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SCHOTTKY COHOMOLOGIES FOR VERTEX ALGEBRAS

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Abstract. Using Schottky procedure of forming a genus g Riemann surface by multiple attaching handles to a complex sphere, we introduce cohomologies for vertex algebras by sewing procedure for coboundary operators.

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1. Introduction

In [H] (see also [Q]) the notion of a cohomology of grading-restricted vertex algebras [K, FHL, FLM, F] was introduced. In such formulation, matrix elements are associated with correlation functions for vertex algebras with formal parameters identified with local coordinates on the complex sphere [Z]. In [H] a coboundary operator for chain-cochain double complexes was introduced in terms of rational functions obtained as matrix elements for such vertex algebras. It is natural then to consider a different construction, when local coordinates for vertex operators are taken on a complex sphere with multiply sewn handles [Y]. This construction leads to a different form of coboundary operators and cohomology for a grading-restricted vertex algebra.

In this paper we introduce the cohomology of vertex algebras with coboundary operators defined by summations of matrix elements associated to multiple a genus g Riemann surface obtained by multiple sewing g handles to the complex sphere in the frames of the Schottky uniformization. Recall that correlation functions for a vertex algebra V [FHL, FLM, Z] on the torus obtained as a result of sewing a sphere to itself, one starts from matrix elements

$$\langle \mathbf{1}_V, Y(v_1,z_1)...Y(v_1,z_n)\mathbf{1}_V\rangle,$$

where $(v_1, \ldots, v_n) \in V$, $(z_1, \ldots, z_n)$ on $\Sigma^{(0)}$, and pass to matrix elements

$$\sum_{\substack{w\in W;\\ k\geq 0}} \rho^k \langle \bar{w}, Y_W(\bar{u},\eta_1)\, Y_W(v_1,z_1)...\, Y_W(v_1,z_n)\, Y_W(u,\eta_2)\, w\rangle,$$

reproducing the trace of product of vertex operators on the torus, where $\bar{w}$ is dual to w with respect to a non-degenerate bilinear form $\langle .,. \rangle$ on W, $Y_W(v_1,z_1) \ldots Y_W(v_1,z_n)$ are vertex operators in a V-module W, and η_1, $\eta_2 \in \mathbb{C}$ are coordinates of points on the sphere where a handle is attached, and ρ as introduced above. In contrust to [H], in this paper we introduce a cohomology theory which is associated on a different auxiliary space. In case of the construciton of [H], that auxiliary space is the Riemann sphere, while in our construction it is a genus g Riemann surface formed

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in the Schottky procedure of multiple sewing handles to the Riemann sphere. Such methodology is widely used in conformal field theory and as it is usually done in the procedure of construction of higher genus partition and correlation functions for vertex operator algebras.

2. Riemann Surfaces from a Sewn Sphere

Let us first recall the set up for self-sewing the complex sphere [Y]. Consider the construction of a torus $\Sigma^{(1)}$ formed by self-sewing a handle to a Riemann sphere $\Sigma^{(0)}$. This is given by Yamada formalism [Y], or so-called ρ-formalism. Let z_1, z_2 be local coordinates in the neighborhood of two separated points p_1 and p_2 on the sphere. Consider two disks

$$|z_a| \leq r_a,$$

for $r_a > 0$ and $a = 1, 2$. Note that r_1, r_2 must be sufficiently small to ensure that the disks do not intersect. Introduce a complex parameter ρ where

$$|\rho| \leq r_1 r_2,$$

and excise the disks

$$\{z_a : |z_a| < |\rho| r_{\bar{a}}^{-1}\} \subset \Sigma^{(0)},$$

to form a twice-punctured sphere

$$\widehat{\Sigma}^{(0)} = \Sigma^{(0)} \backslash \bigcup_{a=1,2} \{z_a : |z_a| < |\rho| r_{\bar{a}}^{-1}\}.$$

We use the convention $\bar{1} = 2$, $\bar{2} = 1$. We define annular regions $\mathcal{A}_a \subset \widehat{\Sigma}^{(g)}$ with

$$\mathcal{A}_a = \{z_a : |\rho| r_{\bar{a}}^{-1} \leq |z_a| \leq r_a\}$$

and identify them as a single region

$$\mathcal{A} = \mathcal{A}_1 \simeq \mathcal{A}_2,$$

via the sewing relation

$$z_1 z_2 = \rho, \tag{2.1}$$

to form a torus

$$\Sigma^{(1)} = \widehat{\Sigma}^{(0)} \backslash \{\mathcal{A}_1 \cup \mathcal{A}_2\} \cup \mathcal{A}.$$

The sewing relation (2.1) can be considered to be a parameterization of a cylinder connecting the punctured Riemann surface to itself.

One can also construct a genus g Riemann surface by simultaneous sewing of g handles to the complex sphere. Now let us recall the Schottky formulation of forming a genus g Riemann surface. We identify g pairs of annuli centred at $A_{\pm i} \in \widehat{\mathbb{C}}$, for $1 \leq 1 \leq g$, and sewing parameters ρ_i satisfying

$$(z - A_{-i})(z' - A_i) = \rho_i, \tag{2.2}$$

provided no two annuli intersect.

We define q, $a_{\pm 1}$, known as Schottky parameters, by

$$\begin{aligned} a_i &= \frac{A_i + qA_{-i}}{1+q}, \\ \chi &= \frac{q}{(1+q)^2}, \end{aligned} \tag{2.3}$$

for $i = \pm 1$. Consider the construction of a torus by sewing a handle to the Riemann sphere $\hat{\mathbb{C}}$ by identifying annular regions centered at $A_{\pm 1} \in \hat{\mathbb{C}}$ via a sewing condition

$$\left(\frac{z - a_{-1}}{z - a_1}\right)\left(\frac{z' - a_1}{z' - a_{-1}}\right) = q. \tag{2.4}$$

Inverting (2.3) we find that $q = C(\chi)$ for Catalan series

$$\begin{aligned} C(\chi) &= \frac{1 - (1 - 4\chi)^{1/2}}{2\chi} - 1 \\ &= \sum_{n \geq 1} \frac{1}{n} \binom{2n}{n+1} \chi^n. \end{aligned} \tag{2.5}$$

