Continual Lie algebras
determined by chain complexes

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CONTINUAL LIE ALGEBRAS DETERMINED BY CHAIN COMPLEXES

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ABSTRACT. Continual Lie algebras are infinite dimensional generalizations of ordinary Lie with continuous set of roots. In this paper we establish a general relation between chain complexes and continual Lie algebras. First, by defining and appropriate product among a chain complex spaces we endow it with the structure of an exterior differential graded algebra $\mathcal{C}$. The natural orthogonality condition with respect to a product bring about to $\mathcal{C}$ the structure of a graded algebra with differential relations and compatibility conditions for elements of chain complex spaces. By considering the union of all relations for independent elements in the tree graph structure of the system, we prove the main result of this paper: a chain complex brings about the structure of a continual Lie algebra with the root space determined by parameters for the complex. This provides a new source of examples of continual Lie algebras. Finally, as an example, we consider the case of Čech-de Rham complex associated to a foliation on a smooth manifold. In a particular case of this chain complex, we derive explicitly the commutation relations for corresponding continual Lie algebra.

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1. INTRODUCTION

Continual Lie algebras introduced in [12] are generalizations of ordinary Lie algebras. Then the notion was generalized and developed in [13]– [17]. These algebras are formulated in such a way that the space of roots is defined by continual sets of vectors. In commutation relations, the generators depend on kernels which are functionals of continual roots. Jacobi identities for continual Lie algebras result in non-trivial functional relations for kernels.

Though the general theory of continual Lie algebras and their representations is missing, there exist many applications, especially in the theory of completely integrable and exactly solvable models [10, 10]. Various applications can be also found in other fields of mathematics [1, 3, 13, 17].

In original papers [12]– [17], various approaches in order to find new classes of examples of continual Lie algebras were studied. In a series of papers [15, 16] they have generalized first known examples and have found new non-trivial and fundamental ones arising from various branches of mathematics, in particular, functional analysis [17], differential geometry [13], algebraic geometry, non-commutative geometry, and mathematical physics [1, 5]. Some of them were have come from physical applications, e.g., equations related to the Ricci flow and cosmology [1, 5]. Though they appear

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in various corners of modern mathematical research, at some stage, the arsenal of new ideas of generation of new examples of continual Lie algebras was exhausted. An intriguing question is how construct a general way which would allow to create continual Lie algebras. In many cases, this problem reduces to the question of finding new appropriate solutions for relations (1.3) (which is a very interesting problem by itself) following from Jacobi identities.

The idea of this work is to use properties of general chain complexes, (with spaces possessing sets of parameters) in order to derive the structure of continual Lie algebras. The plan of the paper is the following. In the next subsection we recall the notion of a continual Lie algebra [12]. In Section 2, starting from a general (infinite) chain complex, we first define a product acting among spaces of a complex. We require Leibniz rule to be fulfilled for (co)boundary operators with respect to this product. In addition to that we assume its skew-symmetry properties. Thus we endow the complex with the structure of an exterior differential graded algebra $\mathcal{C}$.

We then apply the natural condition of orthogonality with respect to a product among the chain complex spaces. For instance, in differential geometry, the orthogonality condition reduces to the integrability condition for differential forms. Further actions of (co)boundary operators, as well as consequences from the definition of the multiplication, generates a system of differential relations for elements of various chain complex spaces with extra compatibility conditions for indexes. This determines the structure of a graded algebra with differential relations for $\mathcal{C}$. By choosing independent elements satisfying the above relations, we then show the main result of this paper: for an exterior graded differential algebra associated to a chain complex, the independent generators, the system of differential relations, and Jacobi identities define the structure of a continual Lie algebra with the root space provided by the spaces parameters of a chain complex.

In Section 3 we consider the example of the Čech-de Rham cochain bicomplex [2] (c.f. Appendix) associated to a smooth manifold foliation [4]. This bicomplex has a deep geometric meaning [6] and is defined for the spaces of differential forms and holonomy embeddings acting between sections of the transversal basis for a foliation. In Appendix we recall the notions of holonomy mappings, transversal sections, and transversal basis for a foliation. The space of differential forms possesses already a structure of bigraded differential algebra with respect to the natural product [2]. For the bicomplex under consideration, we define the second product, satisfying properties required for the construction of a system of differential relations. According to the exposition we give in Section 2, by involving the orthogonality conditions for an arbitrary pairs of the Čech-de Rham bicomplex, we apply the general scheme of construction of a bigraded differential algebra. Then we single out generators and commutation relations for corresponding continual Lie algebra with the space of roots and kernels given by the sets of holonomy embedding. In a particular geometric case associated to a codimension one foliation over a three-dimensional smooth manifold, we start from the integrability condition applied to elements of the same bicomplex space which leads to the appearance of a continual Lie algebra with generators, kernels, and commutation relations explicitly described.

