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**The product of  $W$ -spaces**

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# THE PRODUCT OF $\mathcal{W}$ -SPACES

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ABSTRACT. Let  $V$  be a grading-restricted vertex algebra,  $W$  be its module,  $\overline{W}$  be the algebraic completion of  $W$ , and  $\mathcal{W}_{z_1, \dots, z_n}$  be the space of rational differential forms in  $(z_1, \dots, z_n)$ . Using geometric interpretation in terms of sewing two Riemann spheres with a number of marked points, we introduce a product of elements of two spaces  $\mathcal{W}_{x_1, \dots, x_k}$  and  $\mathcal{W}_{y_1, \dots, y_n}$ , and study its properties. The product takes values in  $\mathcal{W}_{x_1, \dots, x_k; y_1, \dots, y_n}$ . We prove that the product is defined by an absolutely convergent series. In applications, we consider a product of elements of two spaces  $C_m^k(V, W)$  and  $C_{m'}^n(V, W)$  of chain-cochain double complex associated to a grading-restricted vertex algebra  $V$  coherent with the differential of the complex. We prove that the product brings about a map to the space  $C_{m+m'}^{k+n}(V, W)$ , and satisfy an analogue of Leibniz formula.

## 1. INTRODUCTION

The problem of defining a product of  $\mathcal{W}$ -spaces (and in particular,  $C_m^n(V, W)$ -spaces) is very important for the cohomology theory of vertex algebras, continual Lie algebras, the theory of integrable models, as well as for further applications to cohomologies of smooth manifolds [11]. A cohomology theory for grading-restricted vertex algebras was introduced in [6] (see also [10]). Vertex algebras, generalizations of ordinary Lie algebras, are essential in conformal field theory. [3], and it is a rapidly developing field of studies. Algebraic nature of methods applied in this field helps to understand and compute the structure of vertex algebra characters [1–3, 8, 14]. On the other hand, the geometric side of vertex algebra characters is in associating their formal parameters with local coordinates on a complex variety. Depending on geometry, one can obtain various consequences for a vertex algebra and its space of characters, and vice-versa, one can study geometrical property of a manifold by using algebraic nature of a vertex algebra attached.

For purposes [11] of construction of cohomological invariants of vertex algebras it is important to define product of elements of chain-cochain double complex spaces. In that direction, an extremely difficult question of composability with vertex operators occur. The composability of a map with vertex operators represents by it own a convergence problem. For vertex operator operator algebras and modules, the convergence is a consequence of Jacobi identity. For the cohomology theory of vertex algebras, one has to assume that the chain-cochains are composable with vertex operators which assumes the convergence. Especially when we want to compute calculate

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the cohomology of a vertex algebra, we have to deal with the convergence problem first. In case of grading-restricted vertex algebras [6], the difficulty is that chain-cochains are not represented by vertex or intertwining operators. The techniques for vertex operators or intertwining operators in general do not work. The aim of this paper is to Thus we have to develop such new techniques.

For products of spaces of chain-cochains, we propose to involve the geometrical procedure [13] of sewing of Riemann surfaces as auxiliary model spaces in a geometrical interpretation of algebraic products of spaces associated to vertex algebras. Similar to various other structures in the theory of vertex operator algebras, this is not be usual associative product. The product that occur is parametrized by a nonzero complex number  $\epsilon$  identified to the complex parameter of the sewing procedure we involve. More generally, the product is constructed from two Riemann spheres with a collection of marked points, and local coordinates vanishing at these points. The same scheme works, for example, for tensor products of modules which are in fact parametrized by such geometric objects. Because of this, the existence of such products involves the convergence. In addition to that, a vertex operator algebra must satisfy some conditions in order for such convergence to hold.

In this paper we introduce the product of  $\mathcal{W}$ -spaces of rational differential forms for a grading-restricted vertex algebra [6] For the construction of double complexes (cf. Section application, [6]) we make use of linear maps from tensor powers of  $V$  to the space  $\mathcal{W}_{z_1, \dots, z_n}$  to define cochains in vertex algebra cohomology theory. For that purpose, in particular, to define the coboundary operator, we have to compose chain-cochains with vertex operators. However, as mentioned in [6], the images of vertex operator maps in general do not belong to algebras or their modules. Such objects belong to corresponding algebraic completions which constitute one of the most subtle features of the theory of vertex algebras. Because of this, we might not be able to compose vertex operators directly. In order to overcome this problem, one we first writes a series by projecting an element of the algebraic completion of an algebra or a module to its homogeneous components. Then we compose these homogeneous components with vertex operators, and take formal sums. If such formal sums are absolutely convergent, then these operators can be composed and can be used in constructions.

The plan of the paper is the following. In Section 2 we recall [6, 10] the definition of the space of  $\mathcal{W}$ -valued rational forms for a grading-restricted vertex algebras, and remind properties of their elements. In Section 3 we introduce a product for elements of two  $\mathcal{W}_{z_1, \dots, z_n}$ -spaces. In Section 4 we study properties of the resulting product. In Section 5 we recall the definition and properties [6] of spaces  $C_m^n(V, \mathcal{W})$  for the chain-cochain double complex for a grading-restricted vertex algebra. In Section 6 we define the product for  $C_m^n(V, \mathcal{W})$ -spaces and study its properties. In Section 7 we consider the particular case of a short exceptional complex associated to certain  $C_m^n(V, \mathcal{W})$  subspaces. In Appendixes we provide the material needed for construction of the product for  $\mathcal{W}$ -spaces. In Appendix 8 we recall the notion of a quasi-conformal grading-restricted vertex algebra and its modules. In Appendix 9 we describe the geometric procedure of forming a Riemann sphere by sewing two initial Riemann spheres. Finally, Appendix 10 contains the proof of Proposition 10.

2. SPACES OF  $\mathcal{W}$ -VALUED RATIONAL FORMS

2.1.  **$\mathcal{W}$ -valued rational functions.** We define the configuration spaces:

$$F_n\mathbb{C} = \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid z_i \neq z_j, i \neq j\},$$

for  $n \in \mathbb{Z}_+$ . Let  $V$  be a grading-restricted vertex algebra, and  $W$  a grading-restricted generalized  $V$ -module. By  $\overline{W}$  we denote the algebraic completion of  $W$ ,

$$\overline{W} = \prod_{n \in \mathbb{C}} W_{(n)} = (W')^*.$$

Let  $w' \in W'$  be an arbitrary element of  $W'$  dual to  $W$  with respect to a non-degenerate bilinear form  $\langle \cdot, \cdot \rangle$  on  $W$ .

**Definition 1.** A  $\overline{W}$ -valued rational function in  $(z_1, \dots, z_n)$  with the only possible poles at  $z_i = z_j, i \neq j$ , is a map

$$\begin{aligned} f : F_n\mathbb{C} &\rightarrow \overline{W}, \\ (z_1, \dots, z_n) &\mapsto f(z_1, \dots, z_n), \end{aligned}$$

such that for any  $w' \in W'$ ,

$$R(z_1, \dots, z_n) = R(\langle w', f(z_1, \dots, z_n) \rangle), \quad (2.1)$$

is a rational function in  $(z_1, \dots, z_n)$  with the only possible poles at  $z_i = z_j, i \neq j$ . In this paper, such a map is called  $\overline{W}$ -valued rational function in  $(z_1, \dots, z_n)$  with possible other poles. The space of  $\overline{W}$ -valued rational functions is denoted by  $\overline{W}_{z_1, \dots, z_n}$ .

Here  $R(\cdot)$  denotes the following (cf. [6]). Namely, if a meromorphic function  $f(z_1, \dots, z_n)$  on a region in  $\mathbb{C}^n$  can be analytically extended to a rational function in  $(z_1, \dots, z_n)$ , then the notation  $R(f(z_1, \dots, z_n))$  is used to denote such rational function. Note that the set of a grading-restricted vertex algebra elements  $(v_1, \dots, v_n)$  associated with corresponding  $(z_1, \dots, z_n)$  play the role of non-commutative parameters for a function  $f$  in (2.1). Let us introduce the definition of a  $\mathcal{W}_{z_1, \dots, z_n}$ -space:

**Definition 2.** We define the space  $\mathcal{W}_{z_1, \dots, z_n}$  of  $\overline{W}_{z_1, \dots, z_n}$ -valued rational forms  $\mathcal{F}$  with each vertex algebra element entry  $v_i, 1 \leq i \leq n$  of a quasi-conformal grading-restricted vertex algebra  $V$  tensored with power  $\text{wt}(v_i)$ -differential of corresponding formal parameter  $z_i$ , i.e.,

$$\begin{aligned} \mathcal{F}(v_1, z_1; \dots; v_n, z_n) \\ = \Phi \left( dz_1^{\text{wt}(v_1)} \otimes v_1, z_1; \dots; dz_n^{\text{wt}(v_n)} \otimes v_n, z_n \right) \in \mathcal{W}_{z_1, \dots, z_n}. \end{aligned} \quad (2.2)$$

The reason why vertex algebra elements  $(v_1, \dots, v_n)$  are supplied in (2.2) by powers of differentials of corresponding formal parameters is explained in Appendix 10 where we recall a part of the proof given in [11]. We have

**Proposition 1.** *The form (2.2) is invariant with respect to elements*

$$(\rho_1(z_1, \dots, z_n), \dots, \rho_n(z_1, \dots, z_n)),$$

*of the group  $\text{Aut}_{z_1, \dots, z_n} \mathcal{O}^{(n)}$ , i.e., under the changes*

$$z_i \mapsto z'_i = \rho_i(z_1, \dots, z_n),$$

of formal parameters  $(z_1, \dots, z_n)$ .  $\square$

**2.2. Properties of rational functions for  $\mathcal{W}$ -valued elements.** Let  $V$  be a grading-restricted vertex algebra and  $W$  a grading-restricted generalized  $V$ -module (cf. Appendix 8). Let us recall some definitions and facts about matrix elements for a grading-restricted vertex algebra [6]. If a meromorphic function  $f(z_1, \dots, z_n)$  on a domain in  $\mathbb{C}^n$  is analytically extendable to a rational function in  $(z_1, \dots, z_n)$ , we denote this rational function by  $R(f(z_1, \dots, z_n))$ .

**Definition 3.** For  $n \in \mathbb{Z}_+$ , a linear map

$$\mathcal{F}(v_1, z_1; \dots; v_n, z_n) = V^{\otimes n} \rightarrow \mathcal{W}_{z_1, \dots, z_n},$$

is said to have the  $L_V(-1)$ -derivative property if

$$(i) \quad \partial_{z_i} \mathcal{F}(v_1, z_1; \dots; v_n, z_n) = \mathcal{F}(v_1, z_1; \dots; L_V(-1)v_i, z_i; \dots; v_n, z_n), \quad (2.3)$$

for  $i = 1, \dots, n$ ,  $(v_1, \dots, v_n) \in V$ ,  $w' \in W$ , and

$$(ii) \quad \sum_{i=1}^n \partial_{z_i} \mathcal{F}(v_1, z_1; \dots; v_n, z_n) = L_W(-1) \cdot \mathcal{F}(v_1, z_1; \dots; v_n, z_n), \quad (2.4)$$

with some action  $\cdot$  of  $L_W(-1)$  on  $\mathcal{F}(v_1, z_1; \dots; v_n, z_n)$ .

Note that since  $L_W(-1)$  is a weight-one operator on  $W$ , for any  $z \in \mathbb{C}$ ,  $e^{zL_W(-1)}$  is a well-defined linear operator on  $\mathcal{W}_{z_1, \dots, z_n}$ . In [6] we find the following

**Proposition 2.** *Let  $\mathcal{F}$  be a linear map having the  $L_W(-1)$ -derivative property. Then for  $(v_1, \dots, v_n) \in V$ ,  $(z_1, \dots, z_n) \in F_n\mathbb{C}$ ,  $z \in \mathbb{C}$  such that  $(z_1 + z, \dots, z_n + z) \in F_n\mathbb{C}$ ,*

$$e^{zL_W(-1)} \mathcal{F}(v_1, z_1; \dots; v_n, z_n) = \mathcal{F}(v_1, z_1 + z; \dots; v_n, z_n + z), \quad (2.5)$$

and and  $1 \leq i \leq n$  such that

$$(z_1, \dots, z_{i-1}, z_i + z, z_{i+1}, \dots, z_n) \in F_n\mathbb{C},$$

the power series expansion of

$$\mathcal{F}(v_1, z_1; \dots; v_{i-1}, z_{i-1}; v_i, z_i + z; v_{i+1}, z_{i+1}; \dots; v_n, z_n), \quad (2.6)$$

in  $z$  is equal to the power series

$$\mathcal{F}\left(v_1, z_1; \dots; v_{i-1}, z_{i-1}; e^{zL_V(-1)}v_i, z_i; v_{i+1}, z_{i+1}; \dots; v_n, z_n\right), \quad (2.7)$$

in  $z$ . In particular, the power series (2.7) in  $z$  is absolutely convergent to (2.6) in the disk  $|z| < \min_{i \neq j} \{|z_i - z_j|\}$ .

One states

**Definition 4.** A linear map

$$\mathcal{F} : V^{\otimes n} \rightarrow \mathcal{W}_{z_1, \dots, z_n}$$

has the  $L_W(0)$ -conjugation property if for  $(v_1, \dots, v_n) \in V$ ,  $(z_1, \dots, z_n) \in F_n\mathbb{C}$ , and  $z \in \mathbb{C}^\times$ , such that  $(zz_1, \dots, zz_n) \in F_n\mathbb{C}$ ,

$$z^{L_W(0)} \mathcal{F}(v_1, z_1; \dots; v_n, z_n) = \mathcal{F}\left(z^{L_V(0)}v_1, zz_1; \dots; z^{L_V(0)}v_n, zz_n\right). \quad (2.8)$$

One defines the action of  $S_n$  on the space  $\text{Hom}(V^{\otimes n}, \mathcal{W}_{z_1, \dots, z_n})$  of linear maps from  $V^{\otimes n}$  to  $\mathcal{W}_{z_1, \dots, z_n}$  by

$$\sigma(\mathcal{F})(v_1, z_1; \dots; v_n, z_n) = \mathcal{F}(v_{\sigma(1)}, z_{\sigma(1)}; \dots; v_{\sigma(n)}, z_{\sigma(n)}), \quad (2.9)$$

for  $\sigma \in S_n$ , and  $(v_1, \dots, v_n) \in V$ . We will use the notation  $\sigma_{i_1, \dots, i_n} \in S_n$ , to denote the permutation given by  $\sigma_{i_1, \dots, i_n}(j) = i_j$ , for  $j = 1, \dots, n$ .

Finally, the following result was proved in [2]:

**Proposition 3.** *For  $(v_1, \dots, v_n) \in V$ ,  $w \in W$  and  $w' \in W'$ ,*

$$\langle w', Y_W(v_1, z_1) \dots Y_W(v_n, z_n) w \rangle,$$

*is absolutely convergent in the region  $|z_1| > \dots > |z_n| > 0$  to a rational function*

$$R(\langle w', Y_W(v_1, z_1) \dots Y_W(v_n, z_n) w \rangle),$$

*in  $(z_1, \dots, z_n)$  with the only possible poles at  $z_i = z_j$ ,  $i \neq j$ , and  $z_i = 0$ . The following commutativity holds: for  $\sigma \in S_n$ ,*

$$\begin{aligned} R(\langle w', Y_W(v_1, z_1) \dots Y_W(v_n, z_n) w \rangle) \\ = R(\langle w', Y_W(v_{\sigma(1)}, z_{\sigma(1)}) \dots Y_W(v_{\sigma(n)}, z_{\sigma(n)}) w \rangle). \end{aligned}$$

### 3. PRODUCT OF SPACES OF $\mathcal{W}$ -VALUED FORMS

**3.1. Motivation and geometrical interpretation.** The structure of  $\mathcal{W}_{z_1, \dots, z_n}$ -spaces is quite complicated and it is difficult to introduce algebraically a product of its elements. In order to define an appropriate product of two  $\mathcal{W}_{z_1, \dots, z_n}$ -spaces we first have to interpret them geometrically. Basically, a  $\mathcal{W}_{z_1, \dots, z_n}$ -space must be associated with a certain model space, the algebraic  $\mathcal{W}$ -language should be transferred to a geometrical one, two model spaces should be "connected" appropriately, and, finally, a product should be defined.

For two  $\mathcal{W}_{x_1, \dots, x_k}$ - and  $\mathcal{W}_{y_1, \dots, y_n}$ -spaces we first associate formal complex parameters in the sets  $(x_1, \dots, x_k)$  and  $(y_1, \dots, y_n)$  to parameters of two auxiliary spaces. Then we describe a geometric procedure to form a resulting model space by combining two original model spaces. Formal parameters of  $\mathcal{W}_{z_1, \dots, z_{k+n}}$  should be then identified with parameters of the resulting space.

Note that according to our assumption,  $(x_1, \dots, x_k) \in F_k \mathbb{C}$ , and  $(y_1, \dots, y_n) \in F_n \mathbb{C}$ . As it follows from the definition of the configuration space  $F_n \mathbb{C}$  in Subsection 2.1, in the case of coincidence of two formal parameters they are excluded from  $F_n \mathbb{C}$ . In general, it may happen that some number  $r$  of formal parameters of  $\mathcal{W}_{x_1, \dots, x_k}$  coincide with some  $r$  formal parameters of  $\mathcal{W}_{y_1, \dots, y_n}$ . Thus, we require that the set of formal parameters  $(z_1, \dots, z_{k+n-2r})$  for the resulting model space would belong to  $F_{k+n-2r} \mathbb{C}$ . This leads to the fall off of the total number of formal parameters for the resulting model space  $\mathcal{W}_{z_1, \dots, z_{k+n-2r}}$ . In what follows we consider the case when all formal parameters  $(x_1, \dots, x_k)$  differ from formal parameters of  $(y_1, \dots, y_n)$ . This singular case is then can be treated similar to the ordinary one in lower dimension.

**3.2. Definition of the product for  $\mathcal{W}$ -valued rational forms.** Recall the definition (8.15) of the intertwining operator given in Appendix 8. We then formulate

**Definition 5.** For an arbitrary set of a grading-restricted vertex algebra  $V$  elements  $(v_1, \dots, v_n), (v'_1, \dots, v'_n) \in V$ , and  $\mathcal{F}(v_1, x_1; \dots; v_k, x_k) \in \mathcal{W}_{x_1, \dots, x_k}$ ,  $\mathcal{F}(v'_1, y_1; \dots; v'_n, y_n) \in \mathcal{W}_{y_1, \dots, y_n}$ , introduce the  $\epsilon$ -product

$$\cdot_\epsilon : \mathcal{W}_{x_1, \dots, x_k} \times \mathcal{W}_{y_1, \dots, y_n} \rightarrow \mathcal{W}_{x_1, \dots, x_k; y_1, \dots, y_n}, \quad (3.1)$$

for  $(x_1, \dots, x_k; y_1, \dots, y_n) \in F_{k+n}\mathbb{C}$ . For arbitrary  $w' \in W'$ , the product is associated to the form

$$\begin{aligned} & \mathcal{R}(x_1, \dots, x_k; y_1, \dots, y_n; \epsilon) \\ &= \sum_{l \in \mathbb{Z}} \epsilon^l \sum_{u \in V_l} \langle w', Y_{WV}^W(\mathcal{F}(v_1, x_1; \dots; v_k, x_k), \zeta_1) u \rangle \\ & \quad \langle w', Y_{WV}^W(\mathcal{F}(v'_1, y_1; \dots; v'_n, y_n), \zeta_2) \bar{u} \rangle, \end{aligned} \quad (3.2)$$

via (2.1), parametrized by  $\zeta_1, \zeta_2 \in \mathbb{C}$ . The sum is taken over any  $V_l$ -basis  $\{u\}$ , where  $\bar{u}$  is the dual of  $u$  with respect to a non-degenerate bilinear form  $\langle \cdot, \cdot \rangle_\lambda$ , (8.28) over  $V$ , (see Appendix 8).

By the standard reasoning [2, 14], (3.2) does not depend on the choice of a basis of  $u \in V_l$ ,  $l \in \mathbb{Z}$ . In the case when multiplied forms  $\mathcal{F}$  do not contain  $V$ -elements, i.e., for  $\Phi, \Psi \in \mathcal{W}$ , (3.2) defines the product  $\Phi \cdot_\epsilon \Psi$  associated to a rational function:

$$\mathcal{R}(\epsilon) = \sum_{l \in \mathbb{Z}} \epsilon^l \sum_{u \in V_l} \langle w', Y_{WV}^W(\Phi, \zeta_1) u \rangle \langle w', Y_{WV}^W(\Psi, \zeta_2) \bar{u} \rangle, \quad (3.3)$$

which defines  $\mathcal{F}(\epsilon) \in \mathcal{W}$  via  $\mathcal{R}(\epsilon) = \langle w', \mathcal{F}(\epsilon) \rangle$ . As we will see in Section 5, Definition 5 is also supported by Proposition (8).

*Remark 1.* Note that due to (8.15), in Definition 5, and in (3.2) in particular, it is assumed that  $\mathcal{F}(v_1, x_1; \dots; v_k, x_k)$  and  $\mathcal{F}(v'_1, y_1; \dots; v'_n, y_n)$  are composable with the  $V$ -module  $W$  vertex operators  $Y_W(u, -\zeta_1)$  and  $Y_W(\bar{u}, -\zeta_2)$  correspondingly (see Section 5 for the definition of compossability). The product (3.2) is actually defined by sum of products of matrix elements of ordinary  $V$ -module  $W$  vertex operators acting on  $\mathcal{W}_{z_1, \dots, z_n}$  elements. In what follows we will see that, since  $u \in V$  and  $\bar{u} \in V'$  are connected by (8.29),  $\zeta_1$  and  $\zeta_2$  appear in a relation to each other. The form of the product defined above is natural in terms of the theory of characters for vertex operator algebras [3, 12, 14].