We may similarly construct a general genus g Riemann surface by identifying g pairs of annuli. For $i = 1, \ldots, g$ we define Schottky parameters $a_{\pm i}, q_i$ by

$$\begin{aligned} a_{\pm i} &= \frac{A_{\pm i} + qA_{\mp i}}{1+q_i}, \\ \rho_i &= -\frac{q_i(A_{-i} - A_i)^2}{(1+q_i)^2}, \end{aligned} \tag{2.6}$$

where $|q_i| < 1$ is again related to the Catalan series (2.5)

$$\begin{aligned} q_i &= C(\chi_i), \\ \chi_i &= -\frac{\rho_i}{(A_i - A_{-i})^2}. \end{aligned}$$

The Schottky sewing condition has the form

$$\left(\frac{z - a_{-i}}{z - a_i}\right)\left(\frac{z' - a_i}{z' - a_{-i}}\right) = q_i. \tag{2.7}$$

The genus g partition function for a VOA V in the canonical sewing scheme in terms of genus zero $2g$-point correlation functions can be expressed as a convergent series as follows:

$$Z_V^{(g)}(\rho_i, A_{\pm i}) = \langle \mathbf{1}_V, \prod_{i=1}^{g} \sum_{n_i \geq 0} \rho_i^{n_i} \sum_{v_i \in V_{(n)}} Y(v_i, A_{-i}) Y(\overline{v}_i, A_i) \mathbf{1}_V \rangle, \tag{2.8}$$

where $\overline{v}_i$ is dual to v_i [Zo, M, T].

3. Matrix elements in functional form

In this section, let us recall [H, Q] the functional formulation for matrix elements for a grading-restricted vertex algebra (see Appendix 6).

Let V be a grading-restricted vertex algebra and W a grading-restricted generalized V-module. Let $\overline{W}$ be the algebraic completion of W, that is,

$$\overline{W} = \prod_{n\in\mathbb{C}} W_{(n)} = (W')^* .$$

One defines also the configuration space $F\mathbb{C}_n$

$$F\mathbb{C}_n = \{(z_1, ..., z_{n+1}) : z_i \neq z_j, i \neq j\} . \tag{3.1}$$

A $\overline{W}$-valued rational function in $(z_1, \dots, z_n)$ with the only possible poles at $z_i = z_j$, $i \neq j$ is a map

$$\begin{aligned} f : F_n\mathbb{C} &\to \overline{W}, \\ (z_1, \dots, z_n) &\mapsto f(z_1, \dots, z_n), \end{aligned} \tag{3.2}$$

such that for any $w' \in W'$, matrix element

$$\langle w', f(z_1, \dots, z_n)\rangle, \tag{3.3}$$

is a rational function in $(z_1, \dots, z_n)$ with the only possible poles at $z_i = z_j$, $i \neq j$.

By a rational function of $(z_1, \dots, z_n)$, one means a function of $(z_1, \dots, z_n)$ of the form

$$f(z_1, \dots, z_n) = \frac{P(z_1, \dots, z_n)}{Q(z_1, \dots, z_n)}, \tag{3.4}$$

where $P(z_1, \dots, z_n)$ and $Q(z_1, \dots, z_n)$ are polynomials in $(z_1, \dots, z_n)$.

If the polynomials $P(z_1, \dots, z_n)$ and $Q(z_1, \dots, z_n)$ have no common factors, then for a linear factor $g(z_1, \dots, z_n)$ of $Q(z_1, \dots, z_n)$, one says that $f(z_1, \dots, z_n)$ has poles at the set of zeros of $g(z_1, \dots, z_n)$. The maximal power of $g(z_1, \dots, z_n)$ in $Q(z_1, \dots, z_n)$ is called the order of these poles.

By a rational function with the only possible poles at a set of points in $\mathbb{C}^n$, one means in [H] a rational function of the form above such that $P(z_1, \dots, z_n)$ and $Q(z_1, \dots, z_n)$ have no common factors, $Q(z_1, \dots, z_n)$ is a product of linear factors whose zeros are contained in that set of points in $\mathbb{C}^n$.

Denote the space of all $\overline{W}$-valued rational functions in $(z_1, \dots, z_n)$ by $\widetilde{W}_{z_1,\dots,z_n}$. If a meromorphic function $f(z_1, \dots, z_n)$ on a region in C^n can be analytically extended to a rational function in $(z_1, \dots, z_n)$, we will use [H, Q] $R(f(z_1, \dots, z_n))$ to denote this rational function. For each $(z_1, \dots, z_n, \zeta) \in F_{n+1}\mathbb{C}$, $v_1, \dots, v_n \in V$, $w \in W$ and $w' \in W'$, we have an element

$$E(Y_W(v_1, z_1) \cdots Y_W(v_n, z_n) Y^W_{WV}(w, \zeta)\mathbf{1}_V) \in \overline{W}, \tag{3.5}$$

given by

$$\begin{aligned} &\langle w', E(Y_W(v_1, z_1) \cdots Y_W(v_n, z_n) Y^W_{WV}(w, \zeta)\mathbf{1}_V)\rangle \\ &\quad = R(\langle w', Y_W(v_1, z_1) \cdots Y_W(v_n, z_n) Y^W_{WV}(w, \zeta)\mathbf{1}_V\rangle), \end{aligned}$$

where $Y^W_{WV}(w,\zeta)$ is the intertwining operator. It is a linear map

$$\begin{aligned} Y^W_{WV} : W\otimes V &\to W[[z,z^{-1}]],\\ w\otimes v &\mapsto Y^W_{WV}(w,z)v, \end{aligned}$$

defined by

$$Y^W_{WV}(w,z)v = e^{zL(-1)}Y_W(v,-z)w,$$

for $v\in V$ and $w\in W$.

