INSTITUTE OF MATHEMATICS

THE CZECH ACADEMY OF SCIENCES

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Preprint No. 70-2020

PRAHA 2020

THE SIMPLEST COHOMOLOGICAL INVARIANTS FOR VERTEX ALGEBRAS

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ABSTRACT. For the double complex structure of grading-restricted vertex algebra cohomology defined in [5], we introduce a multiplication of elements of double complex spaces. We show that the orthogonality and bi-grading conditions applied on double complex spaces, provide in relation among mappings and actions of co-boundary operators. Thus, we endow the double complex spaces with structure of bi-graded differential algebra. We then introduce the simples cohomology classes for a grading-restricted vertex algebra, and show their independence on the choice of mappings from double complex spaces. We prove that its cohomology class does not depend on mappings representing of the double complex spaces. Finally, we show that the orthogonality relations together with the bi-grading condition bring about generators and commutation relations for a continual Lie algebra.

AMS Classification: 53C12, 57R20, 17B69

1. INTRODUCTION: $\overline{W}$-VALUED RATIONAL FUNCTIONS

In [5] the cohomology theory for a grading-restricted vertex algebra [8] (see Appendix 5) was introduced. The definition of double complex spaces and co-boundary operators, uses an interpretation of vertex algebras in terms of rational functions constructed from matrix elements [7] for a grading-restricted vertex algebra. The notion of composability (see Section 1.2) of double complex space elements with a number of vertex operators, is essentially involved in the formulation. Then the cohomology of such complexes defines in the standard way a cohomology of a grading-restricted vertex algebras. It is an important problem to study possible cohomological classes for vertex algebras. In this paper we do the first steps to discover simplest cohomological invariants associated to the setup described above. For that purpose we first endow the double complex spaces with natural product, derive a counterpart of Leibniz formula for the action of co-boundary operators. Then we introduce the notion of a cohomological class for a vertex algebra. The orthogonality condition of double complex space is then defined. We show that the orthogonality being applied to the double complex spaces leads to relations among mappings and actions of co-boundary operators. The simplest non-vanishing cohomology classes for a grading-restricted vertex algebra is then derived. We show that such classes are independent of the choice of elements of the double complex spaces. Finally, we discuss occurring relations of a vertex algebra double complex relations with a continual Lie algebra [9]. For further applications of

Key words and phrases. Vertex algebras, cohomological invariants, cohomology classes.

material introduced in this paper, we would mention the natural question of searching for more general cohomological invariants for a grading-restricted vertex algebra. Concerning possible applications, one can use the cohomological classes we derive to compute higher cohomologies of grading-restricted vertex algebras.

Let V be a grading-restricted vertex algebra, and W a a grading-restricted generalized V-module (see Appendix 5). One defines the configuration space [5]:

$$F_n\mathbb{C} = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid z_i \neq z_j, i \neq j\},$$

for $n \in \mathbb{Z}_+$.

Definition 1. A $\overline{W}$-valued rational function $\mathcal{F}$ in $(z_1, \ldots, z_n)$ with the only possible poles at $z_i = z_j$, $i \neq j$, is a map

$$\begin{array}{rcl} \mathcal{F} : F_n\mathbb{C} & \to & \overline{W}, \\ (z_1, \ldots, z_n) & \mapsto & \mathcal{F}(z_1, \ldots, z_n), \end{array}$$

such that for any $w' \in W'$,

$$\langle w', \mathcal{F}(z_1, \ldots, z_n)\rangle, \tag{1.1}$$

is a rational function in $(z_1, \ldots, z_n)$ with the only possible poles at $z_i = z_j$, $i \neq j$. Such map is called in what fallows $\overline{W}$-valued rational function in $(z_1, \ldots, z_n)$ with possible other poles. Denote the space of all $\overline{W}$-valued rational functions in $(z_1, \ldots, z_n)$ by $\overline{W}_{z_1,\ldots,z_n}$.

Namely, if a meromorphic function $f(z_1, \ldots, z_n)$ on a region in $\mathbb{C}^n$ can be analytically extended to a rational function in $(z_1, \ldots, z_n)$, then the notation $R(f(z_1, \ldots, z_n))$, is used to denote such rational function. Note that the set of a grading-restricted vertex algebra elements $(v_1, \ldots, v_n)$ associated with corresponding $(z_1, \ldots, z_n)$ play the role of non-commutative parameters for a function $\mathcal{F}$ in (1.1). Let us introduce the definition of a $\mathcal{W}_{z_1,\ldots,z_n}$-space:

Definition 2. We define the space $\mathcal{W}_{z_1,\ldots,z_n}$ of $\overline{W}_{z_1,\ldots,z_n}$-valued rational forms Φ with each vertex algebra element entry v_i, $1 \leq i \leq n$ of a quasi-conformal grading-restricted vertex algebra V tensored with power wt (v_i)-differential of corresponding formal parameter z_i, i.e.,

$$\begin{aligned} &\Phi\,(v_1, z_1; \ldots; v_n, z_n) \\ &\quad = \mathcal{F}\left(dz_1^{\mathrm{wt}\ (v_1)} \otimes v_1, z_1; \ldots; dz_n^{\mathrm{wt}\ (v_n)} \otimes v_n, z_n\right) \in \mathcal{W}_{z_1,\ldots,z_n}. \end{aligned} \tag{1.2}$$

where $\mathcal{F} \in \overline{W}_{z_1,\ldots,z_n}$.

Definition 3. One defines an action of S_n on the space $\mathrm{Hom}(V^{\otimes n}, \mathcal{W}_{z_1,\ldots,z_n})$ of linear maps from $V^{\otimes n}$ to $W_{z_1,\ldots,z_n}$ by

$$\sigma(\Phi)(v_1 \otimes \cdots \otimes v_n)(z_1, \ldots, z_n), = \Phi(v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)})(z_{\sigma(1)}, \ldots, z_{\sigma(n)}), \tag{1.3}$$

for $\sigma \in S_n$ and $v_1, \ldots, v_n \in V$, $\Phi \in \mathcal{W}_{z_1,\ldots,z_n}$. We will use the notation $\sigma_{i_1,\ldots,i_n} \in S_n$, to denote the the permutation given by $\sigma_{i_1,\ldots,i_n}(j) = i_j$, for $j = 1, \ldots, n$.

Definition 4. For $n \in \mathbb{Z}_+$, a linear map

$$\mathcal{F}(v_1, z_1; \ldots; v_n, z_n) = V^{\otimes n} \to \mathcal{W}_{z_1,\ldots,z_n},$$

is said to have the $L_V(-1)$-derivative property if

$$(i) \qquad \partial_{z_i} \mathcal{F}(v_1, z_1; \ldots; v_n, z_n) = \mathcal{F}(v_1, z_1; \ldots; L_V(-1)v_i, z_i; \ldots; v_n, z_n), \tag{1.4}$$

for $i = 1, \ldots, n$, $(v_1, \ldots, v_n) \in V$, $w' \in W$, and

$$(ii) \qquad \sum_{i=1}^{n} \partial_{z_i} \mathcal{F}(v_1, z_1; \ldots; v_n, z_n) = L_W(-1).\mathcal{F}(v_1, z_1; \ldots; v_n, z_n), \tag{1.5}$$

with some action "." of $L_W(-1)$ on $\mathcal{F}(v_1, z_1; \ldots; v_n, z_n)$.

Definition 5. A linear map

$$\mathcal{F} : V^{\otimes n} \to \mathcal{W}_{z_1,\ldots,z_n}$$

has the $L_W(0)$-conjugation property if for $(v_1, \ldots, v_n) \in V$, $(z_1, \ldots, z_n) \in F_n\mathbb{C}$, and $z \in \mathbb{C}^\times$, such that $(zz_1, \ldots, zz_n) \in F_n\mathbb{C}$,

$$z^{L_W(0)} \mathcal{F}\left(v_1, z_1; \ldots; v_n, z_n\right) = \mathcal{F}\left(z^{L_V(0)} v_1, zz_1; \ldots; z^{L_V(0)} v_n, zz_n\right). \tag{1.6}$$

1.1. **E-elements.** For $w \in W$, the $\overline{W}$-valued function $E_W^{(n)}(v_1 \otimes \cdots \otimes v_n; w)$ is given by

$$E_W^{(n)}(v_1 \otimes \cdots \otimes v_n; w)(z_1, \ldots, z_n) = E(Y_W(v_1, z_1) \cdots Y_W(v_n, z_n)w),$$

where an element $E(.) \in \overline{W}$ is given by

$$\langle w', E(.)\rangle = R(\langle w', .\rangle),$$

and $R(.)$ denotes the rationalization in the sense of [5]. Namely, if a meromorphic function $f(z_1, \ldots, z_n)$ on a region in $\mathbb{C}^n$ can be analytically extended to a rational function in $(z_1, \ldots, z_n)$, then the notation $R(f(z_1, \ldots, z_n))$ is used to denote such rational function. One defines

$$E_{WV}^{W;(n)}(w; v_1 \otimes \cdots \otimes v_n) = E_W^{(n)}(v_1 \otimes \cdots \otimes v_n; w),$$

where $E_{WV}^{W;(n)}(w; v_1 \otimes \cdots \otimes v_n)$ is an element of $\overline{W}_{z_1,\ldots,z_n}$. One defines

$$\Phi \circ \left(E_{V;\,\mathbf{1}}^{(l_1)} \otimes \cdots \otimes E_{V;\,\mathbf{1}}^{(l_n)}\right) : V^{\otimes m+n} \to \overline{W}_{z_1,\ldots,z_{m+n}},$$

by

$$\begin{aligned}
&(\Phi \circ (E_{V;\,\mathbf{1}}^{(l_1)} \otimes \cdots \otimes E_{V;\,\mathbf{1}}^{(l_n)}))(v_1 \otimes \cdots \otimes v_{m+n-1}) \\
&\quad = E(\Phi(E_{V;\mathbf{1}}^{(l_1)}(v_1 \otimes \cdots \otimes v_{l_1}) \otimes \cdots E_{V;\mathbf{1}}^{(l_n)}(v_{l_1+\cdots+l_{n-1}+1} \otimes \cdots \otimes v_{l_1+\cdots+l_{n-1}+l_n}))),
\end{aligned}$$

and

$$E_W^{(m)} \circ_{m+1} \Phi : V^{\otimes m+n} \to \overline{W}_{z_1,\ldots,z_{m+n-1}},$$

is given by

$$\begin{aligned}
&(E_W^{(m)} \circ_{m+1} \Phi)(v_1 \otimes \cdots \otimes v_{m+n}) \\
&\quad = E(E_W^{(m)}(v_1 \otimes \cdots \otimes v_m; \Phi(v_{m+1} \otimes \cdots \otimes v_{m+n}))).
\end{aligned}$$

Finally,

$$E^{W;(m)}_{WV} \circ_0 \Phi : V^{\otimes m+n} \to \overline{W}_{z_1,\dots,z_{m+n-1}},$$

is defined by

$$(E^{W;(m)}_{WV} \circ_0 \Phi)(v_1 \otimes \cdots \otimes v_{m+n}) = E(E^{W;(m)}_{WV}(\Phi(v_1 \otimes \cdots \otimes v_n); v_{n+1} \otimes \cdots \otimes v_{n+m})).$$

In the case that $l_1 = \cdots = l_{i-1} = l_{i+1} = 1$ and $l_i = m - n - 1$, for some $1 \le i \le n$, we will use $\Phi \circ_i E^{(l_i)}_{V;\,\mathbf{1}}$ to denote $\Phi \circ (E^{(l_1)}_{V;\,\mathbf{1}} \otimes \cdots \otimes E^{(l_n)}_{V;\,\mathbf{1}})$.