1.1. **Continual Lie algebras.** In this subsection we recall the notion of a continual Lie algebra introduced in [14] which is a generalization of an ordinary Lie algebra. It was then studied in [15, 16]. Suppose E is an associative algebra over $\mathbb{R}$ or $\mathbb{C}$, and

$$K_0, \; K_{\pm 1}, \; K_{0,0} : E \times E \to E,$$

are bilinear mappings. The local Lie part of a continual Lie algebra can be defined as

$$\widehat{\mathcal{G}} = \mathcal{G}_{-1} \oplus \mathcal{G}_0 \oplus \mathcal{G}_{+1},$$

where $\mathcal{G}_i$, $i = 0, \pm 1$, are isomorphic to E and parametrized by its elements. The subspaces $\mathcal{G}_i$ consist of the elements

$$\left\{ X_i(\phi), \phi \in E \right\},$$

i=0, ± 1. The generators $X_i(\phi)$ are subject to commutation relations. For instance, in the principle grading they can have the form

$$[X_0(\phi), X_0(\psi)] = X_0(K_{0,0}(\phi, \psi)),$$

$$[X_0(\phi), X_{\pm 1}(\psi)] = X_{\pm 1}(K_{\pm 1}(\phi, \psi)), \tag{1.1}$$

$$[X_{+1}(\phi), X_{-1}(\psi)] = X_0(K_0(\phi, \psi)),$$

for all ϕ, $\psi \in E$. It is also assumed that Jacobi identities are satisfied. Then the conditions on mappings $\widehat{K} = (K_{0,0}, K_0, K_{\pm 1})$, follow:

$$K_{\pm 1}(K_{0,0}(\phi, \psi), \chi) = K_{\pm 1}(\phi, K_{\pm 1}(\psi, \chi)) - K_{\pm 1}(\psi, K_{\pm 1}(\phi, \chi)),$$

$$K_{0,0}(\psi, K_0(\phi, \chi)) = K_0(K_{+1}(\psi, \phi), \chi)) + K_0(\phi, K_{-1}(\psi, \chi)), \tag{1.2}$$

for all ϕ, ψ, $\chi \in E$. An infinite dimensional algebra

$$\mathcal{G}\left(E; \widehat{K}\right) = \mathcal{G}'\left(E; \widehat{K}\right) / J,$$

is called a continual contragredient Lie algebra, where $\mathcal{G}'(E; \widehat{K})$ is a Lie algebra freely generated by $\widehat{\mathcal{G}}$, and J is the largest homogeneous ideal with trivial intersection with $\mathcal{G}_0$ (consideration of the quotient is equivalent to imposing the Serre relations in an ordinary Lie algebra case) [15, 16]. By introducing a grading operator (as in ordinary Lie algebra case), one can endow $\mathcal{G}$ with a $\mathbb{Z}$-grading

$$\mathcal{G} = \bigoplus_{i \in \mathbb{Z}} \mathcal{G}_n,$$

where elements of subspaces $\mathcal{G}_n$ satisfy the standard grading condition

$$[\mathcal{G}_n, \mathcal{G}_m] \subset \mathcal{G}_{n+m},$$

where $X_n(\phi) \in \mathcal{G}_n$, and higher mapping $K_{n,m}(\phi, \psi)$ are present in commutation relations among $X_n(\phi)$ and $X_n(\psi)$. In general, Jacobi identity for generators that belong to all grading spaces has the form

$$[X_i(\phi), [X_j(\psi), X_k(\theta)]] + [X_j(\psi), [X_k(\theta), X_i(\phi)]] + [X_k(\theta), [X_i(\phi), X_j(\psi)]] = 0,$$

$$K_{i,j+k}\left(\phi, K_{j,k}\left(\psi, \theta\right)\right) + K_{j,k+i}\left(\psi, K_{k,i}\left(\theta, \phi\right)\right) + K_{k,i+j}\left(\theta, K_{i,j}\left(\phi, \psi\right)\right) = 0. \tag{1.3}$$

2. The algebra of differential relations from a chain complex

2.1. **The exterior differential graded algebra.** In this section we formulate the construction which allow to generate an algebra of graded differential relations starting from a chain complex.

Consider a complex of spaces depending on sets of parameters Θ_i, $i \in Z$, and given by

$$\ldots \overset{\delta(i-1)}{\longrightarrow} C(i, \Theta_i) \overset{\delta(i)}{\longrightarrow} C(i+1, \Theta_{i+1}) \overset{\delta(i+1)}{\longrightarrow} \ldots . \tag{2.1}$$

First, we require it to be a a differential graded alggebra, that is, it admits a differential $\delta(i)$ satisfying the chain property

$$\delta(i+1) \circ \delta(i) = 0,$$

for $\in \mathbb{Z}$. Note that usually the index i also plays a role of a parameter counting some objects in in $C(i, \Theta_i)$.

We assume also that there exists a (not necessary associative) product among spaces $C(i, \Theta_i)$,

$$\cdot : C(i, \Theta_i) \cdot C(j, \Theta_j) \to C(f(i,j), \Theta_{g(i,j)}), \tag{2.2}$$

so for elements $\Phi_i \in C(i, \Theta_i)$ and $\Phi_j \in C(j, \Theta_j)$,

$$\Phi_i \cdot \Phi_j = \Phi_{f(i,j)},$$

where $f(i,j)$ and $g(i,j)$ are some functions of indeces i and j.