Let $\Phi : V^{\otimes n}\to \widetilde{W}_{z_1,\dots,z_n}$, be a map composable [H, Q] (see Appendix 8) with m vertex operators. We then define

$$\Phi(E^{(l_1)}_{V;\,\mathbf{1}_V}\otimes\cdots\otimes E^{(l_n)}_{V;\,\mathbf{1}_V}) : V^{\otimes m+n}\to \widetilde{W}_{z_1,\dots,z_{m+n}},$$

by

$$\begin{aligned} &\Phi(E^{(l_1)}_{V;\,\mathbf{1}_V}\otimes\cdots\otimes E^{(l_n)}_{V;\,\mathbf{1}_V})(v_1\otimes\cdots\otimes v_{m+n-1})\\ &\quad=\Phi(E^{(l_1)}_{V;\mathbf{1}_V}(v_1\otimes\cdots\otimes v_{l_1})\otimes\cdots\\ &\qquad\qquad\otimes E^{(l_n)}_{V;\mathbf{1}_V}(v_{l_1+\cdots+l_{n-1}+1}\otimes\cdots\otimes v_{l_1+\cdots+l_{n-1}+l_n})). \end{aligned}\tag{3.6}$$

Finally, for $\zeta\in\mathbb{C}$ we introduce the special action of an E-element of the form (3.5) on Φ by adding of intertwining operators with formal parameters associated to coordinates of insertion of a handle to the sphere:

$$\begin{aligned} &E\left((v_1,z_1)\otimes\ldots\otimes(v_m,z_m);w,\zeta\right)\circ\Phi(v_{m+1}\otimes\cdots\otimes v_{m+n}))\left(z_{m+1},\dots,z_{m+n}\right)\\ &\quad=R(\langle \mathbf{1}_W, Y_W(v_1,z_1)\dots,Y_W(v_m,z_m) \qquad (3.7)\\ &\qquad Y^W_{WV}(\Phi(v_{m+1}\otimes\cdots\otimes v_{m+n}))(z_{m+1},\dots,z_{m+n})Y^W_{WV}(w,\zeta)\mathbf{1}_V,-\zeta)\mathbf{1}_V)\rangle). \end{aligned}$$

This action provides passing from a matrix element to the trace on sewn sphere. Note that this action can be combined with (3.6) (see (4.2)). The action (3.7) allows to define the coboundary operators for bicomplexes constructed for grading-restricted vertex algebras. The idea is to use E-operators involved in [H, Q] in order to define coboundary operators on the self-sewn sphere in terms of original matrix elements.

4. Schottky cohomology of a grading-restricted vertex algebra

In this section in addition to [H, Q], we define the multiple sewn cohomology (associated to the Schottky univformization of a genus g Riemann surface) for a grading-restricted vertex algebra. One can define an action of S_n on the space $\mathrm{Hom}(V^{\otimes n},\widetilde{W}_{z_1,\dots,z_n})$ of linear maps from

$$V^{\otimes n}\to \widetilde{W}_{z_1,\dots,z_n},$$

by

$$(\sigma(\Phi))(v_1\otimes\cdots\otimes v_n)=\sigma(\Phi(v_{\sigma(1)}\otimes\cdots\otimes v_{\sigma(n)})),$$

for $\sigma\in S_n$ and $(v_1,\dots,v_n)\in V$. We will use the notation $\sigma_{i_1,\dots,i_n}\in S_n$ to denote the the permutation given by $\sigma_{i_1,\dots,i_n}(j)=i_j$ for $j=1,\dots,n$.

Recall the definition of shuffles [H]. For $n \in \mathbb{N}$ and $1 \leq s \leq n-1$, let $J_{n;s}$ be the set of elements of S_n which preserve the order of the first s numbers and the order of the last $n-s$ numbers, that is,

$$J_{n,s} = \{\sigma \in S_n \mid \sigma(1) < \cdots < \sigma(s),\ \sigma(s+1) < \cdots < \sigma(n)\}.$$

Elements of $J_{n;s}$ are called shuffles. Let $J_{n;s}^{-1} = \{\sigma \mid \sigma \in J_{n;s}\}$.

Let V be a grading-restricted vertex algebra and W a V-module. For fixed m, and $n \in \mathbb{Z}_+$, let $C_m^n(V,W)$ be the vector spaces of all linear maps from $V^{\otimes m} \to \widetilde{W}_{z_1,\ldots,z_m}$ composable with m vertex operators, satisfying the $L(-1)$-derivative property and the $L(0)$-conjugation property, and such that

$$\sum_{\sigma \in J_{n;s}^{-1}} (-1)^{|\sigma|} \sigma(\Phi(v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)})) = 0. \tag{4.1}$$

Let $C_m^0(V,W) = W$. Then we have

$$C_m^n(V,W) \subset C_{m-1}^n(V,W).$$

Let us denote

$$\begin{aligned} E^{(1)} &= E((\bar{w}_p, \eta_{1,p}) \otimes (v_1, z_1); w_p, \eta_{2,p}), \\ E^{(2)} &= E((\bar{w}_p, \eta_{1,p}); w_p, \eta_{2,p}), \\ \sigma_{n+1,1,\ldots,n}\, E^{(1)} &= (-1)^{n+1} E\left((\bar{w}_p, \eta_{1,p}) \otimes (v_{n+1}, z_{n+1}); w_p, \eta_{2,p}\right). \end{aligned}$$

We then formulate

Proposition 1. *For coordinates $\eta_{1,p}$, $\eta_{2,p} \in \mathbb{C}$ $p = 1, \ldots, g$, of points on the complex sphere, and arbitrary $\zeta_i \in \mathbb{C}$, $i = 1, \ldots, n$ and let $u_p \in V$ be such that*

$$\lim_{\eta_1 \to 0} Y^\dagger(u_p, \eta_1) \mathbf{1}_W = \bar{w}_p.$$

Then the operator $\delta_m^n(\rho_1, \ldots, \rho_g; \eta_{2,1} \ldots, \eta_{2,g})$

$$\begin{aligned} \delta_m^n \Phi &= \prod_{p=1}^{g} \sum_{\substack{w_p \in W_{(k)};\\ k \geq 0}} \rho_p^k \left[\lim_{\eta_{1,p} \to 0} \left[E^{(1)} \circ \Phi \right.\right. \\ &+ \sum_{i=1}^{n} (-1)^i E^{(2)} \circ \Phi(E^{(2)}_{V;\mathbf{1}_V}(v_i, z_i - \zeta_i; v_{i+1}, z_{i+1} - \zeta_i)) \\ &\qquad \left.\left. + (-1)^{n+1} \sigma_{n+1,1,\ldots,n}\, E^{(1)} \circ \Phi \right]\right], \end{aligned} \tag{4.2}$$

for $\Phi \in C_m^n(V,W)$, defines a coboundary operator for the chain bicomplex

$$\begin{aligned} \delta_m^n : C_m^n(V,W) \to C_{m-1}^{n+1}(V,W), \\ \delta_{m-1}^{n+1} \circ \delta_m^n = 0, \end{aligned} \tag{4.3}$$

(here we omit the dependence on $(\rho_1, \ldots, \rho_g)$ and $(\eta_{2,1} \ldots, \eta_{2,g})$).