1.2. **Maps composable with vertex operators.** Since $\overline{W}$-valued rational functions above are valued in $\overline{W}$, for $z \in \mathbb{C}^\times$, $u, v \in V$, $w \in W$, $Y_V(u,z)v \in \overline{V}$, and $Y_W(u,z)v \in \overline{W}$, one might not be able to compose in general a linear map from a tensor power of V to $\overline{W}_{z_1,\dots,z_n}$ with vertex operators. Thus in [5] they consider linear maps from tensor powers of V to $\overline{W}_{z_1,\dots,z_n}$ such that these maps can be composed with vertex operators in the sense mentioned above.

Definition 6. For a V-module $W = \coprod_{n\in\mathbb{C}} W_{(n)}$ and $m \in \mathbb{C}$, let $P_m : \overline{W} \to W_{(m)}$ be the projection from $\overline{W}$ to $W_{(m)}$. Let $\Phi : V^{\otimes n} \to \overline{W}_{z_1,\dots,z_n}$ be a linear map. For $m \in \mathbb{N}$, Φ is said [5] to be composable with m vertex operators if the following conditions are satisfied:

(1) Let $l_1, \dots, l_n \in \mathbb{Z}_+$ such that $l_1 + \cdots + l_n = m + n$, $v_1, \dots, v_{m+n} \in V$ and $w' \in W'$. Set

$$\Psi_i \quad = \quad E^{(l_i)}_V(v_{k_1} \otimes \cdots \otimes v_{k_i}; \mathbf{1}_V)(z_{k_1}, \dots, z_{k_i}),$$

where $k_1 = l_1 + \cdots + l_{i-1} + 1$, ..., $v_{k_i} = l_1 + \cdots + l_{i-1} + l_i$, for $i = 1, \dots, n$. Then there exist positive integers $N^n_m(v_i, v_j)$ depending only on v_i and v_j for $i, j = 1, \dots, k$, $i \neq j$ such that the series

$$\sum_{r_1,\dots,r_n\in\mathbb{Z}} \langle w', (\Phi(P_{r_1}\Psi_1 \otimes \cdots \otimes P_{r_n}\Psi_n))(\zeta_1, \dots, \zeta_n)\rangle,$$

is absolutely convergent when $|z_{l_1+\cdots+l_{i-1}+p} - \zeta_i| + |z_{l_1+\cdots+l_{j-1}+q} - \zeta_i| < |\zeta_i - \zeta_j|$, for $i, j = 1, \dots, k$, $i \neq j$ and for $p = 1, \dots, l_i$ and $q = 1, \dots, l_j$. The sum must be analytically extended to a rational function in $(z_1, \dots, z_{m+n})$, independent of $(\zeta_1, \dots, \zeta_n)$, with the only possible poles at $z_i = z_j$, of order less than or equal to $N^n_m(v_i, v_j)$, for $i, j = 1, \dots, k$, $i \neq j$.

(2) For $v_1, \dots, v_{m+n} \in V$, there exist positive integers $N^n_m(v_i, v_j)$, depending only on v_i and v_j, for $i, j = 1, \dots, k$, $i \neq j$, such that for $w' \in W'$, and $\mathbf{v}_{n,m} = (v_{1+m} \otimes \cdots \otimes v_{n+m})$, $\mathbf{z}_{n,m} = (z_{1+m}, \dots, z_{n+m})$, such that

$$\sum_{q\in\mathbb{C}} \langle w', (E^{(m)}_W(v_1 \otimes \cdots \otimes v_m; P_q((\Phi(\mathbf{v}_{n,m}))(\mathbf{z}_{n,m}))))\rangle,$$

is absolutely convergent when $z_i \neq z_j$, $i \neq j$ $|z_i| > |z_k| > 0$ for $i = 1, \dots, m$, and $k = m+1, \dots, m+n$, and the sum can be analytically extended to a rational function in $(z_1, \dots, z_{m+n})$ with the only possible poles at $z_i = z_j$, of orders less than or equal to $N^n_m(v_i, v_j)$, for $i, j = 1, \dots, k$, $i \neq j$,.

In [5] one finds:

Proposition 1. *The subspace of* $\mathrm{Hom}(V^{\otimes n}, \mathcal{W}_{z_1,\dots,z_n})$ *consisting of linear maps having the* $L(-1)$*-derivative property, having the* $L(0)$*-conjugation property or being composable with* m *vertex operators is invariant under the action of* S_n*.*

2. Chain complexes and cohomologies

Let us recall the definition of shuffles [5].

Definition 7. For $l \in \mathbb{N}$ and $1 \le s \le l-1$, let $J_{l;s}$ be the set of elements of S_l which preserve the order of the first s numbers and the order of the last $l-s$ numbers, i.e.,

$$J_{l,s} = \{\sigma \in S_l \mid \sigma(1) < \cdots < \sigma(s),\ \sigma(s+1) < \cdots < \sigma(l)\}.$$

The elements of $J_{l;s}$ are called shuffles. Let $J_{l;s}^{-1} = \{\sigma \mid \sigma \in J_{l;s}\}$.

Now we introduce the notion of a $C_m^n(V, \mathcal{W})$-space:

Definition 8. Let V be a vertex operator algebra and W a V-module. For $n \in \mathbb{Z}_+$, let $C_0^n(V, \mathcal{W})$ be the vector space of all linear maps from $V^{\otimes n}$ to $\mathcal{W}_{z_1,\dots,z_n}$ satisfying the $L(-1)$-derivative property and the $L(0)$-conjugation property. For m, $n \in \mathbb{Z}_+$, let $C_m^n(V, \mathcal{W})$ be the vector spaces of all linear maps from $V^{\otimes n}$ to $\mathcal{W}_{z_1,\dots,z_n}$ composable with m vertex operators, and satisfying the $L(-1)$-derivative property, the $L(0)$-conjugation property, and such that

$$\sum_{\sigma \in J_{l;s}^{-1}} (-1)^{|\sigma|} \sigma \left(\Phi(v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(l)}) \right) = 0. \tag{2.1}$$

Using a generalization of the construciton of the vertex algebra bundle and coordinate-free formulation of vertex operators in [1] for the case of $\mathcal{W}$-valued forms, we obtain following

Lemma 1. *that an element* (1.2) *of* $C_m^n(V, \mathcal{W})$ *is invariant with respect the group* $\mathrm{Aut}_{z_1,\dots,z_n}\mathcal{O}^{(n)}$ *of* n*-dimensional independent changes of formal parameters*

$$(z_1, \dots, z_n) \mapsto (\rho_1(z_1, \dots, z_n), \dots, \rho_n(z_1, \dots, z_n)).$$

□

We also find in [5]

Proposition 2. *Let* $C_m^0(V, \mathcal{W}) = \mathcal{W}$*. Then we have* $C_m^n(V, \mathcal{W}) \subset C_{m-1}^n(V, \mathcal{W})$*, for* $m \in \mathbb{Z}_+$*.*

In [5] the co-boundary operator for the double complex spaces $C_m^n(V, \mathcal{W})$ was introduced:

$$\delta_m^n : C_m^n(V, \mathcal{W}) \to C_{m-1}^{n+1}(V, \mathcal{W}). \tag{2.2}$$

For $\Phi \in C_m^n(V, \mathcal{W})$, it is given by

$$\delta_m^n(\Phi) = E_W^{(1)} \circ_2 \Phi + \sum_{i=1}^n (-1)^i \Phi \circ_i E_{V;\mathbf{1}}^{(2)} + (-1)^{n+1} \sigma_{n+1,1,\dots,n}(E_W^{(1)} \circ_2 \Phi), \tag{2.3}$$

where $\circ_i$ is defined in Subsection 1. Explicitly, for $v_1, \dots, v_{n+1} \in V$, $w' \in W'$ and $(z_1, \dots, z_{n+1}) \in F_{n+1}\mathbb{C}$,

$$\begin{aligned}
&\langle w', ((\delta^n_m(\Phi))(v_1 \otimes \cdots \otimes v_{n+1}))(z_1, \dots, z_{n+1})\rangle \\
&\quad = R(\langle w', Y_W(v_1, z_1)(\Phi(v_2 \otimes \cdots \otimes v_{n+1}))(z_2, \dots, z_{n+1})\rangle) \\
&\qquad + \sum_{i=1}^{n} (-1)^i R(\langle w', (\Phi(v_1 \otimes \cdots \otimes v_{i-1} \otimes Y_V(v_i, z_i - z_{i+1}) v_{i+1} \\
&\qquad\qquad \otimes \cdots \otimes v_{n+1}))(z_1, \dots, z_{i-1}, z_{i+1}, \dots, z_{n+1})\rangle) \\
&\qquad + (-1)^{n+1} R(\langle w', Y_W(v_{n+1}, z_{n+1})(\Phi(v_1 \otimes \cdots \otimes v_n))(z_1, \dots, z_n)\rangle).
\end{aligned}$$

In the case $n = 2$, there is a subspace of $C^2_0(V, \mathcal{W})$ containing $C^2_m(V, \mathcal{W})$ for all $m \in \mathbb{Z}_+$ such that δ^2_m is still defined on this subspace. Let $C^2_{\frac{1}{2}}(V, \mathcal{W})$ be the subspace of $C^2_0(V, \mathcal{W})$ consisting of elements Φ such that for $v_1, v_2, v_3 \in V$, $w' \in W'$,

$$\begin{aligned}
&\sum_{r\in\mathbb{C}} \big(\langle w', E^{(1)}_W(v_1; P_r((\Phi(v_2 \otimes v_3))(z_2 - \zeta, z_3 - \zeta)))(z_1, \zeta)\rangle \\
&\quad + \langle w', (\Phi(v_1 \otimes P_r((E^{(2)}_V(v_2 \otimes v_3; \mathbf{1}))(z_2 - \zeta, z_3 - \zeta))))(z_1, \zeta)\rangle\big),
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{r\in\mathbb{C}} \big(\langle w', (\Phi(P_r((E^{(2)}_V(v_1 \otimes v_2; \mathbf{1}))(z_1 - \zeta, z_2 - \zeta)) \otimes v_3))(\zeta, z_3)\rangle \\
&\quad + \langle w', E^{W;(1)}_{WV}(P_r((\Phi(v_1 \otimes v_2))(z_1 - \zeta, z_2 - \zeta)); v_3))(\zeta, z_3)\rangle\big)
\end{aligned}$$

are absolutely convergent in the regions $|z_1 - \zeta| > |z_2 - \zeta|, |z_2 - \zeta| > 0$ and $|\zeta - z_3| > |z_1 - \zeta|, |z_2 - \zeta| > 0$, respectively, and can be analytically extended to rational functions in z_1 and z_2 with the only possible poles at $z_1, z_2 = 0$ and $z_1 = z_2$. It is clear that $C^2_m(V, \mathcal{W}) \subset C^2_{\frac{1}{2}}(V, \mathcal{W})$ for $m \in \mathbb{Z}_+$. The co-boundary operator

$$\delta^2_{\frac{1}{2}} : C^2_{\frac{1}{2}}(V, \mathcal{W}) \to C^3_0(V, \mathcal{W}), \tag{2.4}$$

is defined in [5] by

$$\begin{aligned}
&\delta^2_{\frac{1}{2}}(\Phi) = E^{(1)}_W \circ_2 \Phi + \sum_{i=1}^{2} (-1)^i E^{(2)}_{V,\mathbf{1}_V} \circ_i \Phi + E^{W;(1)}_{WV} \circ_2 \Phi, \\
&\langle w', ((\delta^2_{\frac{1}{2}}(\Phi))(v_1 \otimes v_2 \otimes v_3))(z_1, z_2, z_3)\rangle \\
&= R(\langle w', (E^{(1)}_W(v_1; \Phi(v_2 \otimes v_3))(z_1, z_2, z_3)\rangle \\
&\qquad + \langle w', (\Phi(v_1 \otimes E^{(2)}_V(v_2 \otimes v_3; \mathbf{1})))(z_1, z_2, z_3)\rangle) \\
&\quad - R(\langle w', (\Phi(E^{(2)}_V(v_1 \otimes v_2; \mathbf{1})) \otimes v_3))(z_1, z_2, z_3)\rangle \\
&\qquad + \langle w', (E^{W;(1)}_{WV}(\Phi(v_1 \otimes v_2); v_3))(z_1, z_2, z_3)\rangle)
\end{aligned} \tag{2.5}$$

for $w' \in W'$, $\Phi \in C^2_{\frac{1}{2}}(V, \mathcal{W})$, $v_1, v_2, v_3 \in V$ and $(z_1, z_2, z_3) \in F_3\mathbb{C}$.