**3.3. Convergence of the  $\epsilon$ -product and existence of corresponding rational form.** In order to prove convergence of a product of elements of two spaces  $\mathcal{W}_{x_1, \dots, x_k}$  and  $\mathcal{W}_{y_1, \dots, y_n}$  of rational  $\mathcal{W}$ -valued forms, we have to use a geometrical interpretation [8, 13]. Recall that a  $\mathcal{W}_{z_1, \dots, z_n}$ -space is defined by means of matrix elements of the form (2.1). For a vertex algebra  $V$ , this corresponds [2] to a matrix element of a number of  $V$ -vertex operators with formal parameters identified with local coordinates on a Riemann sphere. Geometrically, each space  $\mathcal{W}_{z_1, \dots, z_n}$  can be also associated to a Riemann sphere with a few marked points, and local coordinates vanishing at these points [8]. An extra point can be associated to a center of an annulus used in order



to sew the sphere with another sphere. The product (3.2) has then a geometric interpretation. The resulting model space would also be associated to a Riemann sphere formed as a result of sewing procedure. In Appendix 9 we describe explicitly the geometrical procedure of sewing of two spheres [13].

Let us identify (as in [1,3,8,12–14]) two sets  $(x_1, \dots, x_k)$  and  $(y_1, \dots, y_n)$  of complex formal parameters, with local coordinates of two sets of points on the first and the second Riemann spheres correspondingly. Identify complex parameters  $\zeta_1, \zeta_2$  of (3.2) with coordinates (9.1) of the annulus (9.3). After identification of annulus  $\mathcal{A}_a$  and  $\mathcal{A}_{\bar{a}}$ ,  $r$  coinciding coordinates may occur. This takes into account case of coinciding formal parameters. In this way, we construct the map (3.1).

As we will see in the next subsection, the product is defined by a sum of products of matrix elements [2] associated to each of two spheres. Such sum is supposed to describe a  $\mathcal{W}$ -valued rational differential form defined on a sphere formed as a result of geometrical sewing [13] of two initial spheres. Since two initial spaces  $\mathcal{W}_{x_1, \dots, x_k}$  and  $\mathcal{W}_{y_1, \dots, y_n}$  are defined through rational-valued forms expressed by matrix elements of the form (2.1), it is then proved (Proposition 4), that the resulting product defines a  $\mathcal{W}_{x_1, \dots, x_k; y_1, \dots, y_n}$ -valued rational form by means of an absolute convergent matrix element on the resulting sphere. In the next subsections we prove the existence of such rational form, and absolute convergence of corresponding matrix element. The complex sewing parameter, parametrizing the module space of sewin spheres, parametrizes also the product of  $\mathcal{W}$ -spaces.

In this subsection and the next section we formulate the results of this paper for the  $\epsilon$ -product of  $\mathcal{W}$ -spaces.

**Proposition 4.** *The product (3.2) of elements of the spaces  $\mathcal{W}_{x_1, \dots, x_k}$  and  $\mathcal{W}_{y_1, \dots, y_n}$  corresponds to an absolutely converging in  $\epsilon$  rational form with only possible poles at  $x_i = x_j$ ,  $y_{i'} = y_{j'}$ , and  $x_i = y_j$ ,  $1 \leq i, i' \leq k$ ,  $1 \leq j, j' \leq n$ .*

*Proof.* In order to prove this proposition we use the geometrical interpretation of the product (3.2) in terms of Riemann spheres with marked points (see Appendix 9). We consider two sets of vertex algebra elements  $(v_1, \dots, v_k)$  and  $(v'_1, \dots, v'_k)$ , and two sets of formal complex parameters  $(x_1, \dots, x_k)$ ,  $(y_1, \dots, y_n)$ . Formal parameters are identified with local coordinates of  $k$  points on the Riemann sphere  $\widehat{\Sigma}_1^{(0)}$ , and  $n$  points on  $\widehat{\Sigma}_2^{(0)}$ , with excised annulus  $\mathcal{A}_a$  (see definitions and notations in Appendix 9). Recall the sewing parameter condition  $\zeta_1 \zeta_2 = \epsilon$  (9.4) of the sewing procedure. Then, for (3.2) we obtain

$$\begin{aligned}
& \langle w', \mathcal{F}(v_1, x_1; \dots; v_k, x_k; v'_1, y_1; \dots; v'_n, y_n; \epsilon) \rangle \\
&= \sum_{l \in \mathbb{Z}} \epsilon^l \sum_{u \in V_l} \langle w', Y_{WV}^W(\mathcal{F}(v_1, x_1; \dots; v_k, x_k), \zeta_1) u \rangle \\
& \quad \langle w', Y_{WV}^W(\mathcal{F}(v'_1, y_1; \dots; v'_n, y_n), \zeta_2) \bar{u} \rangle \tag{3.4} \\
&= \sum_{l \in \mathbb{Z}} \epsilon^l \sum_{u \in V_l} \langle w', e^{\zeta_1 L_W(-1)} Y_W(u, -\zeta_1) \mathcal{F}(v_1, x_1; \dots; v_k, x_k) \rangle \\
& \quad \langle w', e^{\zeta_2 L_W(-1)} Y_W(\bar{u}, -\zeta_2) \mathcal{F}(v'_1, y_1; \dots; v'_n, y_n) \rangle.
\end{aligned}$$

Recall from (9.1) (see Appendix 9) that in two sphere  $\epsilon$ -sewing formulation, the complex parameters  $\zeta_a$ ,  $a = 1, 2$  are coordinates inside identified annuluses  $A_a$ , and  $|\zeta_a| \leq r_a$ . Therefore, due to Proposition 3 the matrix elements

$$\widetilde{\mathcal{R}}(x_1, \dots, x_k; \zeta_1) = \langle w', e^{\zeta_1 L_W(-1)} Y_W(u, -\zeta_1) \mathcal{F}(v_1, x_1; \dots; v_k, x_k) \rangle, \quad (3.5)$$

$$\widetilde{\mathcal{R}}(y_1, \dots, y_n; \zeta_2) = \langle w', e^{\zeta_2 L_W(-1)} Y_W(\bar{u}, -\zeta_2) \mathcal{F}(v'_1, y_1; \dots; v'_n, y_n) \rangle, \quad (3.6)$$

are absolutely convergent in powers of  $\epsilon$  with some radii of convergence  $R_a \leq r_a$ , with  $|\zeta_a| \leq R_a$ . The dependence of (3.5) and (3.6) on  $\epsilon$  is expressed via  $\zeta_a$ ,  $a = 1, 2$ . Let us rewrite the product (3.2) as

$$\begin{aligned} & \langle w', \mathcal{F}(v_1, x_1; \dots; v_k, x_k; v'_1, y_1; \dots; v'_n, y_n; \epsilon) \rangle \\ &= \sum_{l \in \mathbb{Z}} \epsilon^l (\langle w', \mathcal{F}(v_1, x_1; \dots; v_k, x_k; v'_1, y_1; \dots; v'_n, y_n) \rangle_l) \\ &= \sum_{l \in \mathbb{Z}} \sum_{u \in V_1} \sum_{m \in \mathbb{C}} \epsilon^{l-m-1} \widetilde{\mathcal{R}}_m(x_1, \dots, x_k; \zeta_1) \widetilde{\mathcal{R}}_m(y_1, \dots, y_n; \zeta_2), \end{aligned} \quad (3.7)$$

as a formal series in  $\epsilon$  for  $|\zeta_a| \leq R_a$ , where and  $|\epsilon| \leq r$  for  $r < r_1 r_2$ . Then we apply Cauchy's inequality to coefficient forms (3.5) and (3.6) to find

$$\left| \widetilde{\mathcal{R}}_m(x_1, \dots, x_k; \zeta_1) \right| \leq M_1 R_1^{-m}, \quad (3.8)$$

with

$$M_1 = \sup_{|\zeta_1| \leq R_1, |\epsilon| \leq r} \left| \widetilde{\mathcal{R}}(x_1, \dots, x_k; \zeta_1) \right|.$$

Similarly,

$$\left| \widetilde{\mathcal{R}}_m(y_1, \dots, y_n; \zeta_2) \right| \leq M_2 R_2^{-m}, \quad (3.9)$$

for

$$M_2 = \sup_{|\zeta_2| \leq R_2, |\epsilon| \leq r} \left| \widetilde{\mathcal{R}}(y_1, \dots, y_n; \zeta_2) \right|.$$

Using (3.8) and (3.9) we obtain for (3.7)

$$\begin{aligned} & |(\langle w', \mathcal{F}(v_1, x_1; \dots; v_k, x_k; v'_1, y_1; \dots; v'_n, y_n) \rangle_l)| \\ & \leq \left| \widetilde{\mathcal{R}}_m(x_1, \dots, x_k; \zeta_1) \right| \left| \widetilde{\mathcal{R}}_m(y_1, \dots, y_n; \zeta_2) \right| \\ & \leq M_1 M_2 (R_1 R_2)^{-m}. \end{aligned} \quad (3.10)$$

Thus, for  $M = \min \{M_1, M_2\}$  and  $R = \max \{R_1, R_2\}$ , such that

$$|\mathcal{R}_l(x_1; \dots, x_k; y_1, \dots, y'_n; \zeta_1, \zeta_2)| \leq M R^{-l+m+1}. \quad (3.11)$$

Thus, we see that (3.2) is absolute convergent as a formal series in  $\epsilon$  is defined for  $|\zeta_a| \leq r_a$ , and  $|\epsilon| \leq r$  for  $r < r_1 r_2$ , with extra poles only at  $x_i = y_j$ ,  $1 \leq i \leq k$ ,  $1 \leq j \leq n$ .  $\square$

Now we show the existence of appropriate  $\mathcal{W}_{x_1, \dots, x_k; y_1, \dots, y_n}$ -valued rational form corresponding to the absolute convergent rational form  $R(x_1, \dots, x_k; y_1, \dots, y_n; \epsilon)$  defining the  $\epsilon$ -product of elements of the spaces  $\mathcal{W}_{x_1, \dots, x_k}$  and  $\mathcal{W}_{y_1, \dots, y_n}$ .

**Lemma 1.** *For all choices of elements of the spaces  $\mathcal{W}_{x_1, \dots, x_k}$  and  $\mathcal{W}_{y_1, \dots, y_n}$  there exists an element  $\mathcal{F}(v_1, x_1; \dots; v_k, x_k; v'_1, y_1; \dots; v'_n, y_n; \epsilon) \in \mathcal{W}_{x_1, \dots, x_k; y_1, \dots, y_n}$  such that the product (3.2) converges to*

$$R(x_1, \dots, x_k; y_1, \dots, y_n; \epsilon) = \langle w', \mathcal{F}(v_1, x_1; \dots; v_k, x_k; v'_1, y_1; \dots; v'_n, y_n; \epsilon) \rangle.$$

*Proof.* In the proof of Proposition 4 we proved the absolute convergence of the product (3.2) to a rational form  $R(x_1, \dots, x_k; y_1, \dots, y_n; \epsilon)$ . The lemma follows from completeness of  $\mathcal{W}_{x_1, \dots, x_k; y_1, \dots, y_n}$  and density of the space of rational differential forms.  $\square$

As we see, the  $\epsilon$ -product is parametrized by a non-zero complex parameter  $\epsilon$ , and a collection of points on auxiliary spheres with formal parameters vanishing at these points. We then have

**Definition 6.** For fixed sets  $(v_1, \dots, v_k), (v'_1, \dots, v'_n) \in V$ ,  $(x_1, \dots, x_k) \in \mathbb{C}$ ,  $(y_1, \dots, y_n) \in \mathbb{C}$ , we call the set of all  $\mathcal{W}_{x_1, \dots, x_k; y_1, \dots, y_n}$ -valued rational forms  $\mathcal{F}(v_1, x_1; \dots; v_k, x_k; v'_1, y_1; \dots; v'_n, y_n; \epsilon)$  defined by (3.2) with the parameter  $\epsilon$  exhausting all possible values, the complete product of the spaces  $\mathcal{W}_{x_1, \dots, x_k}$  and  $\mathcal{W}_{y_1, \dots, y_n}$ .

#### 4. PROPERTIES OF THE $\mathcal{W}$ -PRODUCT

In this section we study properties of the product  $\mathcal{F}(v_1, x_1; \dots; v_k, x_k; v'_1, y_1; \dots; v'_n, y_n; \epsilon)$  of (3.2). Since we assume that  $(x_1, \dots, x_k; y_1, \dots, y_n) \in F_{k+n}\mathbb{C}$ , i.e., coincidences of  $x_i$  and  $y_j$  are excluded by the definition of  $F_{k+n}\mathbb{C}$ . We have

**Definition 7.** We define the action of  $\partial_l = \partial_{z_l} = \partial/\partial z_l$ ,  $1 \leq l \leq k+n$ , the differentiation of  $\mathcal{F}(v_1, x_1; \dots; v_k, x_k; v'_1, y_1; \dots; v'_n, y_n; \epsilon)$  with respect to the  $l$ -th entry of  $(x_1, \dots, x_k; y_1, \dots, y_n)$  as follows

$$\begin{aligned} & \langle w', \partial_l \mathcal{F}(v_1, x_1; \dots; v_k, x_k; v'_1, y_1; \dots; v'_n, y_n; \epsilon) \rangle \\ &= \sum_{m \in \mathbb{Z}} \epsilon^m \sum_{u \in V_m} \langle w', \partial_{x_i}^{\delta_{l,i}} Y_{WV}^W(\mathcal{F}(v_1, x_1; \dots; v_k, x_k), \zeta_1) u \rangle \\ & \quad \langle w', \partial_{y_j}^{\delta_{l,j}} Y_{WV}^W(\mathcal{F}(v'_1, y_1; \dots; v'_n, y_n), \zeta_2) \bar{u} \rangle. \end{aligned} \quad (4.1)$$

**Proposition 5.** *The product (3.2) satisfies the  $L_V(-1)$ -derivative (2.3) and  $L_V(0)$ -conjugation (2.8) properties.*

*Proof.* By using (2.3) for  $\mathcal{F}(v_1, x_1; \dots; v_k, x_k)$  and  $\mathcal{F}(v'_1, y_1; \dots; v'_n, y_n)$ , we consider

$$\begin{aligned}
& \langle w', \partial_l \mathcal{F}(v_1, x_1; \dots; v_k, x_k; v'_1, y_1; \dots; v'_n, y_n; \epsilon) \rangle \\
&= \sum_{m \in \mathbb{Z}} \epsilon^m \sum_{u \in V_m} \langle w', \partial_{x_i}^{\delta_{l,i}} Y_{WV}^W (\mathcal{F}(v_1, x_1; \dots; v_k, x_k), \zeta_1) u \rangle \\
&\quad \langle w', \partial_{y_j}^{\delta_{l,j}} Y_{WV}^W (\mathcal{F}(v'_1, y_1; \dots; v'_n, y_n), \zeta_2) \bar{u} \rangle \\
&= \sum_{m \in \mathbb{Z}} \epsilon^m \sum_{u \in V_m} \langle w', \partial_{x_i}^{\delta_{l,i}} Y_W (u, -\zeta_1) \mathcal{F}(v_1, x_1; \dots; v_k, x_k) u \rangle \\
&\quad \langle w', \partial_{y_j}^{\delta_{l,j}} Y_W (\bar{u}, -\zeta_2) \mathcal{F}(v'_1, y_1; \dots; v'_n, y_n) \rangle \\
&= \sum_{m \in \mathbb{Z}} \epsilon^m \sum_{u \in V_m} \langle w', Y_{WV}^W \left( \partial_{x_i}^{\delta_{l,i}} \mathcal{F}(v_1, x_1; \dots; v_k, x_k), \zeta_1 \right) u \rangle \\
&\quad \langle w', Y_{WV}^W \left( \partial_{y_j}^{\delta_{l,j}} \mathcal{F}(v'_1, y_1; \dots; v'_n, y_n), \zeta_2 \right) \bar{u} \rangle \\
&= \sum_{m \in \mathbb{Z}} \epsilon^m \sum_{u \in V_m} \langle w', Y_{WV}^W \left( \mathcal{F}(v_1, x_1; \dots; (L_V(-1))^{\delta_{l,i}} v_i, x_i; \dots; v_k, x_k), \zeta_1 \right) u \rangle \\
&\quad \langle w', Y_{WV}^W \left( \mathcal{F}(v'_1, y_1; \dots; (L_V(-1))^{\delta_{l,j}} v'_j, y_j; \dots; v'_n, y_n), \zeta_2 \right) \bar{u} \rangle \\
&= \langle w', \mathcal{F}(v_1, x_1; \dots; (L_V(-1))_l; \dots; v'_n, y_n; \epsilon) \rangle, \tag{4.2}
\end{aligned}$$

where  $(L_V(-1))_l$  acts on the  $l$ -th entry of  $(v_1, \dots; v_k; v'_1, \dots; v'_n)$ . Summing over  $l$  we obtain

$$\begin{aligned}
& \sum_{l=1}^{k+n} \partial_l \mathcal{F}(v_1, x_1; \dots; v_k, x_k; v'_1, y_1; \dots; v'_n, y_n; \epsilon) \rangle \\
&= \sum_{l=1}^{k+n} \langle w', \mathcal{F}(v'_1, x_1; \dots; (L_V(-1))_l; \dots; v'_n, y_n; \epsilon) \rangle \\
&= \langle w', L_W(-1) \cdot \mathcal{F}(v_1, x_1; \dots; v_k, x_k; v'_1, y_1; \dots; v'_n, y_n; \epsilon) \rangle. \tag{4.3}
\end{aligned}$$

Due to (2.8), (8.5), (8.29), (8.30), and (8.13), we have

$$\begin{aligned}
& \langle w', \mathcal{F}(z^{L_V(0)} v_1, z x_1; \dots; z^{L_V(0)} v_k, z x_k; z^{L_V(0)} v'_1, z y_1; \dots; z^{L_V(0)} v'_n, z y_n; \epsilon) \rangle \\
&= \sum_{m \in \mathbb{Z}} \epsilon^m \sum_{u \in V_m} \langle w', Y_{WV}^W \left( \mathcal{F}(z^{L_V(0)} v_1, z x_1; \dots; z^{L_V(0)} v_k, z x_k), \zeta_1 \right) u \rangle \\
&\quad \langle w', Y_{WV}^W \left( \mathcal{F}(z^{L_V(0)} v'_1, z y_1; \dots; z^{L_V(0)} v'_n, z y_n), \zeta_2 \right) \bar{u} \rangle \\
&= \sum_{m \in \mathbb{Z}} \epsilon^m \sum_{u \in V_m} \langle w', Y_{WV}^W \left( z^{L_V(0)} \mathcal{F}(v_1, x_1; \dots; v_k, x_k), \zeta_1 \right) u \rangle \\
&\quad \langle w', Y_{WV}^W \left( z^{L_V(0)} \mathcal{F}(v'_1, y_1; \dots; v'_n, y_n), \zeta_2 \right) \bar{u} \rangle
\end{aligned}$$

$$\begin{aligned}
&= \sum_{m \in \mathbb{Z}} \epsilon^m \sum_{u \in V_m} \langle w', e^{\zeta_1 L_{\mathcal{W}}(-1)} Y_{\mathcal{W}}(u, -\zeta_1) z^{L_V(0)} \mathcal{F}(v_1, x_1; \dots; v_k, x_k) \rangle \\
&\quad \langle w', e^{\zeta_2 L_{\mathcal{W}}(-1)} Y_{\mathcal{W}}(\bar{u}, -\zeta_2) z^{L_V(0)} \mathcal{F}(v'_1, y_1; \dots; v'_n, y_n) \rangle \\
&= \sum_{m \in \mathbb{Z}} \epsilon^m \sum_{u \in V_m} \langle w', e^{\zeta_1 L_{\mathcal{W}}(-1)} z^{L_V(0)} Y_{\mathcal{W}}(z^{-L_V(0)} u, -z \zeta_1) \mathcal{F}(v_1, x_1; \dots; v_k, x_k) \rangle \\
&\quad \langle w', e^{\zeta_2 L_{\mathcal{W}}(-1)} z^{L_V(0)} Y_{\mathcal{W}}(z^{-L_V(0)} \bar{u}, -z \zeta_2) \mathcal{F}(v'_1, y_1; \dots; v'_n, y_n) \rangle \\
&= \sum_{m \in \mathbb{Z}} \epsilon^m \sum_{u \in V_m} \langle w', e^{\zeta_1 L_{\mathcal{W}}(-1)} z^{L_{\mathcal{W}}(0)} z^{-\text{wt}u} Y_{\mathcal{W}}(u, -z \zeta_1) \mathcal{F}(v_1, x_1; \dots; v_k, x_k) \rangle \\
&\quad \langle w', e^{\zeta_2 L_{\mathcal{W}}(-1)} z^{L_{\mathcal{W}}(0)} z^{-\text{wt}\bar{u}} Y_{\mathcal{W}}(\bar{u}, -z \zeta_2) \mathcal{F}(v'_1, y_1; \dots; v'_n, y_n) \rangle \\
&= \sum_{m \in \mathbb{Z}} \epsilon^m \sum_{u \in V_m} \langle w', z^{L_{\mathcal{W}}(0)} e^{\zeta_1 L_{\mathcal{W}}(-1)} Y_{\mathcal{W}}(u, -z \zeta_1) \mathcal{F}(v_1, x_1; \dots; v_k, x_k) \rangle \\
&\quad \langle w', z^{L_{\mathcal{W}}(0)} e^{\zeta_2 L_{\mathcal{W}}(-1)} Y_{\mathcal{W}}(\bar{u}, -z \zeta_2) \mathcal{F}(v'_1, y_1; \dots; v'_n, y_n) \rangle \\
&= \sum_{m \in \mathbb{Z}} \epsilon^m \sum_{u \in V_m} \langle w', z^{L_{\mathcal{W}}(0)} Y_{\mathcal{W}V}^{\mathcal{W}}(\mathcal{F}(v_1, x_1; \dots; v_k, x_k), z \zeta_1) u \rangle \\
&\quad \langle w', z^{L_{\mathcal{W}}(0)} Y_{\mathcal{W}V}^{\mathcal{W}}(\mathcal{F}(v'_1, y_1; \dots; v'_n, y_n), z \zeta_2) \bar{u} \rangle \\
&= \sum_{m \in \mathbb{Z}} \epsilon^m \sum_{u \in V_m} \langle w', z^{L_{\mathcal{W}}(0)} Y_{\mathcal{W}V}^{\mathcal{W}}(\mathcal{F}(v_1, x_1; \dots; v_k, x_k), \zeta'_1) u \rangle \\
&\quad \langle w', z^{L_{\mathcal{W}}(0)} Y_{\mathcal{W}V}^{\mathcal{W}}(\mathcal{F}(v'_1, y_1; \dots; v'_n, y_n), \zeta'_2) \bar{u} \rangle \\
&= \langle w', (z^{L_{\mathcal{W}}(0)}) \cdot \mathcal{F}(v_1, x_1; \dots; v_k, x_k; v'_1, y_1; \dots; v'_n, y_n; \epsilon) \rangle.
\end{aligned}$$

With (9.4), we obtain (2.8) for (3.2).  $\square$

*Remark 2.* As we see in the last expressions, the  $L_V(0)$ -conjugation property (2.8) for the product (3.2) includes the action of  $z^{L_V(0)}$ -operator on complex parameters  $\zeta_a$ ,  $a = 1, 2$ .