In (4.2) dagger means the dual vertex operator with respect to the bilinear form on W. With coboundary operator (4.2) one defines the n-th Schottky cohomology $H^n_m(V,W)$ of the bicomplex $(C^n_m(V,W), \delta^n_m)$ with the spaces $C^n_m(V,W)$ composable with m vertex operators to be

$$H^n_m(V,W) = \ker \delta^n_m / \mathrm{im}\ \delta^{n-1}_{m+1}.$$

Remark 1. Using modifications of (4.2) we are able to construct the spectral sequences for grading-restricted vertex operator algebra complexes which can be used in various cohomology construction, in particular, on orbifolds [V1, V2].

Proof. In (4.2) $w_p \in W_{(n)}$, $p = 1, \ldots, g$, and $\bar{w}_p$ are corresponding dual to w_p with respect to a non-degenerate non-vanishing bilinear form on W, and $\eta_{1,p}$, $\eta_{2,p} \in \mathbb{C}$ are complex coordinates of g pair of points on the sphere where g cylinders are attached to form a genus g Riemann surface in Schottky procedure (see Appendix 2). Let $v_1, \ldots, v_{n+1} \in V$, and $(z_1, \ldots, z_{n+1}) \in F_{n+1}\mathbb{C}$, and $\Phi \in C^n_m(V,W)$, and let us denote

$$\begin{aligned}
\Phi_{1,n} &= \Phi(v_1 \otimes \cdots \otimes v_n)(z_1, \ldots, z_n), \\
\Phi_i &= \Phi(v_1 \otimes \cdots \otimes v_{i-1} \otimes (Y_V(v_i, z_i - \zeta_i) Y_V(v_{i+1}, z_{i+1} - \zeta_i)\mathbf{1}_V) \\
&\qquad \otimes v_{i+2} \otimes \cdots \otimes v_{n+1})(z_1, \ldots, z_{i-1}, \zeta_i, z_{i+2}, \ldots, z_{n+1}), \\
\Phi_{2,n+1} &= \Phi(v_2 \otimes \cdots \otimes v_{n+1})(z_2, \ldots, z_{n+1})..
\end{aligned} \tag{4.4}$$

We consider (4.2) with the action defined in (3.7)

$$\begin{aligned}
&\delta^n_m \Phi(v_1 \otimes \cdots \otimes v_{n+1})(z_1, \ldots, z_{n+1}) \\
&= \prod_{p=1}^{g} \sum_{\substack{w_p \in W_{(k)}; \\ k = \mathrm{wt}(w_p) \geq 0}} \rho_p^k \left[\lim_{\eta_{1,p} \to 0} [R(\langle \mathbf{1}_W, Y_W(u_p, \eta_{1,p}) Y_W(v_1, z_1 - \eta_2) \right. \\
&\qquad\qquad Y^W_{WV}(\Phi_{2,n+1} Y^W_{WV}(w_p, \eta_{2,p})\mathbf{1}_V, -\eta_{2,p})\mathbf{1}_V\rangle) \\
&+ \sum_{i=1}^{n} (-1)^i R(\langle \mathbf{1}_W, Y^W_{WV}(\Phi_i Y^W_{WV}(w_p, \eta_{2,p})\mathbf{1}_V, -\eta_{2,p})\mathbf{1}_V\rangle\rangle) \\
&\qquad + (-1)^{n+1} R(\langle \mathbf{1}_W, Y_W(\bar{u}_1, \eta_{1,p}) Y_W(v_{n+1}, z_{n+1} - \eta_{2,p}) \\
&\qquad\qquad \left. Y^W_{WV}(\Phi_{1,n} Y^W_{Wv}(w, \eta_{2,p})\mathbf{1}_V, -\eta_{2,p})\mathbf{1}_V\rangle\rangle)]\right].
\end{aligned}$$

Note that due to

$$Y_V(v_i, z_i - \zeta_i)\, Y_V(v_{i+1}, z_{i+1} - \zeta_i)\mathbf{1}_V = Y_V(v_i, z_i - z_{i+1}) v_{i+1},$$

in (4.4), the last expression is independent of ζ_i. When we take $\zeta_i = z_{i+1}$ for $i = 1, \ldots, n$, we obtain

$$\delta_m^n \Phi = \prod_{p=1}^{g} \sum_{\substack{w_p \in W_{(k)} \\ k\geq 0}} \rho_p^k \Big[\lim_{\eta_1 \to 0} R(\langle Y_W^\dagger(u_p, \eta_{1,p}) \mathbf{1}_W,$$
$$[Y_W(v_1, z_1 - \eta_{2,p}) e^{-\eta_2 L(-1)} \Phi_{2,n+1} e^{\eta_2 L(-1)}$$
$$+ \sum_{i=1}^{n} (-1)^i e^{-\eta_{2,p} L(-1)} \Phi_i e^{\eta_{2,p} L(-1)}$$
$$+ (-1)^{n+1} Y_W(v_{n+1}, z_{n+1} - \eta_{2,p}) e^{-\eta_{2,p} L(-1)}$$
$$\Phi_{1,n} e^{\eta_{2,p} L(-1)}] Y_W(\mathbf{1}_V, -\eta_2) w_p\rangle) \Big]. \tag{4.5}$$

By performing the summation for all $w_p \in W_{(k)}$ to obtain trace function over W. Using the $L(-1)$ property (7.1) of maps Φ we finally find

$$\delta_m^n \Phi(v_1 \otimes \cdots \otimes v_{n+1}))(z_1, \ldots, z_{n+1}) = \prod_{p=1}^{g} \sum_{k\geq 0} \rho_p^k$$
$$\mathrm{Tr}_W [Y_W(v_1, z_1 - \eta_{2,p})\ \Phi_{2,n+1}$$
$$+ \sum_{i=1}^{n} (-1)^i \Phi_i + (-1)^{n+1} Y_W(v_{n+1}, z_{n+1} - \eta_{2,p})\ \Phi_{1,n}].$$