Consider the short sequence of the double complex spaces

$$0 \longrightarrow C^0_3(V, \mathcal{W}) \xrightarrow{\delta^0_3} C^1_2(V, \mathcal{W}) \xrightarrow{\delta^1_2} C^2_{\frac{1}{2}}(V, \mathcal{W}) \xrightarrow{\delta^2_{\frac{1}{2}}} C^3_0(V, \mathcal{W}) \longrightarrow 0, \tag{2.6}$$

of (2.2). The first and last arrows are trivial embeddings and projections.

In [5] we find:

Proposition 3. *For $n \in \mathbb{N}$ and $m \in \mathbb{Z}_+ + 1$, the co-boundary operators* (2.3) *and* (2.5) *satisfy the chain complex conditions, i.e.,*

$$\delta^{n+1}_{m-1} \circ \delta^n_m = 0,$$
$$\delta^2_{\frac{1}{2}} \circ \delta^1_2 = 0.$$

Since

$$\delta^1_2(C^1_2(V, \mathcal{W})) \subset C^2_1(V, \mathcal{W}) \subset C^2_{\frac{1}{2}}(V, \mathcal{W}),$$

the second formula follows from the first one, and

$$\delta^2_{\frac{1}{2}} \circ \delta^1_2 = \delta^2_1 \circ \delta^1_2 = 0.$$

Using the double complexes (2.2) and (2.4), for $m \in \mathbb{Z}_+$ and $n \in \mathbb{N}$, one introduces in [5] the n-th cohomology $H^n_m(V, W)$ of a grading-restricted vertex algebra V with coefficient in W, and composable with m vertex operators to be

$$H^n_m(V, \mathcal{W}) = \ker \delta^n_m / \mathrm{im}\ \delta^{n-1}_{m+1},$$
$$H^2_{\frac{1}{2}}(V, \mathcal{W}) = \ker \delta^2_{\frac{1}{2}} / \mathrm{im}\ \delta^1_2.$$

3. The ϵ-product of $C^n_m(V, \mathcal{W})$-spaces

In this section we introduce definition of the ϵ-product of double complex spaces $C^n_m(V, \mathcal{W})$ with the image in another double complex space coherent with respect to the original differential (2.2), and satisfying the symmetry (2.1), $L_V(0)$-conjugation (1.6), and $L_V(-1)$-derivative (1.4) properties and derive an analogue of Leibniz formula.

3.1. **Motivation and geometrical interpretation.** The structure of $C^n_m(V, \mathcal{W})$-spaces is quite complicated and it is difficult to introduce algebraically a product of its elements. In order to define an appropriate product of two $C^n_m(V, \mathcal{W})$-spaces we first have to interpret them geometrically. Basically, a $C^n_m(V, \mathcal{W})$-space must be associated with a certain model space, the algebraic $\mathcal{W}$-language should be transferred to a geometrical one, two model spaces should be "connected" appropriately, and, finally, a product should be defined.

For two $\mathcal{W}_{x_1,\ldots,x_k}$- and $\mathcal{W}_{y_1,\ldots,y_n}$-spaces we first associate formal complex parameters in the sets $(x_1, \ldots, x_k)$ and $(y_1, \ldots, y_n)$ to parameters of two auxiliary spaces. Then we describe a geometric procedure to form a resulting model space by combining two original model spaces. Formal parameters of $\mathcal{W}_{z_1,\ldots,z_{k+n}}$ should be then identified with parameters of the resulting space.

Note that according to our assumption, $(x_1, \ldots, x_k) \in F_k\mathbb{C}$, and $(y_1, \ldots, y_n) \in F_n\mathbb{C}$. As it follows from the definition of the configuration space $F_n\mathbb{C}$ in Subsection 1, in the case of coincidence of two formal parameters they are excluded from $F_n\mathbb{C}$. In general, it may happen that some number r of formal parameters of $\mathcal{W}_{x_1,\ldots,x_k}$ coincide with some r formal parameters of $\mathcal{W}_{y_1,\ldots,y_n}$ on the whole $\mathbb{C}$ (or on a domain of definition). Then, we exclude one formal parameter from each coinciding pair. We require that the set of formal parameters

$$(z_1, \ldots, z_{k+n-r}) = (\ldots, x_{i_1}, \ldots, x_{i_r}, \ldots; \ldots, \widehat{y}_{j_1}, \ldots, \widehat{y}_{j_r}, \ldots), \tag{3.1}$$

where $\widehat{.}$ denotes the exclusion of corresponding formal parameter for $x_{i_l} = y_{j_l}$, $1 \le l \le r$, for the resulting model space would belong to $F_{k+n-r}\mathbb{C}$. We denote this operation of formal parameters exclusion by $\widehat{R}\, \mathcal{F}(x_1, \ldots, x_k; y_1, \ldots, y_n; \epsilon)$.

Now we formulate the definition of the ϵ-product of two $C^n_m(V, \mathcal{W})$-spaces:

Definition 9. For $\mathcal{F}(v_1, x_1; \ldots; v_k, x_k) \in C^k_m(V, \mathcal{W})$, and $\mathcal{F}(v'_1, y_1; \ldots; v'_n, y_n) \in C^n_{m'}(V, \mathcal{W})$ the product

$$\begin{aligned} &\mathcal{F}(v_1, x_1; \ldots; v_k, x_k) \cdot_\epsilon \mathcal{F}(v'_1, y_1; \ldots; v'_n, y_n) \\ &\qquad \mapsto \widehat{R}\, \mathcal{F}\left(v_1, x_1; \ldots; v_k, x_k; v'_1, y_1; \ldots; v'_n, y_n; \epsilon\right), \end{aligned} \tag{3.2}$$

is a $\mathcal{W}_{z_1,\ldots,z_{k+n-r}}$-valued rational form

$$\begin{aligned} &\langle w', \widehat{R}\, \mathcal{F}(v_1, x_1; \ldots; v_k, x_k; v'_1, y_1; \ldots; v'_n, y_n; \epsilon)\rangle \\ &\quad = \sum_{u \in V} \langle w', Y^W_{WV}\left(\mathcal{F}(v_1, x_1; \ldots; v_k, x_k), \zeta_1\right)\ u\rangle \\ &\quad \langle w', Y^W_{WV}\left(\mathcal{F}(v'_1, y_1; \ldots; {v'}_{i_1}, \widehat{y}_{i_1}; \ldots; \ldots; {v'}_{j_r}, \widehat{y}_{j_r}; \ldots; v'_n, y_n), \zeta_2\right)\ \overline{u}\rangle, \end{aligned} \tag{3.3}$$

via (1.1), parametrized by ζ_1, $\zeta_2 \in \mathbb{C}$, and we exclude all monomials $(x_{i_l} - y_{j_l})$, $1 \le l \le r$, from (3.5). The sum is taken over any V_l-basis $\{u\}$, where $\overline{u}$ is the dual of u with respect to a non-degenerate bilinear form $\langle .\,,.\rangle_\lambda$, (5.8) over V, (see Appendix 5).

Remark 1. Due to the symmetry of the geometrical interpretation describe above, we could exclude from the set $(x_1, \ldots, x_k)$ in (3.5) r formal parameters which belong to coinciding pairs resulting to the same definition of the ϵ-product.

By the standard reasoning [2, 12], (3.5) does not depend on the choice of a basis of $u \in V_l$, $l \in \mathbb{Z}$. In the case when multiplied forms $\mathcal{F}$ do not contain V-elements, i.e., for Φ, $\Psi \in \mathcal{W}$, (3.5) defines the product $\Phi \cdot_\epsilon \Psi$ associated to a rational function:

$$\mathcal{R}(\epsilon) = \sum_{l \in \mathbb{Z}} \epsilon^l \sum_{u \in V_l} \langle w', Y^W_{WV}\left(\Phi, \zeta_1\right)\ u\rangle \langle w', Y^W_{WV}\left(\Psi, \zeta_2\right)\ \overline{u}\rangle, \tag{3.4}$$

which defines $\mathcal{F}(\epsilon) \in \mathcal{W}$ via $\mathcal{R}(\epsilon) = \langle w', \mathcal{F}(\epsilon)\rangle$.

3.2. **Convergence and properties of of the ϵ-product.** In order to prove convergence of a product of elements of two spaces $\mathcal{W}_{x_1,\ldots,x_k}$ and $\mathcal{W}_{y_1,\ldots,y_n}$ of rational $\mathcal{W}$-valued forms, we have to use a geometrical interpretation [7, 11]. Recall that a $\mathcal{W}_{z_1,\ldots,z_n}$-space is defined by means of matrix elements of the form (1.1). For a vertex algebra V, this corresponds [2] to a matrix element of a number of V-vertex operators with formal parameters identified with local coordinates on a Riemann sphere. Geometrically, each space $\mathcal{W}_{z_1,\ldots,z_n}$ can be also associated to a Riemann sphere with a few marked points, and local coordinates vanishing at these points [7]. An extra point can be associated to a center of an annulus used in order to sew the sphere with another sphere. The product (3.5) has then a geometric interpretation. The resulting model space would also be associated to a Riemann sphere formed as a result of sewing procedure. In Appendix 6 we describe explicitly the geometrical procedure of sewing of two spheres [11].

Let us identify (as in [7, 11, 12, 10, 3, 1]) two sets $(x_1, \ldots, x_k)$ and $(y_1, \ldots, y_n)$ of complex formal parameters, with local coordinates of two sets of points on the first and the second Riemann spheres correspondingly. Identify complex parameters ζ_1, ζ_2 of (3.5) with coordinates (6.1) of the annuluses (6.3). After identification of annuluses $\mathcal{A}_a$ and $\mathcal{A}_{\overline{a}}$, r coinciding coordinates may occur. This takes into account case of coinciding formal parameters. In this way, we construct the map (3.2).

As we see in (3.5), the product is defined by a sum of products of matrix elements [2] associated to each of two spheres. Such sum is supposed to describe a $\mathcal{W}$-valued rational differential form defined on a sphere formed as a result of geometrical sewing [11] of two initial spheres. Since two initial spaces $\mathcal{W}_{x_1,\ldots,x_k}$ and $\mathcal{W}_{y_1,\ldots,y_n}$ are defined through rational-valued forms expressed by matrix elements of the form (1.1). We then arrive at the resulting product defines a $\mathcal{W}_{z_1,\ldots,z_{k+n-r}}$-valued rational form by means of an absolute convergent matrix element on the resulting sphere. The complex sewing parameter, parameterizing the module space of sewin spheres, parametrizes also the product of $\mathcal{W}$-spaces.

