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Almost-compact and compact embeddings of variable exponent spaces

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ALMOST-COMPACT AND COMPACT EMBEDDINGS OF VARIABLE EXPONENT SPACES

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Abstract. Let Ω be an open subset of $\mathbb{R}^N$, and let $p, q : \Omega \to [1, \infty]$ be measurable functions. We give a necessary and sufficient condition for the embedding of the variable exponent space $L^{p(\cdot)}(\Omega)$ in $L^{q(\cdot)}(\Omega)$ to be almost compact. This leads to a condition on Ω, p and q sufficient to ensure that the Sobolev space $W^{1,p(\cdot)}(\Omega)$ based on $L^{p(\cdot)}(\Omega)$ is compactly embedded in $L^{q(\cdot)}(\Omega)$; compact embedding results of this type already in the literature are included as special cases.

1. Introduction

Let Ω be an open subset of $\mathbb{R}^N$ and consider the Lebesgue measure on Ω. If $M \subset \Omega$ is measurable we write $|M|$ for its measure. Let $p, q : \Omega \to [1, \infty]$ be measurable. Much attention has been paid in recent years to the variable exponent space $L^{p(\cdot)}(\Omega)$, the space $W^{1,p(\cdot)}(\Omega)$ of Sobolev type based on $L^{p(\cdot)}(\Omega)$ and conditions under which $W^{1,p(\cdot)}(\Omega)$ is embedded in $L^{q(\cdot)}(\Omega)$: we refer to [2, 4, 5] for a comprehensive account of such matters. The compactness of such an embedding is addressed here: we give conditions that are sufficent to ensure compactness yet weak enough for much earlier work on this topic to be included. To do this we first establish necessary and sufficient conditions for the embedding of $L^{p(\cdot)}(\Omega)$ in $L^{q(\cdot)}(\Omega)$ to be almost compact.

Let $\mathcal{M}(\Omega)$ be the family of all measurable functions $u : \Omega \to [-\infty, \infty]$; denote by χ_E the characteristic function of a set $E \subset \Omega$. Given any sequence $\{E_n\}$ of measurable subsets of Ω, we write $E_n \to \emptyset$ a.e. if the characteristic functions χ_{E_n} converge to 0 pointwise almost everywhere in Ω. Let the symbol $|u|$ stand for the modulus of a function u. We recall the definition of a Banach function space: see, for example, [1]. A normed linear space $(X, \|.\|_X)$ is a Banach function space (BFS

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for short) if the following conditions are satisfied:

(1.1) the norm $\|u\|_X$ is defined for all $u \in \mathcal{M}(\Omega)$, and $u \in X$ if and only if $\|u\|_X < \infty$;

(1.2) $\|u\|_X = \|\ |u|\ \|_X$ for every $u \in \mathcal{M}(\Omega)$;

(1.3) if $0 \leqslant u_n \nearrow u$ a.e. in Ω, then $\|u_n\|_X \nearrow \|u\|_X$;

(1.4) if $E \subset \Omega$ is a measurable set of finite measure, then $\chi_E \in X$;

(1.5) for every measurable set $E \subset \Omega$ of finite measure $|E|$, there exists a positive constant C_E such that $\displaystyle\int_E |u(x)|dx \leqslant C_E \|u\|_X$.

If X and Y are Banach function spaces then X is said to be almost-compactly embedded in Y and we write $X \overset{*}{\hookrightarrow} Y$ if, for every sequence $(E_n)_{n\in\mathbb{N}}$ of measurable subsets of Ω such that $E_n \to \emptyset$ a.e., we have

$$\lim_{n\to\infty} \sup_{\|u\|_X \leq 1} \|u\chi_{E_n}\|_Y = 0.$$

We believe this notion to have independent interest. Moreover, as we know from [10], almost compactness results quickly lead to assertions concerning the compactness of the Sobolev embedding.

To explain in a little more detail what is acheived, suppose that Ω is bounded, $p \in C\left(\overline{\Omega}\right)$ and for all $x \in \Omega$, $1 < p_- \leq p(x) \leq p_+ < N$ and

$$p^{\#}(x) = \frac{Np(x)}{N - p(x)}; \tag{1.6}$$

denote by $W_0^{1,p(\cdot)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$. Let $I_{p,q}$ (resp. $I_{p,q,0}$) stand for the embedding of $W^{1,p(\cdot)}(\Omega)$ $\left(\text{resp.} W_0^{1,p(\cdot)}(\Omega)\right)$ in $L^{q(\cdot)}(\Omega)$. Then it is known (see [7]) that $I_{p,q,0}$ is compact if there exists $\varepsilon > 0$ such that $q(x) \leq p^\sharp(x) - \varepsilon$ for all $x \in \Omega$. In [8] the compactness of $I_{2,q,0}$ is studied under more general assumptions: it is supposed that there exists $x_0 \in \Omega$, a small $\eta > 0$, $0 < l < 1$ and $C > 0$ such that $q(x_0) = 2N/(N-2)$ and

$$q(x) \leq \frac{2N}{N-2} - \frac{C}{\left(\log \frac{1}{|x-x_0|}\right)^l}$$

holds for a.e. $x \in \Omega$ with $|x - x_0| \leqslant \eta$. Some generalizations of these results are given in [6] and [9]. In [9] it is assumed that $q(x) = p^\sharp(x)$ on a compact set K, and compactness of $I_{p,q,0}$ is established under some restrictions on K and on the behavior of $p^\sharp(x) - q(x)$ far from K. The principal aim of this paper is to establish compactness of $I_{p,q}$ for a wider class of sets K on which q is allowed to have the same values as p : various examples of Cantor type are given for which this is possible.

First we find a necessary and sufficient condition for this embedding to be almost compact and as an application we establish the compactness of the Sobolev embedding mentioned above under more general conditions than those previously available.