We also have

**Proposition 6.** *For generic elements  $v_i, v'_j \in V$ ,  $1 \leq i \leq k$ ,  $1 \leq j \leq n$ , of a quasi-conformal grading-restricted vertex algebra, the product (3.2) is canonical with respect to the action of the group  $\text{Aut}_{x_1, \dots, x_k; y_1, \dots, y_n} \mathcal{O}^{(k+n)}$  of  $k+n$ -dimensional changes*

$$\begin{aligned}
&(x_1, \dots, x_k; y_1, \dots, y_n) \mapsto (x'_1, \dots, x'_k; y'_1, \dots, y'_n) \\
&= (\rho_1(x_1, \dots, x_k; y_1, \dots, y_n), \dots, \rho_{k+n}(x_1, \dots, x_k; y_1, \dots, y_n)), \quad (4.4)
\end{aligned}$$

of formal parameters.

*Proof.* Note that due to Proposition 1

$$\begin{aligned}
\mathcal{F}(v_1, x'_1; \dots; v_k, x'_k) &= \mathcal{F}(v_1, x_1; \dots; v_k, x_k), \\
\mathcal{F}(v_1, y'_1; \dots; v_n, y'_n) &= \mathcal{F}(v_1, y_1; \dots; v_n, y_n).
\end{aligned}$$

Thus,

$$\begin{aligned}
& \langle w', \mathcal{F}(v_1, x'_1; \dots; v_k, x'_k; v'_1, y'_1; \dots; v'_n, y'_n; \epsilon) \rangle \\
&= \sum_{l \in \mathbb{Z}} \epsilon^l \sum_{u \in V_l} \langle w', Y_{WV}^W(\mathcal{F}(v_1, x'_1; \dots; v_k, x'_k), \zeta_1) u \rangle \\
&\quad \langle w', Y_{WV}^W(\mathcal{F}(v'_1, y'_1; \dots; v'_n, y'_n), \zeta_2) \bar{u} \rangle \\
&= \sum_{l \in \mathbb{Z}} \epsilon^l \sum_{u \in V_l} \langle w', Y_{WV}^W(\mathcal{F}(v_1, x_1; \dots; v_k, x_k), \zeta_1) u \rangle \\
&\quad \langle w', Y_{WV}^W(\mathcal{F}(v'_1, y_1; \dots; v'_n, y_n), \zeta_2) \bar{u} \rangle \\
&= \langle w', \mathcal{F}(v_1, x_1; \dots; v_k, x_k; v'_1, y_1; \dots; v'_n, y_n; \epsilon) \rangle.
\end{aligned}$$

Thus, the product (3.2) is invariant under (4.4).  $\square$

In the geometric interpretation in terms of auxiliary spaces, the definition (3.2) depends on the choice of insertion points  $p_i$ ,  $1 \leq i \leq k$ , with local coordinates  $x_i$  on  $\widehat{\Sigma}_1^{(0)}$ , and  $p'_j$ ,  $1 \leq j \leq k$ , with local coordinates  $y_j$  on  $\widehat{\Sigma}_2^{(0)}$ . Suppose we change the distribution of points among two Riemann spheres. We formulate the following

**Lemma 2.** *For a fixed set  $(\tilde{v}_1, \dots, \tilde{v}_n) \in V$ , of vertex algebra elements, the  $\epsilon$ -product  $\mathcal{F}(\tilde{v}_1, z_1; \dots; \tilde{v}_n, z_n; \epsilon) \in \mathcal{W}_{z_1, \dots, z_n}$ ,*

$$\cdot_\epsilon : \mathcal{W}_{z_1, \dots, z_k} \times \mathcal{W}_{z_{k+1}, \dots, z_n} \rightarrow \mathcal{W}_{z_1, \dots, z_n}, \quad (4.5)$$

*remains the same for elements  $\mathcal{F}(\tilde{v}_1, z_1; \dots; \tilde{v}_k, z_k) \in \mathcal{W}_{z_1, \dots, z_k}$  and  $\mathcal{F}(\tilde{v}_{k+1}, z_{k+1}; \dots; \tilde{v}_n, z_n) \in \mathcal{W}_{z_{k+1}, \dots, z_n}$ , for any  $0 \leq k \leq n$ .*

*Remark 3.* This Lemma is important for the formulation of cohomological invariants associated to grading-restricted vertex algebras on smooth manifolds [11]. In case  $k = 0$ , we obtain from (4.6),

$$\cdot_\epsilon : \mathcal{W} \times \mathcal{W}_{z_1, \dots, z_n} \rightarrow \mathcal{W}_{z_1, \dots, z_n}. \quad (4.6)$$

*Proof.* Let  $\tilde{v}_i \in V$ ,  $1 \leq i \leq k$ ,  $\tilde{v}_j \in V$ ,  $1 \leq j \leq k$ , and  $z_i, z_j$  are corresponding formal parameters. We show that the  $\epsilon$ -product of  $\mathcal{F}(\tilde{v}_1, z_1; \dots; \tilde{v}_k, z_k)$  and  $\mathcal{F}(\tilde{v}_{k+1}, z_{k+1}; \dots; \tilde{v}_n, z_n)$ , i.e., the  $\mathcal{W}_{z_1, \dots, z_{k+n}}$ -valued differential form

$$\mathcal{F}((\tilde{v}_1, z_1; \dots; \tilde{v}_k, z_k); (\tilde{v}_{k+1}, z_{k+1}; \dots; \tilde{v}_n, z_n); \zeta_1, \zeta_2; \epsilon) \quad (4.7)$$

is independent of the choice of  $0 \leq k \leq n$ . Consider

$$\begin{aligned}
& \langle w', \mathcal{F}(\tilde{v}_1, z_1; \dots; \tilde{v}_k, z_k; \tilde{v}_{k+1}, z_{k+1}; \dots; \tilde{v}_n, z_n; \zeta_1, \zeta_2; \epsilon) \rangle \\
&= \sum_{l \in \mathbb{Z}} \epsilon^l \sum_{u \in V_l} \langle w', Y_{WV}^W(\mathcal{F}(\tilde{v}_1, z_1; \dots; \tilde{v}_k, z_k), \zeta_1) u \rangle \\
&\quad \langle w', Y_{WV}^W(\mathcal{F}(\tilde{v}_{k+1}, z_{k+1}; \dots; \tilde{v}_n, z_n), \zeta_2) \bar{u} \rangle.
\end{aligned} \quad (4.8)$$

On the other hand, for  $0 \leq m \leq k$ , consider

$$\begin{aligned}
& \sum_{l \in \mathbb{Z}} \epsilon^l \sum_{u \in V_l} \langle w', Y_{WV}^W(\mathcal{F}(\tilde{v}_1, z_1; \dots; \tilde{v}_m, z_m), \zeta_1) u \rangle \\
&\quad \langle w', Y_{WV}^W(\mathcal{F}(\tilde{v}_{m+1}, z'_{m+1}; \dots; \tilde{v}_k, z'_k; \tilde{v}_{k+1}, z_1; \dots; \tilde{v}_n, z_n), \zeta_2) \bar{u} \rangle \\
&= \langle w', \mathcal{F}(\tilde{v}_1, z_1; \dots; \tilde{v}_m, z_m; \tilde{v}_{m+1}, z'_{m+1}; \dots; \tilde{v}_k, z'_k; \tilde{v}_{k+1}, z_{k+1}; \dots; \tilde{v}_n, z_n) \rangle.
\end{aligned}$$

The last is the  $\epsilon$ -product (3.2) of  $\mathcal{F}(\tilde{v}_1, z_1; \dots; \tilde{v}_m, z_m) \in \mathcal{W}_{z_1, \dots, z_m}$  and  $\mathcal{F}(\tilde{v}_{m+1}, z'_{m+1}; \dots; \tilde{v}_k, z'_k; \tilde{v}_{k+1}, z_1; \dots; \tilde{v}_n, z_n) \in \mathcal{W}_{z'_{m+1}, \dots, z'_k; z_1, \dots, z_n}$ . Let us apply the invariance with respect to a subgroup of  $\text{Aut}_{z_1, \dots, z_{k+n}} \mathcal{O}^{(n)}$ , with  $(z_1, \dots, z_m)$  and  $(z_{k+1}, \dots, z_n)$  remaining unchanged. Then we obtain the same product (4.8).  $\square$

Next, we formulate

**Definition 8.** We define the action of an element  $\sigma \in S_{k+n}$  on the product of  $\mathcal{F}(v_1, x_1; \dots; v_k, x_k) \in \mathcal{W}_{x_1, \dots, x_k}$ , and  $\mathcal{F}(v'_1, y_1; \dots; v'_n, y_n) \in \mathcal{W}_{y_1, \dots, y_n}$ , as

$$\begin{aligned} & \langle w', \sigma(\mathcal{F})(v_1, x_1; \dots; v_k, x_k; v'_1, y_1; \dots; v'_n, y_n; \epsilon) \rangle \\ &= \langle w', \mathcal{F}(v_{\sigma(1)}, x_{\sigma(1)}; \dots; v_{\sigma(k)}, x_{\sigma(k)}; v'_{\sigma(1)}, y_{\sigma(1)}; \dots; v'_{\sigma(n)}, y_{\sigma(n)}; \epsilon) \rangle \\ &= \sum_{u \in V} \langle w', Y_{WV}^W(\mathcal{F}(v_{\sigma(1)}, x_{\sigma(1)}; \dots; v_{\sigma(k)}, x_{\sigma(k)}), \zeta_1) u \rangle \\ & \quad \langle w', Y_{WV}^W(\mathcal{F}(v'_{\sigma(1)}, y_{\sigma(1)}; \dots; v'_{\sigma(n)}, y_{\sigma(n)}), \zeta_2) \bar{u} \rangle. \end{aligned} \quad (4.9)$$

## 5. DOUBLE COMPLEX SPACES $C_m^n(V, \mathcal{W})$

In [6] (see also [10]) a cohomology theory for grading-restricted vertex algebras was introduced. In particular, spaces  $C_m^n(V, \mathcal{W})$ ,  $n \geq 0$ ,  $m \geq 0$ , and differentials  $\delta_m^n$ , for chain-cochain double complex  $(C_m^n(V, \mathcal{W}), \delta_m^n)$  were introduced. In this section we recall the definition and properties of  $C_m^n(V, \mathcal{W})$ , [6].

**5.1.  $E$ -elements.** For  $w \in W$ , the  $W$ -valued function given by

$$E_W^{(n)}(v_1, z_1; \dots; v_n, z_n; w) = E(\omega_W(v_1, z_1) \dots \omega_W(v_n, z_n)w),$$

where an element  $E(\phi)$  is a  $\overline{W}$ -valued rational function,  $\phi \in \overline{W}$  is given by (see notations for  $\omega_W(\cdot, \cdot)$  in Section 5.3)

$$E(\phi) = R(\langle w', \phi \rangle).$$

One defines

$$E_{WV}^{W;(n)}(w; v_1, z_1; \dots; v_n, z_n) = E_W^{(n)}(v_1, z_1; \dots; v_n, z_n; w),$$

where  $E_{WV}^{W;(n)}(w; v_1, z_1; \dots; v_n, z_n)$  is an element of  $\overline{W}_{z_1, \dots, z_n}$ . For  $(z_1, \dots, z_n, \zeta) \in F_{n+1}\mathbb{C}$ ,  $(v_1, \dots, v_n) \in V$ , and  $w \in W$ , set

$$E_W^{(n,1)}(v_1, z_1; \dots; v_n, z_n; w, \zeta) = E(Y_W(v_1, z_1) \dots Y_W(v_n, z_n) Y_{WV}^W(w, \zeta) \mathbf{1}_V).$$

One defines

$$\mathcal{F} \circ \left( E_{V; \mathbf{1}}^{(l_1)} \otimes \dots \otimes E_{V; \mathbf{1}}^{(l_n)} \right) : V^{\otimes m+n} \rightarrow \overline{W}_{z_1, \dots, z_{m+n}},$$

by

$$\begin{aligned} & (\mathcal{F} \circ (E_{V; \mathbf{1}}^{(l_1)} \otimes \dots \otimes E_{V; \mathbf{1}}^{(l_n)}))(v_1 \otimes \dots \otimes v_{m+n-1}) \\ &= E(\mathcal{F}(E_{V; \mathbf{1}}^{(l_1)}(v_1 \otimes \dots \otimes v_{l_1}) \otimes \dots \\ & \quad \otimes E_{V; \mathbf{1}}^{(l_n)}(v_{l_1+\dots+l_{n-1}+1} \otimes \dots \otimes v_{l_1+\dots+l_{n-1}+l_n}))), \end{aligned}$$

and

$$E_W^{(m)} \circ_0 \mathcal{F} : V^{\otimes m+n} \rightarrow \overline{W}_{z_1, \dots, z_{m+n-1}},$$

is given by

$$\begin{aligned} & (E_W^{(m)} \circ_0 \mathcal{F})(v_1 \otimes \dots \otimes v_{m+n}) \\ &= E(E_W^{(m)}(v_1 \otimes \dots \otimes v_m; \mathcal{F}(v_{m+1} \otimes \dots \otimes v_{m+n}))). \end{aligned}$$

Finally,

$$E_{WV}^{W;(m)} \circ_{m+1} \mathcal{F} : V^{\otimes m+n} \rightarrow \overline{W}_{z_1, \dots, z_{m+n-1}},$$

is defined by

$$(E_{WV}^{W;(m)} \circ_{m+1} \mathcal{F})(v_1 \otimes \dots \otimes v_{m+n}) = E(E_{WV}^{W;(m)}(\mathcal{F}(v_1 \otimes \dots \otimes v_n); v_{n+1} \otimes \dots \otimes v_{m+n})).$$

In the case that  $l_1 = \dots = l_{i-1} = l_{i+1} = 1$  and  $l_i = m - n - 1$ , for some  $1 \leq i \leq n$ , we will use  $\mathcal{F} \circ_i E_{V; \mathbf{1}}^{(l_i)}$  to denote  $\mathcal{F} \circ (E_{V; \mathbf{1}}^{(l_1)} \otimes \dots \otimes E_{V; \mathbf{1}}^{(l_n)})$ . Note that our notations differ with that of [6].

**5.2. Maps composable with vertex operators.** Let us recall the definition of maps composable with a number of vertex operators [6].

**Definition 9.** For a  $V$ -module

$$W = \coprod_{n \in \mathbb{C}} W_{(n)},$$

and  $m \in \mathbb{C}$ , let

$$P_m : \overline{W} \rightarrow W_{(m)},$$

be the projection from  $\overline{W}$  to  $W_{(m)}$ . Let

$$\mathcal{F} : V^{\otimes n} \rightarrow \mathcal{W}_{z_1, \dots, z_n},$$

be a linear map. For  $m \in \mathbb{N}$ ,  $\mathcal{F}$  is called [6, 10] composable with  $m$  vertex operators if the following conditions are satisfied:

1) Let  $l_1, \dots, l_n \in \mathbb{Z}_+$  such that  $l_1 + \dots + l_n = m + n$ ,  $v_1, \dots, v_{m+n} \in V$ , and  $w' \in W'$ . Set

$$\Psi_i = E_V^{(l_i)}(v_{k_1}, z_{k_1} - \zeta_i; \dots; v_{k_i}, z_{k_i} - \zeta_i; \mathbf{1}_V), \quad (5.1)$$

where

$$k_1 = l_1 + \dots + l_{i-1} + 1, \quad \dots, \quad k_i = l_1 + \dots + l_{i-1} + l_i, \quad (5.2)$$

for  $i = 1, \dots, n$ . Then there exist positive integers  $N_m^n(v_i, v_j)$  depending only on  $v_i$  and  $v_j$  for  $i, j = 1, \dots, k$ ,  $i \neq j$  such that the series

$$\mathcal{I}_m^n(\mathcal{F}) = \sum_{r_1, \dots, r_n \in \mathbb{Z}} \langle w', \mathcal{F}(P_{r_1} \Psi_1; \zeta_1; \dots; P_{r_n} \Psi_n, \zeta_n) \rangle, \quad (5.3)$$

is absolutely convergent when

$$|z_{l_1 + \dots + l_{i-1} + p} - \zeta_i| + |z_{l_1 + \dots + l_{j-1} + q} - \zeta_j| < |\zeta_i - \zeta_j|, \quad (5.4)$$

for  $i, j = 1, \dots, k$ ,  $i \neq j$  and for  $p = 1, \dots, l_i$  and  $q = 1, \dots, l_j$ . The sum must be analytically extended to a rational function in  $(z_1, \dots, z_{m+n})$ , independent of  $(\zeta_1, \dots, \zeta_n)$ , with the only possible poles at  $z_i = z_j$ , of order less than or equal to  $N_m^n(v_i, v_j)$ , for  $i, j = 1, \dots, k$ ,  $i \neq j$ .



2) For  $v_1, \dots, v_{m+n} \in V$ , and  $(z_1, \dots, z_{n+m}) \in \mathbb{C}$  there exist positive integers  $N_m^n(v_i, v_j)$ , depending only on  $v_i$  and  $v_j$ , for  $i, j = 1, \dots, k$ ,  $i \neq j$ , such that for arbitrary  $w' \in W'$ , and such that the series

$$\mathcal{J}_m^n(\mathcal{F}) = \sum_{q \in \mathbb{C}} \langle w', E_W^{(m)}(v_1, z_1; \dots; v_m, z_m; P_q(\mathcal{F}(v_{m+1}, z_{m+1}; \dots; v_{m+n}, z_{m+n}))) \rangle, \quad (5.5)$$

is absolutely convergent when

$$\begin{aligned} z_i &\neq z_j, \quad i \neq j, \\ |z_i| &> |z_k| > 0, \end{aligned} \quad (5.6)$$

for  $i = 1, \dots, m$ , and  $k = m+1, \dots, m+n$ , and the sum can be analytically extended to a rational function in  $(z_1, \dots, z_{n+m})$  with the only possible poles at  $z_i = z_j$ , of orders less than or equal to  $N_m^n(v_i, v_j)$ , for  $i, j = 1, \dots, k$ ,  $i \neq j$ .

In [6], we the following useful proposition is proven:

**Proposition 7.** *Let  $\mathcal{F} : V^{\otimes n} \rightarrow \mathcal{W}_{z_1, \dots, z_n}$  be composable with  $m$  vertex operators. Then we have:*

- (1) *For  $p \leq m$ ,  $\mathcal{F}$  is composable with  $p$  vertex operators and for  $p, q \in \mathbb{Z}_+$  such that  $p+q \leq m$  and  $l_1, \dots, l_n \in \mathbb{Z}_+$  such that  $l_1 + \dots + l_n = p+n$ ,  $\mathcal{F} \circ (E_{V; \mathbf{1}}^{(l_1)} \otimes \dots \otimes E_{V; \mathbf{1}}^{(l_n)})$  and  $E_W^{(p)} \circ_{p+1} \mathcal{F}$  are composable with  $q$  vertex operators.*
- (2) *For  $p, q \in \mathbb{Z}_+$  such that  $p+q \leq m$ ,  $l_1, \dots, l_n \in \mathbb{Z}_+$  such that  $l_1 + \dots + l_n = p+n$  and  $k_1, \dots, k_{p+n} \in \mathbb{Z}_+$  such that  $k_1 + \dots + k_{p+n} = q+p+n$ , we have*

$$\begin{aligned} &(\mathcal{F} \circ (E_{V; \mathbf{1}}^{(l_1)} \otimes \dots \otimes E_{V; \mathbf{1}}^{(l_n)})) \circ (E_{V; \mathbf{1}}^{(k_1)} \otimes \dots \otimes E_{V; \mathbf{1}}^{(k_{p+n})}) \\ &= \mathcal{F} \circ (E_{V; \mathbf{1}}^{(k_1 + \dots + k_{l_1})} \otimes \dots \otimes E_{V; \mathbf{1}}^{(k_{l_1 + \dots + l_{n-1} + 1 + \dots + k_{p+n})}). \end{aligned}$$

- (3) *For  $p, q \in \mathbb{Z}_+$  such that  $p+q \leq m$  and  $l_1, \dots, l_n \in \mathbb{Z}_+$  such that  $l_1 + \dots + l_n = p+n$ , we have*

$$E_W^{(q)} \circ_{q+1} (\mathcal{F} \circ (E_{V; \mathbf{1}}^{(l_1)} \otimes \dots \otimes E_{V; \mathbf{1}}^{(l_n)})) = (E_W^{(q)} \circ_{q+1} \mathcal{F}) \circ (E_{V; \mathbf{1}}^{(l_1)} \otimes \dots \otimes E_{V; \mathbf{1}}^{(l_n)}).$$

- (4) *For  $p, q \in \mathbb{Z}_+$  such that  $p+q \leq m$ , we have*

$$E_W^{(p)} \circ_{p+1} (E_W^{(q)} \circ_{q+1} \mathcal{F}) = E_W^{(p+q)} \circ_{p+q+1} \mathcal{F}.$$

Finally, in [6] we find the proof of the following. Let now  $P_n : W \rightarrow W_{(n)}$ , for  $n \in \mathbb{C}$  be the projection from  $W$  to  $W_{(n)}$ .