Let us assume that for all $i \in \mathbb{Z}$, Leibniz rule formula for the operator $\delta(i)$ take place, i.e.,

$$\delta(k+l)\left(\Phi_k \cdot \Phi_l\right) = (\delta(k)\Phi_k) \cdot \Phi_l + (-1)^{\deg(\Phi_k)}\Phi_k \cdot \delta(l)\Phi_l, \tag{2.3}$$

where $|\Phi_k|$ is the degree of Φ_k.

Secondly, we assume that the chain complex above possesses some properties of an exterior algebra with respect to the product (2.2). Namely, consider the elements $\Phi_k \in C(k, \Theta_k)$, and $\Phi_l \in C(l, \Theta_l)$. Let us assume that the product (2.2) satisfies the condition

$$\Phi_k \cdot \Phi_l = -\Phi_l \cdot \Phi_k. \tag{2.4}$$

for all k, $l \in \mathbb{Z}$, $\Phi_k \in C(k, \Theta_k)$ and $\Phi_l \in C(l, \Theta_l)$, and for any $p \in \mathbb{Z}$,

$$\Phi_p \cdot \Phi_p = 0. \tag{2.5}$$

Altogether, the complex (2.1) form an exterior differential graded algebra $\mathcal{C}$. In what follows, we skip the sets of parameters Θ_i in notations for spaces of complexes, i.e., we denote $C(i) = C(i, \Theta_i)$.

2.2. **The algebra of graded differential relations.** In this subsection we show how to make an exterior differential graded algebra $\mathcal{C}$ into an graded algebra with differential relations.

For some i and $j \in \mathbb{Z}$, let us introduce the orthogonality condition for a pair $C(i, \Theta_i)$ and $C(j, \Theta_j)$, with respect to the product (2.2). In particular, let us require that for a pair $C(i, \Theta_i)$, $C(j, \Theta_j)$, there exist subspaces $C'(i) \subset C(i, \Theta_i)$ and $C'(j) \subset C(j, \Theta_j)$, such that for any $\Phi_i \in C'(i)$ and $\Phi_j \in C'(j)$,

$$\Phi_i \cdot \delta(j)\Phi_j = 0, \tag{2.6}$$

namely, Φ_i is supposed to be orthogonal to $\delta(j)\Phi_j$ with respect to (2.2).

By applying further differentials to (2.6) (and further consequences of such action), and using properties of a function $f(i,j)$ we obtain relations among elements of spaces $C_i(\Theta_i)$, $i \in \mathbb{Z}$. In particular, taking into account that both sides of such relations belong to the same space, we obtain limitations (depending on the function f) on indexes. In differential geometry, the orthogonality condition (2.6) provides the definition of integrability conditions for differential forms, and leads to the Frobenius theorem (see, e.g., [7]).

Let $\chi \in C(n_0)$, $\Phi \in C(n)$, for any n_0, $n \in \mathbb{Z}$. Due to the property (2.5) of the multiplication (2.2), the orthogonality condition (2.6) applied to Φ and χ, i.e.,

$$\Phi \cdot \delta(n_0)\chi = 0, \tag{2.7}$$

implies that there exists $\alpha_1^{(R)} \in C\left(n_1^{(R)}\right)$, such that

$$\delta_{n_0}\chi = \Phi \cdot \alpha_1^{(R)}. \tag{2.8}$$

Let $r_1^{(R)}$ be the number of common parameters among k and $n_1^{(R)}$ sets of parameters for Φ and $\alpha_1^{(R)}$. Since both sides of the last relation have to belong to the same bicomplex space, the compatibility condition

$$n_0 + 1 = n + n_1^{(R)} - r_1^{(R)}, \tag{2.9}$$

occurs. Acting by $\delta(n_0 + 1)$ on (2.8) we obtain

$$0 = (\delta(n)\Phi) \cdot \alpha_1^{(R)} + (-1)^{n_0+1}\Phi \cdot \delta\left(n_1^{(R)}\right)\alpha_1^{(R)}. \tag{2.10}$$

On the other hand, (2.7) implies that there exists $\alpha_1^{(L)} \in C\left(n_1^{(L)}\right)$, such that

$$\Phi = \alpha_1^{(L)} \cdot \delta_{n_0}\chi, \tag{2.11}$$

with the condition

$$n = n_1^{(L)} + n_0 + 1 - r_1^{(L)}, \tag{2.12}$$

where $r_1^{(l)}$ is the number of common parameters among $n_1^{(L)}$ and $(n_0 + 1)$ for $\alpha_1^{(L)}$ and $\delta_{n_0}\chi$.