2. Preliminaries

Let X and Y be Banach function spaces on an open set Ω of $\mathbb{R}^N$ with norms $\|\cdot\|_X, \|\cdot\|_Y$ respectively. We say that X is embedded in Y, and write $X \hookrightarrow Y$, if there exists $c > 0$ such that $\|u\|_Y \leq c\|u\|_X$ for all $u \in X$. The space X is said to be compactly embedded in Y, and we write $X \hookrightarrow\hookrightarrow Y$, if given any sequence $\{u_n\}_{n\in\mathbb{N}}$ with each $\|u_n\|_X \leq 1$, there is a subsequence $\{u_{n(k)}\} \subset Y$ and a point $u \in Y$ such that $\|u_{n(k)} - u\|_Y \to 0$.

Definition 2.1. Let X be a BFS. The Sobolev space $W^1(X)$ is defined to be the set of all functions $u \in \mathcal{M}(\Omega)$ with

$$\|u\|_{W^1(X)} = \|u\|_X + \|\nabla u\|_X < \infty.$$

The following proposition is proved in [10], see Theorem 3.2.

Proposition 2.2. Let X, Y, Z be BFSs and assume

$$W^1(X) \hookrightarrow Y, \quad Y \stackrel{*}{\hookrightarrow} Z.$$

Then

$$W^1(X) \hookrightarrow\hookrightarrow Z.$$

Now, define variable Lebesgue spaces. Let $\mathcal{E}(\Omega)$ denote the set of all measurable functions $p(\cdot) : \Omega \to [1, \infty)$. Let $p(\cdot) \in \mathcal{E}(\Omega)$. Define for a function $u : \Omega \to \mathbb{R}$ a modular

$$m_{p(\cdot)}(u) = \int_\Omega |u(x)|^{p(x)} dx \tag{2.1}$$

and define the space $L^{p(\cdot)}(\Omega)$ to be the set of all measurable functions u on Ω with a finite norm

$$\|u\|_{p(\cdot)} = \inf\Big\{\lambda > 0;\ m_{p(\cdot)}(u/\lambda) \leq 1\Big\}.$$

We adopt the notation

$$p_- = \inf\{p(x);\ x \in \Omega\},\ \ p_+ = \sup\{p(x);\ x \in \Omega\}\ \text{ and }\ p'(x) = \frac{p(x) - 1}{p(x)}.$$

Define for a function $u : \Omega \to \mathbb{R}$ a non-increasing rearrangement u^* on $[0, \infty)$ by

$$u^*(t) = \inf\{\lambda > 0; |\{x \in \Omega;\ \ |u(x)| > \lambda\}| \leqslant t\},\ \ (t \geqslant 0).$$

Lemma 2.3. *Let $s : \Omega \to \mathbb{R}$ and $\alpha > 1$. Then $(\alpha^{s(\cdot)})^*(t) = \alpha^{s^*(t)}$ for all $t > 0$.*

Proof. We can easily write

$$\begin{aligned}(\alpha^{s(\cdot)})^*(t) &= \inf\{\lambda > 0; |\{x \in \Omega; \alpha^{s(x)} > \lambda\}| \leqslant t\}\\ &= \inf\{\alpha^\mu > 0; |\{x \in \Omega;\ \alpha^{s(x)} > \alpha^\mu\}| \leqslant t\}\\ &= \inf\{\alpha^\mu > 0; |\{x \in \Omega;\ s(x) > \mu\}| \leqslant t\}\\ &= \alpha^{\inf\{\mu>0;|\{x\in\Omega;\ s(x)>\mu\}|\leqslant t\}} = \alpha^{s^*(t)}.\end{aligned}$$

□

In [7] (see Theorem 2.8) the following lemma is proved.

Lemma 2.4. *Let p, $q \in \mathcal{E}(\Omega)$. Then $L^{p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ if and only if $q(x) \leqslant p(x)$ a.e. in Ω.*

Definition 2.5. We say that $p : \Omega \to \mathbb{R}$ satisfies a log-Hölder condition if there is $c > 0$ such that

$$|p(x) - p(y)| \leqslant -\frac{c}{\ln|x-y|}, \quad 0 < |x-y| \leqslant \frac{1}{2}. \tag{2.2}$$

Definition 2.6. We say that $\Omega \in \mathcal{C}^{0,1}$ if there is a finite number of balls $\overline{B(x_k, r_k)}$, $k = 1, 2, \dots, m$ and the same number of bi-Lipschitz mappings $T_k : [0,1]^{N-1} \times [-1,1] \to \overline{B(x_k, r_k)}$ such that for all $k \in \{1, 2, \dots, m\}$,

(i) $x_k \in \partial\Omega$,
(ii) $\bigcup_{k=1}^m B(x_k, r_k) \supset \partial\Omega$,
(iii) $T_k([0,1]^{N-1} \times [-1,0]) = (\mathbb{R}^N \setminus \Omega) \cap \overline{B(x_k, r_k)}$,
(iv) $T_k([0,1]^{N-1} \times [0,1]) = \Omega \cap \overline{B(x_k, r_k)}$,
(v) $T_k([0,1]^{N-1} \times \{0\}) = \partial\Omega \cap \overline{B(x_k, r_k)}$.

Let $\Omega \in \mathcal{C}^{0,1}$ and $M \subset \overline{\Omega}$ be a compact set. Given $p(\cdot)$, $q(\cdot) : \overline{\Omega} \to \mathbb{R}$ we find conditions on M and, moreover, determine how quickly can $q(\cdot)$ tend to $p^{\#}(\cdot)$ near M while preserving the compactness of the embedding of $W^{1,p(\cdot)}(\Omega)$ in $L^{q(\cdot)}(\Omega)$.

3. Almost-compact embedding between variable spaces

We fix in this section a domain $\Omega \subset \mathbb{R}^N$ and functions $p(\cdot)$, $q(\cdot) \in \mathcal{E}(\Omega)$. Adopt the notation $r(x) = \frac{p(x)}{q(x)}$.