Next, we formulate

Definition 10. We define the action of an element $\sigma \in S_{k+n-r}$ on the product of $\mathcal{F}(v_1, x_1; \ldots; v_k, x_k) \in \mathcal{W}_{x_1,\ldots,x_k}$, and $\mathcal{F}(v'_1, y_1; \ldots; v'_n, y_n) \in \mathcal{W}_{y_1,\ldots,y_n}$, as

$$\begin{aligned}
&\langle w', \sigma(\widehat{R}\,\mathcal{F})(v_1, x_1; \ldots; v_k, x_k; v'_1, y_1; \ldots; v'_n, y_n; \epsilon)\rangle \\
&\quad = \langle w', \mathcal{F}(\widetilde{v}_{\sigma(1)}, z_{\sigma(1)}; \ldots; \widetilde{v}_{\sigma(k+n-r)}, z_{\sigma(k+n-r)}; \epsilon)\rangle \\
&\quad = \sum_{u \in V} \langle w', Y^W_{WV}\left(\mathcal{F}(\widetilde{v}_{\sigma(1)}, z_{\sigma(1)}; \ldots; \widetilde{v}_{\sigma(k)}, z_{\sigma(k)}), \zeta_1\right)\ u\rangle \\
&\quad \langle w', Y^W_{WV}\left(\mathcal{F}(\widetilde{v}_{\sigma(k+1)}, z_{\sigma(k+1)}; \ldots; \widetilde{v}_{\sigma(k+n-r)}, z_{\sigma(k+n-r)}), \zeta_2\right)\ \overline{u}\rangle, \quad (3.5)
\end{aligned}$$

where by $(\widetilde{v}_{\sigma(1)}, \ldots, \widetilde{v}_{\sigma(k+n-r)})$ we denote a permutation of

$$(\widetilde{v}_1, \ldots, \widetilde{v}_{k+n-r}) = (v_1, \ldots; v_k; \ldots, \widehat{v}'_{j_1}, \ldots, \widehat{v}'_{j_r}, \ldots). \quad (3.6)$$

Let t be the number of common vertex operators the mappings $\mathcal{F}(v_1, x_1;\ \ldots;$ $v_k, x_k) \in C^k_m(V, \mathcal{W})$ and $\mathcal{F}(v'_1, y_1; \ldots; v'_n, y_n) \in C^n_{m'}(V, \mathcal{W})$, are composable with. The rational form corresponding to the ϵ-product $\widehat{R}\mathcal{F}\,(v_1, x_1; \ldots; v_k, x_k; v'_1, y_1; \ldots; v'_n, y_n; \epsilon)$ converges in ϵ, and satisfies (2.1), $L_V(0)$-conjugation (1.6) and $L_V(-1)$-derivative (1.4) properties. Using Definition 8 of $C^n_m(V, \mathcal{W})$-space and Definition 6 of mappsings composable with vertex operators, we then have

Proposition 4. *For $\mathcal{F}(v_1, x_1; \ldots; v_k, x_k) \in C^k_m(V, \mathcal{W})$ and $\mathcal{F}(v'_1, y_1; \ldots; v'_n, y_n) \in C^n_{m'}(V, \mathcal{W})$, the product $\widehat{R}\mathcal{F}\,(v_1, x_1; \ldots; v_k, x_k; v'_1, y_1; \ldots; v'_n, y_n; \epsilon)$ (3.5) belongs to the space $C^{k+n-r}_{m+m'-t}(V, \mathcal{W})$, i.e.,*

$$\cdot_\epsilon : C^k_m(V, \mathcal{W}) \times C^n_{m'}(V, \mathcal{W}) \to C^{k+n-r}_{m+m'-t}(V, \mathcal{W}). \quad (3.7)$$

□

Remark 2. Note that due to (5.3), in Definition (3.5) it is assumed that $\mathcal{F}(v_1, x_1; \ldots; v_k, x_k)$ and $\mathcal{F}(v'_1, y_1; \ldots; v'_n, y_n)$ are composable with the V-module W vertex operators $Y_W(u, -\zeta_1)$ and $Y_W(\overline{u}, -\zeta_2)$ correspondingly. The product (3.5) is actually defined by a sum of products of matrix elements of ordinary V-module W vertex operators acting on $\mathcal{W}$-elements. The elements $u \in V$ and $\overline{u} \in V'$ are connected by (5.9), and ζ_1, ζ_2 are related by (6.4). The form of the product defined above is natural in terms of the theory of chacaters for vertex operator algebras [10, 3, 12].

Remark 3. For purposes of construction of cohomological invariant, we do not exclude in this paper the case of r pais of common formal parameters $x_i = y_j$, $1 \le i \le k$, $1 \le j \le n$, for $\mathcal{F}(v_1, x_1; \ldots; v_k, x_k) \in C^k_m(V, \mathcal{W})$ and $\mathcal{F}(v'_1, y_1; \ldots; v'_n, y_n) \in C^n_{m'}(V, \mathcal{W})$ in Proposition 1. Such formal parameter pairs are excluded from the right hand side of the map (3.7).

We then have two corollaries:

Corollary 1. *For the spaces $\mathcal{W}_{x_1,\ldots,x_k}$ and $\mathcal{W}_{y_1,\ldots,y_n}$ with the product (3.5) $\mathcal{F} \in \mathcal{W}_{z_1,\ldots,z_{k+n-r}}$, the subspace of* $\mathrm{Hom}(V^{\otimes n}, \mathcal{W}_{z_1,\ldots,z_{k+n-r}}$ *consisting of linear maps having the $L_W(-1)$-derivative property, having the $L_V(0)$-conjugation property or being composable with m vertex operators is invariant under the action of S_{k+n-r}.*

Corollary 2. *For a fixed set $(v_1, \ldots v_k; v_{k+1}, \ldots, v_{k+n}) \in V$ of vertex algebra elements, and fixed $k+n$, and $m+m'$, the ϵ-product $\mathcal{F}(v_1, z_1; \ldots; v_k, z_k; v_{k+1}, z_{k+1}; \ldots; v_{k+n-r}, y_{k+n-r}; \epsilon)$,*

$$\cdot_\epsilon : C^k_m(V, \mathcal{W}) \times C^n_{m'}(V, \mathcal{W}) \to C^{k+n-r}_{m+m'-t}(V, \mathcal{W}),$$

of the spaces $C^k_m(V, \mathcal{W})$ and $C^n_{m'}(V, \mathcal{W})$, for all choices of k, n, m, $m' \ge 0$, is the same element of $C^{k+n-r}_{m+m'-t}(V, \mathcal{W})$ for all possible $k \ge 0$. □

By Lemma 1, elements of the space $C^{k+n-r}_{m+m'-t}$ resulting from the ϵ-product are invariant with respect to changes of formal parameters of the group $\mathrm{Aut}_{z_1,\ldots,z_{k+n-r}}\mathcal{O}^{(k+n-r)}$.

We then have

Definition 11. For fixed sets $(v_1, \ldots, v_k)$, $(v'_1, \ldots, v'_n) \in V$, $(x_1, \ldots, x_k) \in \mathbb{C}$, $(y_1, \ldots, y_n) \in \mathbb{C}$, we call the set of all $\mathcal{W}_{x_1,\ldots,x_k;y_1,\ldots,y_n}$-valued rational forms $\widehat{R}\mathcal{F}(v_1, x_1; \ldots; v_k, x_k; v'_1, y_1; \ldots; v'_n, y_n; \epsilon)$ defined by (3.5) with the parameter ϵ exhausting all possible values, the complete product of the spaces $\mathcal{W}_{x_1,\ldots,x_k}$ and $\mathcal{W}_{y_1,\ldots,y_n}$.

3.3. **Coboundary operator acting on the product space.** In Proposition 4 we proved that the product (3.5) of elements $\mathcal{F}_1 \mathbb{C}^k_m(V, \mathcal{W})$ and $\mathcal{F}_2 \in C^n_{m'}(V, \mathcal{W})$ belongs to $C^{k+n-r}_{m+m'-t}(V, \mathcal{W})$. Thus, the product admits the action ot the differential operator $\delta^{k+n-r}_{m+m'-t}$ defined in (2.2) where r is the number of common formal parameters, and t the number of commpon composable vertex operators for $\mathcal{F}_1$ and $\mathcal{F}_2$. The coboundary operator (2.2) possesses a variation of Leibniz law with respect to the product (3.5). We then have

Proposition 5. *For $\mathcal{F}(v_1, x_1; \ldots; v_k, x_k) \in C^k_m(V, \mathcal{W})$ and $\mathcal{F}(v'_1, y_1; \ldots; v'_n, y_n) \in C^n_{m'}(V, \mathcal{W})$, the action of $\delta^{k+n-r}_{m+m'-t}$ on their product* (3.5) *is given by*

$$\begin{aligned}
&\delta^{k+n-r}_{m+m'-t} \left(\mathcal{F}(v_1, x_1; \ldots; v_k, x_k) \cdot_\epsilon \mathcal{F}(v'_1, y_1; \ldots; v'_n, y_n)\right) \\
&\quad= \left(\delta^k_m \mathcal{F}(\widetilde{v}_1, z_1; \ldots; \widetilde{v}_k, z_k)\right) \cdot_\epsilon \mathcal{F}(\widetilde{v}_{k+1}, z_{k+1}; \ldots; \widetilde{v}_{k+n}, z_{k+n-r}) \\
&+(-1)^k \mathcal{F}(\widetilde{v}_1, z_1; \ldots; \widetilde{v}_k, z_k) \cdot_\epsilon \left(\delta^{n-r}_{m'-t} \mathcal{F}(\widetilde{v}_1, z_{k+1}; \ldots; \widetilde{v}_{k+n-r}, z_{k+n-r})\right),
\end{aligned} \tag{3.8}$$

where we use the notation as in (3.1) *and* (3.6).

Appendix 7 contains the proof of this Proposition.

Remark 4. Checking (2.2) we see that an extra arbitrary vertex algebra element $v_{n+1} \in V$, as well as corresponding extra arbitrary formal parameter z_{n+1} appear as a result of the action of δ^n_m on $\mathcal{F} \in C^n_m(V, \mathcal{W})$ mapping it to $C^{n+1}_{m-1}(V, \mathcal{W})$. In application to the ϵ-product (3.5) these extra arbitrary elements are involved in the definition of the action of $\delta^{k+n-r}_{m+m'-t}$ on $\mathcal{F}(v_1, x_1; \ldots; v_k, x_k) \cdot_\epsilon \mathcal{F}(v'_1, y_1; \ldots; v'_n, y_n)$.

Note that both sides of (3.8) belong to the space $C^{n+n'-r+1}_{m+m'-t+1}(V, W)$. The co-boundary operators δ^n_m and $\delta^{n'}_{m'}$ in (3.8) do not include the number of common vertex algebra elements (and formal parameters), neither the number of common vertex operators corresponding mappings composable with. The dependence on common vertex algebra elements, parameters, and composable vertex operators is taken into account in mappings multiplying the action of co-boundary operators on Φ.