**Proposition 8.** *For  $k, l_1, \dots, l_{n+1} \in \mathbb{Z}_+$  and  $v_1^{(1)}, \dots, v_{l_1}^{(1)}, \dots, v_1^{(n+1)}, \dots, v_{l_{n+1}}^{(n+1)} \in V$ ,  $w \in W$ , and  $w' \in W'$ , the series*

$$\begin{aligned} &\sum_{r_1, \dots, r_n \in \mathbb{Z}, r_{n+1} \in \mathbb{C}} \langle w', E_W^{(n,1)}(P_{r_1}(E_V^{(l_1)}(v_1^{(1)}, z_1^{(1)}; \dots; v_{l_1}^{(1)}, z_{l_1}^{(1)}; \mathbf{1}_V, z_1^{(0)})); \dots; \\ &P_{r_n}(E_V^{(l_n)}(v_1^{(n)}, z_1^{(n)}; \dots; v_{l_n}^{(n)}, z_{l_n}^{(n)}; \mathbf{1}_V, z_n^{(0)})) \\ &P_{r_{n+1}}(E_W^{(l_{n+1})}(v_1^{(n+1)}, z_1^{(n+1)}; \dots; v_{l_{n+1}}^{(n+1)}, z_{l_{n+1}}^{(n+1)}; w, z_{n+1}^{(0)})) \rangle, \end{aligned} \quad (5.7)$$

converges absolutely to

$$\langle w', E_W^{(n)}(v_1^{(1)}, z_1^{(1)} + z_1^{(0)}; \dots; v_{l_1}^{(1)}, z_{l_1}^{(1)} + z_1^{(0)}; \dots; v_1^{(n+1)}, z_1^{(n+1)} + z_{n+1}^{(0)}; v_{l_{n+1}}^{(n+1)}, z_{l_{n+1}}^{(n+1)} + z_{n+1}^{(0)}; w) \rangle,$$

when  $0 < |z_p^{(i)}| + |z_q^{(j)}| < |z_i^{(0)} - z_j^{(0)}|$  for  $i, j = 1, \dots, n+1$ ,  $i \neq j$ ,  $p = 1, \dots, l_i$ ,  $q = 1, \dots, l_j$ .

**5.3. Definition of  $C_m^n(V, \mathcal{W})$ -spaces.** In this subsection we recall the definition of spaces  $C_m^n(V, \mathcal{W})$  given in [6] for a grading-restricted vertex algebra  $V$ . First, recall the definition of shuffles. Let  $S_l$  be the permutation group. For  $l \in \mathbb{N}$  and  $1 \leq s \leq l-1$ , let  $J_{l,s}$  be the set of elements of  $S_l$  which preserve the order of the first  $s$  numbers and the order of the last  $l-s$  numbers, that is,

$$J_{l,s} = \{\sigma \in S_l \mid \sigma(1) < \dots < \sigma(s), \sigma(s+1) < \dots < \sigma(l)\}.$$

The elements of  $J_{l,s}$  are called shuffles, and we use the notation

$$J_{l,s}^{-1} = \{\sigma \mid \sigma \in J_{l,s}\}.$$

For a set of  $n$  elements  $(v_1, \dots, v_n)$  of a grading-restricted vertex algebra  $V$ , we consider linear maps

$$\mathcal{F}(v_1, z_1; \dots; v_n, z_n) : V^{\otimes n} \rightarrow \mathcal{W}_{z_1, \dots, z_n} \quad (5.8)$$

(see Section 2 for the definition of a  $\mathcal{W}_{z_1, \dots, z_n}$  space). Note that similar to considerations of [1], (2.2) can be treated as  $\text{Aut}_{z_1, \dots, z_n} \mathcal{O}^{(n)}$ -torsor of the product of groups of formal parameter transformations. In what follows, according to definitions of Appendix 2, when we write an element  $\mathcal{F}$  of the space  $\mathcal{W}_{z_1, \dots, z_n}$ , we actually have in mind corresponding matrix element  $\langle w', \mathcal{F} \rangle$  that absolutely converges (in a certain domain) to a rational form-valued function  $R(\langle w', \mathcal{F} \rangle)$ . Quite frequently we will write  $\langle w', \mathcal{F} \rangle$  which would denote a rational  $\mathcal{W}$ -valued form. In notations, we would keep tensor products of vertex algebra elements with wt-powers of  $z$ -differentials when it is inevitable only.

Later in the next section we prove, that for arbitrary  $v_i \in V$ ,  $1 \leq i \leq n$ , with formal parameters  $z_i$  an element (2.2) as well as the vertex operators

$$\omega_W(v_i, z_i) = Y_W(dz_i^{\text{wt}(v_i)} \otimes v_i, z_i), \quad (5.9)$$

are invariant with respect to the action of the group  $\text{Aut}_{z_1, \dots, z_n} \mathcal{O}^{(n)}$ . In (5.9) we mean the ordinary vertex operator (as defined in Appendix 8) not affecting the tensor product with corresponding differential.

In [6] one finds:

**Proposition 9.** *The subspace of  $\text{Hom}(V^{\otimes n}, \mathcal{W}_{z_1, \dots, z_n})$  consisting of linear maps having the  $L_V(-1)$ -derivative property, having the  $L_V(0)$ -conjugation property or being composable with  $m$  vertex operators is invariant under the action of  $S_n$ .  $\square$*

We next have

**Definition 10.** For arbitrary set of vertex algebra elements  $v_i, v_j \in V$ , and formal complex parameters  $z_i, z_j, 1 \leq i \leq n, 1 \leq j \leq m, n \geq 0, m \geq 0$ , we denote by  $C_m^n(V, \mathcal{W})$ , the space of all linear maps (5.8)

$$\mathcal{F}(v_1, z_1; \dots; v_n, z_n) : V^{\otimes n} \rightarrow \mathcal{W}_{z_1, \dots, z_n}, \quad (5.10)$$

composable with a  $m$  of vertex operators (5.9) with vertex algebra elements  $v_j$ , with formal parameters  $z_j$ . We assume also that (2.2) satisfy  $L_V(-1)$ -derivative (2.3),  $L_V(0)$ -conjugation (2.8) properties, and the symmetry property with respect to action of the symmetric group  $S_n$ :

$$\sum_{\sigma \in J_{n,1}^{-1}} (-1)^{|\sigma|} \mathcal{F}(v_{\sigma(1)}, z_{\sigma(1)}; \dots; v_{\sigma(n)}, z_{\sigma(n)}) = 0. \quad (5.11)$$

In [11] we prove the following

**Proposition 10.** *For a quasi-conformal grading-restricted vertex algebra  $V$  and its module  $W$ , the definition (10) is canonical, i.e., invariant with respect to the group of  $n$ -dimensional transformations*

$$(z_1, \dots, z_n) \mapsto (z'_1, \dots, z'_n) = (\rho_1(z_1, \dots, z_n), \dots, \rho_n(z_1, \dots, z_n)),$$

of formal parameters  $z_i, 1 \leq i \leq n$ .

In Appendix 10 we recall the proof of Proposition 10.

*Remark 4.* The condition of quasi-conformality is necessary in the proof of invariance of elements of the space  $\mathcal{W}_{z_1, \dots, z_n}$  with respect to a vertex algebraic representation (cf. Appendix 8) of the group  $\text{Aut}_{z_1, \dots, z_n} \mathcal{O}^{(n)}$ . In what follows, we will always assume the quasi-conformality of  $V$  when it concerns the spaces  $C_m^n(V, \mathcal{W})$ .

**5.4. Coboundary operators.** In this subsection we recall [6] the definition of the coboundary operator for the spaces  $C_m^n(V, \mathcal{W})$ ,

$$\begin{aligned} \delta_m^n \mathcal{F} &= \sum_{i=1}^n (-1)^i \mathcal{F}(\omega_V(v_i, z_i - z_{i+1}) v_{i+1}) \\ &+ \omega_W(v_1, z_1) \mathcal{F}(v_2, z_2; \dots; v_n, z_n) \\ &+ (-1)^{n+1} \omega_W(v_{n+1}, z_{n+1}) \mathcal{F}(v_1, z_1; \dots; v_n, z_n). \end{aligned} \quad (5.12)$$

Note that it is assumed that the coboundary operator does not affect  $dz_i^{\text{wt}(v_i)}$ -tensor multipliers in  $\mathcal{F}$ . In [6] the following proposition is proved

**Proposition 11.** *The operator (5.12) obeis*

$$\delta_m^n : C_m^n(V, \mathcal{W}) \rightarrow C_{m-1}^{n+1}(V, \mathcal{W}), \quad (5.13)$$

$$\delta_{m-1}^{n+1} \circ \delta_m^n = 0, \quad (5.14)$$

$$0 \rightarrow C_m^0(V, \mathcal{W}) \xrightarrow{\delta_m^0} C_{m-1}^1(V, \mathcal{W}) \xrightarrow{\delta_{m-1}^1} \dots \xrightarrow{\delta_1^{m-1}} C_0^m(V, \mathcal{W}) \rightarrow 0, \quad (5.15)$$

i.e., provides the chain-cochain complex  $(C_m^n(V, \mathcal{W}), \delta_m^n)$ .  $\square$

6. APPLICATION: THE PRODUCT OF  $C_m^n(V, \mathcal{W})$ -SPACES

In this section we consider an application of the material of Section 3 to double complex spaces  $C_m^n(V, \mathcal{W})$ , (Definition 10) described in previous section. We introduce the product of two double complex spaces with the image in another double complex space coherent with respect to the original differential (5.12), and the symmetry property (5.11). We prove the canonicity of the product, and derive an analogue of Leibniz formula.

**Definition 11.** For  $\mathcal{F}(v_1, x_1; \dots; v_k, x_k) \in C_m^k(V, \mathcal{W})$ , and  $\mathcal{F}(v'_1, y_1; \dots; v'_n, y_n) \in C_{m'}^n(V, \mathcal{W})$  the product

$$\mathcal{F}(v_1, x_1; \dots; v_k, x_k) \cdot_\epsilon \mathcal{F}(v'_1, y_1; \dots; v'_n, y_n) \mapsto \mathcal{F}(v_1, x_1; \dots; v_k, x_k; v'_1, y_1; \dots; v'_n, y_n; \epsilon),$$

is a  $\mathcal{W}_{x_1, \dots, x_k; y_1, \dots, y_n}$ -valued rational form

$$\begin{aligned} & \langle w', \mathcal{F}(v_1, x_1; \dots; v_k, x_k; v'_1, y_1; \dots; v'_n, y_n; \epsilon) \rangle \\ &= \langle w', \mathcal{F}(v_1, x_1; \dots; v_k, x_k) \cdot_\epsilon \mathcal{F}(v'_1, y_1; \dots; v'_n, y_n) \rangle \\ &= \sum_{u \in V} \langle w', Y_{WV}^W(\mathcal{F}(v_1, x_1; \dots; v_k, x_k), \zeta_1) u \rangle \\ & \quad \langle w', Y_{WV}^W(\mathcal{F}(v'_1, y_1; \dots; v'_n, y_n), \zeta_2) \bar{u} \rangle, \end{aligned} \quad (6.1)$$

defined by (3.2).

*Remark 5.* Let  $t$  be the number of common vertex operators the mappings  $\mathcal{F}(v_1, x_1; \dots; v_k, x_k) \in C_m^k(V, \mathcal{W})$  and  $\mathcal{F}(v'_1, y_1; \dots; v'_n, y_n) \in C_{m'}^n(V, \mathcal{W})$ , are composable with. Similar to the case of common formal parameters, this case is separately treated with a decrease to  $m + m' - t$  of number of composable vertex operators. In what follows, we exclude this case from considerations.

The action of  $\sigma \in S_{k+n}$  on the product  $\mathcal{F}(v_1, x_1; \dots; v_k, x_k; v'_{k+1}, y_1; \dots; v'_n, y_n; \epsilon)$  (6.1) is given by (2.9). We then have

**Proposition 12.** For  $\mathcal{F}(v_1, x_1; \dots; v_k, x_k) \in C_m^k(V, \mathcal{W})$  and  $\mathcal{F}(v'_1, y_1; \dots; v'_n, y_n) \in C_{m'}^n(V, \mathcal{W})$ , the product  $\mathcal{F}(v_1, x_1; \dots; v_k, x_k; v'_1, y_1; \dots; v'_n, y_n; \epsilon)$  (6.1) belongs to the space  $C_{m+m'}^{k+n}(V, \mathcal{W})$ , i.e.,

$$\cdot_\epsilon : C_m^k(V, \mathcal{W}) \times C_{m'}^n(V, \mathcal{W}) \rightarrow C_{m+m'}^{k+n}(V, \mathcal{W}).$$

*Proof.* In Proposition 4 we proved that  $\mathcal{F}(v_1, x_1; \dots; v_k, x_k; v'_1, y_1; \dots; v'_n, y_n; \epsilon) \in \mathcal{W}_{x_1, \dots, x_k; y_1, \dots, y_n}$ . It is clear that

$$\cdot_\epsilon : C_l^k(V, \mathcal{W}) \times C_l^n(V, \mathcal{W}) \rightarrow C_l^{k+n}(V, \mathcal{W}),$$

for some  $l$ . First, we show that (5.11) for  $\sigma \in S_{k+n}$ ,

$$\sum_{\sigma \in J_{k+n; s}^{-1}} (-1)^{|\sigma|} \mathcal{F}(v_{\sigma(1)}, x_{\sigma(1)}; \dots; v_{\sigma(k)}, x_{\sigma(k)}; v'_{\sigma(k+1)}, y_{\sigma(k+1)}; \dots; v'_{\sigma(n)}, y_{\sigma(n)}) = 0.$$

For arbitrary  $w' \in W'$ , we have

$$\begin{aligned}
& \sum_{\sigma \in J_{k+n;s}^{-1}} (-1)^{|\sigma|} \langle w', \mathcal{F}(v_{\sigma(1)}, x_{\sigma(1)}; \dots; v_{\sigma(k)}, x_{\sigma(k)}; v'_{\sigma(1)}, y_{\sigma(1)}; \dots; v'_{\sigma(n)}, y_{\sigma(n)}) \rangle \\
&= \sum_{\sigma \in J_{k+n;s}^{-1}} (-1)^{|\sigma|} \sum_{u \in V} \langle w', Y_{WV}^W(\mathcal{F}(v_{\sigma(1)}, x_{\sigma(1)}; \dots; v_{\sigma(k)}, x_{\sigma(k)}), \zeta_1) u \rangle \\
&\quad \langle w', Y_{WV}^W(\mathcal{F}(v'_{\sigma(1)}, y_{\sigma(1)}; \dots; v'_{\sigma(n)}, y_{\sigma(n)}), \zeta_2) \bar{u} \rangle \\
&= \sum_{u \in V} \sum_{\sigma \in J_{k+n;s}^{-1}} (-1)^{|\sigma|} \langle w', e^{\zeta_1 L_W(-1)} Y_W(u, -\zeta_1) \mathcal{F}(v_{\sigma(1)}, x_{\sigma(1)}; \dots; v_{\sigma(k)}, x_{\sigma(k)}) \rangle \\
&\quad \langle w', e^{\zeta_2 L_W(-1)} Y_W(\bar{u}, -\zeta_2) \mathcal{F}(v'_{\sigma(1)}, y_{\sigma(1)}; \dots; v'_{\sigma(n)}, y_{\sigma(n)}) \rangle \\
&= \sum_{u \in V} \langle w', e^{\zeta_1 L_W(-1)} Y_W(u, -\zeta_1) \sum_{\sigma \in J_{k;s}^{-1}} (-1)^{|\sigma|} \mathcal{F}(v_{\sigma(1)}, x_{\sigma(1)}; \dots; v_{\sigma(k)}, x_{\sigma(k)}) \rangle \\
&\quad \langle w', e^{\zeta_2 L_W(-1)} Y_W(\bar{u}, -\zeta_2) \mathcal{F}(v'_{\sigma(1)}, y_{\sigma(1)}; \dots; v'_{\sigma(n)}, y_{\sigma(n)}) \rangle \\
&+ \sum_{u \in V} \langle w', e^{\zeta_1 L_W(-1)} Y_W(u, -\zeta_1) \mathcal{F}(v_{\sigma(1)}, x_{\sigma(1)}; \dots; v_{\sigma(k)}, x_{\sigma(k)}) \rangle \\
&\quad \langle w', e^{\zeta_2 L_W(-1)} Y_W(\bar{u}, -\zeta_2) \sum_{\sigma \in J_{n;s}^{-1}} (-1)^{|\sigma|} \mathcal{F}(v'_{\sigma(1)}, y_{\sigma(1)}; \dots; v'_{\sigma(n)}, y_{\sigma(n)}) \rangle = 0,
\end{aligned}$$

since,  $J_{k+n;s}^{-1} = J_{k;s}^{-1} \times J_{n;s}^{-1}$ , and due to the fact that  $\mathcal{F}(v_1, x_1; \dots; v_k, x_k)$  and  $\mathcal{F}(v'_1, y_1; \dots; v'_n, y_n)$  satisfy (2.9).

Next, we show that  $\mathcal{F}(v_1, x_1; \dots; v_k, x_k; v'_1, y_1; \dots; v'_n, y_n; \epsilon)$  (6.1) is composable with  $m + m'$  vertex operators. Recall that  $\mathcal{F}(v_1, x_1; \dots; v_k, x_k)$  is composable with  $m$  vertex operators, and  $\mathcal{F}(v'_1, y_1; \dots; v'_n, y_n)$  is composable with  $m'$  vertex operators. For  $\mathcal{F}(v_1, x_1; \dots; v_k, x_k)$  we have:

1) Let  $l_1, \dots, l_k \in \mathbb{Z}_+$  such that  $l_1 + \dots + l_k = k + m$ , and  $v_1, \dots, v_{k+m} \in V$ , and arbitrary  $w' \in W'$ . Set

$$\Psi_i = E_V^{(l_i)}(v_{k_1}, x_{k_1} - \zeta_i; \dots; v_{k_i}, x_{k_i} - \zeta_i; \mathbf{1}_V), \quad (6.2)$$

where

$$k_1 = l_1 + \dots + l_{i-1} + 1, \quad \dots, \quad k_i = l_1 + \dots + l_{i-1} + l_i, \quad (6.3)$$

for  $i = 1, \dots, k$ . Then the series

$$\mathcal{I}_m^k(\mathcal{F}) = \sum_{r_1, \dots, r_k \in \mathbb{Z}} \langle w', \mathcal{F}(P_{r_1} \Psi_1; \zeta_1; \dots; P_{r_k} \Psi_k, \zeta_k) \rangle, \quad (6.4)$$

is absolutely convergent when

$$|x_{l_1 + \dots + l_{i-1} + p} - \zeta_i| + |x_{l_1 + \dots + l_{j-1} + q} - \zeta_i| < |\zeta_i - \zeta_j|, \quad (6.5)$$

for  $i, j = 1, \dots, k$ ,  $i \neq j$  and for  $p = 1, \dots, l_i$  and  $q = 1, \dots, l_j$ . There exist positive integers  $N_m^k(v_i, v_j)$ , depending only on  $v_i$  and  $v_j$  for  $i, j = 1, \dots, k$ ,  $i \neq j$ , such that the sum is analytically extended to a rational function in  $(x_1, \dots, x_{k+m})$ , independent

of  $(\zeta_1, \dots, \zeta_k)$ , with the only possible poles at  $x_i = x_j$ , of order less than or equal to  $N_m^k(v_i, v_j)$ , for  $i, j = 1, \dots, k, i \neq j$ .

For  $\mathcal{F}(v'_1, y_1; \dots; v'_n, y_n)$  we have:

1') Let  $l'_1, \dots, l'_n \in \mathbb{Z}_+$  such that  $l'_1 + \dots + l'_n = n + m'$ ,  $v'_1, \dots, v'_{n+m'} \in V$  and arbitrary  $w' \in W'$ . Set

$$\Psi'_{i'} = E_V^{(l'_{i'})}(v'_{k'_{i'}}, y_{k'_{i'}} - \zeta'_{i'}; \dots; v'_{k'_{i'}}, y_{k'_{i'}} - \zeta'_{i'}; \mathbf{1}_V), \quad (6.6)$$

where

$$k'_1 = l'_1 + \dots + l'_{i'-1} + 1, \quad \dots, \quad k'_{i'} = l'_1 + \dots + l'_{i'-1} + l'_{i'}, \quad (6.7)$$

for  $i' = 1, \dots, n$ . Then the series

$$\mathcal{I}_{m'}^n(\mathcal{F}) = \sum_{r'_1, \dots, r'_n \in \mathbb{Z}} \langle w', \mathcal{F}(P_{r'_1} \Psi'_1; \zeta'_1; \dots; P_{r'_n} \Psi'_n, \zeta'_n) \rangle, \quad (6.8)$$

is absolutely convergent when

$$|y_{l'_1 + \dots + l'_{j'-1} + p'} - \zeta'_{i'}| + |y_{l'_1 + \dots + l'_{j'-1} + q'} - \zeta'_{i'}| < |\zeta'_{i'} - \zeta'_{j'}|, \quad (6.9)$$

for  $i', j' = 1, \dots, n, i' \neq j'$  and for  $p' = 1, \dots, l'_i$  and  $q' = 1, \dots, l'_j$ . There exist positive integers  $N_{m'}^n(v'_{i'}, v'_{j'})$ , depending only on  $v'_{i'}$  and  $v'_{j'}$ , for  $i, j = 1, \dots, n, i' \neq j'$ , such that the sum is analytically extended to a rational function in  $(y_1, \dots, y_{n+m'})$ , independent of  $(\zeta'_1, \dots, \zeta'_n)$ , with the only possible poles at  $y_{i'} = y_{j'}$ , of order less than or equal to  $N_{m'}^n(v'_{i'}, v'_{j'})$ , for  $i', j' = 1, \dots, n, i' \neq j'$ .