Consequently applying corresponding δ operators to (2.7), (2.8) and (2.11) we obtain the system of relations:

$$0 \;=\; \Phi \cdot \delta(n_0)\chi, \qquad (1)$$

$$(1) \;\Rightarrow\; \Phi \;=\; \alpha_1^{(L)} \cdot \delta(n_0)\chi, \qquad (2)$$
$$\delta.(2) \;\Rightarrow\; \delta(n)\Phi \;=\; \left(\delta\left(n_1^{(L)}\right)\alpha_1^{(L)}\right) \cdot \delta(n_0)\chi, \;\; (2') \;\Rightarrow 0,$$

$$\delta.(1) \;\Rightarrow\; 0 \;=\; (\delta(n)\Phi) \cdot \delta(n_0)\chi, \qquad (3)$$

$$(1) \;\Rightarrow\; \delta(n_0)\chi \;=\; \Phi \cdot \alpha_1^{(R)}.$$
$$0 \;=\; \delta(n)\Phi \cdot \alpha_1^{(R)} + (-1)^n \Phi \cdot \delta\left(n_1^{(R)}\right)\alpha_1^{(R)} \;\Rightarrow 0,$$

$$\text{(2.13)}$$

$$(3) \;\Rightarrow\; \delta(n_0)\chi \;=\; (\delta(n)\Phi) \cdot \alpha_2^{(RR)},$$
$$(4): 0 \;=\; (\delta(n)\Phi) \cdot \delta\left(n_2^{(RR)}\right)\alpha_2^{(RR)} \Rightarrow \left(\alpha_3^{(RRR)}\right) \Rightarrow \dots$$
$$\Downarrow \qquad\qquad\qquad (3')$$
$$\left(\alpha_3^{(RRL)}\right) \Rightarrow \dots$$
$$\Downarrow$$
$$\dots$$

$$\text{(2.14)}$$

$$(3) \;\Rightarrow\; \delta(n)\Phi \;=\; \alpha_2^{(LL)} \cdot \delta(n_0)\chi,$$
$$(5): 0 \;=\; \delta\alpha_2^{(LL)} \cdot \delta(n_0)\chi \Rightarrow \left(\alpha_3^{(LLR)}\right) \Rightarrow \dots$$
$$\Downarrow \qquad\qquad\qquad (3'')$$
$$\left(\alpha_3^{(LLL)}\right) \Rightarrow \dots$$
$$\Downarrow$$
$$\dots \qquad\qquad\qquad\qquad \text{(2.15)}$$

where we obtain (infinite) sequences (2.14) and (2.15) of pairs of relations for $\alpha_i^{(K_i)} \in C\left(n_i^{(K_i)}\right)$, $i \geq 1$. Recall that we denote such relations as $\left(\alpha_i^{(K_i)}\right)$. Corresponding indexes $n_i^{(K_i)}$ satisfy (in addition to (2.9) and (2.12)) relations for the sequence starting from (4):

$$n_0 = n + n_i^{(RRK_{i+2})} - r_i^{(RRK_{i+2})}, \qquad (2.16)$$

for the sequence starting from (5):

$$n = n_0 + n_j^{(LLK_{j+2})} - r_j^{(LLK_{j+2})}, \qquad (2.17)$$

$i, j \geq 2$.

Let us explain the notations. In (2.13) we obtain (infinite) sequences of pairs of differential relations the form (2.8)–(2.10). The (infinite) sequence of pair of relations has a tree graph structure with two sequences outgoing from one point. At each point we call one branch "left" and another branch as "right" marking corresponding pair by L or R. We denote such pairs by $\left(\alpha_i^{(K_i)}\right)$, where $\alpha_i^{(K_i)}$ is an element of $C\left(n_i^{(K_i)}\right)$ involved in differential relations, and (K_i) is a sequence of i entries each is either L or R for $i \geq 1$.

One can easily see that, not all elements in (2.13) are independent. For instance, from (2.9) and (2.12) we obtain

$$\left(n_1^{(L)} - r_1^{(L)}\right) = -\left(n_1^{(R)} - r_1^{(R)}\right) = n - n_0 - 1.$$

From (2.8)-(2.11), and from (3′)-(3″) of (2.13) we infer that $\alpha_1^{(L)}$, $\alpha_1^{(R)}$, and $\alpha_i^{(LLK_i)}$, $\alpha_i^{(RRK_i)}$ are related by a conjugation with respect to the product (2.2):

$$\Phi = \alpha_1^{(L)} \cdot \left(\Phi \cdot \alpha_1^{(R)}\right),$$

$$\delta(n_0)\chi = \left(\alpha_i^{(LLK_i)} \cdot \delta(n_0)\ \chi\right) \cdot \alpha_i^{(RRK_i)}.$$

Similar relations applies among other elements $\alpha_i^{(K_i)}$. The sequence of relations (2.13) cancels when one of relations (2.9)–(2.12) or (2.16)–(2.17) for a sequence of pairs of equations is not fulfilled. The natural grading for $\mathcal{G}(\Theta_{(n(i))})$ is given by the condition that both sides of differential relations in (2.13) belong to the same chain complex space.

Remark 1. In this paper we consider the simplest form (2.1) of a complex. In general, for more complicated actions of δ, such that

$$\delta(i) : C(i) \to C(i + k(i)),$$

where $k(i)$ depends on $i \in \mathbb{Z}$ (see, e.g., [8] for non-trivial actions of certain δ operators among chain complex spaces for vertex algebras. Consideration of such complexes will be given by the author elsewhere). corresponding differential relations as well as compatibility relations could be different from (2.13)

As an upshot, the orthogonality condition (2.6) for all choices of n_0, $n \in \mathbb{Z}$, and the conditions (2.16)–(2.17), applied to the chain complex (2.1) bring about the structure of a graded algebra with differential relations (2.13) with respect to the multiplication (2.4).

