Ergodic hypothesis for open fluid systems

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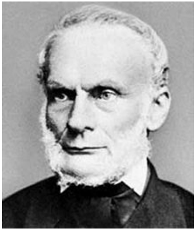
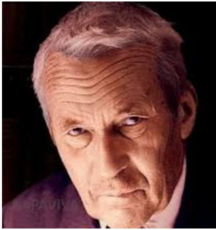
Motivation



Rudolf Clasius
1822–1888

Basic principles of thermodynamics of *closed* **systems**
The energy of the world is constant; its entropy tends to a maximum

Turbulence - ergodic hypothesis for *open* **systems**
Time averages along trajectories of the flow converge, for large enough times, to an ensemble average given by a certain probability measure



Andrey Nikolaevich Kolmogorov
1903–1987

Weak version of ergodic hypothesis

Dynamical system

$$\mathbf{U}(t,\cdot):[0,\infty)\times X\to X$$

Ergodic hypothesis

$$\frac{1}{T}\int_0^T \delta_{\mathbf{U}}\ \mathrm{d}t \to \mathcal{V} \text{ in } \mathfrak{P}\left[C_{\rm loc}([0,\infty);X)\right]$$

$\mathcal{V}$ invariant measure or stationary solution

$$\frac{1}{T}\int_0^T F(\mathbf{U})\ \mathrm{d}t \to \int F(\mathbf{V})\ \mathrm{d}\mathcal{V}$$

F Borel measurable on $C_{\rm loc}([0,\infty);X)$

Krylov–Bogolyubov method

$$\frac{1}{T_n}\int_0^{T_n} F(\mathbf{U})\ \mathrm{d}t \to \int F(\mathbf{V})\ \mathrm{d}\mathcal{V}$$

for some $T_n\to\infty$

Barotropic Navier–Stokes system

Field equations

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}) + \varrho \nabla_x F + \varrho \mathbf{g}$$

Constitutive equations

- barotropic pressure–density EOS $p = p(\varrho)$
- Newton's rheological law

$$\mathbb{S} = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{d} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) + \eta \mathrm{div}_x \mathbf{u} \mathbb{I}, \ \mu > 0, \ \eta \geq 0$$

- Gravitational and external force

$$\varrho \mathbf{g}, \ \nabla_x F, \ F = F(x)$$

Energy

$$E(\varrho, \mathbf{m}) \equiv \frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) - \varrho F, \ P'(\varrho)\varrho - P(\varrho) = p(\varrho), \ \mathbf{m} = \varrho \mathbf{u}$$

Energetically open system - boundary conditions

In/out flow boundary conditions

$$\mathbf{u} = \mathbf{u}_b \text{ on } \partial\Omega$$

$$\Gamma_{\rm in} = \left\{ x \in \partial\Omega \ \middle| \ \mathbf{u}_b(x) \cdot \mathbf{n}(x) < 0 \right\}, \ \Gamma_{\rm out} = \left\{ x \in \partial\Omega \ \middle| \ \mathbf{u}_b(x) \cdot \mathbf{n}(x) \geq 0 \right\}$$

Density (pressure) on the inflow boundary

$$\varrho = \varrho_b \text{ on } \Gamma_{\rm in}$$

Energy balance

$$\frac{\rm d}{{\rm d}t} \int_\Omega \frac{1}{2} \varrho |\mathbf{u} - \mathbf{u}_b|^2 + P(\varrho) \ {\rm d}x + \int_\Omega \mathbb{S} : \nabla_x \mathbf{u} \ {\rm d}x{\rm d}t$$

$$+ \int_{\Gamma_{\rm in}} P(\varrho_b) \mathbf{u}_b \cdot \mathbf{n} \ {\rm d}S_x + \int_{\Gamma_{\rm out}} P(\varrho) \mathbf{u}_b \cdot \mathbf{n} \ {\rm d}S_x$$

$$= (\leq) - \int_\Omega [\varrho \mathbf{u} \otimes \mathbf{u} + p(\varrho) \mathbb{I}] : \nabla_x \mathbf{u}_b \ {\rm d}x + \frac{1}{2} \int_\Omega \varrho \mathbf{u} \cdot \nabla_x |\mathbf{u}_b|^2 \ {\rm d}x{\rm d}t$$

$$+ \int_\Omega \mathbb{S} : \nabla_x \mathbf{u}_b \ {\rm d}x{\rm d}t + \int_\Omega \varrho \nabla_x F \cdot (\mathbf{u} - \mathbf{u}_b) \ {\rm d}x$$

Energetically open system - stochastic forcing

Homogeneous/periodic boundary conditions

$$\mathbf{u}|_{\partial\Omega} = 0, \text{ or } \Omega = \mathcal{T}^d$$

Stochastic forcing

$$\varrho \mathbf{g} \approx \varrho \sum_{k=1}^{\infty} \mathbf{F}_k(x, \varrho, \mathbf{u}) \mathrm{d} W_k$$

$$W = \{W_k\}_{k=1}^{\infty} \text{ cylindrical Wiener process}$$

$$|\mathbf{F}_k(x, \varrho, \mathbf{u})| + |\nabla_{\varrho, \mathbf{u}} \mathbf{F}_k(x, \varrho, \mathbf{u})| \leq f_k(1 + |\mathbf{u}|^{\alpha}) \text{ for some } \alpha \in [0, 1)$$

$$\sum_{k \geq 0} f_k^2 < \infty$$

Abstract setting

George Roger Sell 1937–2015

Space of entire trajectories

$$\mathcal{T} = C_{\rm loc}(R; X), \ \ t \in (-\infty, \infty)$$

ω**–limit set**

$$\omega[\mathbf{U}(\cdot, X_0)] \subset \mathcal{T}$$

$$\omega[\mathbf{U}(\cdot, X_0)] = \left\{ \mathbf{V} \in \mathcal{T} \ \middle| \ \mathbf{U}(\cdot + t_n, X_0) \to \mathbf{V} \text{ in } \mathcal{T} \text{ as } t_n \to \infty \right\}$$