Now let us consider the first condition of Definition 9 of composability for the product (6.1) of  $\mathcal{F}(v_1, x_1; \dots; v_k, x_k)$  and  $\mathcal{F}(v'_1, y_1; \dots; v'_n, y_n)$  with a number of vertex operators. Then we obtain for  $\mathcal{F}(v_1, x_1; \dots; v_k, x_k; v'_1, y_1; \dots; v'_n, y_n; \epsilon)$  the following. We redefine the notations for the set

$$\begin{aligned} & (v''_1, \dots, v''_k; v''_{k+1}, \dots, v''_{k+m}; v''_{k+m+1}, \dots, v''_{k+n+m+m'}; v_{n+1}, \dots, v'_{n+m'}) \\ & = (v_1, \dots, v_k; v_{k+1}, \dots, v_{k+m}; v'_1, \dots, v'_n; v'_{n+1}, \dots, v'_{n+m'}), \\ & (z_1, \dots, z_k; z_{k+1}, \dots, z_{k+n}) = (x_1, \dots, x_k; y_1, \dots, y_n), \end{aligned}$$

of vertex algebra  $V$  elements. Introduce  $l''_1, \dots, l''_{k+n} \in \mathbb{Z}_+$ , such that  $l''_1 + \dots + l''_{k+n} = k + n + m + m'$ . Define

$$\Psi''_i = E_V^{(l''_i)}(v''_{k'_{i''}}, z_{k'_{i''}} - \zeta''_{i''}; \dots; v''_{k'_{i''}}, z_{k'_{i''}} - \zeta''_{i''}; \mathbf{1}_V), \quad (6.10)$$

where

$$k''_1 = l''_1 + \dots + l''_{i''-1} + 1, \quad \dots, \quad k''_{i''} = l''_1 + \dots + l''_{i''-1} + l''_{i''}, \quad (6.11)$$

for  $i'' = 1, \dots, k + n$ , and we take

$$(\zeta''_1, \dots, \zeta''_{k+n}) = (\zeta_1, \dots, \zeta_k; \zeta'_1, \dots, \zeta'_n).$$

Then we consider

$$\mathcal{I}_{m+m'}^{k+n}(\mathcal{F}) = \sum_{r''_1, \dots, r''_{k+n} \in \mathbb{Z}} \langle w', \mathcal{F}(P_{r''_1} \Psi''_1; \zeta''_1; \dots; P_{r''_{k+n}} \Psi''_{k+n}, \zeta''_{k+n}) \rangle, \quad (6.12)$$

and prove it is absolutely convergent with some conditions.

The condition

$$|z_{l_1''+\dots+l_{i-1}''+p''} - \zeta_i''| + |z_{l_1''+\dots+l_{j-1}''+q''} - \zeta_j''| < |\zeta_i'' - \zeta_j''|, \quad (6.13)$$

of absolute convergence for (6.12) for  $i'', j'' = 1, \dots, k+n$ ,  $i \neq j$  and for  $p'' = 1, \dots, l_i''$  and  $q'' = 1, \dots, l_j''$ , follows from the conditions (6.5) and (6.19). The action of  $e^{\zeta^{L\mathcal{W}(-1)}} Y_W(\cdot, \cdot)$ ,  $a = 1, 2$ , in

$$\begin{aligned} \langle w', e^{\zeta_1^{L\mathcal{W}(-1)}} Y_W(u, -\zeta) \sum_{r_1, \dots, r_k \in \mathbb{Z}} \mathcal{F}(P_{r_1} \Psi_1; \zeta_1; \dots; P_{r_k} \Psi_k, \zeta_k) \rangle, \\ \langle w', e^{\zeta_2^{L\mathcal{W}(-1)}} Y_W(\bar{u}, -\tilde{\zeta}) \sum_{r'_1, \dots, r'_n \in \mathbb{Z}} \mathcal{F}(P_{r'_1} \Psi'_1; \zeta_1; \dots; P_{r'_n} \Psi'_n, \zeta'_n) \rangle, \end{aligned}$$

does not affect the absolute convergency of (6.4) and (6.8). We obtain

$$\begin{aligned} |\mathcal{I}_{m+m'}^{k+n}(\mathcal{F})| &= \\ &= \left| \sum_{r''_1, \dots, r''_{k+n} \in \mathbb{Z}} \langle w', \mathcal{F}(P_{r''_1} \Psi''_1; \zeta''_1; \dots; P_{r''_{k+n}} \Psi''_{k+n}, \zeta''_{k+n}) \rangle \right| \\ &= \left| \sum_{u \in V} \langle w', Y_{V\mathcal{W}}^W \left( \sum_{r_1, \dots, r_k \in \mathbb{Z}} \mathcal{F}(P_{r_1} \Psi_1; \zeta_1; \dots; P_{r_k} \Psi_k, \zeta_k), \zeta \right) u \right. \\ &\quad \left. \langle w', Y_{V\mathcal{W}}^W \left( \sum_{r'_1, \dots, r'_n \in \mathbb{Z}} \mathcal{F}(P_{r'_1} \Psi'_1; \zeta'_1; \dots; P_{r'_n} \Psi'_n, \zeta'_n), \tilde{\zeta} \right) \bar{u} \right| \\ &\leq |\mathcal{I}_m^k(\mathcal{F})| |\mathcal{I}_{m'}^n(\mathcal{F})|. \end{aligned}$$

Thus, we infer that (6.12) is absolutely convergent. Recall that the maximal orders of possible poles of (6.12) are  $N_m^k(v_i, v_j)$ ,  $N_{m'}^n(v'_{i'}, v'_{j'})$  at  $x_i = x_j$ ,  $y_{i'} = y_{j'}$ . From the last expression we infer that there exist positive integers  $N_{m+m'}^{k+n}(v''_{i''}, v''_{j''})$  for  $i, j = 1, \dots, k$ ,  $i \neq j$ ,  $i', j' = 1, \dots, n$ ,  $i' \neq j'$ , depending only on  $v''_{i''}$  and  $v''_{j''}$  for  $i'', j'' = 1, \dots, k+n$ ,  $i'' \neq j''$  such that the series (6.12) can be analytically extended to a rational function in  $(x_1, \dots, x_k; y_1, \dots, y_n)$ , independent of  $(\zeta''_1, \dots, \zeta''_{k+n})$ , with extra possible poles at and  $x_i = y_j$ , of order less than or equal to  $N_{m+m'}^{k+n}(v''_{i''}, v''_{j''})$ , for  $i'', j'' = 1, \dots, n$ ,  $i'' \neq j''$ .

Let us proceed with the second condition of composability. For  $\mathcal{F}(v_1, x_1; \dots; v_k, x_k) \in C_m^k(V, \mathcal{W})$ , and  $(v_1, \dots, v_{k+m}) \in V$ ,  $(x_1, \dots, x_{k+m}) \in \mathbb{C}$ , we have

2) For arbitrary  $w' \in W'$ , the series

$$\mathcal{J}_m^k(\mathcal{F}) = \sum_{q \in \mathbb{C}} \langle w', E_W^{(m)} \left( v_1, x_1; \dots; v_m, x_m; P_q(\mathcal{F}(v_{m+1}, x_{m+1}; \dots; v_{m+k}, x_{m+k})) \right) \rangle, \quad (6.14)$$

is absolutely convergent when

$$\begin{aligned} x_i \neq x_j, \quad i \neq j, \\ |x_i| > |x_{k'}| > 0, \end{aligned} \quad (6.15)$$

for  $i = 1, \dots, m$ , and  $k' = m+1, \dots, k+m$ , and the sum can be analytically extended to a rational function in  $(x_1, \dots, x_{k+m})$  with the only possible poles at  $x_i = x_j$ , of orders less than or equal to  $N_m^k(v_i, v_j)$ , for  $i, j = 1, \dots, k$ ,  $i \neq j$ .

2') For  $\mathcal{F}(v'_1, y_1; \dots; v'_n, y_n) \in C_{m'}^n(V, \mathcal{W})$ ,  $(v'_1, \dots, v'_{n+m'}) \in V$ , and  $(y_1, \dots, y_{n+m'}) \in \mathbb{C}$ , the series

$$\begin{aligned} \mathcal{J}_{m'}^n(\mathcal{F}) &= \sum_{q \in \mathbb{C}} \langle w', E_W^{(m')} (v'_1, y_1; \dots; v'_{m'}, y_{m'}; \\ &P_q(\mathcal{F}(v'_{m'+1}, y_{m'+1}; \dots; v'_{m'+n}, y_{m'+n}))) \rangle, \end{aligned} \quad (6.16)$$

is absolutely convergent when

$$\begin{aligned} y_{i'} &\neq y_{j'}, \quad i' \neq j', \\ |y_{i'}| &> |y_{j''}| > 0, \end{aligned} \quad (6.17)$$

for  $i' = 1, \dots, m'$ , and  $k'' = m'+1, \dots, n+m'$ , and the sum can be analytically extended to a rational function in  $(y_1, \dots, y_{n+m'})$  with the only possible poles at  $y_{i'} = y_{j'}$ , of orders less than or equal to  $N_{m'}^n(v'_{i'}, v'_{j'})$ , for  $i', j' = 1, \dots, n$ ,  $i' \neq j'$ .

2'') Thus, for the product (6.1) we obtain  $(v''_1, \dots, v''_{k+n+m+m'}) \in V$ , and  $(z_1, \dots, z_{k+n+m+m'}) \in \mathbb{C}$ , we find positive integers  $N_{m+m'}^{k+n}(v''_i, v''_j)$ , depending only on  $v''_i$  and  $v''_j$ , for  $i'', j'' = 1, \dots, k+n$ ,  $i'' \neq j''$ , such that for arbitrary  $w' \in W'$ . First we note

**Lemma 3.**

$$\begin{aligned} &\sum_{q \in \mathbb{C}} \langle w', E_W^{(m+m')} (v''_1, z_1; \dots; v''_{m+m'}, z_{m+m'}; \\ &P_q(\mathcal{F}(v''_{m+m'+1}, z_{m+m'+1}; \dots; v''_{m+m'+k+n}, z_{m+m'+k+n}))) \rangle \\ &= \sum_{u \in V} \langle w', E_W^{(m)} (v_{k+1}, x_{k+1}; \dots; v_{k+m}, x_{k+m}; \\ &P_q(Y_{WV}^W(\mathcal{F}(v_1, x_1; \dots; v_k, x_k), \zeta_1) u)) \rangle \\ &\langle w', E_W^{(m')} (v'_{n+1}, y_{n+1}; \dots; v'_{n+m'}, y_{n+m'}; \\ &P_q(Y_{WV}^W(\mathcal{F}(v'_1, y_1; \dots; v'_n, y_n), \zeta_2) \bar{u})) \rangle. \end{aligned}$$

*Proof.* Consider

$$\begin{aligned} &\sum_{u \in V} \langle w', E_W^{(m+m')} (v''_1, z_1; \dots; v''_{m+m'}, z_{m+m'}; \\ &P_q(Y_{WV}^W(\mathcal{F}(v''_{m+m'+1}, z_{m+m'+1}; \dots; v''_{m+m'+k}, z_{m+m'+k}), \zeta_1) u)) \rangle \\ &\langle w', E_W^{(m+m')} (v''_1, z_1; \dots; v''_{m+m'}, z_{m+m'}; \\ &P_q(Y_{WV}^W(\mathcal{F}(v''_{m+m'+k+1}, z_{m+m'+k+1}; \dots; \\ &v''_{m+m'+k+n}, z_{m+m'+k+n}), \zeta_2) \bar{u})) \rangle \end{aligned}$$



$$\begin{aligned}
&= \sum_{q \in \mathbb{C}} \sum_{u \in V} \langle w', E_W^{(m+m')} \left( v_1'', z_1; \dots; v_{m+m'}'', z_{m+m'}; \right. \\
&\quad \left. P_q \left( e^{\zeta_1 L_W(-1)} Y_W(u, -\zeta_1) \mathcal{F}(v_{m+m'+1}'', z_{m+m'+1}; \dots; v_{m+m'+k}'', z_{m+m'+k}) \right) \right) \rangle \\
&\quad \langle w', E_W^{(m+m')} \left( v_1'', z_1; \dots; v_{m+m'}'', z_{m+m'}; \right. \\
&\quad \left. P_q \left( e^{\zeta_2 L_W(-1)} Y_W(\bar{u}, -\zeta_2) \mathcal{F}(v_{m+m'+k+1}'', z_{m+m'+k+1}; \dots; \right. \right. \\
&\quad \quad \left. \left. v_{m+m'+k+n}'', z_{m+m'+k+n}) \right) \right) \rangle.
\end{aligned}$$

The action of exponentials  $e^{\zeta_a L_W(-1)}$ ,  $a = 1, 2$ , of the differential operator  $L_W(-1)$ , and  $W$ -module vertex operators  $Y_W(u, -\zeta_1)$ ,  $Y_W(u, -\zeta_2)$  shifts the grading index  $q$  of  $W_q$ -subspaces by  $\alpha \in \mathbb{C}$  which can be later rescaled to  $q$ . Thus, we can rewrite the last expression as

$$\begin{aligned}
&= \sum_{q \in \mathbb{C}} \sum_{u \in V} \langle w', E_W^{(m+m')} \left( v_1'', z_1; \dots; v_{m+m'}'', z_{m+m'}; \right. \\
&\quad \left. e^{\zeta_1 L_W(-1)} Y_W(u, -\zeta_1) P_{q+\alpha} \left( \mathcal{F}(v_{m+m'+1}'', z_{m+m'+1}; \dots; v_{m+m'+k}'', z_{m+m'+k}) \right) \right) \rangle \\
&\quad \langle w', E_W^{(m+m')} \left( v_1'', z_1; \dots; v_{m+m'}'', z_{m+m'}; \right. \\
&\quad \left. e^{\zeta_2 L_W(-1)} Y_W(\bar{u}, -\zeta_2) P_{q+\alpha} \left( \mathcal{F}(v_{m+m'+k+1}'', z_{m+m'+k+1}; \dots; v_{m+m'+k+n}'', z_{m+m'+k+n}) \right) \right) \rangle \\
&= \sum_{q \in \mathbb{C}} \sum_{u \in V} \langle w', E_W^{(m+m')} \left( v_1'', z_1; \dots; v_{m+m'}'', z_{m+m'}; \right. \\
&\quad \left. Y_{WV}^W \left( P_{q+\alpha} \left( \mathcal{F}(v_{m+m'+1}'', z_{m+m'+1}; \dots; v_{m+m'+k}'', z_{m+m'+k}) \right), \zeta_1 \right) u \right) \rangle \\
&\quad \langle w', E_W^{(m+m')} \left( v_1'', z_1; \dots; v_{m+m'}'', z_{m+m'}; \right. \\
&\quad \left. Y_{WV}^W \left( P_{q+\alpha} \left( \mathcal{F}(v_{m+m'+k+1}'', z_{m+m'+k+1}; \dots; v_{m+m'+k+n}'', z_{m+m'+k+n}), -\zeta_2 \right) \bar{u} \right) \right) \rangle \\
&= \sum_{q \in \mathbb{C}} \sum_{\tilde{w} \in W} \langle w', E_W^{(m+m')} \left( v_1'', z_1; \dots; v_{m+m'}'', z_{m+m'}; \tilde{w} \right) \rangle \\
&\quad \sum_{u \in V} \langle w', Y_{WV}^W \left( P_{q+\alpha} \left( \mathcal{F}(v_{m+m'+1}'', z_{m+m'+1}; \dots; v_{m+m'+k}'', z_{m+m'+k}), -\zeta_1 \right) u \right) \rangle \\
&\quad \langle \tilde{w}', E_W^{(m+m')} \left( v_1'', z_1; \dots; v_{m+m'}'', z_{m+m'}; \tilde{w} \right) \rangle \\
&\quad \langle w', Y_{WV}^W \left( P_{q+\alpha} \left( \mathcal{F}(v_{m+m'+k+1}'', z_{m+m'+k+1}; \dots; v_{m+m'+k+n}'', z_{m+m'+k+n}), -\zeta_2 \right) \bar{u} \right) \rangle
\end{aligned}$$

$$= \sum_{q \in \mathbb{C}} \langle w', E_W^{(m+m')} \left( v''_1, z_1; \dots; v''_{m+m'}, z_{m+m'}; \right. \\ \left. P_{q+\alpha} \left( \mathcal{F}(v''_{m+m'+1}, z_{m+m'+1}; \dots; v''_{m+m'+k}, z_{m+m'+k}; \right. \right. \\ \left. \left. v''_{m+m'+k+1}, z_{m+m'+k+1}; \dots; v''_{m+m'+k+n}, z_{m+m'+k+n} \right) \right) \rangle.$$

Now note that, according to Proposition 6, as an element of  $\mathcal{W}_{z_1, \dots, z_{k+n+m+m'}}$

$$\langle w', E_W^{(m+m')} \left( v''_1, z_1; \dots; v''_{m+m'}, z_{m+m'}; \right. \\ \left. P_{q+\alpha} \left( \mathcal{F}(v''_{m+m'+1}, z_{m+m'+1}; \dots; v''_{m+m'+k}, z_{m+m'+k}; \right. \right. \\ \left. \left. v''_{m+m'+k+1}, z_{m+m'+k+1}; \dots; v''_{m+m'+k+n}, z_{m+m'+k+n} \right) \right) \rangle, \quad (6.18)$$

is invariant with respect to the action of  $\sigma \in S_{k+n+m+m'}$ . Thus we are able to use this invariance to show that (6.18) is reduced to

$$\langle w', E_W^{(m+m')} \left( v''_{k+1}, z_{k+1}; \dots; v''_{k+1+m}, z_{k+1+m}; v''_{n+1}, z_{n+1}; \dots; v''_{n+1+m'}, z_{n+1+m'}; \right. \\ \left. P_{q+\alpha} \left( \mathcal{F}(v''_1, z_1; \dots; v''_k, z_k; v''_{k+1}, z_{k+1}; \dots; v''_{k+n}, z_{k+n}) \right) \right) \rangle \\ = \langle w', E_W^{(m+m')} \left( v_{k+1}, x_{k+1}; \dots; v_{k+1+m}, x_{k+1+m}; v'_{n+1}, y_{n+1}; \dots; v'_{n+1+m'}, y_{n+1+m'}; \right. \\ \left. P_{q+\alpha} \left( \mathcal{F}(v_1, x_1; \dots; v_k, x_k; v'_1, y_1; \dots; v'_n, y_n) \right) \right) \rangle.$$

Similarly, since

$$\langle w', E_W^{(m)} \left( v''_1, z_1; \dots; v''_{m+m'}, z_{m+m'}; \right. \\ \left. P_q \left( Y_{WV}^W \left( \mathcal{F}(v''_{m+m'+1}, z_{m+m'+1}; \dots; v''_{m+m'+k}, z_{m+m'+k}), \zeta_1 \right) u \right) \right) \rangle, \\ \langle w', E_W^{(m')} \left( v''_1, z_1; \dots; v''_{m+m'}, z_{m+m'}; \right. \\ \left. P_q \left( Y_{WV}^W \left( \mathcal{F}(v''_{m+m'+k+1}, z_{m+m'+k+1}; \dots; v''_{m+m'+k+n}, z_{m+m'+k+n}), \zeta_2 \right) \bar{u} \right) \right) \rangle.$$

correspond to elements of  $\mathcal{W}_{z_1, \dots, z_{m+m'+k}}$  and  $\mathcal{W}_{z_{m+m'+k+1}, \dots, z_{m+m'+k+n}}$ , we use Proposition 6 again and obtain

$$\langle w', E_W^{(m)} \left( v_{k+1}, x_{k+1}; \dots; v_{k+m}, x_{k+m}; P_q \left( Y_{WV}^W \left( \mathcal{F}(v_1, x_1; \dots; v_k, x_k), \zeta_1 \right) u \right) \right) \rangle \\ \langle w', E_W^{(m')} \left( v'_{n+1}, y_{n+1}; \dots; v'_{n+m'}, y_{n+m'}; P_q \left( Y_{WV}^W \left( \mathcal{F}(v'_1, y_1; \dots; v'_n, y_n), \zeta_2 \right) \bar{u} \right) \right) \rangle,$$

correspondingly. Thus, the assertion of Lemma follows.  $\square$

Under conditions

$$z_{i''} \neq z_{j''}, \quad i'' \neq j'', \\ |z_{i''}| > |z_{k''}| > 0, \quad (6.19)$$

for  $i'' = 1, \dots, m + m'$ , and  $k''' = m + m' + 1, \dots, m + m' + k + n$ , let us introduce

$$\begin{aligned} \mathcal{J}_{m+m'}^{k+n}(\mathcal{F}) &= \sum_{q \in \mathbb{C}} \langle w', E_W^{(m+m')} (v_1'', z_1; \dots; v_{m+m'}'', z_{m+m'}; \\ &\quad P_q(\mathcal{F}(v_{m+m'+1}'', z_{m+m'+1}; \dots; v_{m+m'+k+n}'', z_{m+m'+k+n}); \epsilon) \rangle. \end{aligned} \quad (6.20)$$