As we can see, the system of relations (2.13) has a tree structure. "Left" and "right" directions have a mixture of dependent elements. Let us denote $n(i) = n_i^{(K_i)}$. For n_0, $n \in \mathbb{Z}$, let $I(n_0, n)$ be the set of sequences of indexes $(n(i))$, $i \geq 0$, marking all paths (K_i) in the tree structure of (2.13), describing "left" or "right" choice at each point. Let

$$I = \bigcup_{n_0, n \in \mathbb{Z}} I(n_0, n),$$

be the space of all sequences over the tree graph for a complex (2.1). Denote by $I_0 \subset I$ the subset of such paths that include pairs of independent elements only. Then we

are able to single out generators and commutation relations of a a generalization of an ordinary Lie algebra with a continual space of roots, (see subsection (1.1) [14]).

2.3. **Construction of a continual Lie algebra from a chain complex.** Let us further assume the spaces $C(i)$, $i \in \mathbb{Z}$ admit also an an ordinary (not necessary commutative) product $\Phi\Psi$ among elements for $\Phi_k \in C(k)$, and $\Phi_l \in C(l)$, for all k, $l \in \mathbb{Z}$.

Then, as a product (2.2), satisfying conditions (2.4) and (2.3), one can take

$$\Phi_k \cdot \Phi_l = [\Phi_k, \Phi_l] = \Phi_k\Phi_l - \Phi_l\Phi_k, \tag{2.18}$$

where brackets mean the ordinary commutator. It is known that the introduction of the commutator with respect to the original multiplication of an algebra transfers it into a Lie algebra when Jacobi conditions are satisfied. In our case we show that the exterior differential algebra $\mathcal{C}$ defined above being supplied with the orthogonality conditions deliver the structure of a continual Lie algebra,

To complete this section, we formulate the main result of this paper:

Proposition 1. *For the set I_0 of all pairs of independent elements $\alpha_i^{(K_i)}$, the orthogonality condition* (2.6)*, the generators*

$$\left\{ \chi, \ \delta(n_0)\chi, \ \Phi, \ \delta(n)\Phi, \ \alpha_i^{(K_i)}, \ \delta(n(i))\alpha_i^{(K_i)} \right\},$$

$n(i) = n_i^{(K_i)}$*, the relations* (2.13)*, and Jacobi identities* (1.3) *form a continual Lie algebra $\mathcal{G}(\Theta_{(n(i))})$ with the root space depending on a set of parameters $\Theta_{n(i)}$, of spaces $C\left(n(i), \Theta_{n(i)}\right)$.*

Though the structure of the system (2.13) may seem to be not very complicated, actual properties of corresponding continual Lie algebra depends on properties of the spaces of a specific the bicomplex (2.1) and the nature of parameters Θ_i. For a fixed choice of a path of independent functions/differential equation in the system (2.13), there exists a variety of choices how to identify generators of a continual Lie algebra with generators of the differential algebra. Therefore, the actual form of commutation relations for corresponding continual Lie algebra varies accordingly. One can also chose various ways how to define a grading for each specific $\mathcal{G}(\Theta_{(n(i))})$, for generators of a continual Lie algebra (see subsection 1.1) resulting from (2.13). The structure of the product (2.2) together with the condition (2.3), and the action of the differentials δ provide Jacobi identities for generators on the continual Lie algebra $\mathcal{G}(\Theta_{(n(i))})$, and, simultaneously, apply conditions of the form (1.3) to elements of the parameter spaces $\Theta_{(n(i))}$, $i \geq 0$. In the next Section we specify the above construction and Proposition (1) for the case of double cochain complex (5.1)–(5.2) associated to differential forms [2]. We derive explicitly the generators and commutation relations for corresponding continual Lie algebras.

3. An example: double complex associated with foliations

Recall the formulation of the Čech-de Rham cohomology given in [2] for foliations on smooth manifolds (see Appendix). In this case the spaces in (2.1) are differential

forms $C^{n,m}(\mathcal{F})$ defined on a foliation, and the coboundary operators is given by

$$\delta^{p,q} = (-1)^p d + \delta^q,$$

where d and δ operators are defined in of Appendix. The ordinary product for differential forms $\omega^{n,m}(h_1, \ldots, h_m) \in C^{n,m}(\mathcal{F})$ is given by (5.3). The product (2.2) required for the formulation of Section 2 is provided by the commutator (2.18). As it was mentioned in [2], the bicomplex (5.1)–(5.2) possess the structure of a differential bigraded algebra with respect to the ordinary product (5.3). According to explanations of Section 2, for the bicomplex (5.1)–(5.2) generate the exterior differential graded algebra with relations (2.13) with respect to the product (2.18).