Lemma 3.1. *Let Ω be bounded and $q(\cdot) \in \mathcal{E}(\Omega)$, $q_+ < \infty$. Then*

$$\|u\|_{q(\cdot)} \leqslant (m_{q(\cdot)}(u))^{1/q_+} \quad \textit{provided} \quad m_{q(\cdot)}(u) \leqslant 1, \tag{3.1}$$

$$\|u\|_{q(\cdot)} \geqslant (m_{q(\cdot)}(u))^{1/q_-} \quad \textit{provided} \quad m_{q(\cdot)}(u) \leqslant 1, \tag{3.2}$$

$$\|u\|_{q(\cdot)} \leqslant (m_{q(\cdot)}(u))^{1/q_-} \quad \textit{provided} \quad m_{q(\cdot)}(u) \geqslant 1, \tag{3.3}$$

$$\|u\|_{q(\cdot)} \geqslant (m_{q(\cdot)}(u))^{1/q_+} \quad \textit{provided} \quad m_{q(\cdot)}(u) \geqslant 1. \tag{3.4}$$

Proof. Set $a = m_{q(\cdot)}(u)$ and assume $a \leqslant 1$. Then

$$\int_\Omega \Big(\frac{|u(x)|}{a^{1/q_+}}\Big)^{q(x)} dx \leqslant \int_\Omega \Big(\frac{|u(x)|}{a^{1/q(x)}}\Big)^{q(x)} dx = \int_\Omega \frac{|u(x)|^{q(x)}}{a} dx = 1$$

which gives $\|u\|_{q(\cdot)} \leqslant a^{1/q_+}$ and proves (3.1). The assertions (3.2), (3.3) and (3.4) can be proved analogously. $\square$

Lemma 3.2. *Let Ω be bounded and suppose that $q(x) \leqslant p(x) \leqslant p_+ < \infty$ for all $x \in \Omega$. If $\|u\|_{p(\cdot)} \leqslant 1$ then*

$$\|u\|_{p(\cdot)}^{q_+} \leqslant \|\, |u(\cdot)|^{q(\cdot)} \|_{r(\cdot)} \leqslant \|u\|_{p(\cdot)}^{q_-}.$$

Proof. Assume first $0 < a := \|u\|_{p(\cdot)} < 1$. Then

$$1 = \int_\Omega \Big(\frac{|u(x)|}{a}\Big)^{p(x)} dx > \int_\Omega |u(x)|^{p(x)} dx. \tag{3.5}$$

Denote $b = \|\, |u(\cdot)|^{q(\cdot)} \|_{r(\cdot)}$. Then

$$1 = \int_\Omega \Big(\frac{|u(x)|^{q(x)}}{b}\Big)^{r(x)} dx = \int_\Omega \Big(\frac{|u(x)|}{b^{1/q(x)}}\Big)^{p(x)} dx.$$

If $b > 1$ then

$$1 = \int_\Omega \Big(\frac{|u(x)|}{b^{1/q(x)}}\Big)^{p(x)} dx < \int_\Omega |u(x)|^{p(x)} dx \overset{(3.5)}{<} 1$$

which is a contradiction. So $b \leqslant 1$. Consequently,

$$\int_\Omega \Big(\frac{|u(x)|}{b^{1/q_+}}\Big)^{p(x)} dx \leqslant 1 = \int_\Omega \Big(\frac{|u(x)|}{b^{1/q(x)}}\Big)^{p(x)} dx \leqslant \int_\Omega \Big(\frac{|u(x)|}{b^{1/q_-}}\Big)^{p(x)} dx$$

which gives $b^{1/q_+} \geqslant \|u\|_{p(\cdot)} \geqslant b^{1/q_-}$ and finally

$$\|u\|_{p(\cdot)}^{q_+} \leqslant \| \, |u(\cdot)|^{q(\cdot)} \|_{r(\cdot)} \leqslant \|u\|_{p(\cdot)}^{q_-}.$$

Assume now $0 < \|u\|_{p(\cdot)} \leqslant 1$. Choose $\varepsilon > 0$. Then $\|\frac{u}{1+\varepsilon}\|_{p(\cdot)} < 1$ and so,

$$\Big\| \, \frac{|u(\cdot)|}{1+\varepsilon} \Big\|_{p(\cdot)}^{q_+} \leqslant \Big\| \, \Big(\frac{|u(\cdot)|}{1+\varepsilon}\Big)^{q(\cdot)} \Big\|_{r(\cdot)} \leqslant \Big\| \, \frac{|u(\cdot)|}{1+\varepsilon} \Big\|_{p(\cdot)}^{q_-}.$$

Since $\frac{1}{(1+\varepsilon)^{q_-}} \geqslant \frac{1}{(1+\varepsilon)^{q(x)}} \geqslant \frac{1}{(1+\varepsilon)^{q_+}}$ we have

$$\frac{1}{(1+\varepsilon)^{q_+}} \Big\| \, |u(\cdot)|^{q(\cdot)} \Big\|_{r(\cdot)} \leqslant \Big\| \, \Big(\frac{|u(\cdot)|}{1+\varepsilon}\Big)^{q(\cdot)} \Big\|_{r(\cdot)} \leqslant \Big\| \, \frac{|u(\cdot)|}{1+\varepsilon} \Big\|_{p(\cdot)}^{q_-} \leqslant \frac{1}{(1+\varepsilon)^{q_-}} \|u\|_{p(\cdot)}^{q_-}$$

and

$$\frac{1}{(1+\varepsilon)^{q_-}} \Big\| \, |u(\cdot)|^{q(\cdot)} \Big\|_{r(\cdot)} \geqslant \Big\| \, \Big(\frac{|u(\cdot)|}{1+\varepsilon}\Big)^{q(\cdot)} \Big\|_{r(\cdot)} \geqslant \Big\| \, \frac{|u(\cdot)|}{1+\varepsilon} \Big\|_{p(\cdot)}^{q_+} \geqslant \frac{1}{(1+\varepsilon)^{q_+}} \|u\|_{p(\cdot)}^{q_+}$$

which proves

$$(1+\varepsilon)^{q_- - q_+} \|u\|_{p(\cdot)}^{q_+} \leqslant \| \, |u(\cdot)|^{q(\cdot)} \|_{r(\cdot)} \leqslant (1+\varepsilon)^{q_+ - q_-} \|u\|_{p(\cdot)}^{q_-}.$$