Using Lemma 3 we obtain

$$\begin{aligned} &|\mathcal{J}_{m+m'}^{k+n}(\mathcal{F})| \\ &= \left| \sum_{q \in \mathbb{C}} \langle w', E_W^{(m+m')} (v_1'', z_1; \dots; v_{m+m'}'', z_{m+m'}; \\ &\quad P_q(\mathcal{F}(v_{m+m'+1}'', z_{m+m'+1}; \dots; v_{m+m'+k+n}'', z_{m+m'+k+n}); \epsilon) \rangle \right| \\ &= \left| \sum_{q \in \mathbb{C}} \sum_{u \in V} \langle w', E_W^{(m)} (v_{k+1}, x_{k+1}; \dots; v_{k+m}, x_{k+m}; \\ &\quad P_q(Y_{WV}^W(\mathcal{F}(v_1, x_1; \dots; v_k, x_k), \zeta_1) u) \rangle \right. \\ &\quad \left. \langle w', E_W^{(m')} (v_{n+1}', y_{n+1}; \dots; v_{n+m'}', y_{n+m'}; \\ &\quad P_q(Y_{WV}^W(\mathcal{F}(v_1', y_1; \dots; v_n', y_n), \zeta_2) \bar{u}) \rangle \right| \\ &\leq |\mathcal{J}_m^k(\mathcal{F})| |\mathcal{J}_{m'}^n(\mathcal{F})|, \end{aligned}$$

where we have used the invariance of (6.1) with respect to  $\sigma \in S_{m+m'+k+n}$ . According to Proposition 8  $\mathcal{J}_m^k(\mathcal{F})$  and  $\mathcal{J}_{m'}^n(\mathcal{F})$  in the last expression are absolute convergent. Thus, we infer that  $\mathcal{J}_{m+m'}^{k+n}(\mathcal{F})$  is absolutely convergent, and the sum (6.12) is analytically extendable to a rational function in  $(z_1, \dots, z_{k+n+m+m'})$  with the only possible poles at  $x_i = x_j$ ,  $y_i' = y_j'$ , and at  $x_i = y_{j'}$ , i.e., the only possible poles at  $z_{i''} = z_{j''}$ , of orders less than or equal to  $N_{m+m'}^{k+n}(v_{i''}', v_{j''}')$ , for  $i'', j'' = 1, \dots, k''', i'' \neq j''$ . This finishes the proof of the proposition.  $\square$

Now we prove the following

**Corollary 1.** For  $\mathcal{F}(v_1, x_1; \dots; v_k, x_k) \in C_m^k(V, \mathcal{W})$  and  $\mathcal{F}(v_1', y_1; \dots; v_n', y_n) \in C_{m'}^n(V, \mathcal{W})$ , the product

$$\begin{aligned} &\mathcal{F}(v_1, x_1; \dots; v_k, x_k; v_1', y_1; \dots; v_n', y_n; \epsilon) \\ &= \mathcal{F}(v_1, x_1; \dots; v_k, x_k) \cdot_\epsilon \mathcal{F}(v_1', y_1; \dots; v_n', y_n), \end{aligned} \quad (6.21)$$

is canonical with respect to the action

$$\begin{aligned} &(x_1, \dots, x_k; y_1, \dots, y_n) \mapsto (x_1', \dots, x_k'; y_1', \dots, y_n') \\ &= (\rho_1(x_1, \dots, x_k; y_1, \dots, y_n), \dots, \rho_{k+n}(x_1, \dots, x_k; y_1, \dots, y_n)), \end{aligned} \quad (6.22)$$

of elements the group  $\text{Aut}_{x_1, \dots, x_k; y_1, \dots, y_n} \mathcal{O}^{(k+n)}$ .

*Proof.* In Subsection 4 we have proved that the product (3.2) belongs to  $W_{x_1, \dots, x_k; y_1, \dots, y_n}$ , and is invariant with respect to the group  $\text{Aut}_{x_1, \dots, x_k; y_1, \dots, y_n} \mathcal{O}^{(k+n)}$ . Similar as in the proof of Proposition 10, vertex operators  $\omega_V(v_i, x_i)$ ,  $1 \leq i \leq m$ , composable with  $\mathcal{F}(v_1, x_1; \dots; v_k, x_k)$ , and vertex operators  $\omega_V(v_j, y_j)$ ,  $1 \leq j \leq m'$ , composable with  $\mathcal{F}(v'_1, y_1; \dots; v'_n, y_n)$ , are also invariant with respect to  $(\rho_1(x_1, \dots, x_k; y_1, \dots, y_n), \dots, \rho_{k+n}(x_1, \dots, x_k; y_1, \dots, y_n)) \in \text{Aut}_{x_1, \dots, x_k; y_1, \dots, y_n} \mathcal{O}^{(k+n)}$ .  $\square$

Since the product of  $\mathcal{F}(v_1, x_1; \dots; v_k, x_k) \in C_m^k(V, \mathcal{W})$  and  $\mathcal{F}(v'_1, y_1; \dots; v'_n, y_n) \in C_{m'}^n(V, \mathcal{W})$  results in an element of  $C_{m+m'}^{k+n}(V, \mathcal{W})$ , then, similar to Proposition 9 [6], the following corollary follows directly from Proposition (12) and Definition 8:

**Corollary 2.** *For the spaces  $\mathcal{W}_{x_1, \dots, x_k}$  and  $\mathcal{W}_{y_1, \dots, y_n}$  with the product (3.2)  $\mathcal{F} \in \mathcal{W}_{x_1, \dots, x_k; y_1, \dots, y_n}$ , the subspace of  $\text{Hom}(V^{\otimes n}, \mathcal{W}_{x_1, \dots, x_k; y_1, \dots, y_n})$  consisting of linear maps having the  $L_W(-1)$ -derivative property, having the  $L_V(0)$ -conjugation property or being composable with  $m$  vertex operators is invariant under the action of  $S_{k+n}$ .*

Finally, we have the following

**Corollary 3.** *For a fixed set  $(v_1, \dots, v_k; v_{k+1}, \dots, v_{k+n}) \in V$  of vertex algebra elements, and fixed  $k+n$ , and  $m+m'$ , the  $\epsilon$ -product  $\mathcal{F}(v_1, z_1; \dots; v_k, z_k; v_{k+1}, z_{k+1}; \dots; v_{k+n}, y_{k+n}; \epsilon)$ ,*

$$\cdot_\epsilon : C_m^k(V, \mathcal{W}) \times C_{m'}^n(V, \mathcal{W}) \rightarrow C_{m+m'}^{k+n}(V, \mathcal{W}),$$

*of the spaces  $C_m^k(V, \mathcal{W})$  and  $C_{m'}^n(V, \mathcal{W})$ , for all choices of  $k, n, m, m' \geq 0$ , is the same element of  $C_{m+m'}^{k+n}(V, \mathcal{W})$  for all possible  $k \geq 0$ .*

*Proof.* In Proposition 4 we have proved that the result of the maps belongs to  $W_{x_1, \dots, x_k; y_1, \dots, y_n}$ , for all  $k, n \geq 0$ , and fixed  $k+n$ . As in proof of Proposition 12, by checking conditions for the forms (6.4) and (6.8), we see by Proposition 7, the product  $\mathcal{F}(v_1, x_1; \dots; v_k, x_k; v'_1, y_1; \dots; v'_n, y_n)$  is composable with fixed  $m+m'$ .  $\square$

**6.1. Coboundary operator acting on the product space.** In Proposition 12 we proved that the product (6.1) of elements of spaces  $C_m^k(V, \mathcal{W})$  and  $C_{m'}^n(V, \mathcal{W})$  belongs to  $C_{m+m'}^{k+n}(V, \mathcal{W})$ . Thus, the product admits the action of the differential operator  $\delta_{m+m'}^{k+n}$  defined in (5.12). The co-boundary operator (5.12) possesses a variation of Leibniz law with respect to the product (6.1). Indeed, we state here

**Proposition 13.** *For  $\mathcal{F}(v_1, x_1; \dots; v_k, x_k) \in C_m^k(V, \mathcal{W})$  and  $\mathcal{F}(v'_1, y_1; \dots; v'_n, y_n) \in C_{m'}^n(V, \mathcal{W})$ , the action of  $\delta_{m+m'}^{k+n}$  on their product (6.1) is given by*

$$\begin{aligned} & \delta_{m+m'}^{k+n} (\mathcal{F}(v_1, x_1; \dots; v_k, x_k) \cdot_\epsilon \mathcal{F}(v'_1, y_1; \dots; v'_n, y_n)) \\ &= (\delta_m^k \mathcal{F}(v_1, x_1; \dots; v_k, x_k)) \cdot_\epsilon \mathcal{F}(v'_1, y_1; \dots; v'_n, y_n) \\ &+ (-1)^k \mathcal{F}(v_1, x_1; \dots; v_k, x_k) \cdot_\epsilon (\delta_{m'}^n \mathcal{F}(v'_1, y_1; \dots; v'_n, y_n)). \end{aligned} \quad (6.23)$$

*Remark 6.* Checking (5.12) we see that an extra arbitrary vertex algebra element  $v_{n+1} \in V$ , as well as corresponding extra arbitrary formal parameter  $z_{n+1}$  appear as a result of the action of  $\delta_m^n$  on  $\mathcal{F} \in C_m^n(V, \mathcal{W})$  mapping it to  $C_{m-1}^{n+1}(V, \mathcal{W})$ . In application to the  $\epsilon$ -product (6.1) these extra arbitrary elements are involved in the definition of the action of  $\delta_{m+m'}^{k+n}$  on  $\mathcal{F}(v_1, x_1; \dots; v_k, x_k) \cdot_\epsilon \mathcal{F}(v'_1, y_1; \dots; v'_n, y_n)$ .

*Proof.* According to (5.12), the action of  $\delta_{m+m'}^{k+n}$  on  $\mathcal{F}(v_1, x_1; \dots; v_k, x_k; v'_1, y_1; \dots; v'_n, y_n; \epsilon)$  is given by

$$\begin{aligned}
& \langle w', \delta_{m+m'}^{k+n} \mathcal{F}(v_1, x_1; \dots; v_k, x_k; v'_1, y_1; \dots; v'_n, y_n; \epsilon) \rangle \\
&= \langle w', \sum_{i=1}^k (-1)^i \mathcal{F}(v_1, x_1; \dots; v_{i-1}, x_{i-1}; \omega_V(v_i, x_i - x_{i+1})v_{i+1}, x_{i+1}; v_{i+2}, x_{i+2}; \\
&\quad \dots; v_k, x_k; v'_1, y_1; \dots; v'_n, y_n; \epsilon) \rangle \\
&\quad + \sum_{i=1}^n (-1)^i \langle w', \mathcal{F}(v_1, x_1; \dots; v_k, x_k; v'_1, y_1; \dots; v'_{i-1}, y_{i-1}; \\
&\quad \quad \omega_V(v'_i, y_i - y_{i+1}; v'_{i+1}, y_{i+1}; v'_{i+2}, y_{i+2}; \dots; v'_n, y_n; \epsilon) \rangle \\
&\quad + \langle w', \omega_W(v_1, x_1) \mathcal{F}(v_2, x_2; \dots; v_k, x_k; v'_1, y_1; \dots; v'_n, y_n; \epsilon) \rangle \\
&\quad + \langle w', (-1)^{k+n+1} \omega_W(v'_{n+1}, y_{n+1}) \mathcal{F}(v_1, x_1; \dots; v_k, x_k; v'_1, y_1; \dots; v'_n, y_n; \epsilon) \rangle \\
&= \sum_{u \in V} \langle w', \sum_{i=1}^k (-1)^i Y_{VW}^W(\mathcal{F}(v_1, x_1; \dots; v_{i-1}, x_{i-1}; \omega_V(v_i, x_i - x_{i+1})v_{i+1}, x_{i+1}; \\
&\quad v_{i+2}, x_{i+2}; \dots; v_k, x_k), \zeta_1) u \rangle \langle w', Y_{VW}^W(\mathcal{F}(v'_1, y_1; \dots; v'_n, y_n), \zeta_2) \bar{u} \rangle \\
&\quad + \sum_{u \in V} \sum_{i=1}^n (-1)^i \langle w', Y_{VW}^W(\mathcal{F}(v_1, x_1; \dots; v_k, x_k), \zeta_1) u \rangle \\
&\quad \quad \langle w', Y_{VW}^W(\mathcal{F}(v'_1, y_1; \dots; v'_{i-1}, y_{i-1}; \\
&\quad \quad \omega_V(v'_i, y_i - y_{i+1}; v'_{i+1}, y_{i+1}; v'_{i+2}, y_{i+2}; \dots; v'_n, y_n), \zeta_2) \bar{u} \rangle \\
&\quad + \sum_{u \in V} \langle w', Y_{VW}^W(\omega_W(v_1, x_1) \mathcal{F}(v_2, x_2; \dots; v_k, x_k), \zeta_1) u \rangle \\
&\quad \quad \langle w', Y_{VW}^W(\mathcal{F}(v'_1, y_1; \dots; v'_n, y_n), \zeta_2) \bar{u} \rangle \\
&\quad + \sum_{u \in V} \langle w', Y_{VW}^W((-1)^{k+1} \omega_W(v_{k+1}, x_{k+1}) \mathcal{F}(v_1, x_1; \dots; v_k), \zeta_1) u \rangle \\
&\quad \quad \langle w', Y_{VW}^W(\mathcal{F}(v_k, v'_1, y_1; \dots; v'_n, y_n), \zeta_2) \bar{u} \rangle
\end{aligned}$$

$$\begin{aligned}
& - \sum_{u \in V} \langle w', (-1)^{k+1} \langle w', Y_{VW}^W(\omega_W(v_{k+1}, x_{k+1}) \mathcal{F}(v_1, x_1; \dots; v_k, x_k), \zeta_1) u \rangle \\
& \quad \langle w', Y_{VW}^W(\mathcal{F}(v'_1, y_1; \dots; v'_n, y_n), \zeta_2) \bar{u} \rangle \\
& + \sum_{u \in V} \langle w', Y_{VW}^W(\mathcal{F}(v_1, x_1; \dots; v_k, x_k), \zeta_1) u \rangle \\
& \quad \langle w', Y_{VW}^W(\omega_W(v'_1, y_1) \mathcal{F}(v'_2, y_2; \dots; v'_n, y_n), \zeta_2) \bar{u} \rangle \\
& \sum_{u \in V} \langle w', Y_{VW}^W(\mathcal{F}(v_1, x_1; \dots; v_k, x_k), \zeta_1) \rangle \\
& \quad \langle w', Y_{VW}^W(\omega_W(v'_1, y_1) \mathcal{F}(v'_2, y_2; \dots; v'_n, y_n), \zeta_2) \rangle \\
& = \sum_{u \in V} \langle w', Y_{VW}^W(\delta_m^k \mathcal{F}(v_1, x_1; \dots; v_k, x_k), \zeta_1) u \rangle \\
& \quad \langle w', Y_{VW}^W(\mathcal{F}(v'_1, y_1; \dots; v'_n, y_n), \zeta_2) \bar{u} \rangle \\
& + (-1)^k \sum_{u \in V} \langle w', Y_{VW}^W(\mathcal{F}(v_1, x_1; \dots; v_k, x_k), \zeta_1) u \rangle \\
& \quad \langle w', Y_{VW}^W(\delta_{m'}^n \mathcal{F}(v'_1, y_1; \dots; v'_n, y_n), \zeta_2) \bar{u} \rangle \\
& = \langle w', \delta_m^k \mathcal{F}(v_1, x_1; \dots; v_k, x_k) \cdot_\epsilon \langle w', \mathcal{F}(v'_1, y_1; \dots; v'_n, y_n) \rangle \\
& + (-1)^k \langle w', \mathcal{F}(v_1, x_1; \dots; v_k, x_k) \cdot_\epsilon \delta_{m'}^n \mathcal{F}(v'_1, y_1; \dots; v'_n, y_n) \rangle,
\end{aligned}$$

since,

$$\begin{aligned}
& \sum_{u \in V} \langle w', (-1)^{k+1} Y_{VW}^W(\omega_W(v_{k+1}, x_{k+1}) \mathcal{F}(v_1, x_1; \dots; v_k, x_k), \zeta_1) u \rangle \\
& \quad \langle w', Y_{VW}^W(\mathcal{F}(v'_1, y_1; \dots; v'_n, y_n), \zeta_2) \bar{u} \rangle \\
& = \sum_{u \in V} \langle w', (-1)^{k+1} e^{\zeta_1 L_W(-1)} Y_W(u, -\zeta_1) \omega_W(v_{k+1}, x_{k+1}) \mathcal{F}(v_1, x_1; \dots; v_k, x_k) \rangle \\
& \quad \langle w', Y_{VW}^W(\mathcal{F}(v'_1, y_1; \dots; v'_n, y_n), \zeta_2) \bar{u} \rangle \\
& = \sum_{u \in V} \langle w', (-1)^{k+1} e^{\zeta_1 L_W(-1)} \omega_W(v_{k+1}, x_{k+1}) Y_W(u, -\zeta_1) \mathcal{F}(v_1, x_1; \dots; v_k, x_k) \rangle \\
& \quad \langle w', Y_{VW}^W(\mathcal{F}(v'_1, y_1; \dots; v'_n, y_n), \zeta_2) \bar{u} \rangle \\
& = \sum_{u \in V} \langle w', (-1)^{k+1} \omega_W(v_{k+1}, x_{k+1} + \zeta_1) e^{\zeta_1 L_W(-1)} Y_W(u, -\zeta_1) \mathcal{F}(v_1, x_1; \dots; v_k, x_k) \rangle \\
& \quad \langle w', Y_{VW}^W(\mathcal{F}(v'_1, y_1; \dots; v'_n, y_n), \zeta_2) \bar{u} \rangle \\
& = \sum_{v \in V} \sum_{u \in V} \langle v', (-1)^{k+1} \omega_W(v_{k+1}, x_{k+1} + \zeta_1) w \rangle \\
& \quad \langle w', e^{\zeta_1 L_W(-1)} Y_W(u, -\zeta_1) \mathcal{F}(v_1, x_1; \dots; v_k, x_k) \rangle \\
& \quad \langle w', Y_{VW}^W(\mathcal{F}(v'_1, y_1; \dots; v'_n, y_n), \zeta_2) \bar{u} \rangle
\end{aligned}$$

$$\begin{aligned}
&= \sum_{u \in V} \langle w', e^{\zeta_1 L_W(-1)} Y_W(u, -\zeta_1) \mathcal{F}(v_1, x_1; \dots; v_k, x_k) \rangle \\
&\quad \sum_{v \in V} \langle v', (-1)^{k+1} \omega_W(v_{k+1}, x_{k+1} + \zeta_1) w \rangle \langle w', Y_{VW}^W(\mathcal{F}(v'_1, y_1; \dots; v'_n, y_n), \zeta_2) \bar{u} \rangle \\
&= \sum_{u \in V} \langle w', Y_{VW}^W(\mathcal{F}(v_1, x_1; \dots; v_k, x_k), \zeta_1) u \rangle \\
&\quad \langle w', (-1)^{k+1} \omega_W(v_{k+1}, x_{k+1} + \zeta_1) Y_{VW}^W(\mathcal{F}(v'_1, y_1; \dots; v'_n, y_n), \zeta_2) \bar{u} \rangle \\
&= \sum_{u \in V} \langle w', Y_{VW}^W(\mathcal{F}(v_1, x_1; \dots; v_k, x_k), \zeta_1) u \rangle \\
&\quad \langle w', (-1)^{k+1} \omega_W(v_{k+1}, x_{k+1} + \zeta_1) e^{\zeta_2 L_W(-1)} Y_W(\bar{u}, -\zeta_2) \mathcal{F}(v'_1, y_1; \dots; v'_n, y_n) \rangle \\
&= \sum_{u \in V} \langle w', Y_{VW}^W(\mathcal{F}(v_1, x_1; \dots; v_k, x_k), \zeta_1) u \rangle \\
&\quad \langle w', (-1)^{k+1} e^{\zeta_2 L_W(-1)} Y_W(\bar{u}, -\zeta_2) \omega_W(v_{k+1}, x_{k+1} + \zeta_1 - \zeta_2) \mathcal{F}(v'_1, y_1; \dots; v'_n, y_n) \rangle \\
&\quad = \sum_{u \in V} \langle w', Y_{VW}^W(\mathcal{F}(v_1, x_1; \dots; v_k, x_k), \zeta_1) u \rangle \\
&\quad \langle w', Y_{VW}^W(\omega_W(v'_1, y_1) \mathcal{F}(v'_2, y_2; \dots; v'_n, y_n), \zeta_2) \bar{u} \rangle,
\end{aligned}$$

due to locality (8.7) of vertex operators, and arbitrariness of  $v_{k+1} \in V$  and  $x_{k+1}$ , we can always put

$$\omega_W(v_{k+1}, x_{k+1} + \zeta_1 - \zeta_2) = \omega_W(v'_1, y_1),$$

for  $v_{k+1} = v'_1$ ,  $x_{k+1} = y_1 + \zeta_2 - \zeta_1$ .  $\square$

Finally, we have the following

**Corollary 4.** *The multiplication (6.1) extends the chain-cochain complex (5.13)–(5.15) structure to all products  $C_m^k(V, \mathcal{W}) \times C_{m'}^n(V, \mathcal{W})$ ,  $k, n \geq 0$ ,  $m, m' \geq 0$ .  $\square$*

**Corollary 5.** *The product (6.1) and the product operator (5.12) endow the space  $C_m^k(V, \mathcal{W}) \times C_m^n(V, \mathcal{W})$ ,  $k, n \geq 0$ ,  $m, m' \geq 0$ , with the structure of a bi-graded differential algebra  $\mathcal{G}(V, \mathcal{W}, \cdot, \delta_m^n)$ .  $\square$*

## 7. EXAMPLE: EXCEPTIONAL COMPLEX

In addition to the double complex  $(C_m^n(V, \mathcal{W}), \delta_m^n)$  provided by (5.13)–(5.15), there exists an exceptional short double complex  $(C_{ex}^2(V, \mathcal{W}), \delta_{ex}^2)$ . In [6] we have