3.1. **General case.** Let us recall, that we assume in this case non-negative indexes for all bicomples spaces C^{n_i,m_i}. Let $\chi \in C^{n_0,m_0}$, $\Phi \in C^{n,m}$. The orthogonality condition (2.6) with respect ot the product (2.18) leads to systems of the form (2.13) when applied to the double complex (2.1). In particular, for $\chi \in C^{n_0,m_0}$ and $\Phi \in C^{n_0,m_0}$, we obtain the system (2.13) of relations for elements χ, $\delta^{n_0,m_0}\chi$, Φ, $\delta^{n,m}\Phi$, $\alpha_i^{(K_i)} \in C^{n_i^{(K_i)},m_i^{(K_i)}}$, $\delta^{n(i),m(i)}\alpha_i^{(K_i)}$, $i \geq 1$, for $n \geq 0$, $m \geq 0$. Let $r_i^{(K_i)}$ and $t_i^{(K_i)}$ be numbers of common degrees and transversal sections of C^{n_i,m_i} for the forms $\chi \in C^{n_0,m_0}$ and $\Phi \in C^{n_0,m_0}$. Then the compatibility conditions (2.9)–(2.12) and (2.16)–(2.17) for indexes n, m, n_0, m_0, $n_i^{(K_i)}$, $m_i^{(K_i)}$, satisfy the relations in vector form:

$$(n_0 + 1, m_0 + 1) = (n, m) + \left(n_1^{(R)}, m_1^{(R)}\right) - \left(r_1^{(R)}, t_1^{(R)}\right), \qquad (3.1)$$

for (2) in (2.13):

$$(n, m) = \left(n_1^{(L)}, m_1^{(L)}\right) + (n_0 + 1, m_0 + 1) - \left(r_1^{(L)}, t_1^{(L)}\right), \qquad (3.2)$$

For the sequence starting from (4) in (2.13) we have:

$$(n_0, m_0) = (n, m) + \left(n_i^{(RRK_i)}, m_j^{(RRK_j)}\right) - \left(r_i^{(RRK_i)}, t_j^{(RRK_j)}\right), \qquad (3.3)$$

for the sequence starting from (5) in (2.13):

$$(n, m) = (n_0, m_0) + \left(n_i^{(LLK_i)}, m_j^{(LLK_j)}\right) - \left(r_i^{(LLK_i)}, t_j^{(LLK_j)}\right). \qquad (3.4)$$

$i, j \geq 2$. Note that it is assumed that all resulting indexes compatibility conditions are non-negative. For the general complex (2.1), $i \in \mathbb{Z}$. For the bicomplex (5.1)–(5.2) we have p, q are non-negative. Since we assume that $0 \leq r_i^{(LLK_i)} \leq n_i^{(LLK_i)}$, and $0 \leq t^{(LLK_i)} \leq m_i^{(LLK_i)}$, from the compatibility conditiions (3.1)–(3.2) and (3.3)–(3.4) we see that, depending on the signs of $n - n_0 - \delta_{i,1}$, $m - m_0 - \delta_{i,1}$, and only one branch of systems of the form (2)–(2′) exists at each vertex of the three graph associated to the double complex (5.1)–(5.2).

Recall that, according to Theorem 1 of [2], the definition of the spaces $C^{n,m}$ do not depend on the choice of the transversal basis $\mathcal{U}$, still it depends on parameters of foliation $\mathcal{F}$. Nevertheless, differential forms $\omega^{n,m}$ do depend on holonomy mappings (see Appendix) h_j, $j \geq 0$, and play the role of extra parameters in the consideration.

As explained at the end of Section 2, due to Proposition (1), a path (K_i), $i \geq 1$, defining the generators $\left\{ \chi, \ \delta^{n_0,m_0}\chi, \ \Phi, \ \delta^{n,m}\Phi, \ \alpha_i^{(K_i)}, \ \delta^{n(i),m(i)}\alpha_i^{(K_i)} \right\}$, $n(i) = n_i^{(K_i)}$, $m(i) = m_i^{(K_j)}$, and relations for independent elements of $C^{n,m}$ in (2.13), form a continual Lie algebra $\mathcal{G}(\mathcal{F})$ with the space of roots provided by the holonomy mappings and of parameters of the foliation $\mathcal{F}$.

3.2. Double cochain complex: Godbillon-Vey type example.

In this subsection we provide the explicit example for Proposition (1) in the case of the orthogonality condition (2.6) applied to the particular case when, in the consideration of the precious subsection, $\Phi = \chi$. In particular, in differential geometry, the case of a foliation $\mathcal{F}$ of codimension one defined by a one-form on a three-dimensional manifold, and the formulation of the Gadbillon-Vey cohomology class, are included in this consideration.

We require the orthogonality for $\chi \in C^{n,m}(\mathcal{F})$ within its own bicomplex space, i.e., to satisfy the condition

$$\chi \cdot \delta^{n,m}\chi = 0. \tag{3.5}$$

Thus, for $\alpha_1^{(R)} \in C^{n',m'}(\mathcal{F})$ one has

$$\delta^{n,m}\chi = \chi \cdot \alpha_1^{(R)}, \tag{3.6}$$

and $n+1 = n+n'-r$, $m+1 = m+m'-t$, and (3.6) is possible only when $n' = r+1$, $m' = t+1$, $0 \leq r \leq n$, $0 \leq t \leq n$, and $\alpha_1^{(R)} \in C^{r+1,t+1}(\mathcal{F})$. If we requite from (3.5) that for $\alpha_1^{(L)} \in C^{\alpha,\beta}(\mathcal{F})$,

$$\chi = \alpha_1^{(L)} \cdot \delta^{n,m}\chi,$$

then $n = \alpha + n + 1 - r'$, $\alpha = r' - 1$, i.e., α is smaller than the common degree which is not possible and thus such $\alpha_1^{(L)}$ does not exists.