Finally, we have the following

Corollary 3. *The multiplication* (3.5) *extends the chain-cochain complex structure of Proposition 3 to all products $C^k_m(V, \mathcal{W}) \times C^n_{m'}(V, \mathcal{W})$, k, $n \geq 0$, m, $m' \geq 0$.* □

Corollary 4. *The product* (3.5) *and the product operator* (2.2) *endow the space $C^k_m(V, \mathcal{W}) \times C^n_m(V, \mathcal{W})$, k, $n \geq 0$, m, $m' \geq 0$, with the structure of a bi-graded differential algebra $\mathcal{G}(V, \mathcal{W}, \cdot_\epsilon, \delta^{k+n-r}_{m+m'-t})$.* □

For elements of the spaces $C^2_{ex}(V, \mathcal{W})$ we have the following

Corollary 5. *The product of elements of the spaces $C^2_{ex}(V, \mathcal{W})$ and $C^n_m(V, \mathcal{W})$ is given by* (3.5),

$$\cdot_\epsilon : C^2_{ex}(V, \mathcal{W}) \times C^n_m(V, \mathcal{W}) \to C^{n+2-r}_m(V, \mathcal{W}), \tag{3.9}$$

and, in particular,

$$\cdot_\epsilon : C^2_{ex}(V, \mathcal{W}) \times C^2_{ex}(V, \mathcal{W}) \to C^{4-r}_0(V, \mathcal{W}).$$

□

3.4. **The commutator.** Let us consider the mappings $\Phi(v_1, z_1 ; \ldots; v_n, z_k) \in C^k_m(V, \mathcal{W})$, and $\Psi(v_{k+1}, z_{k+1}; \ldots; v_{k+n}, z_{k+n}) \in C^n_{m'}(V, \mathcal{W})$, with have r common vertex algebra elements (and, correspondingly, r formal variables), and t common vertex operators mappings Φ and Ψ are composable with. Note that when applying the co-boundary operators (2.3) and (2.5) to a map $\Phi(v_1, z_1; \ldots; v_n, z_n) \in C^n_m(V, \mathcal{W})$,

$$\delta^n_m : \Phi(v_1, z_1; \ldots; v_n, z_n) \to \Phi(v'_1, z'_1; \ldots; v'_{n+1}, z'_{n+1}) \in C^{n+1}_{m-1}(V, \mathcal{W}),$$

one does not necessary assume that we keep the same set of vertex algebra elements/formal parameters and vertex operators composable with for $\delta^n_m \Phi$, though it might happen that some of them could be common with Φ.

Let us define an extra product (related to the ϵ-product) the product of Φ and Ψ,

$$\Phi \cdot \Psi : V^{\otimes (k+n-r)} \to \mathcal{W}_{z_1,\ldots,z_{k+n-r}}, \tag{3.10}$$

$$\Phi \cdot \Psi = [\Phi, \cdot_\epsilon \Psi] = \Phi \cdot_\epsilon \Psi - \Psi \cdot_\epsilon \Phi, \tag{3.11}$$

where brackets denote ordinary commutator in $\mathcal{W}_{z_1,\ldots,z_{k+n-r}}$. Due to the properties of the maps $\Phi \in C^k_m(V, \mathcal{W})$ and $\Psi \in C^n_{m'}(V, \mathcal{W})$, the map $(\Phi \cdot_\epsilon \Psi)$ belongs to the space $C^{k+n-r}_{m+m'-t}(V, \mathcal{W})$. For $k = n$ and

$$\Psi(v_{n+1}, z_{n+1}; \ldots; v_{2n}, z_{2n}) = \Phi(v_1, z_1; \ldots; v_n, z_n),$$

we obtain from (3.11) and (3.5) that

$$\Phi(v_1, z_1; \ldots; v_n, z_n) \cdot \Phi(v_1, z_1; \ldots; v_n, z_n) = 0. \tag{3.12}$$

4. The invariants

In this section we provide the main result of the paper by deriving the simplest cohomological invariants associated to the short double complex (2.4) for a grading-restricted vertex algebra.

Let us give first some further definitions. In this section we skip the dependence on vertex algebra elements and formal parameters in notations for elements of $C^m_n(V, \mathcal{W})$.

Definition 12. In analogy with differential forms, we call a map $\Phi \in C^n_m(V, \mathcal{W})$ closed if

$$\delta^n_m \Phi = 0.$$

For $m \geq 1$, we call it exact if there exists $\Psi \in C^{n+1}_{m-1}(V, \mathcal{W})$ such that

$$\Psi = \delta^n_m \Phi.$$

Definition 13. For $\Phi \in C^n_m(V, \mathcal{W})$ we call the cohomology class of mappings $[\Phi]$ the set of all closed forms that differs from Φ by an exact mapping, i.e., for $\chi \in C^{n-1}_{m+1}$,

$$[\Phi] = \Phi + \delta^{n-1}_{m+1} \chi,$$

(we assume that both parts of the last formula belongs to the same space $C^n_m(V, \mathcal{W})$).

Under a natural extra condition, the short double complex (2.6) allows us to establish relations among elements of double complex spaces. In particular, we require that for a pair of double complex spaces $C^{n_1}_{k_1}(V, \mathcal{W})$ and $C^{n_2}_{k_2}(V, \mathcal{W})$ there exist subspaces ${C'}^{n_1}_{k_1}(V, \mathcal{W}) \subset C^{n_1}_{k_1}(V, \mathcal{W})$ and ${C'}^{n_2}_{k_2}(V, \mathcal{W}) \subset C^{n_2}_{k_2}(V, \mathcal{W})$ such that for $\Phi_1 \in {C'}^{n_1}_{k_1}(V, \mathcal{W})$ and $\Phi_2 \in {C'}^{n_2}_{k_2}(V, \mathcal{W})$,

$$\Phi_1 \cdot \delta^{n_2}_{k_2} \Phi_2 = 0, \tag{4.1}$$

namely, Φ_1 supposed to be orthogonal to $\delta^{n_2}_{k_2} \Phi_2$ (i.e., commutative with respect to the product (3.11)). We call this *the orthogonality condition* for mappings and actions of co-boundary operators for a double complex. It is easy to see that the assumption to belong to the same double complex space for both sides of the equations following from the orthogonality condition applies the bi-grading condition on double complex spaces. Note that in the case of differential forms considered on a smooth manifold, the Frobenius theorem for a distribution provides the orthogonality condition. In this Section we derive algebraic relations occurring from the orthogonality condition on the short double complex (2.6). We formulate

Proposition 6. *The orthogonality condition for the short double complex sequence* (2.6) *determines the cohomological classes:*

$$\left[\left(\delta^1_2\Phi\right)\cdot\Phi\right],\ \left[\left(\delta^0_3\chi\right)\cdot\chi\right],\ \left[\left(\delta^1_t\alpha\right)\cdot\alpha\right], \tag{4.2}$$

for $0\le t\le 2$, *with non-vanishing* $\left(\delta^1_2\Phi\right)\cdot\Phi$, $\left(\delta^0_3\chi\right)\cdot\chi$, *and* $\left(\delta^1_t\alpha\right)\cdot\alpha$. *These classes are independent on the choice of* $\Phi\in C^1_2(V,\mathcal{W})$, $\chi\in C^0_3(V,\mathcal{W})$, *and* $\alpha\in C^1_t(V,\mathcal{W})$.

Remark 5. A cohomology class with vanishing $\left(\delta^1_2\Phi\right)\cdot\Phi\cdot\alpha$ is given by $\left[\left(\delta^1_2\Phi\right)\cdot\Phi\cdot\alpha\right]$.

Proof. Let us consider two maps $\chi\in C^0_3(V,\mathcal{W})$, $\Phi\in C^1_2(V,\mathcal{W})$. We require them to be orthogonal, i.e.,

$$\Phi\cdot\delta^0_3\chi=0. \tag{4.3}$$

Thus, there exists $\alpha\in C^n_m(V,\mathcal{W})$, such that

$$\delta^0_3\chi=\Phi\cdot\alpha, \tag{4.4}$$

and $1=1+n-r$, $2=2+m-t$, i.e., $n=r$, which leads to $r=1$; $m=t$, $0\le t\le 2$, i.e., $\alpha\in C^1_t(V,\mathcal{W})$. All other orthogonality conditions for the short sequence (2.6) does not allow relations of the form (4.4).

Consider now (4.3). We obtain, using (3.8)

$$\delta^{2-r'}_{4-t'}(\Phi\cdot\delta^0_3\chi)=\left(\delta^1_2\Phi\right)\cdot\delta^0_3\chi+\Phi\cdot\delta^1_2\delta^0_3\chi=\left(\delta^1_2\Phi\right)\cdot\delta^0_3\chi=\left(\delta^1_2\Phi\right)\cdot\Phi\cdot\alpha.$$

Thus

$$0=\delta^{3-r'}_{3-t'}\delta^{2-r'}_{4-t'}(\Phi\cdot\delta^0_3\chi)=\delta^{3-r'}_{3-t'}\left(\left(\delta^1_2\Phi\right)\cdot\Phi\cdot\alpha.\right),$$

and $\left(\left(\delta^1_2\Phi\right)\cdot\Phi\cdot\alpha\right)$ is closed. At the same time, from (4.3) it follows that

$$0=\delta^1_2\Phi\cdot\delta^0_3\chi-\Phi\cdot\delta^1_2\delta^0_3\chi=\left(\Phi\cdot\delta^0_3\chi\right).$$

Thus

$$\delta^1_2\Phi\cdot\delta^0_3\chi=\delta^1_2\Phi\cdot\Phi\cdot\alpha=0.$$

Consider (4.4). Acting by δ^1_2 and substituting back we obtain

$$0=\delta^1_2\delta^0_3\chi=\delta^1_2(\Phi\cdot\alpha)=\delta^1_2(\Phi)\cdot\alpha-\Phi\cdot\delta^1_t\alpha.$$

thus

$$\delta^1_2(\Phi)\cdot\alpha=\Phi\cdot\delta^1_t\alpha.$$

The last equality trivializes on applying δ^3_{t+1} to both sides.

Let us show now the non-vanishing property of $\left(\left(\delta^1_2\Phi\right)\cdot\Phi\right)$. Indeed, suppose $\left(\delta^1_2\Phi\right)\cdot\Phi=0$. Then there exists $\gamma\in C^n_m(V,\mathcal{W})$, such that $\delta^1_2\Phi=\gamma\cdot\Phi$. Both sides of the last equality should belong to the same double complex space but one can see that it is not possible. Thus, $\left(\delta^1_2\Phi\right)\cdot\Phi$ is non-vanishing. One proves in the same way that $\left(\delta^0_3\chi\right)\cdot\chi$ and $\left(\delta^1_t\alpha\right)\cdot\alpha$ do not vanish too. Now let us show that $\left[\left(\delta^1_2\Phi\right)\cdot\Phi\right]$ is invariant, i.e., it does not depend on the choice of $\Phi \in C^1_2(V,\mathcal{W})$. Substitute Φ by $(\Phi+\eta) \in C^1_2(V,\mathcal{W})$. We have

$$\begin{aligned} \left(\delta^1_2\left(\Phi+\eta\right)\right)\cdot(\Phi+\eta) &= \left(\delta^1_2\Phi\right)\cdot\Phi + \left(\left(\delta^1_2\Phi\right)\cdot\eta - \Phi\cdot\delta^1_2\eta\right) \\ &+ \left(\Phi\cdot\delta^1_2\eta + \delta^1_2\eta\cdot\Phi\right) + \left(\delta^1_2\eta\right)\cdot\eta. \end{aligned} \tag{4.5}$$

Since

$$\left(\Phi\cdot\delta^1_2\eta + \left(\delta^1_2\eta\right)\cdot\Phi\right) = \Phi\delta^1_2\eta - (\delta^1_2\eta)\Phi + \left(\delta^1_2\eta\right)\Phi - \Phi\,\delta^1_2\eta = 0,$$

then (4.5) represents the same cohomology class $\left[\left(\delta^1_2\Phi\right)\cdot\Phi\cdot\alpha\right]$. The same folds for $\left[\left(\delta^0_3\chi\right)\cdot\chi\right]$, and $\left[\left(\delta^1_t\alpha\right)\cdot\alpha\right]$. □

Remark 6. Due to Proposition 1, all chahomological classes are invariant with respect to correponding group $\mathrm{Aut}_{z_1,\ldots,z_n}\mathcal{O}^{(n)}$ changes of formal parameters.

