When are surjective algebra homomorphisms of $\mathscr{B}(X)$ automatically injective?

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Some notation & motivation

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Can we drop the injectivity assumption in Eidelheit's Theorem?...

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Let X and Y be Banach spaces, let $\psi \colon \mathscr{B}(X) \to \mathscr{B}(Y)$ be a surjective (continuous) algebra homomorphism. Is ψ automatically injective?

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Let X be infinite-dimensional such that $\mathscr{B}(X)$ has a character $\iff \exists \varphi \colon \mathscr{B}(X) \to \mathbb{C}$ unital (surjective, continuous) algebra homomorphism.

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Example

The following Banach spaces X are such that $\mathscr{B}(X)$ has a character:

- The James space J_p (where $1 < p < \infty$), the Semadeni space $C[0, \omega_1]$, any hereditarily indecomposable space (Gowers–Maurey, Argyros–Haydon, ...);

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- Mankiewicz's separable and superreflexive space X_M, Gowers' space $\mathcal{G}$, Tarbard's indecomposable but not H.I. space X_∞, the space $C(K_0)$ where K_0 is a connected "Koszmider" space, the Motakis–Puglisi–Zisimopoulou space X_K.

In examples of the second type the character is obtained from a commutative quotient of $\mathscr{B}(X)$.

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Remark

The same argument works if we replace $\mathcal{H}$ with c_0 or ℓ_p, where $1 \leqslant p < \infty$. Indeed if X is one of the above, then by the Gohberg–Markus–Feldman Theorem the ideal lattice of $\mathscr{B}(X)$ is given by

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Definition

A Banach space X has the *SHAI property* (Surjective Homomorphisms Are Injective) if for every non-zero Banach space Y every surjective algebra homomorphism $\psi\colon \mathscr{B}(X) \to \mathscr{B}(Y)$ is injective.

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Consequently, if X has the SHAI property, Y is non-zero and there is a surjective algebra homomorphism $\psi\colon \mathscr{B}(X) \to \mathscr{B}(Y)$, then

$$\mathscr{B}(X) \cong \mathscr{B}(Y) \iff X \cong Y.$$

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(The answer to the question is YES, but a different approach is needed.)

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$T \in \mathscr{B}(X)$ is *inessential* if $I_X - ST$ is Fredholm, or equivalently

$$\dim(\mathrm{Ker}(I_X - ST)) < \infty, \quad \mathrm{codim}(\mathrm{Ran}(I_X - ST)) < \infty$$

for all $S \in \mathscr{B}(X)$.

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Fact

The set $\mathscr{E}(X)$ of inessential operators is a proper, closed, two-sided ideal of $\mathscr{B}(X)$ if X is infinite-dimensional.

For an infinite-dimensional X the chain

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Proof.

(Sketch.) Under the hypothesis $\mathscr{B}(X)$ cannot have finite-codimensional proper two-sided ideals. □

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In both cases $\mathrm{Ker}(\psi) = \mathscr{E}(X)$ thus $\mathscr{B}(X)/\mathscr{E}(X) \cong \mathscr{B}(Y)$. Note that LHS is simple because $\mathscr{E}(X)$ is maximal, but RHS is not simple as Y is infinite-dimensional. A contradiction. □

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a contradiction. Consequently $P \in \mathrm{Ker}(\psi)$ must hold.

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Main ingredient:

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Alternative proof: $\mathscr{B}(X)/\mathscr{K}(X)$ does not have minimal idempotents.

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Thus there is no Banach space Y with $\mathscr{B}(\ell_2(\lambda))/\mathscr{J} \cong \mathscr{B}(Y)$, as minimal idempotents in $\mathscr{B}(Y)$ are precisely the rank one idempotents.

The long sequence spaces

Theorem (H.–Kania)

Let λ be an infinite cardinal. Then $c_0(\lambda)$, $\ell_\infty^c(\lambda)$ and $\ell_p(\lambda)$ (for $1 \leqslant p < \infty$) have the SHAI property.

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Ingredients of the proof.

Definition

Let X and Y be Banach spaces. Let $\mathscr{S}_Y(X)$ be a subset of $\mathscr{B}(X)$ defined by

$$T \notin \mathscr{S}_Y(X) \Longleftrightarrow \exists W \subseteq X \text{ subspace with } W \cong Y \text{ such that } T|_W \text{ is bounded below.}$$

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$\mathscr{S}_Y(X)$ is called the set of Y*-singular operators on* X.

Ingredients of the proof (con't).

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The spaces $c_0(\lambda)$, $\ell_\infty^c(\lambda)$ and $\ell_p(\lambda)$ (where $1 \leqslant p < \infty$) are complementably homogeneous.