Let us now prove the convergence of (4.2). For that purpose we make a connection with the proofs of Propositions (4.1) and (4.4) in [T] stating convergence for a similar expression for a genus g n-point correlation function in the case of the Heisenberg vertex operator algebra. In order to prove the convergence of (4.5) let us use the construction of Schottky uniformization of the Riemann sphere to form a genus g Riemann surface as a auxillary space defining the differential (4.5). In this regard, let us associate the pairs of complex variables $\eta_{1,p}$, $\eta_{1,p}$ to local coordinates of pairs of points on the Riemann sphere. Each pair of coordinates describes two points to which a hangle corresponding to $1 \leq p \leq g$ in (4.5) is attached.

According to the expression of δ_m^n (4.5) and the definition of a $\overline{W}$-valued rational function (3.2), the differential δ_m^n is given by the power series in ρ_p, $1 \leq p \leq g$, with converging [H] $\overline{W}$-valued rational functions as coefficients. Each coefficient in (4.5) is given by the function

$$Z_{p,k}^{(0)}(z_1, \ldots, z_{n+1}, \eta_{2,p}) = \frac{F_{p,k}(z_1, \ldots, z_{n+1}, \eta_{2,p})}{Q_{p,k}(z_1, \ldots, z_{n+1}, \eta_{2,p})}, \tag{4.6}$$

where F_p and Q_p are polynomials in $(z_1, \ldots, z_{n+1}, \eta_{2,p})$. Thus, we can represent (4.5) in the following form

$$\delta_m^n \Phi = \prod_{p=1}^{g} \sum_{k\geq 0} \rho_p^k\ Z_{p,k}^{(0)}(z_1, \ldots, z_{n+1}, \eta_{2,p}). \tag{4.7}$$

Recall [FHL, Z] $u_i \in U$, $1 \le i \le n$ inserted at points with local coordinates z_i the ordinary n-point correlation function on a Riemann surface is defined by

$$Z^{(0)}(u_1, z_1; \ldots; u_n, z_n) = \langle \mathbf{1}_U, Y(u_1, z_1) \ldots \; Y(u_n, z_n)\mathbf{1}_U \rangle,$$

where $\langle \cdot, \cdot \rangle$ is a invariant bilinear form on U, and $Z^{(0)}(u_1, z_1; \ldots; u_n, z_n)$ it is given by a rational function of z_i. In [T] the genus g n-point correlation function is defined for a vertex operator algebra which has the form

$$Z^{(g)}(u_1, z_1; \ldots; u_n, z_n) = \prod_{p=1}^{g} \sum_{k \ge 0} \rho_p^{\mathrm{wt}(b_p)} Z^{(0)}(u_1, z_1; \ldots; u_n, z_n; \bar{b}_{-p}, y_{-p}, b_p, y_p),$$

where y_{-p}, $y_p \in \mathbb{C}$. For each $k \ge 0$, $1 \le p \le g$, we find a set of the Heisenberg vertex algebra U element $b_{2,p,k} \in U_{(k)}$, its dual $\bar{b}_{1,p,k}$, $1 \le p \le g$, and $u_{i,p,k} \in U$, $1 \le i \le n+1$, $k \ge 0$, such that the the right hand side of expression (4.7) is equal to a genus g $n+1$ point function (41) of [T], i.e.,

$$\prod_{p=1}^{g} \sum_{k \ge 0} \rho_p^k \; Z^{(0)}_{p,k}(u_{1,p,k}, z_1; \ldots; u_{n+1,p,k}, z_{n+1}; \; \bar{b}_{1,p,k}, \eta_{1,p}, \; b_{2,p,k}, \eta_{2,p}).$$

Using the MacMahon Master Theorem (2.1), and the Theorem (3.3) (the convergence of a infinite determinant depending on $\eta_{1,p}$, $\eta_{2,p}$, $1 \le p \le g$, and ρ_p), it is proven in Propositions (4.1) and (4.5) of [T] that the last expression is convergent. Thus, (4.5) is also convergent.

Now let us prove, using the Schottky uniformization, that the limiting function of the convergent sum (4.7) is analytically extendable to a $\overline{W}$-valued function defined on the configuration space $F\mathbb{C}_{n+1} = \{(z_1, ..., z_{n+1}) : z_i \ne z_j, i \ne j\}$. The element $\Phi \in C^n_m(V, W)$ is defined on the configuration space $F\mathbb{C}_n$. Due to the property of the original differential operator δ^n_m of [H], for every $w_p \in W_{(k)}$, each summand in (4.7) defines a rational function $Z^{(0)}_{p,k}(u_{1,p,k}, z_1; \ldots; u_{n+1,p,k}, z_{n+1}; \; \bar{b}_{1,p,k}, \eta_{1,p}, \; b_{2,p,k}, \eta_{2,p})$ on the configuration space $F\mathbb{C}_{n+1}$.