Tending $\varepsilon \to 0^+$ we obtain

$$\|u\|_{p(\cdot)}^{q_+} \leqslant \| \, |u(\cdot)|^{q(\cdot)} \|_{r(\cdot)} \leqslant \|u\|_{p(\cdot)}^{q_-}.$$

□

Lemma 3.3. *Let Ω be bounded, $p(\cdot)$, $q(\cdot) \in \mathcal{E}(\Omega)$, and suppose that $q(x) \leqslant p(x) \leqslant p_+ < \infty$ for all $x \in \Omega$. Assume that for any sequence $\{E_n\}_{n\in\mathbb{N}}$ of measurable subsets of Ω such that $|E_n| \to 0$, we have*

$$\|\chi_E\|_{r'(\cdot)} \to 0.$$

Then $L^{p(\cdot)}(\Omega) \overset{*}{\hookrightarrow} L^{q(\cdot)}(\Omega)$.

Proof. Let $E_n \subset \Omega$, $|E_n| \to 0$. Then by Lemma 3.1 we obtain

$$\begin{aligned} &\lim_{n\to\infty} \sup\{\|u\chi_{E_n}\|_{q(\cdot)};\ \|u\|_{p(\cdot)} \leqslant 1\} \\ &\leqslant \lim_{n\to\infty} \sup\{\max\{(m_{q(\cdot)}(u\chi_{E_n}))^{1/q_+}, (m_{q(\cdot)}(u\chi_{E_n}))^{1/q_-}\};\ \|u\|_{p(\cdot)} \leqslant 1\}. \end{aligned}$$

If $\|u\|_{p(\cdot)} \leqslant 1$ we obtain by the Hölder inequality and Lemma 3.2

$$\begin{aligned} m_{q(\cdot)}(u\chi_{E_n}) &= \int_\Omega |u(x)\chi_{E_n}(x)|^{q(x)} dx \leqslant c\|\chi_{E_n}\|_{r'(\cdot)} \| \, |u(\cdot)|^{q(\cdot)} \|_{r(\cdot)} \\ &\leqslant c\|\chi_{E_n}\|_{r'(\cdot)} \|u\|_{p(\cdot)}^{q_-}. \end{aligned}$$

This gives

$$\begin{aligned}
&\lim_{n\to\infty} \sup\{\|u\chi_{E_n}\|_{q(\cdot)};\ \|u\|_{p(\cdot)} \leqslant 1\}\\
&\leqslant c \lim_{n\to\infty} \sup\{\max\{\|\chi_{E_n}\|_{r'(\cdot)}^{1/q_+}\|u\|_{p(\cdot)}^{q_-/q_+},\ \|\chi_{E_n}\|_{r'(\cdot)}^{1/q_-}\|u\|_{p(\cdot)}\};\ \|u\|_{p(\cdot)} \leqslant 1\}\\
&\leqslant c \lim_{n\to\infty} \max\{\|\chi_{E_n}\|_{r'(\cdot)}^{1/q_+},\ \|\chi_{E_n}\|_{r'(\cdot)}^{1/q_-}\} = 0.
\end{aligned}$$

□

Theorem 3.4. *Let Ω be bounded, $p(\cdot)$, $q(\cdot) \in \mathcal{E}(\Omega)$, and suppose that $q(x) \leqslant p(x) \leqslant p_+ < \infty$ for all $x \in \Omega$. Denote $s(x) = \frac{1}{p(x)-q(x)}$. Assume*

$$\int_0^{|\Omega|} a^{s^*(t)} dt < \infty. \tag{3.6}$$

for all $a > 1$. Then $L^{p(\cdot)}(\Omega) \overset{*}{\hookrightarrow} L^{q(\cdot)}(\Omega)$.

Proof. Let $E_n \subset \Omega$, $|E_n| \to 0$. Assume that there is $\alpha > 0$ such that

$$\|\chi_{E_n}\|_{r'(\cdot)} \geqslant \alpha$$

for all n. Without loss of generality we can assume $\alpha < 1$. Then

$$\alpha \leqslant \inf\Big\{\lambda > 0; \int_\Omega \Big|\frac{\chi_{E_n}(x)}{\lambda}\Big|^{r'(x)} dx \leqslant 1\Big\} = \inf\Big\{\lambda > 0; \int_\Omega \Big|\frac{\chi_{E_n}(x)}{\lambda}\Big|^{p(x)s(x)} dx \leqslant 1\Big\}.$$

Choose $0 < \beta < \alpha$. Then we obtain by Lemma 2.3

$$1 < \int_\Omega \Big|\frac{\chi_{E_n}(x)}{\beta}\Big|^{p(x)s(x)} dx \leqslant \int_{E_n} \Big(\frac{1}{\beta^{p_+}}\Big)^{s(x)} dx = \int_0^{|E_n|} \Big(\frac{1}{\beta^{p_+}}\Big)^{s^*(t)} dt.$$

Since by the assumption

$$\int_0^{|E_n|} \Big(\frac{1}{\beta^{p_+}}\Big)^{s^*(t)} dt \to 0 \quad \text{for} \quad n \to \infty$$

we have a contradiction. So, $\|\chi_{E_n}\|_{r'(\cdot)} \to 0$. By Lemma 3.3 we have $L^{p(\cdot)}(\Omega) \overset{*}{\hookrightarrow} L^{q(\cdot)}(\Omega)$. □

We remark that the condition (3.6) was first introduced in Corollary 2.7 of [3].

Lemma 3.5. *Let Ω be bounded, $p(\cdot)$, $q(\cdot) \in \mathcal{E}(\Omega)$, and suppose that $q(x) \leqslant p(x) \leqslant p_+ < \infty$ for all $x \in \Omega$. Assume that there exist an $\alpha > 0$ and a sequence $E_n \subset \Omega$ with $|E_n| \to 0$ such that for all n*

$$\|\chi_{E_n}\|_{r'(\cdot)} \geqslant \alpha.$$

Then $L^{p(\cdot)}(\Omega) \overset{*}{\not\hookrightarrow} L^{q(\cdot)}(\Omega)$.