Necessary ingredients

- **Dissipativity** – ultimate boundedness of trajectories
- **Compactness** – in appropriate spaces

Dissipativity in the sense of Lewinson

Bounded absorbing set – deterministic case

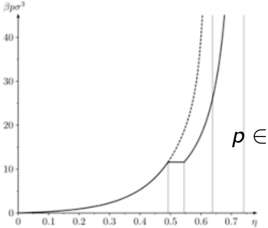
$$\limsup_{t\to\infty} \int_\Omega E(\varrho, \mathbf{m}) \ \mathrm{d}x \leq \mathcal{E}_\infty$$

Bounded absorbing set – stochastic case

$$\limsup_{t\to\infty} \mathbb{E}\left[\left(\int_\Omega E(\varrho, \mathbf{m}) \ \mathrm{d}x\right)^m\right] \leq \mathcal{E}_\infty$$

boundedness of m moments $m \leq 4$

Harp sphere pressure EOS



$$p \in C[0, \overline{\varrho}) \cap C^1(0, \overline{\varrho}), \ p'(\varrho) \geq 0 \text{ for } \varrho > 0, \ \lim_{\varrho \to \overline{\varrho}-} p(\varrho) = \infty$$

Asymptotic compactness

Generating sequence

$$\varrho_n = \varrho(t+\tau_n),\ \mathbf{m}_n = \mathbf{m}(t+\tau_n)$$

Convergence in the trajectory space (deterministic case)

$$\varrho_n \to \varrho,\ \mathbf{m}_n \to \mathbf{m} \text{ in } C_{\rm loc, weak}(R, L^2(\Omega) \times L^2(\Omega; R^d))$$

Convergence in law in the stochastic case (Skorokhod representation theorem)

$$\varrho_n \approx \widetilde{\varrho}_n \to \varrho,\ \mathbf{m}_n \approx \widetilde{m}_n \to \mathbf{m} \text{ in } C_{\rm loc, weak}(R, L^2(\Omega) \times L^2(\Omega; R^d)) \ a.s.$$

The limit solution

$$\varrho, \mathbf{m} \text{ is a solution of the Navier–Stokes system}$$

Principal problem

$$\text{compactness of } \{\varrho_n\} \ - \text{no information on the "initial data"}$$

Vanishing oscillation defect, I

Compactness of densities:

$$\varrho_n \equiv \varrho(\cdot + T_n) \to \varrho \text{ in } C_{\rm weak,loc}(R; L^\gamma(\Omega))$$

$$\varrho_n \log(\varrho_n) \to \overline{\varrho \log(\varrho)} \geq \varrho \log(\varrho)$$

$$\textbf{oscillation defect: } D(t) \equiv \int_\Omega \overline{\varrho \log(\varrho)} - \varrho \log(\varrho) \ {\rm d}x \geq 0$$

Renormalized equation:

$$\partial_t(\varrho \log(\varrho)) + {\rm div}_x(\varrho \log(\varrho)\mathbf{u}) + \varrho {\rm div}_x \mathbf{u} = 0$$

$$\frac{\rm d}{{\rm d}t} D + \int_\Omega \left[\overline{\varrho {\rm div}_x \mathbf{u}} - \varrho {\rm div}_x \mathbf{u} \right] \ {\rm d}x = 0, \ 0 \leq D \leq \overline{D}, \ t \in R$$

Lions' identity

$$\overline{\varrho {\rm div}_x \mathbf{u}} - \varrho {\rm div}_x \mathbf{u} = \overline{p(\varrho)\varrho} - \overline{p(\varrho)} \ \varrho \geq 0$$

Vanishing oscillation defect, II

Crucial differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t} D + \Psi(D) \leq 0,\ 0 \leq D \leq \overline{D},\ t \in R$$

$$\Psi \in C(R),\ \Psi(0) = 0,\ \Psi(Z)Z > 0 \text{ for } Z \neq 0$$

$$\Rightarrow$$

$$D \equiv 0$$

Relevant references

- **Bounded absorbing set**
  Deterministic case: J. Březina, EF, A. Novotný, CPDE 2020 (to appear)

  Stochastic case: D. Breit, EF, M. Hofmanová (preprint in preparation)
- **Asymptotic compactness**
  Deterministic case: F. Fanelli, EF, M. Hofmanová (archiv preprint, submitted in TAMS)

  Stochastic case:
  D. Breit, EF, M. Hofmanová, Stochastically forced compressible fluid flows, De Gruyter 2018

  D. Breit, EF, M. Hofmanová (preprint in preparation)