The orthogonality condition for a double complex sequence (2.6), together with the action of co-boundary operators (2.2) and (2.4), and the multiplication formulas (3.11)–(3.8), define a differential bi-graded algebra depending on vertex algebra elements and formal parameters. In particular, for the short sequence (2.6), we obtain in this way the generators and commutation relations for a continual Lie algebra $\mathcal{G}(V)$ (a generalization of ordinary Lie algebras with continual space of roots, c.f. [9]) with the continual root space represented by a grading-restricted vertex algebra V.

Lemma 2. *For the short sequence* (2.6) *we get a continual Lie algebra* $\mathcal{G}(V)$ *with generators*

$$\left\{\Phi(v_1),\ \chi,\ \alpha(v_2),\ \delta^1_2\Phi(v_1),\ \delta^0_3\chi,\ \delta^1_t\alpha(v_2), 0\le t\le 2\right\}, \tag{4.6}$$

and commutation relations for a continual Lie algebra $\mathcal{G}(V)$

$$\begin{aligned} \Phi\cdot\delta^1_t\alpha &= \alpha\cdot\delta^1_2\Phi \neq 0, \\ \delta^0_3\chi &= \Phi\cdot\alpha, \end{aligned} \tag{4.7}$$

with all other relations being trivial. The sum of cohomological classes (4.2) *provides an invariant of* $\mathcal{G}(V)$.

Proof. Recall that $\Phi(v_1)(z_1) \in C^1_2(V,\mathcal{W})$, $\chi \in C^0_3(V,\mathcal{W})$, $\alpha \in C^1_t(V,\mathcal{W})$, $0\le t\le 2$. One easily checks the commutation relations coming from the orthogonality and bi-grading conditions. Further applications of (2.2), (2.4), and (4.1) to (2.6) lead to trivial results. $\Phi\cdot\delta^1_t\alpha \neq 0$ is proven by contradiction. It is easy to check Jacobi identities for (4.6) and (4.7). With a redefinition

$$\begin{aligned} H &= \delta^0_3\chi, \\ H^* &= \chi, \\ X_+(v_1) &= \Phi(v_1), \\ X_-(v_2) &= \alpha(v_2), \\ Y_+(v_1) &= \delta^1_2\Phi(v_1), \\ Y_-(v_2) &= \delta^1_t\alpha(v_2), \end{aligned} \tag{4.8}$$

the commutation relations (4.7) become:

$$\begin{aligned} [X_+(v_1), X_-(v_2)] &= H, \\ [X_+(v_1), Y_-(v_1)] &= [X_-(v_2), Y_+(v_1)], \end{aligned}$$

i.e., the orthogonality condition brings about a representation of an affinization [8] of continual counterpart of the Lie algebra sl_2. Vertex algebra elements in (4.8) play the role of roots belonging to continual non-commutative root space given by a vertex algebra V. □

Acknowledgments

The author would like to thank Y.-Z. Huang, H. V. Lê, and P. Somberg for related discussions. Research of the author was supported by the GACR project 18-00496S and RVO: 67985840.

5. Appendix: Grading-restricted vertex algebras and their modules

In this section, following [5] we recall basic properties of grading-restricted vertex algebras and their grading-restricted generalized modules, useful for our purposes in later sections. We work over the base field $\mathbb{C}$ of complex numbers. A vertex algebra $(V, Y_V, \mathbf{1})$, cf. [8], consists of a $\mathbb{Z}$-graded complex vector space

$$V = \bigoplus_{n\in\mathbb{Z}} V_{(n)}, \quad \dim V_{(n)} < \infty \text{ for each } n \in \mathbb{Z},$$

and linear map

$$Y_V : V \to \text{End } (V)[[z, z^{-1}]],$$

for a formal parameter z and a distinguished vector $\mathbf{1_V} \in V$. The evaluation of Y_V on $v \in V$ is the vertex operator

$$Y_V(v) \equiv Y_V(v, z) = \sum_{n\in\mathbb{Z}} v(n) z^{-n-1},$$

with components

$$(Y_V(v))_n = v(n) \in \text{End } (V),$$

where $Y_V(v, z)\mathbf{1} = v + O(z)$. Now we describe further restrictions [5], defining a grading-restricted vertex algebra:

(1) Grading-restriction condition: $V_{(n)}$ is finite dimensional for all $n \in \mathbb{Z}$, and $V_{(n)} = 0$ for $n \ll 0$.
(2) Lower-truncation condition: For $u, v \in V$, $Y_V(u, z)v$ contains only finitely many negative power terms, that is, $Y_V(u, z)v \in V((z))$ (the space of formal Laurent series in z with coefficients in V).
(3) Identity property: Let $\mathbf{1}_V$ be the identity operator on V. Then

$$Y_V(\mathbf{1}_V, z) = \text{Id}_V.$$

(4) Creation property: For $u \in V$, $Y_V(u, z)\mathbf{1}_V \in V[[z]]$ and

$$\lim_{z\to 0} Y_V(u, z)\mathbf{1}_V = u.$$

(5) Duality: For $u_1, u_2, v \in V$, $v' \in V' = \coprod_{n\in\mathbb{Z}} V^*_{(n)}$ ($V^*_{(n)}$ denotes the dual vector space to $V_{(n)}$ and $\langle .,. \rangle$ the evaluation pairing $V' \otimes V \to \mathbb{C}$), the series $\langle v', Y_V(u_2, z_2)Y_V(u_1, z_1)v\rangle$, and $\langle v', Y_V(Y_V(u_1, z_1-z_2)u_2, z_2)v\rangle$, are absolutely convergent in the regions $|z_1| > |z_2| > 0$, $|z_2| > |z_1| > 0$, $|z_2| > |z_1 - z_2| > 0$, respectively, to a common rational function in z_1 and z_2 with the only possible poles at $z_1 = 0 = z_2$ and $z_1 = z_2$.

One assumes the existence of Virasoro vector $\omega \in V$: its vertex operator $Y(\omega, z) = \sum_{n\in\mathbb{Z}} L(n)z^{-n-2}$ is determined by Virasoro operators $L(n) : V \to V$ fulfilling (notice that with abuse of notation we denote $L_V(n) = L(n)$)

$$[L(m), L(n)] = (m-n)L(m+n) + \frac{c}{12}(m^3 - m)\delta_{m+b,0}\mathrm{Id_V},$$

(c is called the *central charge* of V). The grading operator is given by $L(0)u = nu, \quad u \in V_{(n)}$, ($n$ is called the weight of u and denoted by wt (u)).

(6) $L_V(0)$-bracket formula: Let $L_V(0) : V \to V$ be defined by $L_V(0)v = nv$ for $v \in V_{(n)}$. Then

$$[L_V(0), Y_V(v, z)] = Y_V(L_V(0)v, z) + z\frac{d}{dz}Y_V(v, z),$$

for $v \in V$.

(7) $L_V(-1)$-derivative property: Let $L_V(-1) : V \to V$ be the operator given by

$$L_V(-1)v = \mathrm{Res}_z z^{-2}Y_V(v, z)\mathbf{1} = Y_{(-2)}(v)\mathbf{1},$$

for $v \in V$. Then for $v \in V$,

$$\frac{d}{dz}Y_V(u, z) = Y_V(L_V(-1)u, z) = [L_V(-1), Y_V(u, z)].$$

Correspondingly, a grading-restricted generalized V-module is a vector space W equipped with a vertex operator map

$$Y_W : V \otimes W \to W[[z, z^{-1}]],$$

$$u \otimes w \quad \mapsto \quad Y_W(u, w) \equiv Y_W(u, z)w = \sum_{n\in\mathbb{Z}} (Y_W)_n(u, w)z^{-n-1},$$

and linear operators $L_W(0)$ and $L_W(-1)$ on W satisfying conditions similar as in the definition for a grading-restricted vertex algebra. In particular,

(1) Grading-restriction condition: The vector space W is $\mathbb{C}$-graded, that is, $W = \coprod_{\alpha\in\mathbb{C}} W_{(\alpha)}$, such that $W_{(\alpha)} = 0$ when the real part of α is sufficiently negative.
(2) Lower-truncation condition: For $u \in V$ and $w \in W$, $Y_W(u, z)w$ contains only finitely many negative power terms, that is, $Y_W(u, z)w \in W((z))$.
(3) Identity property: Let Id_W be the identity operator on W, $Y_W(\mathbf{1}, z) = \mathrm{Id}_W$.
(4) Duality: For u_1, $u_2 \in V$, $w \in W$, $w' \in W' = \coprod_{n\in\mathbb{Z}} W^*_{(n)}$ (W' is the dual V-module to W), the series

$$\begin{aligned} &\langle w', Y_W(u_1, z_1)Y_W(u_2, z_2)w\rangle, \\ &\langle w', Y_W(u_2, z_2)Y_W(u_1, z_1)w\rangle, \\ &\langle w', Y_W(Y_V(u_1, z_1 - z_2)u_2, z_2)w\rangle, \end{aligned} \tag{5.1}$$

are absolutely convergent in the regions $|z_1| > |z_2| > 0$, $|z_2| > |z_1| > 0$, $|z_2| > |z_1 - z_2| > 0$, respectively, to a common rational function in z_1 and z_2 with the only possible poles at $z_1 = 0 = z_2$ and $z_1 = z_2$.

The locality

$$Y_W(v_1, z_1) Y_W(v_2, z_2) \sim Y_W(v_2, z_2) Y_W(v_1, z_1),$$

and associativity

$$Y_W(v_1, z_1) Y_W(v_2, z_2) \sim Y_W(Y_V v_1, z_1 - z_2) v_2, z_2),$$

properties for the vertex operators in a V-module W follow from the Jacobi identity [8].

(5) $L_W(0)$-bracket formula: For $v \in V$,

$$[L_W(0), Y_W(v, z)] = Y_W(L(0)v, z) + z\frac{d}{dz} Y_W(v, z).$$

(6) $L_W(0)$-grading property: For $w \in W_{(\alpha)}$, there exists $N \in \mathbb{Z}_+$ such that $(L_W(0) - \alpha)^N w = 0$.