Proof. Let $E_n \subset \Omega$, $\|\chi_{E_n}\|_{p'(\cdot)} \geqslant \alpha$. Fix n. Without loss of generality we can assume $\alpha \leqslant 1$. Set

$$u_n(x) = \frac{\chi_{E_n}(x)}{\|\chi_{E_n}\|_{r'(\cdot)}^{\frac{r'(x)}{r(x)q(x)}}}.$$

Clearly,

$$\int_\Omega |u_n(x)|^{p(x)}dx = \int_\Omega \left(\frac{\chi_{E_n}(x)}{\|\chi_{E_n}\|_{r'(\cdot)}^{\frac{r'(x)}{r(x)q(x)}}} \right)^{p(x)} dx = \int_\Omega \frac{\chi_{E_n}(x)}{\|\chi_{E_n}\|_{r'(\cdot)}^{r'(x)}}dx$$
$$= \int_\Omega \left(\frac{\chi_{E_n}(x)}{\|\chi_{E_n}\|_{r'(\cdot)}} \right)^{r'(x)} dx = 1.$$

Thus

$$\|u_n\|_{p(\cdot)} = 1.$$

Then by Lemma 3.1 we obtain

$$\sup\{\|u\chi_{E_n}\|_{q(\cdot)};\ \|u\|_{p(\cdot)} \leqslant 1\} \geqslant \min\{(m_{q(\cdot)}(u_n\chi_{E_n}))^{1/q_+},\ (m_{q(\cdot)}(u_n\chi_{E_n}))^{1/q_-}\}.$$

Further

$$m_{q(\cdot)}(u_n\chi_{E_n}) = \int_\Omega |u_n(x)\chi_{E_n}(x)|^{q(x)}dx = \int_\Omega \chi_{E_n}(x) \left(\frac{\chi_{E_n}(x)}{\|\chi_{E_n}\|_{r'(\cdot)}^{\frac{r'(x)}{r(x)q(x)}}} \right)^{q(x)} dx$$
$$= \int_\Omega \frac{\chi_{E_n}(x)}{\|\chi_{E_n}\|_{r'(\cdot)}^{\frac{r'(x)}{r(x)}}}dx = \|\chi_{E_n}\|_{r'(\cdot)} \int_\Omega \frac{\chi_{E_n}(x)}{\|\chi_{E_n}\|_{r'(\cdot)}^{1+\frac{r'(x)}{r(x)}}}dx = \|\chi_{E_n}\|_{r'(\cdot)} \int_\Omega \frac{\chi_{E_n}(x)}{\|\chi_{E_n}\|_{r'(\cdot)}^{r'(x)}}dx$$
$$= \|\chi_{E_n}\|_{r'(\cdot)} \int_\Omega \left(\frac{\chi_{E_n}(x)}{\|\chi_{E_n}\|_{r'(\cdot)}} \right)^{r'(x)} dx = \|\chi_{E_n}\|_{r'(\cdot)} \geqslant \alpha.$$

Then

$$\begin{aligned}\sup\{\|u\chi_{E_n}\|_{q(\cdot)};\ \|u\|_{p(\cdot)} \leqslant 1\} &\geqslant \min\{(m_{q(\cdot)}(u_n\chi_{E_n}))^{1/q_+},\ (m_{q(\cdot)}(u_n\chi_{E_n}))^{1/q_-}\} \\ &\geqslant \min\{\alpha^{1/q_+}, \alpha^{1/q_-}\} = \alpha^{1/q_-}\end{aligned}$$

which proves the lemma. □

Lemma 3.6. *Let $E \subset \Omega$, $g : \Omega \to [0, \infty)$ and assume that*

$$\inf\{g(x); x \in E\} \geqslant \sup\{g(x); x \in \Omega \setminus E\}.$$

Then

$$(g\chi_E)^*(t) = g^*(t)\chi_{(0,|E|)}(t).$$

Proof. Trivial. □

Theorem 3.7. *Let Ω be bounded, $p(\cdot)$, $q(\cdot) \in \mathcal{E}(\Omega)$, and suppose that $q(x) \leqslant p(x) \leqslant p_+ < \infty$ for all $x \in \Omega$. Denote $s(x) = \frac{1}{p(x)-q(x)}$. Assume that there is $a > 1$ such that*

$$\int_0^{|\Omega|} a^{s^*(t)}dt = \infty. \tag{3.7}$$

Then $L^{p(\cdot)}(\Omega) \overset{*}{\not\hookrightarrow} L^{q(\cdot)}(\Omega)$.

Proof. Define $E_n = \{x \in \Omega; s(x) \geqslant n\}$. Assume for a moment that there exists n_0 such that $|E_{n_0}| = 0$. Then

$$s(x) = \frac{1}{p(x) - q(x)} < n_0$$

almost everywhere and so we have for any $a > 1$

$$\int_0^{|\Omega|} a^{s^*(t)} dt \leqslant \int_0^{|\Omega|} a^{n_0} dt = a^{n_0} |\Omega| < \infty$$

which is a contradiction with the assumption. So, $|E_n| > 0$ for all n. Fix n and assume

$$\max\{\|\chi_{E_n}\|_{r'(\cdot)}^{p+}, \|\chi_{E_n}\|_{r'(\cdot)}^{p-}\} \leqslant \frac{1}{a}. \tag{3.8}$$

Then

$$1 = \int_\Omega \left(\frac{\chi_{E_n}(x)}{\|\chi_E\|_{r'(\cdot)}} \right)^{r'(x)} dx = \int_\Omega \left(\frac{\chi_{E_n}(x)}{\|\chi_{E_n}\|_{r'(\cdot)}^{p(x)}} \right)^{s(x)} dx \tag{3.9}$$

$$\geqslant \int_{E_n} \left(\frac{1}{\max\{\|\chi_{E_n}\|_{r'(\cdot)}^{p+}, \|\chi_{E_n}\|_{r'(\cdot)}^{p-}\}} \right)^{s(x)} dx \geqslant \int_{E_n} a^{s(x)} dx.$$