Let us identify the formal parameters $(z_1, ..., z_{n+1})$ with local coordinates on the initial Riemann sphere where summands over k in (4.5) are defined. Chose pairs of points with local coordinates in the sewing handle annulus (as described in Section 2) given by complex parameters $\eta_{1,p}$, $\eta_{2,p}$, $1 \le p \le g$, for insertion of g handles. Recall the sewing relation (2.2) for (Section 2) for the Schottky uniformization parameters, identifying the annuli for each handle. By the construction, it is assumed that annular regions are not intersecting. Let $\mathcal{U}_p$ be open domains in the first annulu of $1 \le p \le g$ pairs. The sewing relation (2.2) identifies coordinates on $\mathcal{U}'_p$ on the second annuli. Thus, the sewing relation (2.2) defines the extention of the function given by (4.5) on domains covering the original Riemann sphere to domains covering the resulting genus g Riemann surface. Now let z_i and z_j be any pair of $n+1$ parameteres of the function (4.5) satisfying the configuration space $F\mathbb{C}_{n+1}$ condition (3.1) on the original Riemann sphere. Let z'_i and z'_j corresponding parameters for the extention of (4.5) on the genus g Riemann surface resulting from the Schottky uniformization. By substituting (z_i, z_j) and (z'_i, z'_j) into (2.2) and expressing for (z'_i, z'_j), we obtain that $z_i = z_j$ which contradicts the configuration space $F\mathbb{C}_{n+1}$ condition (3.1). We infer that the Schottky uniformization preserves the condition for configuration space for coordinates for the whole (4.7). Thus, the Schottky uniformization defines an analytic continuation for $\overline{W}$-valued function given by the action (4.5) on the whole resulting Riemann surface. Similar, considerations prove that the limiting function of (4.5) is a $\bar{W}$-valued rational function with the only possible poles at $z_i = z_j$, $1 \leq i < j \leq n+1$. Indeed, due to properties of the initial differential δ^n_m, the summands $Z^{(0)}(v_1, z_1; \ldots; v_{n+1} z_{n+1}; \eta_{2,p})$ in (4.5) have only possible polew at $z_i = z_j$, $1 \leq i < j \leq n+1$. As above, using the sewing condition (2.2) it follows that only poles at $z'_i = z'_j$, $1 \leq i < j \leq n+1$, can appear for the $\overline{W}$-valued function resulting from (4.5).

By Proposition 3.10 of [H] (see Appendix 9 in this paper), the summand in (4.5) is composable with $m-1$ vertex operators and has the $L(-1)$-derivative property and the $L(0)$-conjugation property. Thus the operators $\delta^n_m(\Phi)$ also satisfy these properties, and $\delta^n_m \Phi \in C^{n+1}_{m-1}(V, W)$ and δ^n_m is a map with image in $C^{n+1}_{m-1}(V, W)$. The shuffle condition (4.1) insures [H] that $\delta^n_m(\Phi) \in C^n_m$ and δ^n_m is indeed a map whose image is included in the kernel of $\delta^{n-1}_{m+1}.C^{n+1}_{m-1}$. The proof of (4.3) is then provided by cancellation of combinations of E-elements as in [H] in the summands of (4.5) in (4.2). $\square$

5. Conclusions

The notion of multiple sewn Schottky cohomology for grading-restricted vertex algebras is constructed in order to enrich the structure of cohomology of vertex algebras. We propose new formula for the coboundary operators depending on g sewing parameters and leading to another sophisticated structure of cohomology spaces. Taking into account the above definitions and construction, we would like to develop a theory [G] of characteristic classes for vertex algebras.

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6. Appendix: Grading-restricted vertex algebras and modules

In this section, we recall [H, Q] the definitions of grading-restricted vertex algebra and grading-restricted generalized module. The description is over the field $\mathbb{C}$ of complex numbers. A vertex algebra $(V, Y_V, \mathbf{1}_V)$, [K] consists of a $\mathbb{Z}$-graded complex vector space $V = \coprod_{n\in\mathbb{Z}} V_{(n)}$, where $\dim V_{(n)} < \infty$ for each $n \in \mathbb{Z}$, a linear map $Y_V : V \to \mathrm{End}(V)[[z, z^{-1}]]$, for a formal parameter z and a distinguished vector $\mathbf{1}_V$. For each $v \in V$, the image under the map Y_V is the vertex operator $Y_V(v, z) = \sum_{n\in\mathbb{Z}} v(n) z^{-n-1}$, with modes $(Y_V)_n = v(n) \in \mathrm{End}(V)$, where $Y_V(v, z)\mathbf{1} = v + O(z)$.

We recall here definitions introduced in [H, Q]. A grading-restricted vertex algebra satisfies the following conditions:

(1) Grading-restriction condition: For $n \in \mathbb{Z}$, $\dim V_{(n)} < \infty$, and when n is sufficiently negative, $V_{(n)} = 0$.
(2) Lower-truncation condition for vertex operators: For $u, v \in V$, $Y_V(u, x)v$ contain only finitely many negative power terms, that is, $Y_V(u, x)v \in V((x))$ (the space of formal Laurent series in x with coefficients in V and with finitely many negative power terms).
(3) Identity property: Let 1_V be the identity operator on V. Then $Y_V(\mathbf{1}, x) = 1_V$.
(4) Creation property: For $u \in V$, $Y_V(u, x)\mathbf{1} \in V[[x]]$ and $\lim_{x\to 0} Y_V(u, x)\mathbf{1} = u$.
(5) Duality: For $u_1, u_2, v \in V$, $v' \in V' = \coprod_{n\in\mathbb{Z}} V^*_{(n)}$, the series

$$\begin{gathered}\langle v', Y_V(u_1, z_1)Y_V(u_2, z_2)v\rangle,\\ \langle v', Y_V(u_2, z_2)Y_V(u_1, z_1)v\rangle,\\ \langle v', Y_V(Y_V(u_1, z_1 - z_2)u_2, z_2)v\rangle,\end{gathered}$$

are absolutely convergent in the regions $|z_1| > |z_2| > 0$, $|z_2| > |z_1| > 0$, $|z_2| > |z_1 - z_2| > 0$, respectively, to a common rational function in z_1 and z_2 with the only possible poles at $z_1, z_2 = 0$ and $z_1 = z_2$.
(6) $L_V(0)$-bracket formula: Let $L_V(0) : V \to V$ be defined by $L_V(0)v = nv$ for $v \in V_{(n)}$. Then

$$[L_V(0), Y_V(v, x)] = Y_V(L_V(0)v, x) + x\frac{d}{dx}Y_V(v, x),$$

for $v \in V$.
(7) $L_V(-1)$-derivative property: Let $L_V(-1) : V \to V$ be the operator given by

$$L_V(-1)v = \mathrm{Res}_x x^{-2}Y_V(v, x)\mathbf{1} = Y_{(-2)}(v)\mathbf{1},$$

for $v \in V$. Then for $v \in V$,

$$\frac{d}{dx}Y_V(u, x) = Y_V(L_V(-1)u, x) = [L_V(-1), Y_V(u, x)]. \tag{6.1}$$

We denote $(v) = k$ for $v \in V_{(k)}$. One also defines a special operation $o(v) = v_{(wt v-1)}$. One also has

$$Y_V(\mathbf{1}_V, z) = 1, \quad \lim_{z\to 0} Y(u, z)\mathbf{1}_V = u.$$

Correspondingly, a grading-restricted generalized V-module is a vector space W equipped with a vertex operator map

$$\begin{aligned} Y_W : V \otimes W &\to W[[x, x^{-1}]],\\ u \otimes w &\mapsto Y_W(u, x)w = \sum_{n\in\mathbb{Z}} (Y_W)_n(u)wx^{-n-1}\end{aligned}$$

and linear operators $L_W(0)$ and $L_W(-1)$ on W satisfying the following conditions:

(1) Grading-restriction condition: The vector space W is $\mathbb{C}$-graded, that is, $W = \coprod_{n\in\mathbb{C}} W_{(n)}$, such that $W_{(n)} = 0$ when the real part of n is sufficiently negative.
(2) Lower-truncation condition for vertex operators: For $u \in V$ and $w \in W$, $Y_W(u, x)w$ contain only finitely many negative power terms, that is, $Y_W(u, x)w \in W((x))$.

(3) Identity property: Let 1_W be the identity operator on W. Then $Y_W(\mathbf{1}, x) = 1_W$.
(4) Duality: For $u_1, u_2 \in V$, $w \in W$, $w' \in W' = \coprod_{n\in\mathbb{Z}} W^*_{(n)}$, the series

$$\langle w', Y_W(u_1, z_1)Y_W(u_2, z_2)w\rangle,$$
$$\langle w', Y_W(u_2, z_2)Y_W(u_1, z_1)w\rangle,$$
$$\langle w', Y_W(Y_V(u_1, z_1 - z_2)u_2, z_2)w\rangle$$

are absolutely convergent in the regions $|z_1| > |z_2| > 0$, $|z_2| > |z_1| > 0$, $|z_2| > |z_1 - z_2| > 0$, respectively, to a common rational function in z_1 and z_2 with the only possible poles at $z_1, z_2 = 0$ and $z_1 = z_2$.
(5) $L_W(0)$-bracket formula: For $v \in V$,

$$[L_W(0), Y_W(v, x)] = Y_W(L(0)v, x) + x\frac{d}{dx}Y_W(v, x).$$

(6) $L_W(0)$-grading property: For $w \in W_{(n)}$, there exists $N \in \mathbb{Z}_+$ such that $(L_W(0) - n)^N w = 0$.
(7) $L_W(-1)$-derivative property: For $v \in V$,

$$\frac{d}{dx}Y_W(u, x) = Y_W(L_V(-1)u, x) = [L_W(-1), Y_W(u, x)].$$

Note also the $L(-1)$-translation property [K] of vertex operators which we will make use later

$$Y_W(u, z) = e^{-\zeta L(-1)}Y_W(u, z + \zeta)e^{\zeta L(-1)}. \tag{6.2}$$

where $\zeta \in \mathbb{C}$.

7. Appendix: Properties of matrix elements for grading-restricted vertex algebra

Let us recall some facts about matrix elements for a grading-restricted vertex algebra [H, Q]. For a function of $V^{\otimes n}$ inside the matrix element, $L(-1)$-derivative property means

$$\frac{\partial}{\partial z_i}\langle w', (Y(v_1 \otimes \cdots \otimes v_n))(z_1, \ldots, z_n)\rangle$$
$$= \langle w', (Y(v_1 \otimes \cdots \otimes v_{i-1} \otimes L_V(-1)v_i \otimes v_{i+1} \otimes \cdots \otimes v_n))(z_1, \ldots, z_n)\rangle,$$

for $i = 1, \ldots, n$, $v_1, \ldots, v_n \in V$ and $w' \in W'$ and (ii)

$$\left(\frac{\partial}{\partial z_1} + \cdots + \frac{\partial}{\partial z_n}\right)\langle w', (Y(v_1 \otimes \cdots \otimes v_n))(z_1, \ldots, z_n)\rangle$$
$$= \langle w', L_W(-1)(Y(v_1 \otimes \cdots \otimes v_n))(z_1, \ldots, z_n)\rangle,$$

and $v_1, \ldots, v_n \in V$, $w' \in W'$. Note that since $L_W(-1)$ is a weight-one operator on W, for any $z \in \mathbb{C}$, $e^{zL_W(-1)}$ is a well-defined linear operator on $\overline{W}$.

One has [H, Q] the following property. Let Y be a linear map having the $L(-1)$-derivative property. Then for $v_1, \ldots, v_n \in V$, $w' \in W'$, $(z_1, \ldots, z_n) \in F_n\mathbb{C}$, $z \in \mathbb{C}$

such that $(z_1+z, \dots, z_n+z) \in F_n\mathbb{C}$,

$$\langle w', e^{zL_W(-1)}(Y(v_1\otimes\cdots\otimes v_n))(z_1,\dots,z_n)\rangle$$
$$= \langle w', (Y(v_1\otimes\cdots\otimes v_n))(z_1+z,\dots,z_n+z)\rangle, \tag{7.1}$$

and for $v_1,\dots,v_n\in V$, $w'\in W'$, $(z_1,\dots,z_n)\in F_n\mathbb{C}$, $z\in\mathbb{C}$ and $1\le i\le n$ such that

$$(z_1,\dots,z_{i-1},z_i+z,z_{i+1},\dots,z_n)\in F_n\mathbb{C},$$

the power series expansion of

$$\langle w', (Y(v_1\otimes\cdots\otimes v_n))(z_1,\dots,z_{i-1},z_i+z,z_{i+1},\dots,z_n)\rangle, \tag{7.2}$$

in z is equal to the power series

$$\langle w', (Y(v_1\otimes\cdots\otimes v_{i-1}\otimes e^{zL(-1)}v_i\otimes v_{i+1}\otimes\cdots\otimes v_n))(z_1,\dots,z_n)\rangle, \tag{7.3}$$

in z. In particular, the power series (7.3) in z is absolutely convergent to (7.2) in the disk $|z|<\min_{i\neq j}\{|z_i-z_j|\}$.