(7) $L_W(-1)$-derivative property: For $v \in V$,

$$\frac{d}{dz} Y_W(u, z) = Y_W(L_V(-1)u, z) = [L_W(-1), Y_W(u, z)].$$

For $v \in V$, and $w \in W$, the intertwining operator

$$\begin{aligned} &Y^W_{WV} : V \to W, \\ &v \mapsto Y^W_{WV}(w, z) v, \end{aligned} \tag{5.2}$$

is defined by

$$Y^W_{WV}(w, z) v = e^{z L_W(-1)} Y_W(v, -z) w. \tag{5.3}$$

5.1. **Non-degenerate invariant bilinear form on V.** The subalgebra

$$\{L_V(-1), L_V(0), L_V(1)\} \cong SL(2, \mathbb{C}),$$

associated with Möbius transformations on z naturally acts on V, (cf., e.g. [8]). In particular,

$$\gamma_\lambda = \begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix} : z \mapsto w = -\frac{\lambda^2}{z}, \tag{5.4}$$

is generated by

$$T_\lambda = \exp\left(\lambda L_V(-1)\right)\ \exp\left(\lambda^{-1} L_V(1)\right)\ \exp\left(\lambda L_V(-1)\right),$$

where

$$T_\lambda Y(u, z) T_\lambda^{-1} = Y\left(\exp\left(-\frac{z}{\lambda^2} L_V(1)\right) \left(-\frac{z}{\lambda}\right)^{-2L_V(0)} u, -\frac{\lambda^2}{z}\right). \tag{5.5}$$

In our considerations (cf. Appendix 6) of Riemann sphere sewing, we use in particular, the Möbius map

$$z \mapsto z' = \epsilon / z,$$

associated with the sewing condition (6.4) with

$$\lambda = -\xi \epsilon^{\frac{1}{2}}, \tag{5.6}$$

with $\xi \in \{\pm\sqrt{-1}\}$. The adjoint vertex operator [8, 2] is defined by

$$Y^\dagger(u,z) = \sum_{n\in\mathbb{Z}} u^\dagger(n) z^{-n-1} = T_\lambda Y(u,z) T_\lambda^{-1}. \tag{5.7}$$

A bilinear form $\langle ., . \rangle_\lambda$ on V is invariant if for all $a, b, u \in V$, if

$$\langle Y(u,z)a, b\rangle_\lambda = \langle a, Y^\dagger(u,z)b\rangle_\lambda, \tag{5.8}$$

i.e.

$$\langle u(n)a, b\rangle_\lambda = \langle a, u^\dagger(n)b\rangle_\lambda.$$

Thus it follows that

$$\langle L_V(0)a, b\rangle_\lambda = \langle a, L_V(0)b\rangle_\lambda, \tag{5.9}$$

so that

$$\langle a, b\rangle_\lambda = 0, \tag{5.10}$$

if $wt(a) \neq wt(b)$ for homogeneous a, b. One also finds

$$\langle a, b\rangle_\lambda = \langle b, a\rangle_\lambda.$$

The form $\langle ., . \rangle_\lambda$ is unique up to normalization if $L_V(1)V_1 = V_0$. Given any V basis $\{u^\alpha\}$ we define the dual V basis $\{\overline{u}^\beta\}$ where

$$\langle u^\alpha, \overline{u}^\beta\rangle_\lambda = \delta^{\alpha\beta}.$$

6. Appendix: A sphere formed from sewing of two spheres

The matrix element for a number of vertex operators of a vertex algebra is usually associated [2, 3, 10] with a vertex algebra character on a sphere. We extrapolate this notion to the case of $\mathcal{W}_{z_1,\dots,z_n}$ spaces. In Section 3 we explained that a space $\mathcal{W}_{z_1,\dots,z_n}$ can be associated with a Riemann sphere with marked points, while the product of two such spaces is then associated with a sewing of such two spheres with a number of marked points and extra points with local coordinates identified with formal parameters of $\mathcal{W}_{x_1,\dots,x_k}$ and $\mathcal{W}_{y_1,\dots,y_n}$. In order to supply an appropriate geometric construction for the product, we use the ϵ-sewing procedure (described in this Appendix) for two initial spheres to obtain a matrix element associated with (3.2).

Remark 7. In addition to the ϵ-sewing procedure of two initial spheres, one can alternatively use the self-sewing procedure [11] for the sphere to get, at first, the torus, and then by sending parameters to appropriate limit by shrinking genus to zero. As a result, one obtains again the sphere but with a different parameterization. In the case of spheres, such a procedure consideration of the product of $\mathcal{W}$-spaces so we focus in this paper on the ϵ-formalizm only.

In our particular case of $\mathcal{W}$-values rational functions obtained from matrix elements (1.1) two initial auxiliary spaces we take Riemann spheres $\Sigma_a^{(0)}$, $a = 1$, 2, and the resulting space is formed by the sphere $\Sigma^{(0)}$ obtained by the procedure of sewing $\Sigma_a^{(0)}$. The formal parameters $(x_1, \dots, x_k)$ and $(y_1, \dots, y_n)$ are identified with local coordinates of k and n points on two initial spheres $\Sigma_a^{(0)}$, $a = 1, 2$ correspondingly. In the ϵ sewing procedure, some r points among $(p_1, \dots, p_k)$ may coincide with points among $(p'_1, \ldots, p'_n)$ when we identify the annuluses (6.3). This corresponds to the singular case of coincidence of r formal parameters.

Consider the sphere formed by sewing together two initial spheres in the sewing scheme referred to as the ϵ-formalism in [11]. Let $\Sigma^{(0)}_a$, $a = 1, 2$ be to initial spheres. Introduce a complex sewing parameter ϵ where

$$|\epsilon| \le r_1 r_2,$$

Consider k distinct points on $p_i \in \Sigma^{(0)}_1$, $i = 1, \ldots, k$, with local coordinates $(x_1, \ldots, x_k) \in F_k\mathbb{C}$, and distinct points $p_j \in \Sigma^{(0)}_2$, $j = 1, \ldots, n$, with local coordinates $(y_1, \ldots, y_n) \in F_n\mathbb{C}$, with

$$|x_i| \ge |\epsilon|/r_2,$$
$$|y_i| \ge |\epsilon|/r_1.$$

Choose a local coordinate $z_a \in \mathbb{C}$ on $\Sigma^{(0)}_a$ in the neighborhood of points $p_a \in \Sigma^{(0)}_a$, $a = 1, 2$. Consider the closed disks

$$|\zeta_a| \le r_a,$$

and excise the disk

$$\{\zeta_a,\ |\zeta_a| \le |\epsilon|/r_{\overline{a}}\} \subset \Sigma^{(0)}_a, \tag{6.1}$$

to form a punctured sphere

$$\widehat{\Sigma}^{(0)}_a = \Sigma^{(0)}_a \backslash \{\zeta_a, |\zeta_a| \le |\epsilon|/r_{\overline{a}}\}.$$

We use the convention

$$\overline{1} = 2, \quad \overline{2} = 1. \tag{6.2}$$

Define the annulus

$$\mathcal{A}_a = \{\zeta_a, |\epsilon|/r_{\overline{a}} \le |\zeta_a| \le r_a\} \subset \widehat{\Sigma}^{(0)}_a, \tag{6.3}$$

and identify $\mathcal{A}_1$ and $\mathcal{A}_2$ as a single region $\mathcal{A} = \mathcal{A}_1 \simeq \mathcal{A}_2$ via the sewing relation

$$\zeta_1 \zeta_2 = \epsilon. \tag{6.4}$$

In this way we obtain a genus zero compact Riemann surface

$$\Sigma^{(0)} = \left\{\widehat{\Sigma}^{(0)}_1 \backslash \mathcal{A}_1\right\} \cup \left\{\widehat{\Sigma}^{(0)}_2 \backslash \mathcal{A}_2\right\} \cup \mathcal{A}.$$

This sphere form a suitable geometrical model for the construction of a product of $\mathcal{W}$-valued rational forms in Section 3.

7. Appendix: proof of Proposition 5

Proof. For a vertex operator $Y_{V,W}(v, z)$ let us introduce a notation $\omega_{V,W} = Y_{V,W}(v, z)\ dz^{\mathrm{wt} v}$. Let us use notations (3.1) and (3.6). According to (2.2), the action of $\delta^{k+n-r}_{m+m'-t}$ on $\widehat{R}\mathcal{F}(v_1, x_1; \ldots; v_k, x_k; v'_1, y_1; \ldots; v'_k, y_n; \epsilon)$ is given by

$$\begin{aligned}
&\langle w', \delta^{k+n-r}_{m+m'-t} \widehat{R}\ \mathcal{F}(v_1, x_1; \ldots; v_k, x_k; v'_1, y_1; \ldots; v'_n, y_n; \epsilon)\rangle \\
&\quad = \langle w', \sum_{i=1}^{k} (-1)^i\ \widehat{R}\ \mathcal{F}(\widetilde{v}_1, z_1; \ldots; \widetilde{v}_{i-1}, z_{i-1};\ \omega_V(\widetilde{v}_i, z_i - z_{i+1})\widetilde{v}_{i+1}, z_{i+1};\ \widetilde{v}_{i+2}, z_{i+2}; \\
&\qquad\qquad \ldots; \widetilde{v}_k, z_k; \widetilde{v}_{k+1}, z_{k+1}; \ldots; \widetilde{v}_{k+n}, z_{k+n}; \epsilon)\rangle
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{n-r}(-1)^i \, \langle w', \mathcal{F}\left(\widetilde{v}_1, z_1; \ldots; \widetilde{v}_k, z_k; \widetilde{v}_{k+1}, z_{k+1}; \ldots; \widetilde{v}_{k+i-1}, z_{k+i-1};\right.\\
&\qquad \omega_V\left(\widetilde{v}_{k+i}, z_{k+i}-z_{k+i+1}\right) \widetilde{v}_{k+i+1}, z_{k+i+1};\\
&\qquad\qquad \left.\widetilde{v}_{k+i+2}, z_{k+i+2}; \ldots; \widetilde{v}_{k+n-r}, z_{k+n-r}; \epsilon\right)\rangle\\
&+\langle w', \omega_W\left(\widetilde{v}_1, z_1\right) \; \mathcal{F}(\widetilde{v}_2, z_2; \ldots; \widetilde{v}_k, z_k; \widetilde{v}_{k+1}, z_{k+1}; \ldots; \widetilde{v}_{k+n-r}, z_{k+n-r}; \epsilon)\rangle\\
&+\langle w, (-1)^{k+n+1-r}\omega_W(\widetilde{v}_{k+n-r+1}, z_{k+n-r+1})\\
&\qquad \mathcal{F}(\widetilde{v}_1, z_1; \ldots; \widetilde{v}_k, z_k; \widetilde{v}_{k+1}, z_{k+1}; \ldots; \widetilde{v}_{k+n-r}, z_{k+n-r}; \epsilon)\rangle\\
&=\sum_{u\in V}\langle w', \sum_{i=1}^{k}(-1)^i \, Y^W_{VW}(\mathcal{F}(\widetilde{v}_1, z_1; \ldots; \widetilde{v}_{i-1}, z_{i-1}; \; \omega_V(\widetilde{v}_i, z_i-z_{i+1})\widetilde{v}_{i+1}, z_{i+1};\\
&\qquad \widetilde{v}_{i+2}, z_{i+2}; \ldots; \widetilde{v}_k, z_k), \zeta_1)u\rangle\\
&\qquad\qquad \langle w', Y^W_{VW}(\mathcal{F}(\widetilde{v}_{k+1}, z_{k+1}; \ldots; \widetilde{v}_{k+n-r}, z_{k+n-r}), \zeta_2)\overline{u}\rangle\\
&+\sum_{u\in V}\sum_{i=1}^{n-r}(-1)^i \, \langle w', Y^W_{VW}(\mathcal{F}\left(\widetilde{v}_1, z_1; \ldots; \widetilde{v}_k, z_k\right), \zeta_1)u\rangle\\
&\qquad \langle w', Y^W_{VW}(\mathcal{F}(\widetilde{v}_{k+1}, z_{k+1}; \ldots; \widetilde{v}_{k+i-1}, z_{k+i-1};\\
&\qquad\quad \omega_V(\widetilde{v}_i, z_{k+i}-z_{k+i+1}) \, \widetilde{v}_{k+i+1}, z_{k+i+1}; \widetilde{v}_{k+i+2}, z_{k+i+2};\\
&\qquad\qquad \ldots; \widetilde{v}_{k+n-r}, z_{k+n-r}), \zeta_2)\overline{u}\rangle\\
&+\sum_{u\in V}\langle w', Y^W_{VW}(\omega_W\left(\widetilde{v}_1, z_1\right) \; \mathcal{F}(\widetilde{v}_2, z_2; \ldots; \widetilde{v}_k, z_k), \zeta_1)u\rangle\\
&\qquad \langle w', Y^W_{VW}(\mathcal{F}(\widetilde{v}_{k+1}, z_{k+1}; \ldots; \widetilde{v}_{k+n-r}, z_{k+n-r}), \zeta_2)\overline{u}\rangle\\
&+\sum_{u\in V}\langle w', Y^W_{VW}((-1)^{k+1}\omega_W\left(\widetilde{v}_{k+1}, z_{k+1}\right) \; \mathcal{F}(\widetilde{v}_1, z_1; \ldots; \widetilde{v}_k, z_k), \zeta_1)u\rangle\\
&\qquad \langle w', Y^W_{VW}(\mathcal{F}(\widetilde{v}_{k+2}, z_{k+2}; \ldots; \widetilde{v}_{k+n-r}, z_{k+n-r}), \zeta_2)\overline{u}\rangle\\
&-\sum_{u\in V}\langle w', (-1)^{k+1}\langle w', Y^W_{VW}(\omega_W\left(\widetilde{v}_{k+1}, z_{k+1}\right) \; \mathcal{F}(\widetilde{v}_1, z_1; \ldots; \widetilde{v}_k, z_k), \zeta_1)u\rangle\\
&\qquad \langle w', Y^W_{VW}(\mathcal{F}(\widetilde{v}_{k+2}, z_{k+2}; \ldots; \widetilde{v}_{k+n-r}, z_{k+n-r}), \zeta_2)\overline{u}\rangle\\
&+\sum_{u\in V}\langle w', Y^W_{VW}(\mathcal{F}(\widetilde{v}_1, z_1; \ldots; \widetilde{v}_k, z_k), \zeta_1)u\rangle\\
&\qquad \langle w', Y^W_{VW}(\omega_W(\widetilde{v}_{k+n-r+1}, z_{k+n-r+1})\\
&\qquad\qquad \mathcal{F}(\widetilde{v}_{k+1}, z_{k+1}; \ldots; \widetilde{v}_{k+n-r}, z_{k+n-r}), \zeta_2)\overline{u}\rangle\\
&-\sum_{u\in V}\langle w', Y^W_{VW}(\mathcal{F}(\widetilde{v}_1, z_1; \ldots; \widetilde{v}_k, z_k), \zeta_1))\rangle\\
&\qquad \langle w', Y^W_{VW}(\omega_W(\widetilde{v}_{k+n-r+1}, z_{k+n-r+1})\\
&\qquad\qquad \mathcal{F}(\widetilde{v}_{k+1}, z_{k+1}; \ldots; \widetilde{v}_{k+n-r}, z_{k+n-r}), \zeta_2))\rangle
\end{aligned}
$$