Now, by the definition of E_n we have that $s(x) \geqslant n$ on E_n and $s(x) < n$ on $\Omega \setminus E_n$. This gives us $a^{s(x)} \geqslant a^n$ on E_n and $a^{s(x)} < a^n$ on $\Omega \setminus E_n$. Then we have by Lemma 3.6

$$(a^{s(\cdot)} \chi_{E_n}(\cdot))^*(t) = (a^{s(\cdot)})^*(t) \chi_{(0,|E_n|)}(t) = a^{s^*(t)} \chi_{(0,|E_n|)}(t)$$

which gives with (3.9)

$$1 \geqslant \int_{E_n} a^{s(x)} dx = \int_\Omega a^{s(x)} \chi_{E_n}(x) dx = \int_0^{|\Omega|} (a^{s(\cdot)} \chi_{E_n}(\cdot))^*(t) dt$$

$$= \int_0^{|\Omega|} a^{s^*(t)} \chi_{(0,|E_n|)}(t) dt = \int_0^{|E_n|} a^{s^*(t)}(t) dt = \infty$$

which is a contradiction. So, our assumption (3.8) is false and we have

$$\max\{\|\chi_{E_n}\|_{r'(\cdot)}^{p+}, \|\chi_{E_n}\|_{r'(\cdot)}^{p-}\} > \frac{1}{a}$$

which yields

$$\|\chi_{E_n}\|_{r'(\cdot)} > \min\{a^{-1/p_+}, a^{-1/p_-}\} := b > 0.$$

Thus, we have $\|\chi_{E_n}\|_{r'(\cdot)} \geqslant b > 0$ for any n and Lemma 3.5 gives us $L^{p(\cdot)}(\Omega) \overset{*}{\not\hookrightarrow} L^{q(\cdot)}(\Omega)$. □

Consider a special case. Let $K \subset \Omega$ be compact with $|K| = 0$. Denote $d_K(x) = \operatorname{dist}(x, K)$. Set

$$K(t) = \{x \in \Omega; \mathrm{d}_K(x) < t\}. \tag{3.10}$$

Denote

$$\varphi(t) = |K(t)|, \ t \in [0, \operatorname{diam}(\Omega)]. \tag{3.11}$$

Let $\omega : [0, \operatorname{diam}(\Omega)] \to \mathbb{R}$ be a decreasing continuous non-negative function, $\omega_0 := \omega(\operatorname{diam}(\Omega))$. Let ω^{-1} denote the inverse function to ω.

Lemma 3.8. *Let Ω be bounded, $p(\cdot)$, $q(\cdot) \in \mathcal{E}(\Omega)$, and suppose that $q(x) \leqslant p(x) \leqslant p_+ < \infty$ for all $x \in \Omega$. Assume that there is $c > 0$ such that*

$$s(x) = \frac{1}{p(x) - q(x)} \leqslant c\, \omega(\mathrm{d}_K(x)),\ x \in \Omega,$$
$$\int_{\omega_0}^{\infty} \varphi(\omega^{-1}(y)) a^y dy < \infty \text{ for all } a > 1.$$

Then $L^{p(\cdot)}(\Omega) \overset{*}{\hookrightarrow} L^{q(\cdot)}(\Omega)$.

Proof. Let $a > 1$. Then

$$\begin{aligned}
&\int_0^{|\Omega|} a^{s^*(t)} dt = \int_\Omega a^{s(x)} dx \leqslant \int_\Omega a^{c\omega(\mathrm{d}_K(x))} dx = \int_0^\infty |\{x; a^{c\omega(\mathrm{d}_K(x))} > \lambda\}| d\lambda \\
&= \int_0^{a^{c\omega_0}} |\{x; a^{c\omega(\mathrm{d}_K(x))} > \lambda\}| d\lambda + \int_{a^{c\omega_0}}^\infty |\{x; a^{c\omega(\mathrm{d}_K(x))} > \lambda\}| d\lambda \\
&= a^{c\omega_0}|\Omega| + c \ln a \int_{\omega_0}^\infty |\{x; a^{c\omega(\mathrm{d}_K(x))} > a^{cy}\}| a^{cy} dy \\
&= a^{c\omega_0}|\Omega| + c \ln a \int_{\omega_0}^\infty |\{x; \omega(\mathrm{d}_K(x)) > y\}| a^{cy} dy \\
&= a^{c\omega_0}|\Omega| + c \ln a \int_{\omega_0}^\infty |\{x; \mathrm{d}_K(x)) < \omega^{-1}(y)\}| a^{cy} dy \\
&= a^{c\omega_0}|\Omega| + c \ln a \int_{\omega_0}^\infty \varphi(\omega^{-1}(y)) a^{cy} dy < \infty.
\end{aligned}$$

By Theorem 3.4 we have $L^{p(\cdot)}(\Omega) \overset{*}{\hookrightarrow} L^{q(\cdot)}(\Omega)$. □

4. Examples of Cantor sets

Let $\{a_k\}_{k\in\mathbb{N}}$ be a given sequence of positive real numbers with

$$\sum_{k=1}^{\infty} a_k = 1. \tag{4.1}$$

Construct a generalized Cantor set by the following process. Set $K_0 = [0, 1]$. Omit in the first step from K_0 a centered interval of length a_1 to obtain a set K_1. We write

$$K_1 = K_0 \setminus \Big(\frac{1 - a_1}{2}, \frac{1 + a_1}{2}\Big) = \Big[0, \frac{1 - a_1}{2}\Big] \cup \Big[\frac{1 + a_1}{2}, 1\Big] := J_0 \cup J_1.$$