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The set $\mathscr{S}_{E_\kappa}(E_\lambda)$ is a closed, non-zero, proper two-sided ideal in $\mathscr{B}(E_\lambda)$ for every infinite cardinal $\kappa \leq \lambda$. In particular $\mathscr{S}_{E_\lambda}(E_\lambda)$ is maximal.

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Let λ and κ be uncountable cardinals with $\lambda \geqslant \kappa$, and suppose that κ is not a successor of any cardinal number. Then

$$\mathscr{S}_{E_\kappa}(E_\lambda) = \overline{\bigcup_{\alpha<\kappa} \mathscr{S}_{E_\alpha}(E_\lambda)}.$$

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Let λ and κ be infinite cardinals with $\lambda \geqslant \kappa$. Let $T \in \mathscr{B}(E_\lambda)$ be such that $T \notin \mathscr{S}_{E_\kappa}(E_\lambda)$. Then

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The proof that E_λ has SHAI uses:

- Transfinite induction on the cardinals $\kappa \leqslant \lambda$;
- the above 3 theorems;
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The proof that E_λ has SHAI uses:

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- and the Dichotomy Result II. In this context, $\overline{\mathscr{G}}_{E_\kappa}(E_\lambda) \subseteq \mathrm{Ker}(\psi)$, where $\psi\colon \mathscr{B}(E_\lambda) \to \mathscr{B}(Y)$ is some surjective, non-injective algebra hom.

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We *claim* that $\mathscr{S}_{E_\lambda}(E_\lambda) \subseteq \mathrm{Ker}(\psi)$. We consider three cases:

1. $\lambda = \omega$;
2. λ is a successor cardinal;
3. λ is uncountable and not a successor cardinal.

Proof (con't.)

(1) If $\lambda = \omega$ then $E_\lambda = c_0$ or $E_\lambda = \ell_p$, where $p \in [1, \infty]$. Then Dichotomy Result I yields

$$\mathscr{S}_{E_\lambda}(E_\lambda) = \mathscr{E}(E_\lambda) \subseteq \mathrm{Ker}(\psi).$$

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Intermezzo: Fun times around Zakopane

Figure: Descending from Kasprowy Wierch, 2018 Summer

Further results, remarks

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Further results, remarks

Recall that so far that all examples of Banach spaces X which lack SHAI have the property that there exists a character $\varphi\colon \mathscr{B}(X) \to \mathbb{C}$. (Or finite sums thereof, we can quotient to $M_n(\mathbb{C})$.)

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We can have infinite-dimensional targets for surjective, non-injective algebra homomorphisms:

Theorem (H.)

Let Y be a separable, reflexive Banach space. Let

$$X_Y := \left\{ f \in C\big([0, \omega_1]; Y\big) \colon f(\omega_1) = 0_Y \right\}.$$

There exists a surjective, non-injective algebra homomorphism

$$\psi\colon \mathscr{B}(X_Y) \to \mathscr{B}(Y).$$

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Theorem (Kania–Koszmider–Laustsen, Trans. Lond. Math. Soc., 2014)

For every $T \in \mathscr{B}(C_0[0,\omega_1))$ there exists a unique $\varphi(T) \in \mathbb{C}$ such that there exists a club ($\iff$ closed and unbounded) subset $D \subseteq [0,\omega_1)$ such that:

$$(Tf)(\alpha) = \varphi(T)f(\alpha) \quad (\alpha \in D,\ f \in C_0[0,\omega_1)).$$

Moreover, $\varphi : \mathscr{B}(C_0[0,\omega_1)) \to \mathbb{C};\ T \mapsto \varphi(T)$ is a character.

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Moreover, $\varphi : \mathscr{B}(C_0[0,\omega_1)) \to \mathbb{C};\ T \mapsto \varphi(T)$ *is a character.*

Note that the club subset in the statement is never unique.

Some remarks

- The character
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- Partial structure of the lattice of closed two-sided ideals of $\mathscr{B}(C_0[0,\omega_1))$ is given in [Kania–Laustsen, Proc. Amer. Math. Soc., 2015], in particular
$$\mathscr{E}(C_0[0,\omega_1)) = \mathscr{K}(C_0[0,\omega_1)) \subsetneq \mathcal{M}_{LW}.$$

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$$\Big[\mu = \xi\Big] \iff \Big[\langle f\otimes x, \mu\rangle = \langle f\otimes x, \xi\rangle \quad \big(x \in Y,\ f \in C_0[0,\omega_1]\big)\Big].$$

Some remarks (con't.)