**Lemma 4.** *For  $n = 2$ , there exists a subspace  $C_{ex}^0(V, \mathcal{W})$*

$$C_m^2(V, \mathcal{W}) \subset C_{ex}^2(V, \mathcal{W}) \subset C_0^2(V, \mathcal{W}),$$

for all  $m \geq 1$ , with the action of coboundary operator  $\delta_m^2$  defined.  $\square$

Let us recall some facts about the exceptional complex [6]. Consider the space  $C_0^2(V, \mathcal{W})$ . It consist of  $\mathcal{W}_{z_1, z_2}$ -elements with zero vertex operators composable. The space  $C_0^2(V, \mathcal{W})$  contains elements of  $\mathcal{W}_{z_1, z_2}$  so that the action of  $\delta_0^2$  is zero. Nevertheless, as for  $\mathcal{J}_m^n(\Phi)$  in (5.5), Definition 9, let us consider sum of projections

$$P_r : \mathcal{W}_{z_i, z_j} \rightarrow W_r,$$

for  $r \in \mathbb{C}$ , and  $(i, j) = (1, 2), (2, 3)$ , so that the condition (5.5) is satisfied for some elements similar to the action (5.5) of  $\delta_0^2$ . Separating the first two and the second two summands in (5.12), we find that for a subspace of  $C_0^2(V, \mathcal{W})$  (which we denote as  $C_{ex}^2(V, \mathcal{W})$ ), for  $v_1, v_2, v_3 \in V$ , and arbitrary  $w' \in W'$ ,  $\zeta \in \mathbb{C}$ , the following elements

$$\begin{aligned} & G_1(z_1, z_2, z_3) \\ &= \sum_{r \in \mathbb{C}} \left( \langle w', E_W^{(1)}(v_1, z_1; P_r(\mathcal{F}(v_2, z_2 - \zeta; v_3, z_3 - \zeta))) \rangle \right. \\ & \quad \left. + \langle w', \mathcal{F}(v_1, z_1; P_r(E_V^{(2)}(v_2, z_2 - \zeta; v_3, z_3 - \zeta; \mathbf{1}_V), \zeta)) \rangle \right) \\ &= \sum_{r \in \mathbb{C}} \left( \langle w', \omega_W(v_1, z_1) P_r(\mathcal{F}(v_2, z_2 - \zeta; v_3, z_3 - \zeta)) \rangle \right. \\ & \quad \left. + \langle w', \mathcal{F}(v_1, z_1; P_r(\omega_V(v_2, z_2 - \zeta) \omega_V(v_3, z_3 - \zeta) \mathbf{1}_V), \zeta) \rangle \right), \end{aligned} \tag{7.1}$$

and

$$\begin{aligned} & G_2(z_1, z_2, z_3) \\ &= \sum_{r \in \mathbb{C}} \left( \langle w', \mathcal{F}(P_r(E_V^{(2)}(v_1, z_1 - \zeta, v_2, z_2 - \zeta; \mathbf{1}_V)), \zeta; v_3, z_3) \rangle \right. \\ & \quad \left. + \langle w', E_{WV}^{W;(1)}(P_r(\mathcal{F}(v_1, z_1 - \zeta; v_2, z_2 - \zeta), \zeta; v_3, z_3)) \rangle \right) \\ &= \sum_{r \in \mathbb{C}} \left( \langle w', \mathcal{F}(P_r(\omega_V(v_1, z_1 - \zeta) \omega_V(v_2, z_2 - \zeta) \mathbf{1}_V), \zeta); v_3, z_3) \rangle \right. \\ & \quad \left. + \langle w', \omega_V(v_3, z_3) P_r(\mathcal{F}(v_1, z_1 - \zeta; v_2, z_2 - \zeta)) \rangle \right), \end{aligned} \tag{7.2}$$

are absolutely convergent in the regions

$$\begin{aligned} & |z_1 - \zeta| > |z_2 - \zeta|, \\ & |z_2 - \zeta| > 0, \\ & |\zeta - z_3| > |z_1 - \zeta|, \\ & |z_2 - \zeta| > 0, \end{aligned}$$

where  $z_i$ ,  $1 \leq i \leq 3$ . These functions can be analytically extended to rational form-valued functions in  $z_1$  and  $z_2$  with the only possible poles at  $z_1, z_2 = 0$ , and  $z_1 = z_2$ . Note that (7.1) and (7.2) constitute the first two and the last two terms of (5.12) correspondingly. According to Proposition 7 (cf. Appendix 5.2),  $C_m^2(V, \mathcal{W})$  is a subspace of  $C_{ex}^2(V, \mathcal{W})$ , for  $m \geq 0$ , and  $\mathcal{F} \in C_m^2(V, \mathcal{W})$  are composable with  $m$  vertex operators. Then we have



**Definition 12.** The coboundary operator

$$\delta_{ex}^2 : C_{ex}^2(V, \mathcal{W}) \rightarrow C_0^3(V, \mathcal{W}), \quad (7.3)$$

is defined by

$$\begin{aligned} \delta_{ex}^2 \mathcal{F} &= \langle w', \omega_W(v_1, z_1) \mathcal{F}(v_2, z_2; v_3, z_3) \rangle \\ &- \langle w', \mathcal{F}(\omega_V(v_1, z_1) \omega_V(v_2, z_2) \mathbf{1}_V; v_3, z_3) \rangle \\ &\quad + \langle w', \mathcal{F}(v_1, z_1; \omega_V(v_2, z_2) \omega_V(v_3, z_3) \mathbf{1}_V) \rangle \\ &\quad + \langle w', \omega_W(v_3, z_3) \mathcal{F}(v_1, z_1; v_2, z_2) \rangle, \end{aligned} \quad (7.4)$$

for arbitrary  $w' \in W'$ ,  $\mathcal{F} \in C_{ex}^2(V, \mathcal{W})$ ,  $(v_1, v_2, v_3) \in V$  and  $(z_1, z_2, z_3) \in F_3\mathbb{C}$ .

In [6] we also find

**Proposition 14.** *The operator (7.4) provides the chain-cochain complex*

$$\delta_{ex}^2 \circ \delta_2^1 = 0,$$

$$0 \longrightarrow C_3^0(V, \mathcal{W}) \xrightarrow{\delta_3^0} C_2^1(V, \mathcal{W}) \xrightarrow{\delta_2^1} C_{ex}^2(V, \mathcal{W}) \xrightarrow{\delta_{ex}^2} C_0^3(V, \mathcal{W}) \longrightarrow 0. \quad (7.5)$$

□

Since

$$\delta_2^1 C_2^1(V, \mathcal{W}) \subset C_1^2(V, \mathcal{W}) \subset C_{ex}^2(V, \mathcal{W}),$$

the second formula follows from the first one, and

$$\delta_{ex}^2 \circ \delta_2^1 = \delta_1^2 \circ \delta_2^1 = 0.$$

For elements of the spaces  $C_{ex}^2(V, \mathcal{W})$  we have the following

**Corollary 6.** *The product of elements of the spaces  $C_{ex}^2(V, \mathcal{W})$  and  $C_m^n(V, \mathcal{W})$  is given by (6.1),*

$$\cdot_\epsilon : C_{ex}^2(V, \mathcal{W}) \times C_m^n(V, \mathcal{W}) \rightarrow C_m^{n+2}(V, \mathcal{W}), \quad (7.6)$$

and, in particular,

$$\cdot_\epsilon : C_{ex}^2(V, \mathcal{W}) \times C_{ex}^2(V, \mathcal{W}) \rightarrow C_0^4(V, \mathcal{W}).$$

*Proof.* The fact that the number of formal parameters is  $n + 2$  in the product (6.1) follows from Proposition (4). Consider the product (6.1) for  $C_{ex}^2(V, \mathcal{W})$  and  $C_m^n(V, \mathcal{W})$ . It is clear that, similar to considerations of the proof of Proposition 12, the total number  $m$  of vertex operators the product  $\mathcal{F}$  is composable to remains the same. □

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## 8. APPENDIX: GRADING-RESTRICTED VERTEX ALGEBRAS AND THEIR MODULES

In this section, following [6] we recall basic properties of grading-restricted vertex algebras and their grading-restricted generalized modules, useful for our purposes in later sections. We work over the base field  $\mathbb{C}$  of complex numbers.

**Definition 13.** A vertex algebra  $(V, Y_V, \mathbf{1}_V)$ , (cf. [9]), consists of a  $\mathbb{Z}$ -graded complex vector space

$$V = \coprod_{n \in \mathbb{Z}} V_{(n)}, \quad \dim V_{(n)} < \infty,$$

for each  $n \in \mathbb{Z}$ , and linear map

$$Y_V : V \rightarrow \text{End}(V)[[z, z^{-1}]],$$

for a formal parameter  $z$  and a distinguished vector  $\mathbf{1}_V \in V$ . The evaluation of  $Y_V$  on  $v \in V$  is the vertex operator

$$Y_V(v) \equiv Y_V(v, z) = \sum_{n \in \mathbb{Z}} v(n)z^{-n-1}, \quad (8.1)$$

with components  $(Y_V(v))_n = v(n) \in \text{End}(V)$ , where  $Y_V(v, z)\mathbf{1}_V = v + O(z)$ .

**Definition 14.** A grading-restricted vertex algebra satisfies the following conditions:

- (1) Grading-restriction condition:  $V_{(n)}$  is finite dimensional for all  $n \in \mathbb{Z}$ , and  $V_{(n)} = 0$  for  $n \ll 0$ ;
- (2) Lower-truncation condition: For  $u, v \in V$ ,  $Y_V(u, z)v$  contains only finitely many negative power terms, that is,

$$Y_V(u, z)v \in V((z)),$$

(the space of formal Laurent series in  $z$  with coefficients in  $V$ );

- (3) Identity property: Let  $\text{Id}_V$  be the identity operator on  $V$ . Then

$$Y_V(\mathbf{1}_V, z) = \text{Id}_V;$$

- (4) Creation property: For  $u \in V$ ,

$$Y_V(u, z)\mathbf{1}_V \in V[[z]],$$

and

$$\lim_{z \rightarrow 0} Y_V(u, z)\mathbf{1}_V = u;$$

- (5) Duality: For  $u_1, u_2, v \in V$ ,

$$v' \in V' = \coprod_{n \in \mathbb{Z}} V_{(n)}^*,$$

where  $V_{(n)}^*$  denotes the dual vector space to  $V_{(n)}$  and  $\langle \cdot, \cdot \rangle$  the evaluation pairing  $V' \otimes V \rightarrow \mathbb{C}$ , the series

$$\langle v', Y_V(u_1, z_1)Y_V(u_2, z_2)v \rangle, \quad (8.2)$$

$$\langle v', Y_V(u_2, z_2)Y_V(u_1, z_1)v \rangle, \quad (8.3)$$

$$\langle v', Y_V(Y_V(u_1, z_1 - z_2)u_2, z_2)v \rangle, \quad (8.4)$$

are absolutely convergent in the regions

$$|z_1| > |z_2| > 0,$$

$$|z_2| > |z_1| > 0,$$

$$|z_2| > |z_1 - z_2| > 0,$$

respectively, to a common rational function in  $z_1$  and  $z_2$  with the only possible poles at  $z_1 = 0 = z_2$  and  $z_1 = z_2$ ;

(6)  $L_V(0)$ -bracket formula: Let  $L_V(0) : V \rightarrow V$ , be defined by

$$L_V(0)v = nv, \quad n = \text{wt}(v),$$

for  $v \in V_{(n)}$ . Then

$$[L_V(0), Y_V(v, z)] = Y_V(L_V(0)v, z) + z \frac{d}{dz} Y_V(v, z), \quad (8.5)$$

for  $v \in V$ .

(7)  $L_V(-1)$ -derivative property: Let

$$L_V(-1) : V \rightarrow V,$$

be the operator given by

$$L_V(-1)v = \text{Res}_z z^{-2} Y_V(v, z) \mathbf{1}_V = Y_{(-2)}(v) \mathbf{1}_V,$$

for  $v \in V$ . Then for  $v \in V$ ,

$$\frac{d}{dz} Y_V(u, z) = Y_V(L_V(-1)u, z) = [L_V(-1), Y_V(u, z)]. \quad (8.6)$$

In addition to that, we recall here the following definition (cf. [1]):

**Definition 15.** A grading-restricted vertex algebra  $V$  is called conformal of central charge  $c \in \mathbb{C}$ , if there exists a non-zero conformal vector (Virasoro vector)  $\omega \in V_{(2)}$  such that the corresponding vertex operator

$$Y_V(\omega, z) = \sum_{n \in \mathbb{Z}} L_V(n) z^{-n-2},$$

is determined by modes of Virasoro algebra  $L_V(n) : V \rightarrow V$  satisfying

$$[L_V(m), L_V(n)] = (m - n)L_V(m + n) + \frac{c}{12}(m^3 - m)\delta_{m+n,0} \text{Id}_V.$$

**Definition 16.** A vector  $A$  which belongs to a module  $W$  of a quasi-conformal grading-restricted vertex algebra  $V$  is called primary of conformal dimension  $\Delta(A) \in \mathbb{Z}_+$  if

$$\begin{aligned} L_W(k)A &= 0, \quad k > 0, \\ L_W(0)A &= \Delta(A)A. \end{aligned}$$

### 8.1. Grading-restricted generalized $V$ -module.

**Definition 17.** A grading-restricted generalized  $V$ -module is a vector space  $W$  equipped with a vertex operator map

$$\begin{aligned} Y_W : V \otimes W &\rightarrow W[[z, z^{-1}]], \\ u \otimes w &\mapsto Y_W(u, w) \equiv Y_W(u, z)w = \sum_{n \in \mathbb{Z}} (Y_W)_n(u, w)z^{-n-1}, \end{aligned}$$

and linear operators  $L_W(0)$  and  $L_W(-1)$  on  $W$  satisfying the following conditions:

- (1) Grading-restriction condition: The vector space  $W$  is  $\mathbb{C}$ -graded, that is,

$$W = \coprod_{\alpha \in \mathbb{C}} W_{(\alpha)},$$

such that  $W_{(\alpha)} = 0$  when the real part of  $\alpha$  is sufficiently negative;

- (2) Lower-truncation condition: For  $u \in V$  and  $w \in W$ ,  $Y_W(u, z)w$  contains only finitely many negative power terms, that is,  $Y_W(u, z)w \in W((z))$ ;  
(3) Identity property: Let  $\text{Id}_W$  be the identity operator on  $W$ . Then

$$Y_W(\mathbf{1}_V, z) = \text{Id}_W;$$

- (4) Duality: For  $u_1, u_2 \in V$ ,  $w \in W$ ,

$$w' \in W' = \coprod_{n \in \mathbb{Z}} W_{(n)}^*,$$

$W'$  denotes the dual  $V$ -module to  $W$  and  $\langle \cdot, \cdot \rangle$  their evaluation pairing, the series

$$\langle w', Y_W(u_1, z_1)Y_W(u_2, z_2)w \rangle, \quad (8.7)$$

$$\langle w', Y_W(u_2, z_2)Y_W(u_1, z_1)w \rangle, \quad (8.8)$$

$$\langle w', Y_W(Y_V(u_1, z_1 - z_2)u_2, z_2)w \rangle, \quad (8.9)$$

are absolutely convergent in the regions

$$|z_1| > |z_2| > 0,$$

$$|z_2| > |z_1| > 0,$$

$$|z_2| > |z_1 - z_2| > 0,$$

respectively, to a common rational function in  $z_1$  and  $z_2$  with the only possible poles at  $z_1 = 0 = z_2$  and  $z_1 = z_2$ .

- (5)  $L_W(0)$ -bracket formula: For  $v \in V$ ,

$$[L_W(0), Y_W(v, z)] = Y_W(L_V(0)v, z) + z \frac{d}{dz} Y_W(v, z);$$

- (6)  $L_W(0)$ -grading property: For  $w \in W_{(\alpha)}$ , there exists  $N \in \mathbb{Z}_+$  such that

$$(L_W(0) - \alpha)^N w = 0; \quad (8.10)$$

- (7)  $L_W(-1)$ -derivative property: For  $v \in V$ ,

$$\frac{d}{dz} Y_W(u, z) = Y_W(L_V(-1)u, z) = [L_W(-1), Y_W(u, z)]. \quad (8.11)$$

The translation property of vertex operators

$$Y_W(u, z) = e^{-z' L_W(-1)} Y_W(u, z + z') e^{z' L_W(-1)}, \quad (8.12)$$

for  $z' \in \mathbb{C}$ , follows from (8.11). For  $a \in \mathbb{C}$ , the conjugation property with respect to the grading operator  $L_W(0)$  is given by

$$a^{L_W(0)} Y_W(v, z) a^{-L_W(0)} = Y_W(a^{L_W(0)} v, az). \quad (8.13)$$

For  $v \in V$ , and  $w \in W$ , the intertwining operator

$$\begin{aligned} Y_{WV}^W : V &\rightarrow W, \\ v &\mapsto Y_{WV}^W(w, z)v, \end{aligned} \quad (8.14)$$

is defined by

$$Y_{WV}^W(w, z)v = e^{z L_W(-1)} Y_W(v, -z)w. \quad (8.15)$$

**8.2. Group of automorphisms of formal parameters.** Let us recall some further facts from [1] relating generators of Virasoro algebra with the group of automorphisms in complex dimension one. Let us represent an element of  $\text{Aut}_z \mathcal{O}^{(1)}$  by the map

$$z \mapsto \rho = \rho(z), \quad (8.16)$$

given by the power series

$$\rho(z) = \sum_{k \geq 1} a_k z^k, \quad (8.17)$$

$\rho(z)$  can be represented in an exponential form

$$f(z) = \exp \left( \sum_{k > -1} \beta_k z^{k+1} \partial_z \right) (\beta_0)^{z \partial_z} .z, \quad (8.18)$$

where we express  $\beta_k \in \mathbb{C}$ ,  $k \geq 0$ , through combinations of  $a_k$ ,  $k \geq 1$ . A representation of Virasoro algebra modes in terms of differential operators is given by [9]

$$L_W(m) \mapsto -\zeta^{m+1} \partial_\zeta, \quad (8.19)$$

for  $m \in \mathbb{Z}$ . By expanding (8.18) and comparing to (10.1) we obtain a system of equations which, can be solved recursively for all  $\beta_k$ . In [1],  $v \in V$ , they derive the formula

$$[L_W(n), Y_W(v, z)] = \sum_{m \geq -1} \frac{1}{(m+1)!} (\partial_z^{m+1} z^{m+1}) Y_W(L_V(m)v, z), \quad (8.20)$$

of a Virasoro generator commutation with a vertex operator. Given a vector field

$$\beta(z) \partial_z = \sum_{n \geq -1} \beta_n z^{n+1} \partial_z, \quad (8.21)$$

which belongs to local Lie algebra of  $\text{Aut} \mathcal{O}^{(1)}$ , one introduces the operator

$$\beta = - \sum_{n \geq -1} \beta_n L_W(n).$$

We conclude from (8.21) with the following

**Lemma 5.**

$$[\beta, Y_W(v, z)] = \sum_{m \geq -1} \frac{1}{(m+1)!} (\partial_z^{m+1} \beta(z)) Y_W(L_V(m)v, z). \quad (8.22)$$

The formula (8.22) is used in [1] in order to prove invariance of vertex operators multiplied by conformal weight differentials in case of primary states, and in generic case.

Let us give some further definitions:

**Definition 18.** A conformal grading-restricted vertex algebra is a conformal vertex algebra  $V$ , such that its module  $W$  is equipped with an action of the Virasoro algebra and hence its Lie subalgebra  $\text{Der}_0 \mathcal{O}^{(n)}$  given by the Lie algebra of  $\text{Aut } \mathcal{O}^{(n)}$ .

**Definition 19.** A grading-restricted vertex algebra  $V$ -module  $W$  is called quasi-conformal if it carries an action of local Lie algebra of  $\text{Aut } \mathcal{O}$  such that commutation formula (8.22) holds for any  $v \in V$ , the element  $L_W(-1) = -\partial_z$ , as the translation operator  $T$ ,

$$L_W(0) = -z\partial_z,$$

acts semi-simply with integral eigenvalues, and the Lie subalgebra of the positive part of local Lie algebra of  $\text{Aut } \mathcal{O}^{(n)}$  acts locally nilpotently.

Recall [1] the exponential form  $f(\zeta)$  (8.18) of the coordinate transformation (8.16)  $\rho(z) \in \text{Aut } \mathcal{O}^{(1)}$ . A quasi-conformal vertex algebra possesses the formula (8.22), thus it is possible by using the identification (8.19), to introduce the linear operator representing  $f(\zeta)$  (8.18) on  $\mathcal{W}_{z_1, \dots, z_n}$ ,

$$P(f(\zeta)) = \exp \left( \sum_{m>0} (m+1) \beta_m L_V(m) \right) \beta_0^{L_W(0)}, \quad (8.23)$$

(note that we have a different normalization in it). In [1] it was shown that the action of an operator similar to (8.23) on a vertex algebra element  $v \in V_n$  contains finitely many terms, and subspaces

$$V_{\leq m} = \bigoplus_{n \geq K}^m V_n,$$

are stable under all operators  $P(f)$ ,  $f \in \text{Aut } \mathcal{O}^{(1)}$ . In [1] they proved the following

**Lemma 6.** *The assignment*

$$f \mapsto P(f),$$

*defines a representation of  $\text{Aut } \mathcal{O}^{(1)}$  on  $V$ ,*

$$P(f_1 * f_2) = P(f_1) P(f_2),$$

*which is the inductive limit of the representations  $V_{\leq m}$ ,  $m \geq K$ .*

Similarly, (8.23) provides a representation operator on  $\mathcal{W}_{z_1, \dots, z_n}$ .