$$= \sum_{u\in V} \langle w',\ Y^W_{VW}(\delta^k_m \mathcal{F}(\widetilde{v}_1, z_1; \ldots; \widetilde{v}_k, z_k), \zeta_1) u\rangle$$
$$\langle w', Y^W_{VW}(\mathcal{F}(\widetilde{v}_{k+1}, z_{k+1}; \ldots; \widetilde{v}_{k+n-r}, z_{k+n-r}), \zeta_2)\overline{u}\rangle$$
$$+(-1)^k \sum_{u\in V} \langle w', Y^W_{VW}(\mathcal{F}(\widetilde{v}_1, z_1; \ldots; \widetilde{v}_k, z_k), \zeta_1) u\rangle$$
$$\langle w', Y^W_{VW}(\delta^{n-r}_{m'-t} \mathcal{F}(\widetilde{v}_{k+1}, z_{k+1}; \ldots; \widetilde{v}_{k+n-r}, z_{k+n-r}), \zeta_2)\overline{u}\rangle$$

$$= \langle w', \delta^k_m \mathcal{F}(\widetilde{v}_1, z_1; \ldots; \widetilde{v}_k, z_k) \cdot_\epsilon \langle w', \mathcal{F}(\widetilde{v}_{k+1}, z_{k+1}; \ldots; \widetilde{v}_{k+n-r}, z_{k+n-r})\rangle$$
$$+(-1)^k \langle w', \mathcal{F}(\widetilde{v}_1, z_1; \ldots; \widetilde{v}_k, z_k) \cdot_\epsilon \delta^{n-r}_{m'-t} \mathcal{F}(\widetilde{v}_{k+1}, z_{k+1}; \ldots; \widetilde{v}_{k+n-r}, z_{k+n-r})\rangle,$$

since,

$$\sum_{u\in V} \langle w', (-1)^{k+1} Y^W_{VW}(\omega_W\, (\widetilde{v}_{k+1}, z_{k+1})\ \mathcal{F}(\widetilde{v}_1, z_1; \ldots; \widetilde{v}_k, z_k), \zeta_1) u\rangle$$
$$\langle w', Y^W_{VW}(\mathcal{F}(\widetilde{v}_{k+2}, z_{k+2}; \ldots; \widetilde{v}_{k+n-r}, z_{k+n-r}), \zeta_2)\overline{u}\rangle$$

$$= \sum_{u\in V} \langle w', (-1)^{k+1} e^{\zeta_1 L_W(-1)} Y_W(u, -\zeta_1)\ \omega_W\, (\widetilde{v}_{k+1}, z_{k+1})\ \mathcal{F}(\widetilde{v}_1, z_1; \ldots; \widetilde{v}_k, z_k)\rangle$$
$$\langle w', Y^W_{VW}(\mathcal{F}(\widetilde{v}_{k+2}, z_{k+2}; \ldots; \widetilde{v}_{k+n-r}, z_{k+n-r}), \zeta_2)\overline{u}\rangle$$

$$= \sum_{u\in V} \langle w', (-1)^{k+1} e^{\zeta_1 L_W(-1)} \omega_W\, (\widetilde{v}_{k+1}, z_{k+1})\, Y_W(u, -\zeta_1)\ \mathcal{F}(\widetilde{v}_1, z_1; \ldots; \widetilde{v}_k, z_k)\rangle$$
$$\langle w', Y^W_{VW}(\mathcal{F}(\widetilde{v}_{k+2}, z_{k+2}; \ldots; \widetilde{v}_{k+n-r}, z_{k+n-r}), \zeta_2)\overline{u}\rangle$$

$$= \sum_{u\in V} \langle w', (-1)^{k+1}\ \omega_W\, (\widetilde{v}_{k+1}, z_{k+1} + \zeta_1)\ e^{\zeta_1 L_W(-1)} Y_W(u, -\zeta_1)\ \mathcal{F}(\widetilde{v}_1, z_1; \ldots; \widetilde{v}_k, z_k)\rangle$$
$$\langle w', Y^W_{VW}(\mathcal{F}(\widetilde{v}_{k+2}, z_{k+2}; \ldots; \widetilde{v}_{k+n-r}, z_{k+n-r}), \zeta_2)\overline{u}\rangle$$

$$= \sum_{v\in V} \sum_{u\in V} \langle v', (-1)^{k+1}\ \omega_W\, (\widetilde{v}_{k+1}, z_{k+1} + \zeta_1)\, w\rangle$$
$$\langle w', e^{\zeta_1 L_W(-1)} Y_W(u, -\zeta_1)\ \mathcal{F}(\widetilde{v}_1, z_1; \ldots; \widetilde{v}_k, z_k)\rangle$$
$$\langle w', Y^W_{VW}(\mathcal{F}(\widetilde{v}_{k+2}, z_{k+2}; \ldots; \widetilde{v}_{k+n-r}, z_{k+n-r}), \zeta_2)\overline{u}\rangle$$

$$= \sum_{u\in V} \langle w', e^{\zeta_1 L_W(-1)} Y_W(u, -\zeta_1)\ \mathcal{F}(\widetilde{v}_1, z_1; \ldots; \widetilde{v}_k, z_k)\rangle$$
$$\sum_{v\in V} \langle v', (-1)^{k+1}\ \omega_W\, (\widetilde{v}_{k+1}, z_{k+1} + \zeta_1)\, w\rangle$$
$$\langle w', Y^W_{VW}(\mathcal{F}(\widetilde{v}_{k+2}, z_{k+2}; \ldots;$$
$$\widetilde{v}_{k+n-r}, z_{k+n-r}), \zeta_2)\overline{u}\rangle$$

$$= \sum_{u\in V} \langle w', Y^W_{VW}(\mathcal{F}(\widetilde{v}_1, z_1; \ldots; \widetilde{v}_k, z_k), \zeta_1) u \ \rangle$$
$$\langle w', (-1)^{k+1} \ \omega_W \ (\widetilde{v}_{k+1}, z_{k+1} + \zeta_1)$$
$$Y^W_{VW}(\mathcal{F}(\widetilde{v}_{k+2}, z_{k+2}; \ldots; \widetilde{v}_{k+n-r}, z_{k+n-r}), \zeta_2)\overline{u}\rangle$$

$$= \sum_{u\in V} \langle w', Y^W_{VW}(\mathcal{F}(\widetilde{v}_1, z_1; \ldots; \widetilde{v}_k, z_k), \zeta_1) u \ \rangle$$
$$\langle w', (-1)^{k+1} \ \omega_W \ (\widetilde{v}_{k+1}, z_{k+1} + \zeta_1)$$
$$e^{\zeta_2 L_W(-1)} Y_W(\overline{u}, -\zeta_2) \ \mathcal{F}(\widetilde{v}_{k+2}, z_{k+2}; \ldots; \widetilde{v}_{k+n-r}, z_{k+n-r})\rangle$$

$$= \sum_{u\in V} \langle w', Y^W_{VW}(\mathcal{F}(\widetilde{v}_1, z_1; \ldots; \widetilde{v}_k, z_k), \zeta_1) u \ \rangle$$
$$\langle w', (-1)^{k+1} \ \ e^{\zeta_2 L_W(-1)} \ Y_W(\overline{u}, -\zeta_2) \ \omega_W \ (\widetilde{v}_{k+1}, z_{k+1} + \zeta_1 - \zeta_2)$$
$$\mathcal{F}(\widetilde{v}_{k+2}, z_{k+2}; \ldots; \widetilde{v}_{k+n-r}, z_{k+n-r})\rangle$$

$$= \sum_{u\in V} \langle w', Y^W_{VW}(\mathcal{F}(\widetilde{v}_1, z_1; \ldots; \widetilde{v}_k, z_k), \zeta_1) u\rangle$$
$$\langle w', Y^W_{VW}(\omega_W(\widetilde{v}_{k+1}, z_{k+1}) \ \mathcal{F}(\widetilde{v}_{k+2}, z_{k+2}; \ldots; \widetilde{v}_{k+n-r}, z_{k+n-r}), \zeta_2)\overline{u}\rangle,$$

due to locality (5.1) of vertex opertors, and arbitrarness of $\widetilde{v}_{k+1} \in V$ and z_{k+1}, we can always put

$$\omega_W \ (\widetilde{v}_{k+1}, z_{k+1} + \zeta_1 - \zeta_2) = \omega_W(\widetilde{v}_{k+2}, z_{k+2}),$$

for $\widetilde{v}_{k+1} = \widetilde{v}_{k+2}$, $z_{k+2} = z_{k+1} + \zeta_2 - \zeta_1$. □

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