- $C[0,\omega_1]\hat{\otimes}_\varepsilon Y \overset{(1)}{\cong} C([0,\omega_1];Y)$, so we can may identify elements of the form $f \otimes x$ with $f(\cdot)x$.
- Recall $X_Y := \{f \in C([0,\omega_1];Y) \colon f(\omega_1) = 0_Y\}$. Fix $\mu, \xi \in X_Y^*$, then

$$\left[\mu = \xi\right] \iff \left[\langle f \otimes x, \mu\rangle = \langle f \otimes x, \xi\rangle \quad \left(x \in Y,\ f \in C_0[0,\omega_1)\right)\right].$$

- From the above and the Hahn–Banach Separation Theorem it follows that

$$X_Y \overset{(1)}{\cong} C_0[0,\omega_1)\hat{\otimes}_\varepsilon Y.$$

Some remarks (con't.)

- By a result of Rudin we have

$$C[0,\omega_1]^* \overset{(1)}{\cong} \ell_1(\omega_1^+) := \left\{ g\colon [0,\omega_1] \to \mathbb{C} \colon \sum_{\alpha<\omega_1^+} |g(\alpha)| < \infty \right\},$$

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- $C[0,\omega_1]^* \overset{(1)}{\cong} \ell_1(\omega_1^+)$ has the Radon–Nikodým Property.
- Thus

$$\begin{aligned} C([0,\omega_1]; Y)^* &\overset{(1)}{\cong} (C[0,\omega_1]\hat{\otimes}_\varepsilon Y)^* \overset{(1)}{\cong} C[0,\omega_1]^* \hat{\otimes}_\pi Y^* \\ &\overset{(1)}{\cong} \ell_1(\omega_1^+)\hat{\otimes}_\pi Y^* \overset{(1)}{\cong} \ell_1(\omega_1^+; Y^*). \end{aligned}$$

Proof of the Theorem

Fix $S \in \mathscr{B}(X_Y)$, $x \in Y$ and $\psi \in Y^*$. For any $f \in C_0[0, \omega_1)$ we can define the map

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It is clear that $S_x^\psi f$ is a continuous map, moreover by $S(f \otimes x) \in X_Y$ we also have $(S_x^\psi f)(\omega_1) = 0$, consequently $S_x^\psi f \in C_0[0,\omega_1)$.

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Consequently, by the K–K–L Theorem there is a club subset $D_{x,\psi} \subseteq [0,\omega_1)$ such that

$$(S_x^\psi)^* \delta_\alpha = \varphi(S_x^\psi) \delta_\alpha \quad (\alpha \in D_{x,\psi}).$$

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$$\tilde{\Theta}_S : Y \times Y^* \to \mathbb{C}; \quad (x, \psi) \mapsto \varphi(S_x^\psi),$$

and we have

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$$\begin{aligned}(S_{x+\lambda y}^\psi f)(\alpha) &= \langle (S(f \otimes (x + \lambda y)))(\alpha), \psi \rangle \\ &= \langle (S(f \otimes x))(\alpha), \psi \rangle + \lambda \langle (S(f \otimes y))(\alpha), \psi \rangle \\ &= (S_x^\psi f)(\alpha) + \lambda (S_y^\psi f)(\alpha),\end{aligned}$$

proving $S_{x+\lambda y}^\psi = S_x^\psi + \lambda S_y^\psi$.

Proof of the Theorem (con't.)

Since φ is linear,

$$\begin{aligned}\tilde{\Theta}_S(x+\lambda y,\psi) &= \varphi(S^{\psi}_{x+\lambda y}) \\ &= \varphi(S^{\psi}_x + \lambda S^{\psi}_y) \\ &= \varphi(S^{\psi}_x) + \lambda\varphi(S^{\psi}_y) \\ &= \tilde{\Theta}_S(x,\psi) + \lambda\tilde{\Theta}_S(y,\psi)\end{aligned}$$

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Let $\kappa_Y\colon Y \to Y^{**}$ denote the canonical embedding. By reflexivity of Y the map

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As a countable intersection of club subsets is a club subset, we have that

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Fix $S \in \mathscr{B}(X_Y)$, $\alpha \in D^S$ and $f \in C_0[0, \omega_1)$. Define the maps

$$g_{(S,f,\alpha)}\colon Y \times Y^* \to \mathbb{C}; \quad (x, \psi) \mapsto \langle S(f \otimes x), \delta_\alpha \otimes \psi\rangle,$$
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Thus we can reformulate the above equation as

$$g_{(S,f,\alpha)}(x,\psi) = h_{(S,f,\alpha)}(x,\psi) \quad ((x,\psi) \in \mathcal{Q}\times\mathcal{R}).$$

Proof of the Theorem (con't.)