In the second step we omit from J_0 and J_1 centered intervals of length $a_2/2$ to obtain K_2. Then

$$\begin{aligned}
&K_2 = K_1 \setminus \Big(\Big(\frac{1 - a_1 - a_2}{2^2}, \frac{1 - a_1 + a_2}{2^2}\Big) \cup \Big(\frac{3 + a_1 - a_2}{2^2}, \frac{3 + a_1 + a_2}{2^2}\Big)\Big) \\
&= \Big[0, \frac{1 - a_1 - a_2}{2^2}\Big] \cup \Big[\frac{1 - a_1 + a_2}{2^2}, \frac{1 - a_1}{2}\Big] \cup \Big[\frac{1 + a_1}{2}, \frac{3 + a_1 - a_2}{2^2}\Big] \cup \Big[\frac{3 + a_1 + a_2}{2^2}, 1\Big] \\
&:= J_{00} \cup J_{01} \cup J_{10} \cup J_{11}.
\end{aligned}$$

We follow this process step by step to obtain sets K_n. Then K_n consists of 2^n intervals J_α, $\alpha \in \{0,1\}^n$. Clearly,

$$|J_\alpha| = 2^{-n}\Big(1 - \sum_{k=1}^{n} a_k\Big). \tag{4.2}$$

Set

$$K = \bigcap_{n=1}^{\infty} K_n.$$

Clearly, K is a compact set and for each n

$$|K| \leqslant |K_n|$$

which gives with (4.2)

$$|K| \leqslant \sum_{\alpha\in\{0,1\}^n} |J_\alpha| = 2^n 2^{-n}\Big(1 - \sum_{k=1}^{n} a_k\Big) = 1 - \sum_{k=1}^{n} a_k.$$

Using (4.1) we have

$$|K| = 0.$$

Now, we will be interested in the behavior of the function $|K(t)|$.

Lemma 4.1. *The function $|K(\cdot)|$ is non-increasing and* $\lim_{t\to 0_+} |K(t)| = 0$.

Proof. The monotonicity of $|K(\cdot)|$ is clear. Moreover, $K(t) \searrow K$ and $K(1) < \infty$ since K is compact. It is easily seen that $\lim_{t\to 0_+} |K(t)| = 0$. □

Lemma 4.2. *For each $n \in \mathbb{N}$ let r_n, ε_n be given by*

$$r_n = 1 - \sum_{k=1}^{n} a_k;$$

$$\varepsilon_n = 2^{-n}\Big(1 - \sum_{k=1}^{n} a_k\Big) = 2^{-n} r_n.$$

Then

$$r_n \leqslant |K(\varepsilon_n)| \leqslant 4r_n.$$

Proof. Clearly, $\varepsilon_n = |J_\alpha|$ for $\alpha \in \{0,1\}^n$ and so,

$$K(\varepsilon_n) \supset \bigcup_{\alpha\in\{0,1\}^n} J_\alpha$$

which gives

$$|K(\varepsilon_n)| \geqslant \sum_{\alpha\in\{0,1\}^n} |J_\alpha| = 2^n \varepsilon_n = r_n.$$

For the right-hand inequality, denote by M the set of all endpoints of intervals J_α, $\alpha \in \{0,1\}^n$. The number of these points is $2(1 + 2 + 2^2 + \cdots + 2^{n-1}) = 2(2^n - 1)$ and

$$K(\varepsilon_n) \subset \bigcup_{x\in M} (x - \varepsilon_n, x + \varepsilon_n).$$

Then

$$|K(\varepsilon_n)| \leqslant \sum_{x\in M} 2\varepsilon_n = 2(2^n-1)2\varepsilon_n \leqslant 4.2^n\varepsilon_n = 4r_n.$$

□

One important case is obtained by choosing $a_k = \frac{a^{k-1}}{(a+1)^k}$ where $a>0$. When $a=2$ we obtain the classical Cantor set.

Lemma 4.3. *Let* $a_k = \frac{a^{k-1}}{(a+1)^k}$ *and set*

$$s = \frac{\ln(\frac{a}{a+1})}{\ln(\frac{a}{2(a+1)})}.$$

Then there are positive constants c_1, c_2 *such that*

$$c_1 t^s \leqslant |K(t)| \leqslant c_2 t^s,\ \ t\in[0,\operatorname{diam}(\Omega)].$$

Proof. Let $q = \frac{a}{a+1}$. Then $s = \frac{\ln q}{\ln(q/2)}$. Clearly,

$$r_n = 1 - \sum_{k=1}^{n} \frac{a^{k-1}}{(a+1)^k} = \Big(\frac{a}{a+1}\Big)^n = q^n,$$
$$\varepsilon_n = 2^{-n}\Big(\frac{a}{a+1}\Big)^n = 2^{-n}q^n$$

and

$$r_{n+1} = qr_n, \quad \varepsilon_{n+1} = \frac{q}{2}\varepsilon_n.$$

It is easy to see that $0<s<1$.

Fix $t\in[\varepsilon_{n+1},\varepsilon_n]$. By Lemmas 4.1 and 4.2 we know that

$$qr_n = r_{n+1} \leqslant |K(\varepsilon_{n+1})| \leqslant |K(t)| \leqslant |K(\varepsilon_n)| \leqslant 4r_n. \tag{4.3}$$

Since $q/2\varepsilon_n \leqslant t \leqslant \varepsilon_n$, we have

$$\ln q/2 + n\ln q/2 = \ln q/2 + \ln\varepsilon_n \leqslant \ln t \leqslant \ln\varepsilon_n = n\ln q/2$$

which gives

$$\frac{\ln t}{\ln q/2} \leqslant n \leqslant \frac{\ln t - \ln q/2}{\ln q/2}.$$

This implies that

$$t^{\frac{\ln q}{\ln q/2}} = q^{\frac{\ln t}{\ln q/2}} \leqslant r_n \leqslant q^{\frac{\ln t-\ln q/2}{\ln q/2}} = \frac{1}{q}\ t^{\frac{\ln q}{\ln q/2}}.$$

By (4.3) we obtain

$$qt^s \leqslant |K(t)| \leqslant \frac{4}{q}\ t^s.$$

□

We recall the definition of the Riemann function

$$\zeta(s) = \sum_{k=1}^{\infty} k^{-s},\ \ s>1.$$

Lemma 4.4. *Let $a_k = \frac{k^{-s}}{\zeta(s)}$. Then there are positive constants c_1, c_2 such that*

$$\frac{c_1}{(\ln(\mathrm{e}/t))^{s-1}} \leqslant |K(t)| \leqslant \frac{c_2}{(\ln(\mathrm{e}/t))^{s-1}}, \ t \in [0, \mathrm{diam}(\Omega)].$$