8.3. **Non-degenerate invariant bilinear form on  $V$ .** The subalgebra

$$\{L_V(-1), L_V(0), L_V(1)\} \cong SL(2, \mathbb{C}),$$

associated with Möbius transformations on  $z$  naturally acts on  $V$ , (cf., e.g. [9]). In particular,

$$\gamma_\lambda = \begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix} : z \mapsto w = -\frac{\lambda^2}{z}, \quad (8.24)$$

is generated by

$$T_\lambda = \exp(\lambda L_V(-1)) \exp(\lambda^{-1} L_V(1)) \exp(\lambda L_V(-1)),$$

where

$$T_\lambda Y(u, z) T_\lambda^{-1} = Y\left(\exp\left(-\frac{z}{\lambda^2} L_V(1)\right) \left(-\frac{z}{\lambda}\right)^{-2L_V(0)} u, -\frac{\lambda^2}{z}\right). \quad (8.25)$$

In our considerations (cf. Appendix 9) of Riemann sphere sewing, we use in particular, the Möbius map

$$z \mapsto z' = \epsilon/z,$$

associated with the sewing condition (9.4) with

$$\lambda = -\xi \epsilon^{\frac{1}{2}}, \quad (8.26)$$

with  $\xi \in \{\pm\sqrt{-1}\}$ . The adjoint vertex operator [2, 9] is defined by

$$Y^\dagger(u, z) = \sum_{n \in \mathbb{Z}} u^\dagger(n) z^{-n-1} = T_\lambda Y(u, z) T_\lambda^{-1}. \quad (8.27)$$

A bilinear form  $\langle \cdot, \cdot \rangle_\lambda$  on  $V$  is invariant if for all  $a, b, u \in V$ , if

$$\langle Y(u, z)a, b \rangle_\lambda = \langle a, Y^\dagger(u, z)b \rangle_\lambda, \quad (8.28)$$

i.e.

$$\langle u(n)a, b \rangle_\lambda = \langle a, u^\dagger(n)b \rangle_\lambda.$$

Thus it follows that

$$\langle L_V(0)a, b \rangle_\lambda = \langle a, L_V(0)b \rangle_\lambda, \quad (8.29)$$

so that

$$\langle a, b \rangle_\lambda = 0, \quad (8.30)$$

if  $wt(a) \neq wt(b)$  for homogeneous  $a, b$ . One also finds

$$\langle a, b \rangle_\lambda = \langle b, a \rangle_\lambda.$$

The form  $\langle \cdot, \cdot \rangle_\lambda$  is unique up to normalization if  $L_V(1)V_1 = V_0$ . Given any  $V$  basis  $\{u^\alpha\}$  we define the dual  $V$  basis  $\{\bar{u}^\beta\}$  where

$$\langle u^\alpha, \bar{u}^\beta \rangle_\lambda = \delta^{\alpha\beta}.$$

## 9. APPENDIX: A SPHERE FORMED FROM SEWING OF TWO SPHERES

The matrix element for a number of vertex operators of a vertex algebra is usually associated [2, 3, 12] with a vertex algebra character on a sphere. We extrapolate this notion to the case of  $\mathcal{W}_{z_1, \dots, z_n}$  spaces. In Section 3 we explained that a space  $\mathcal{W}_{z_1, \dots, z_n}$  can be associated with a Riemann sphere with marked points, while the product of two such spaces is then associated with a sewing of such two spheres with a number of marked points and extra points with local coordinates identified with formal parameters of  $\mathcal{W}_{x_1, \dots, x_k}$  and  $\mathcal{W}_{y_1, \dots, y_n}$ . In order to supply an appropriate geometric construction for the product, we use the  $\epsilon$ -sewing procedure (described in this Appendix) for two initial spheres to obtain a matrix element associated with (3.1).

*Remark 7.* In addition to the  $\epsilon$ -sewing procedure of two initial spheres, one can alternatively use the self-sewing procedure [13] for the sphere to get, at first, the torus, and then by sending parameters to appropriate limit by shrinking genus to zero. As a result, one obtains again the sphere but with a different parameterization. In the case of spheres, such a procedure consideration of the product of  $\mathcal{W}$ -spaces so we focus in this paper on the  $\epsilon$ -formalism only.

In our particular case of  $\mathcal{W}$ -values rational functions obtained from matrix elements (2.1) two initial auxiliary spaces we take Riemann spheres  $\Sigma_a^{(0)}$ ,  $a = 1, 2$ , and the resulting space is formed by the sphere  $\Sigma^{(0)}$  obtained by the procedure of sewing  $\Sigma_a^{(0)}$ . The formal parameters  $(x_1, \dots, x_k)$  and  $(y_1, \dots, y_n)$  are identified with local coordinates of  $k$  and  $n$  points on two initial spheres  $\Sigma_a^{(0)}$ ,  $a = 1, 2$  correspondingly. In the  $\epsilon$  sewing procedure, some  $r$  points among  $(p_1, \dots, p_k)$  may coincide with points among  $(p'_1, \dots, p'_n)$  when we identify the annuluses (9.3). This corresponds to the singular case of coincidence of  $r$  formal parameters.

Consider the sphere formed by sewing together two initial spheres in the sewing scheme referred to as the  $\epsilon$ -formalism in [13]. Let  $\Sigma_a^{(0)}$ ,  $a = 1, 2$  be to initial spheres. Introduce a complex sewing parameter  $\epsilon$  where

$$|\epsilon| \leq r_1 r_2,$$

Consider  $k$  distinct points on  $p_i \in \Sigma_1^{(0)}$ ,  $i = 1, \dots, k$ , with local coordinates  $(x_1, \dots, x_k) \in F_k \mathbb{C}$ , and distinct points  $p_j \in \Sigma_2^{(0)}$ ,  $j = 1, \dots, n$ , with local coordinates  $(y_1, \dots, y_n) \in F_n \mathbb{C}$ , with

$$|x_i| \geq |\epsilon|/r_2,$$

$$|y_j| \geq |\epsilon|/r_1.$$

Choose a local coordinate  $z_a \in \mathbb{C}$  on  $\Sigma_a^{(0)}$  in the neighborhood of points  $p_a \in \Sigma_a^{(0)}$ ,  $a = 1, 2$ . Consider the closed disks

$$|\zeta_a| \leq r_a,$$

and excise the disk

$$\{\zeta_a, |\zeta_a| \leq |\epsilon|/r_a\} \subset \Sigma_a^{(0)}, \quad (9.1)$$



to form a punctured sphere

$$\widehat{\Sigma}_a^{(0)} = \Sigma_a^{(0)} \setminus \{\zeta_a, |\zeta_a| \leq |\epsilon|/r_a\}.$$

We use the convention

$$\bar{1} = 2, \quad \bar{2} = 1. \quad (9.2)$$

Define the annulus

$$\mathcal{A}_a = \{\zeta_a, |\epsilon|/r_a \leq |\zeta_a| \leq r_a\} \subset \widehat{\Sigma}_a^{(0)}, \quad (9.3)$$

and identify  $\mathcal{A}_1$  and  $\mathcal{A}_2$  as a single region  $\mathcal{A} = \mathcal{A}_1 \simeq \mathcal{A}_2$  via the sewing relation

$$\zeta_1 \zeta_2 = \epsilon. \quad (9.4)$$

In this way we obtain a genus zero compact Riemann surface

$$\Sigma^{(0)} = \{\widehat{\Sigma}_1^{(0)} \setminus \mathcal{A}_1\} \cup \{\widehat{\Sigma}_2^{(0)} \setminus \mathcal{A}_2\} \cup \mathcal{A}.$$

This sphere form a suitable geometrical model for the construction of a product of  $\mathcal{W}$ -valued rational forms in Section 3.

## 10. APPENDIX: THE PROOF OF PROPOSITION 10

In this Appendix we recall [11] the proof of Proposition 10, namely, we prove that Definition 3 is independent of the choice of formal parameters. Let us first recall definitions required for that. Let

$$\text{Aut}_{z_1, \dots, z_n} \mathcal{O}^{(n)} = \text{Aut}_{\mathbb{C}}[[z_1, \dots, z_n]],$$

be the group of formal automorphisms of  $n$ -dimensional formal power series algebra  $\mathbb{C}[[z_1, \dots, z_n]]$ .

The  $\mathbb{Z}$ -grading on  $W$  is given by the operator  $L_W(0)$  bounded from below,

$$W = \bigoplus_{k > k_0} W_k,$$

for some  $k_0 \in \mathbb{Z}$ . Since the vector fields  $z^{k+1} \partial_z$  with  $k \in \mathbb{N}$  act on  $W$  as the operators  $-L_W(k)$  of degree  $-k$ , the action of the Lie subalgebra  $\text{Der}_+ \mathcal{O}^{(n)}$  is locally nilpotent. Furthermore,  $z \partial_z$  acts as the grading operator  $L_W(0)$ , which is diagonalizable with integral eigenvalues. Thus, the action of  $\text{Der} \mathcal{O}^{(n)}$  on a conformal vertex algebra  $V$  can be exponentiated to an action of  $\text{Aut}_{z_1, \dots, z_n} \mathcal{O}^{(n)}$ .

We write an element of  $\text{Aut}_{z_1, \dots, z_n} \mathcal{O}^{(n)}$  as

$$\begin{aligned} (z_1, \dots, z_n) \rightarrow \rho &= (\rho_1, \dots, \rho_n), \\ \rho_i &= \rho_i(z_1, \dots, z_n), \end{aligned}$$

for  $i = 1, \dots, n$ , where  $\rho_i$  are defined by elements of  $\mathfrak{m} \in \mathcal{O}^{(n)}$

$$\rho_i(z_1, \dots, z_n) = \sum_{\substack{i_1 \geq 0, \dots, i_n \geq 0, \\ \sum_{j=1}^n i_j \geq 1}} a_{(i_1, \dots, i_k)} z_1^{i_1} \cdots z_k^{i_k}, \quad a_{(i_1, \dots, i_k)} \in \mathbb{C} \quad (10.1)$$

and the images of  $\rho_i$ ,  $i = 1, \dots, n$ , in the finite dimensional  $\mathbb{C}$ -vector space  $\mathfrak{m}/\mathfrak{m}^2$  are linearly independent. Let us denote

$$\begin{aligned}\mathbf{v} &= (v_1 \otimes \dots \otimes v_n), \\ \mathbf{z} &= (z_1, \dots, z_n), \\ w_i &= \rho_i(z_1, \dots, z_n), \\ \mathbf{w} &= (w_1, \dots, w_n).\end{aligned}$$

The natural object that turns to be invariant with respect to the action of the group  $\text{Aut}_{z_1, \dots, z_n} \mathcal{O}^{(n)}$  is given by the matrix element of the  $n$ -vector

$$\langle w', \bar{\Phi}(\mathbf{v}, \mathbf{z} \, d\mathbf{z}) \rangle = \langle w', [\Phi(v_1, z_1 \, dz_{i(1)}; \dots; v_n, z_n \, dz_{i(n)})] \rangle, \quad (10.2)$$

containing  $n$   $\mathcal{F}$ -entries, where  $i(j)$  denotes the cyclic permutation of  $(1, \dots, n)$  starting with  $j$ . In the main text we use (2.2) which is related to (10.4). Due to (2.8), (10.4) can be written in the form

$$\begin{aligned}\langle w', \bar{\Phi}(\mathbf{d}\mathbf{z} \, \mathbf{v})(\mathbf{z}) \rangle &= \langle w', \left[ (dz_{i(j)})^{-L(0)w} \Phi \left( \left( (dz_{i(j)})^{L_0^{(v)}} \mathbf{v} \right) (\mathbf{z}) \right) \right] \rangle \\ &= \langle w', \left[ (dz_{i(j)})^{-L(0)w} \Phi \left( (dz_{i(j)})^{\text{wt}(v_j)} \mathbf{v} \right) (\mathbf{z}) \right] \rangle,\end{aligned}$$

coherent with the one-dimensional case of [1] and containing  $\text{wt}(v)$ -differentials. The idea to use torsors [1] is to represent the action of  $\rho$  (10.1) of the group  $\text{Aut}_{z_1, \dots, z_n} \mathcal{O}^{(n)}$  on formal parameters  $\mathbf{z}$  in vectors  $(\mathbf{v}, \mathbf{z})$  to the action by  $V$ -operators on vertex algebra states  $\mathbf{v}$ . Recall the standard representation of the Virasoro mode [9]

$$z_j^{m+1} \partial_{z_j} \mapsto -L_m, \quad m \in \mathbb{Z}.$$

In order to represent the action of the group  $\text{Aut}_{z_1, \dots, z_n} \mathcal{O}^{(n)}$  on the variables  $(z_1, \dots, z_n)$  of  $\bar{\mathcal{F}}$  (10.4) on  $(v_1, \dots, v_n)$ , we have to transfer (as in  $n = 1$  case of [1]) to an exponential form of (10.1). The coefficients  $\beta_{r_1, \dots, r_n}^{(j)} \in \mathbb{C}$  are recursively found [5] in terms of coefficients  $a_{r_1, \dots, r_n}^{(i)}$  of (10.1). We introduce the linear operators

$$R(\rho_1, \dots, \rho_n) : V^{\otimes n} \rightarrow V^{\otimes n},$$

and define the action

$$\bar{\Phi}(\mathbf{v}, \mathbf{w} \, d\mathbf{w}) = R(\rho_1, \dots, \rho_n) \bar{\Phi}(\mathbf{v}, \mathbf{z} \, d\mathbf{w}). \quad (10.3)$$

*Proof.* Consider the vector

$$\bar{\Phi} = [\Phi(v_1, w_1 \, dw_{i(1)}; \dots; v_n, w_n \, dw_{i(n)})]. \quad (10.4)$$

Note that

$$dw_j = \sum_{i=1}^n \frac{\partial \rho_j}{\partial z_i} dz_i, \quad \partial_{z_i} \rho_j = \frac{\partial \rho_j}{\partial z_i}, \quad (10.5)$$

(as in [1], we skip the complex conjugated part  $d\bar{z}_i$ ). By definition (10.3) of the action of  $\text{Aut}_{z_1, \dots, z_n} \mathcal{O}^{(n)}$ , and due to (10.5) by rewriting  $dw_i$ , we have

$$\begin{aligned} \bar{\Phi}(\mathbf{v}, \mathbf{w} \, \mathbf{d}\mathbf{w}) &= R(\rho_1, \dots, \rho_n) \left[ \Phi(v_1, z_1 dw_{i(1)}; \dots; v_n, z_n dw_{i(n)}) \right] \\ &= R(\rho_1, \dots, \rho_n) \\ &\quad \left[ \Phi \left( v_1, z_1 \sum_{j=1}^n \partial_j \rho_{i(1)} dz_j; \dots; v_n, z_n \sum_{j=1}^n \partial_j \rho_{i(n)} dz_j \right) \right]. \end{aligned}$$

By using (2.3) and linearity of the mapping  $\Phi$ , we obtain from the last equation

$$\bar{\Phi}(\mathbf{v}, \mathbf{w} \, \mathbf{d}\mathbf{w}) = \left[ \Phi(v_1, z_1 dz_{i(1)}; \dots; v_n, z_n dz_{i(n)}) \right], \quad (10.6)$$

with

$$R(\rho_1, \dots, \rho_n) = \left[ \hat{\partial}_J \rho_{i(I)} \right] = \begin{bmatrix} \hat{\partial}_J \rho_{i_1(I)} \\ \hat{\partial}_J \rho_{i_2(I)} \\ \dots \\ \hat{\partial}_J \rho_{i_n(I)} \end{bmatrix}. \quad (10.7)$$

The index operator  $J$  takes the value of index  $z_j$  of arguments in the vector (10.6), while the index operator  $I$  takes values of index of differentials  $dz_i$  in each entry of the vector  $\bar{\Phi}$  (10.4). Thus, the index operator  $i(I) = (i_1, \dots, i_n(I))$  is given by consequent cycling permutations of  $I$ . Taking into account the property (2.3), we define the operator

$$\hat{\partial}_J \rho_a = \exp \left( - \sum_{(r_1 \dots r_n), \sum_{i=1}^n r_i \geq 1} r_J \beta_{r_1, \dots, r_n}^{(a)} \zeta_1^{r_1} \dots \zeta_J^{r_J} \dots \zeta_n^{r_n} L_{(W)-1} \right), \quad (10.8)$$

which contains index operators  $J$  as index of a dummy variable  $\zeta_J$  turning into  $z_j$ ,  $j = 1, \dots, n$ . (10.8) acts on each argument of mappings  $\Phi$  in the vector  $\bar{\Phi}$  (10.4). Due to the definition of a grading-restricted vertex algebra, the action of operators  $R(\rho_1, \dots, \rho_n)$  for  $i = 1, \dots, n$ , on  $v \in V$  results in a sum of finitely many terms. Similar to [1], for  $n = 1$ , one proves

**Lemma 7.** *The mappings*

$$\rho(z_1, \dots, z_n) \mapsto R(\rho_1, \dots, \rho_n),$$

for  $i, j = 1, \dots, n$ , define a representation of  $\text{Aut}_{z_1, \dots, z_n} \mathcal{O}^{(n)}$  on  $V^{\otimes n}$  by

$$R(\rho \circ \rho') = R(\rho) R(\rho'),$$

for  $\rho, \rho' \in \text{Aut}_{z_1, \dots, z_n} \mathcal{O}^{(n)}$ .

We then conclude with

$$\left[ \bar{\Phi}(v_1, z_1 dz_1; \dots; v_n, \dots, z_n dz_n) \right]. \quad (10.9)$$

Thus the vector  $\bar{\Phi}$  (10.4) is invariant, i.e.,

$$\bar{\Phi}(\mathbf{v}, \mathbf{w} \, \mathbf{d}\mathbf{w}) = \bar{\Phi}(\mathbf{v}, \mathbf{z} \, \mathbf{d}\mathbf{z}). \quad (10.10)$$

Recall that the construction of the double complex spaces  $C_m^n(V, \mathcal{W})$  assumes that  $\Phi \in C_m^n(V, \mathcal{W})$  is composable with  $m$  vertex operators. In one-dimensional complex case, [1] they proved that a vertex operator multiplied to the wt  $(v_i)$ -power of the differential  $Y_W(v_i, z_i) dz_i^{\text{wt}(v_i)}$  is invariant with respect to the action of the group  $\text{Aut}_{z_i} \mathcal{O}^{(1)}$ . Here we prove that  $Y_W(v_i, z_i) dz_i^{\text{wt}(v_i)}$  is invariant with respect to the change of the local coordinates  $z_i \mapsto w_i(z_1, \dots, z_n)$ .

Let  $(z_1, \dots, z_n)$  be an open ball  $D_{\mathbf{z}_0}^{(n)}$  of local formal coordinates around a fixed-value  $\mathbf{z}_0$  of  $(z_1, \dots, z_n)$ . Define a wt  $(v_i)$ -differential on  $D_{\mathbf{z}_0}^{(n)}$  with values in  $\text{End}(W)_{\mathbf{z}_0}$  as follows: identify  $\text{End}(W)_{\mathbf{z}_0}$  with  $\text{End} W$  using the formal parameters  $(z_1, \dots, z_n)$ , and set

$$\omega_{i,x} = Y_W(v_i, z_i) dz_i^{\text{wt}(v_i)}.$$

Let

$$(w_1, \dots, w_n) = (\rho_1(z_1, \dots, z_n), \dots, \rho_n(z_1, \dots, z_n)),$$

be another set of formal parameters on an  $n$ -dimensional ball  $D_{\mathbf{z}_0}^{(n)}$ . Let us express the set of wt  $(v_i)$ -differentials on  $D_{\mathbf{z}_0}^{(n)}$ ,

$$Y_W(v_i, w_i) dw_i^{\text{wt}(v_i)},$$

$i = 1, \dots, n$ , in terms of the parameters  $(z_1, \dots, z_n)$ . We would like to show that it coincides with the set of wt  $(v_i)$ -differentials  $Y_W(v_i, z_i) dz_i^{\text{wt}(v_i)}$ .

Recall the notion of torsors (Section 8). Consider a vector  $(v_i, z_1, \dots, z_n) \in W_{\mathbf{z}_0}$  with  $v_i \in V$ . Then the same vector equals

$$(R_i^{-1}(\rho_1, \dots, \rho_n) v_i, w_1, \dots, w_n),$$

i.e., it is identified with

$$R_i^{-1}(\rho_1, \dots, \rho_n) v_i \in V,$$

using the formal parameters  $(w_1, \dots, w_n)$ . Here  $R_i(\rho_1, \dots, \rho_n)$  is an operator representing transformation of  $z_i \rightarrow w_i$ , as an action on  $V$ . Therefore if we have an operator on  $W_{\mathbf{z}_0}$  which is equal to a  $\text{Aut} \mathcal{O}^{(n)}$ -torsor  $S$  under the identification  $\text{End} W_{\mathbf{z}_0} \in \text{End} W$  using the formal parameters  $(w_1, \dots, w_n)$ , then this operator equals

$$R_i(\rho_1, \dots, \rho_n) S R_i^{-1}(\rho_1, \dots, \rho_n),$$

under the identification  $\text{End} W_{\mathbf{z}_0} \in \text{End} W^{(i)}$  using the combined parameters  $(v_i, z_1, \dots, z_n)$ . Thus, in terms of  $(v_i, z_1, \dots, z_n)$ , the differential  $Y_W(v_i, w_i) dw_i^{\text{wt}(v_i)}$  becomes

$$Y_W(v_i, z_i) dz_i^{\text{wt}(v_i)} = R_i(\rho) Y_W(v_i, \rho(z_1, \dots, z_n)) R_i^{-1}(\rho) dw_i^{\text{wt}(v_i)}.$$

According to Definition (10), elements  $\Phi$  are composable with  $m$  vertex operators. Thus we see that (10.4) is a canonical object of  $C_m^n(V, \mathcal{W})$ . We have proved that elements of the spaces  $C_m^n(V, \mathcal{W})$  are independent on the choice of formal parameters.  $\square$

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