+2p-1}\delta_{ab}. \tag{7.7}$$

Denote by

$$\widetilde{R} = R\Delta,$$

and the formal inverse $(I - \widetilde{R})^{-1}$ is given by

$$\left(I - \widetilde{R}\right)^{-1} = \sum_{k \geqslant 0} \widetilde{R}^{\,k}. \tag{7.8}$$

Define $\chi(x) = (\chi_a(x;\ell))$ and

$$o(u;\boldsymbol{v},\boldsymbol{y}) = (o_a(u;\boldsymbol{v},\boldsymbol{y};\ell)),$$

are finite row and column vectors indexed by $a \in \mathcal{I}$, $0 \leqslant \ell \leqslant 2p-2$ with

$$\chi_a(x;\ell) = \rho_a^{-\frac{\ell}{2}} (p(x) + \widetilde{p}(x)(I - \widetilde{R})^{-1} R)_a(\ell), \tag{7.9}$$

$$o_a(\ell) = o_a(u;\boldsymbol{v},\boldsymbol{y};\ell) = \rho_a^{\frac{\ell}{2}} X_a(\ell), \tag{7.10}$$

and where

$$\widetilde{p}(x) = p(x)\Delta.$$

$\psi_p(x,y)$ is defined by

$$\psi_p(x,y) = \psi_p^{(0)}(x,y) + \widetilde{p}(x)(I - \widetilde{R})^{-1} q(y). \tag{7.11}$$

For each $a \in \mathcal{I}_+$ we define a vector

$$\theta_a(x) = (\theta_a(x;\ell)),$$

indexed by $0 \leqslant \ell \leqslant 2p-2$ with components

$$\theta_a(x;\ell) = \chi_a(x;\ell) + (-1)^p \rho_a^{p-1-\ell} \chi_{-a}(x;2p-2-\ell). \tag{7.12}$$

Now define the following vectors of formal differential forms

$$P(x) = p(x)\; dx^p,$$
$$Q(y) = q(y)\; dy^{1-p}, \tag{7.13}$$

with

$$\widetilde{P}(x) = P(x)\Delta.$$

Then with

$$\Psi_p(x,y) = \psi_p(x,y)\ dx^p\ dy^{1-p}, \tag{7.14}$$

we have

$$\Psi_p(x,y) = \Psi_p^{(0)}(x,y) + \widetilde{P}(x)(I - \widetilde{R})^{-1}Q(y). \tag{7.15}$$

Defining

$$\Theta_a(x;\ell) = \theta_a(x;\ell)\ dx^p, \tag{7.16}$$

and

$$O_a(u;\boldsymbol{v},\boldsymbol{y};\ell) = o_a(u;\boldsymbol{v},\boldsymbol{y};\ell)\ \boldsymbol{dy}^{\mathrm{wt}(\boldsymbol{v})}, \tag{7.17}$$

Remark 4. The $\Theta_a(x)$, and $\Psi_p(x,y)$ coefficients depend on $p = \mathrm{wt}(u)$ but are otherwise independent of the vertex operator algebra V. Note that for a 1-point function, (2.6) implies

$$\mathcal{F}_V^{(g)}(u,x) = \sum_{a=1}^{g} \Theta_a(x)\ O_a(u). \tag{7.18}$$

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