As $g_{(S,f,\alpha)}$ and $h_{(S,f,\alpha)}$ are continuous functions between metric spaces, density of $\mathcal{Q} \times \mathcal{R}$ in $Y \times Y^*$ implies that

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In other words, for any $S \in \mathscr{B}(X_Y)$ there exists a club subset $D^S \subseteq [0,\omega_1)$ such that

$$\langle f \otimes x, S^*(\delta_\alpha \otimes \psi)\rangle = \langle f \otimes x, \delta_\alpha \otimes (\Theta(S)^*\psi)\rangle$$

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Therefore we obtain that

$$S^*(\delta_\alpha \otimes \psi) = \delta_\alpha \otimes (\Theta(S)^*\psi). \tag{1}$$

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for $i \in \{1, 2\}$, all $\alpha \in D_i^S$ and all $\psi \in Y^*$. Let $\alpha \in D_1^S \cap D_2^S$, $x \in Y$ and $\psi \in Y^*$ be fixed. Then

$$\langle \Theta_1(S)x, \psi \rangle = \langle \mathbf{1}_{[0,\alpha]} \otimes x, \delta_\alpha \otimes (\Theta_1(S)^* \psi) \rangle$$

Proof of the Theorem (con't.)

We show that for any $S \in \mathscr{B}(X_Y)$ the operator $\Theta(S)$ is determined by equation (1). Indeed, suppose $\Theta_1(S), \Theta_2(S) \in \mathscr{B}(Y)$ are such that there exist club subsets $D_1^S, D_2^S \subseteq [0, \omega_1)$ with the property that

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for $i \in \{1, 2\}$, all $\alpha \in D_i^S$ and all $\psi \in Y^*$. Let $\alpha \in D_1^S \cap D_2^S$, $x \in Y$ and $\psi \in Y^*$ be fixed. Then

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and thus $\Theta_1(S) = \Theta_2(S)$.

We are now prepared to prove that Θ is an algebra homomorphism.

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We show that Θ is multiplicative. Let $S, T \in \mathscr{B}(X_Y)$ be fixed.

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Linearity can be shown with analogous reasoning.

For any $S \in \mathscr{B}(X_Y)$ we have $\|\Theta(S)\| = \|\tilde{\Theta}_S\| \leq \|S\|$, thus $\|\Theta\| \leq 1$.

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We now show that Θ is surjective. We show more: There exists a norm one algebra homomorphism

$$\Lambda\colon \mathscr{B}(Y) \to \mathscr{B}(X_Y) \quad \text{with} \quad \Theta \circ \Lambda = \mathrm{id}_{\mathscr{B}(Y)}.$$

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$$((P \otimes_\varepsilon A)(g \otimes x))(\omega_1) = (Pg)(\omega_1)Ax = 0$$

holds for any $g \in C[0,\omega_1]$ and $x \in Y$, since $Pg \in C_0[0,\omega_1)$.

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Thus by linearity and continuity of $P \otimes_\varepsilon A$ in fact

$$((P \otimes_\varepsilon A)u)(\omega_1) = 0 \quad (u \in C[0, \omega_1] \hat{\otimes}_\varepsilon Y),$$

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and thus $\Theta(S) = A$. In particular, we obtain $\Theta(I_{X_Y}) = I_Y$, with $\|\Theta\| \leq 1$ this yields $\|\Theta\| = 1$.

Proof of the Theorem (con't.)

Also, the above shows that the map

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satisfies $\Theta \circ \Lambda = \mathrm{id}_{\mathscr{B}(Y)}$.

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It remains to prove that Θ is not injective. For assume towards a contradiction it is; then $\mathscr{B}(X_Y)$ and $\mathscr{B}(Y)$ are isomorphic as Banach algebras.

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It remains to prove that Θ is not injective. For assume towards a contradiction it is; then $\mathscr{B}(X_Y)$ and $\mathscr{B}(Y)$ are isomorphic as Banach algebras. By Eidelheit's Theorem this is equivalent to saying that X_Y and Y are isomorphic as Banach spaces.

Proof of the Theorem (con't.)

Also, the above shows that the map

$$\Lambda : \mathscr{B}(Y) \to \mathscr{B}(X_Y); \quad A \mapsto (P \otimes_\varepsilon A)|_{X_Y}$$

satisfies $\Theta \circ \Lambda = \mathrm{id}_{\mathscr{B}(Y)}$. It is immediate that Λ is linear with $\|\Lambda\| \leq 1$. Also, $\Lambda(I_Y) = I_{X_Y}$ holds by $I_{X_Y} = (P \otimes_\varepsilon I_Y)|_{X_Y}$, consequently $\|\Lambda\| = 1$. The map Λ is an algebra homomorphism plainly because $P \in \mathscr{B}(C[0, \omega_1])$ is an idempotent. Indeed,

$$(P \otimes_\varepsilon A)(P \otimes_\varepsilon B) = P \otimes_\varepsilon (AB) \quad (A, B \in \mathscr{B}(Y)).$$

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OK, the very last slide, really

Thank you for your attention :)

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Sources

- B. Horváth, "When are full representations of algebras of operators on Banach spaces automatically faithful?", Studia Mathematica (2020), available on the arXiv;
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