Then, as a result of (2.13), we obtain the system of differential relations:

$$\begin{aligned} 0 &= \chi \cdot \delta^{n,m}\chi, \qquad (1), \\ 0 &= (\delta^{n,m}\chi) \cdot \delta^{n,m}\chi, \qquad (3) \\ \delta^{n,m}\chi &= \chi \cdot \alpha_1^{(R)}. \\ 0 &= \delta^{n,m}\chi \cdot \alpha_1^{(R)} + (-1)^n \chi \cdot \delta^{r+1,t+1}\alpha_1^{(R)}, \end{aligned} \tag{3.7}$$

and the rest of the system (2.13) collapses since further its branches follows from (3) which is trivial.

Let us denote $\mathbf{h}_n = (h_1, \ldots, h_n)$, an n-tuple of holonomy mappings (see Appendix). In this case, for forms

$$\left\{ \chi, \ \delta^{n,m}\chi, \ \alpha_1^{(R)}, \ \delta^{r+1,t+1}\alpha_1^{(L)} \right\},$$

we obtain the following continual Lie algebra by identifying the differential forms with generators of $\mathcal{G}(\mathcal{F})$ as

$$
\begin{aligned}
X_+ &= \chi, \\
X_- &= \delta^{n,m}\chi, \\
H &= \alpha_1^{(R)}, \\
H^* &= \delta^{r+1,t+1}\alpha_1^{(R)},
\end{aligned}
$$

and the commutation relations (in a non-principal grading):

$$
\begin{aligned}
[X_+(\mathbf{h}_n), X_-(\mathbf{h}'_n)] &= 0, \\
[X_+(\mathbf{h}_n), H(\mathbf{h}_{r+1})] &= X_-\left(K_{+1,0}\left(\mathbf{h}_n, \mathbf{h}_{r+1}\right)\right), \\
[X_-(\mathbf{h}_{n+1}), H(\mathbf{h}_{r+1})] &+ (-1)^n [X_+(\mathbf{h}_n), H^*(\mathbf{h}_{r+2})] = 0.
\end{aligned}
\tag{3.8}
$$

Note that an element of $C^{n,m}(\mathcal{F})$ is an n-form $\omega(h_1, \ldots, h_n)$ depending on n holonomy maps. Thus the space of continual roots is provided by the space of holonomy embeddings (see Appendix). Taking into account (3.7), we find the kernels

$$
\begin{aligned}
K_{+1,-1}\left(\mathbf{h}_n, \mathbf{h}_{r+1}\right) &= 0, \\
K_{+1,0}\left(\mathbf{h}_n, \mathbf{h}_{r+1}\right) &= \mathbf{h}_{n+1}, \\
K_{0,-1}\left(\mathbf{h}_n, \mathbf{h}_{r+1}\right) &= \mathbf{h}_{n+2}, \\
K_{0,1}\left(\mathbf{h}_n, \mathbf{h}_{r+1}\right) &= \mathbf{h}_{n+2}.
\end{aligned}
$$

It is easy to check that Jacobi identities for generators (3.8) are fulfilled.

In the case of codimension one foliation defines by a one-form, χ, $\alpha_1^{(R)}$, $\chi \in C^{1,m}(\mathcal{F})$. Recall, [7], that the Godbillon-Vey cohomology class is given by $\left[\alpha_1^{(R)} \wedge \delta^{1,m}\alpha_1^{(R)}\right]$. The construction above clarifies the Lie-algebraic meaning of this cohomology class.

4. Conclusion

In conclusion, we would like to mention a few directions of development and further applications of the material presented in this paper. We propose a way how to associate a continual Lie algebra to a chain complex. Thus the properties, in particular, Jacobi identities, kernels, and relations of resulting continual Lie algebras depend on the kind of chain complex spaces as well as on set of their parameters. One can think of introducing various types of products suitable for the construction of more complicated than (2.13) systems of differential relations. In our particular case (Section 2), in order to make connection with continual Lie algebras, we have chosen the commutator (2.18) (with respect to the original product defined in bicomplex spaces) as the simplest natural product. One could think of other possibilities which would be coherent with the orthogonality condition (2.6).

In our exposition, the standard form of chain complexes was involved. Nevertheless, one can consider more complicated setups, in particular, complexes where differentials act in non-trivial ways with respect to indexes of spaces (c.f. examples in [8]). That would lead to alternative forms of systems of differential relations as well

as compatibility conditions. What could be especially interesting, is to treat multiple chain complexes containing combinations of few chain-cochains.

The example of the Čhech-de Rham cohomology that we study in Section 3 comes from the differential geometry of foliations. In classics, the orthogonality condition applied to elements and their differentials of one particular bicomplex space, boiles down to the integrability condition and leads to the Frobenius theorem. Then it delivers the Godbillon-Vey cohomological class whose geometric meaning is not studied completely yet [6]. As for further applications in differential geometry, starting from the orthogonality condition, it would be interesting to find other cohomological invariants related to the Čech-de Rham bicomplex for foliations, and to understand their geometric meaning. The constructions considered in this paper can be also used for the cohomology theory of smooth manifolds, in particular, in various approaches to the construction of cohomological classes (cf., in partiucar, [9]).