8. Appendix: Definition of maps composable with vertex operators

Next we give a definition of a map composable [H, Q] with vertex operators. For a V-module $W=\coprod_{\theta\in\mathbb{C}}W_{(\theta)}$ and $\chi\in\mathbb{C}$, let $P_\chi:\overline{W}\to W_{(\chi)}$ be the projection from $\overline{W}$ to $W_{(\chi)}$. Let $\Phi: V^{\otimes n}\to \widetilde{W}_{z_1,\dots,z_n}$ be a linear map. For $m\in\mathbb{N}$, Φ is said [H, Q] to be composable with m vertex operators if the following conditions are satisfied:

(1) Let $l_1,\dots,l_n\in\mathbb{Z}_+$ such that $l_1+\cdots+l_n=m+n$, $v_1,\dots,v_{m+n}\in V$ and $w'\in W'$. Set

$$\Psi_i = (E_V^{(l_i)}(v_{l_1+\cdots+l_{i-1}+1}\otimes\cdots\otimes v_{l_1+\cdots+l_{i-1}+l_i};\mathbf{1}))(z_{l_1+\cdots+l_{i-1}+1}-\zeta_i,\dots,z_{l_1+\cdots+l_{i-1}+l_i}-\zeta_i) \tag{8.1}$$

for $i=1,\dots,n$. Then there exist positive integers $N(v_i,v_j)$ depending only on v_i and v_j for $i,j=1,\dots,k$, $i\neq j$ such that the series

$$\sum_{r_1,\dots,r_n\in\mathbb{Z}}\langle w', (\Phi(P_{r_1}\Psi_1\otimes\cdots\otimes P_{r_n}\Psi_n))(\zeta_1,\dots,\zeta_n)\rangle,$$

is absolutely convergent when

$$|z_{l_1+\cdots+l_{i-1}+p}-\zeta_i|+|z_{l_1+\cdots+l_{j-1}+q}-\zeta_i|<|\zeta_i-\zeta_j|$$

for $i,j=1,\dots,k$, $i\neq j$ and for $p=1,\dots,l_i$ and $q=1,\dots,l_j$. and the sum can be analytically extended to a rational function in $z_1,\dots,z_{m+n}$, independent of $\zeta_1,\dots,\zeta_n$, with the only possible poles at $z_i=z_j$ of order less than or equal to $N(v_i,v_j)$ for $i,j=1,\dots,k$, $i\neq j$.

(2) For $v_1,\dots,v_{m+n}\in V$, there exist positive integers $N(v_i,v_j)$ depending only on v_i and v_j for $i,j=1,\dots,k$, $i\neq j$ such that for $w'\in W'$,

$$\sum_{\theta\in\mathbb{C}}\langle w', (E_W^{(m)}(v_1\otimes\cdots\otimes v_m;$$
$$P_\theta((\Phi(v_{m+1}\otimes\cdots\otimes v_{m+n}))(z_{m+1},\dots,z_{m+n})))(z_1,\dots,z_m)\rangle$$

is absolutely convergent when $z_i \neq z_j$, $i \neq j$ $|z_i| > |z_k| > 0$ for $i = 1, \ldots, m$ and $k = m+1, \ldots, m+n$ and the sum can be analytically extended to a rational function in $z_1, \ldots, z_{m+n}$ with the only possible poles at $z_i = z_j$ of orders less than or equal to $N(v_i, v_j)$ for $i, j = 1, \ldots, k$, $i \neq j$.

9. Appendix: Properties of maps composable with a number of vertex operators

Here we recall proposition 3.10 from [H]:

Proposition 2. *Let $\Phi : V^{\otimes n} \to \widetilde{W}_{z_1,\ldots,z_n}$ be composable with m vertex operators. Then we have:*

(1) *For $p \leq m$, Φ is composable with p vertex operators and for $p, q \in \mathbb{Z}_+$ such that $p+q \leq m$ and $l_1, \ldots, l_n \in \mathbb{Z}_+$ such that $l_1 + \cdots + l_n = p+n$, $\Phi \circ (E^{(l_1)}_{V;\,\mathbf{1}} \otimes \cdots \otimes E^{(l_n)}_{V;\,\mathbf{1}})$ and $E^{(p)}_W \circ_{p+1} \Phi$ are composable with q vertex operators.*

(2) *For $p, q \in \mathbb{Z}_+$ such that $p+q \leq m$, $l_1, \ldots, l_n \in \mathbb{Z}_+$ such that $l_1 + \cdots + l_n = p+n$ and $k_1, \ldots, k_{p+n} \in \mathbb{Z}_+$ such that $k_1 + \cdots + k_{p+n} = q+p+n$, we have*
$$(\Phi \circ (E^{(l_1)}_{V;\,\mathbf{1}} \otimes \cdots \otimes E^{(l_n)}_{V;\,\mathbf{1}})) \circ (E^{(k_1)}_{V;\,\mathbf{1}} \otimes \cdots \otimes E^{(k_{p+n})}_{V;\,\mathbf{1}})$$
$$= \Phi \circ (E^{(k_1+\cdots+k_{l_1})}_{V;\,\mathbf{1}} \otimes \cdots \otimes E^{(k_{l_1+\cdots+l_{n-1}+1}+\cdots+k_{p+n})}_{V;\,\mathbf{1}}).$$

(3) *For $p, q \in \mathbb{Z}_+$ such that $p+q \leq m$ and $l_1, \ldots, l_n \in \mathbb{Z}_+$ such that $l_1 + \cdots + l_n = p+n$, we have*
$$E^{(q)}_W \circ_{q+1} (\Phi \circ (E^{(l_1)}_{V;\,\mathbf{1}} \otimes \cdots \otimes E^{(l_n)}_{V;\,\mathbf{1}})) = (E^{(q)}_W \circ_{q+1} \Phi) \circ (E^{(l_1)}_{V;\,\mathbf{1}} \otimes \cdots \otimes E^{(l_n)}_{V;\,\mathbf{1}}).$$

(4) *For $p, q \in \mathbb{Z}_+$ such that $p+q \leq m$, we have*
$$E^{(p)}_W \circ_{p+1} (E^{(q)}_W \circ_{q+1} \Phi) = E^{(p+q)}_W \circ_{p+q+1} \Phi.$$

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