Wide applications are awating new examples of continual Lie algebras in the field of integrable models [11, 10]. The cases of non-commutative fields used to define continual Lie algebras would also be useful for in non-commutative geometry.

Acknowledgments

5. Appendix: Čech-de Rham complex for foliations

In this Appendix we recall [2] the notion of the basis of transversal sections, and Čech-de Rham complex for a foliation.

Let M be a manifold of dimension n, equipped with a foliation $\mathcal{F}$ of co-dimension l [2]. A transversal section of $\mathcal{F}$ is an embedded l-dimensional submanifold $U \subset M$ which is everywhere transverse to the leaves. If α is a path between two points x and y on the same leaf, and if U and V are transversal sections through x and y, then α defines a transport along the leaves from a neighborhood of x in U to a neighborhood of y in V, hence a germ of a diffeomorphism $hol(\alpha) : (U, x) \longrightarrow (V, y)$, called the holonomy of the path α. Two homotopic paths always define the same holonomy.

If such a transport is defined in all of U and embeds U into V, this embedding $h : U \hookrightarrow V$ is sometimes also denoted by $hol(\alpha) : U \hookrightarrow V$. Embeddings of this form will be called holonomy embeddings.

Transversal sections U through a point x are neighborhoods of the leaf through x in the leaf space. One defines a transversal basis for $(M, \mathcal{F})$ as a family $\mathcal{U}$ of transversal sections $U \subset M$ with the property that, if V is any transversal section through a given point $y \in M$, there exists a holonomy embedding $h : U \hookrightarrow V$ with $U \in \mathcal{U}$ and $y \in h(U)$. A transversal section is a l-disk given by a chart for the foliation. One then constructs a transversal basis $\mathcal{U}$ out of a basis $\tilde{\mathcal{U}}$ of M by domains of foliation charts $\phi_U : \tilde{U} \xrightarrow{\sim} \mathbb{R}^{n-l} \times U$, $\tilde{U} \in \tilde{\mathcal{U}}$, with $U = \mathbb{R}^l$. Note that each inclusion $\tilde{U} \hookrightarrow \tilde{V}$ between opens of $\tilde{\mathcal{U}}$ induces a holonomy embedding $h_{U,V} : U \longrightarrow V$ defined by the condition that the plaque in $\tilde{U}$ through x is contained in the plaque in $\tilde{V}$ through $h_{U,V}(x)$.

Let us recall the construction of the Čech-de Rham cohomology in [2]. Let $\mathcal{U}$ be a family of transversal sections of $\mathcal{F}$. Such a family is called a transversal basis if for each transversal section V of $\mathcal{F}$ and each point $y \in V$, there exists a section $U \in \mathcal{U}$ and a holonomy embedding $h : U \to V$ such that $y \in h(U)$.

Consider the double complex

$$C^{p,q}(\mathcal{F}) = \prod_{U_0 \xrightarrow{h_1} \cdots \xrightarrow{h_p} U_p} \Omega^q(U_0), \tag{5.1}$$

where the product ranges over all p-tuples of holonomy embeddings h_i, $0 \le i \le p$, between transversal sections from a fixed transversal basis $\mathcal{U}$, and Ω^q is the space of differential forms of order q. The vertical differential is defined as

$$(-1)^p d : C^{p,q}(\mathcal{F}) \to C^{p,q+1}(\mathcal{F}),$$

where d is the usual de Rham differential. The horizontal differential

$$\delta : C^{p,q}(\mathcal{F}) \to C^{p+1,q}(\mathcal{F}),$$

is given by

$$\delta = \sum_{i=0}^{k+1} (-1)^i \delta_i,$$

where

$$\delta_i \omega(h_1, \ldots, h_{k+1}) = \begin{cases} h_1^* \omega(h_2, \ldots, h_{k+1}) & \text{if } i = 0, \\ \omega(h_1, \ldots, h_{i+1} h_i, \ldots, h_{k+1}) & \text{if } 0 < i < k+1, \\ \omega(h_1, \ldots, h_k) & \text{if } i = k+1. \end{cases} \tag{5.2}$$

The usual product

$$(\omega \eta)(h_1, \ldots, h_{n+n'}) = (-1)^{nn'} \omega(h_1, \ldots, h_n) \ (h_1^* \ldots h_n^*) . \eta(h_{n+1}, \ldots h_{n+n'}), \tag{5.3}$$

for $\omega \in C^{n,m}$ and $\eta \in C^{n',m'}$ and h_i^* being the dual to h_i Thus $(\omega \, \eta)(h_1, \ldots, h_{n+n'}) \in C^{n+n',m+m'}$, and the product (5.3) delivers the structure of a bigraded differential algebra.

The cohomology of this complex is called the Čech-de Rham cohomology $\check{H}_{\mathcal{U}}^*(M/\mathcal{F})$ of the leaf space $M/\mathcal{F}$ with respect to the transversal basis $\mathcal{U}$.

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