Proof. Clearly,

$$r_n = 1 - \frac{1}{\zeta(s)} \sum_{k=1}^{n} k^{-s} = \frac{1}{\zeta(s)} \sum_{k=n+1}^{\infty} k^{-s}.$$

It is easy to see that

$$\frac{1}{\zeta(s)(s-1)(n+1)^{s-1}} = \frac{1}{\zeta(s)} \int_{n+1}^{\infty} x^{-s} ds \leqslant r_n$$
$$\leqslant \frac{1}{\zeta(s)} \int_{n}^{\infty} x^{-s} ds = \frac{1}{\zeta(s)(s-1)n^{s-1}}.$$

This gives for $n \geqslant 2$

$$2^{1-s} \leqslant \frac{r_{n+1}}{r_n} \leqslant 1, \quad 2^{-s} \leqslant \frac{\varepsilon_{n+1}}{\varepsilon_n} \leqslant \frac{1}{2}.$$

Fix $t \in [\varepsilon_{n+1}, \varepsilon_n]$. Then

$$2^{1-s} r_n = r_{n+1} \leqslant |K(\varepsilon_{n+1})| \leqslant |K(t)| \leqslant |K(\varepsilon_n)| \leqslant 4 r_n. \tag{4.4}$$

We know

$$2^{-s}\varepsilon_n \leqslant \varepsilon_{n+1} \leqslant t \leqslant \varepsilon_n$$

which gives

$$\frac{2^{-n}(n+1)^{1-s}}{\zeta(s)(s-1)} \leqslant \varepsilon_n \leqslant \frac{2^{-n} n^{1-s}}{\zeta(s)(s-1)}$$

and consequently

$$\frac{1}{2^s 2^{s-1}} \frac{2^{-n} n^{1-s}}{\zeta(s)(s-1)} \leqslant t \leqslant \frac{2^{-n} n^{1-s}}{\zeta(s)(s-1)}.$$

So, there are constants b_1, b_2 such that

$$b_1 2^{-n} n^{1-s} \leqslant t \leqslant b_2 2^{-n} n^{1-s}.$$

It yields

$$\ln b_1 - n \ln 2 - (s-1)n \leqslant \ln b_1 - n \ln 2 - (s-1) \ln n$$
$$\leqslant \ln t$$
$$\leqslant \ln b_2 - n \ln 2 - (s-1) \ln n \leqslant \ln b_2 - n \ln 2$$

and so

$$\frac{\ln(b_1/t)}{\ln 2 + s - 1} \leqslant n \leqslant \frac{\ln(b_2/t)}{\ln 2}.$$

Then

$$|K(t)| \leqslant 4 r_n = \frac{4}{\zeta(s)(s-1)n^{s-1}} \leqslant \frac{4(\ln 2 + s - 1)^{s-1}}{\zeta(s)(s-1)(\ln(b_1/t))^{s-1}} \leqslant \frac{c_2}{(\ln(\mathrm{e}/t))^{s-1}}$$

Finally,

$$
\begin{aligned}
|K(t)| &\geqslant r_{n+1} \geqslant \frac{4}{\zeta(s)(s-1)(n+2)^{s-1}} \geqslant \frac{1}{\zeta(s)(s-1)}\Big(\frac{n}{n+2}\Big)^{s-1}\frac{1}{n^{s-1}} \\
&\geqslant \frac{1}{3^{s-1}\zeta(s)(s-1)n^{s-1}} \geqslant \frac{(\ln 2)^{s-1}}{3^{s-1}\zeta(s)(s-1)(\ln(b_2/t))^{s-1}} \geqslant \frac{c_1}{(\ln(\mathrm{e}/t))^{s-1}}.
\end{aligned}
$$

□

Lemma 4.5. *Define a function* $\eta(s) = \sum_{k=1}^{\infty} \frac{1}{(k+1)\ln^s(k+1)}$, $s > 1$. *Choose* $a_k = \frac{1}{\eta(s)}\frac{1}{(k+1)\ln^s(k+1)}$. *Then there are positive constants* c_1, c_2 *and* b *such that*

$$c_1(\ln\ln(b/t))^{1-s} \leqslant |K(t)| \leqslant c_2(\ln\ln(b/t))^{1-s}.$$

Proof. The proof is analogous to the previous one. Clearly,

$$r_n = \sum_{k=n+1}^{\infty} \frac{1}{(k+1)\ln^s(k+1)}, \quad \varepsilon_n = 2^{-n} r_n.$$

By the integral criterion we have estimate

$$\frac{1}{2(\varepsilon-1)\ln^{s-1} n} \leqslant r_n \leqslant \frac{1}{(\varepsilon-1)\ln^{s-1} n}, \qquad n \geqslant 2.$$

Fix $t \in [\varepsilon_{n+1}, \varepsilon_n]$. Then

$$\frac{1}{2(\varepsilon-1)2^{n+1}\ln^{s-1}(n+1)} \leqslant 2^{-n-1} r_{n+1} = \varepsilon_{n+1} \leqslant t \leqslant \varepsilon_n = 2^{-n} r_n \leqslant \frac{1}{(\varepsilon-1)2^n \ln^{s-1} n}.$$

Since $2^{n+1}\ln^{s-1}(n+1)$ is comparable with $2^n \ln^{s-1} n$ for large n we can take positive contant b_1, b_2 such that

$$\frac{b_1}{2^n \ln^{s-1} n} \leqslant t \leqslant \frac{b_2}{2^n \ln^{s-1} n}. \tag{4.5}$$

By Lemma 4.2 we have

$$r_{n+1} \leqslant |K(t)| \leqslant r_n \tag{4.6}$$

and so there are two positive constants d_1, d_2 with

$$\frac{d_1}{\ln^{s-1} n} \leqslant |K(t)| \leqslant \frac{d_2}{\ln^{s-1} n}.$$

By (4.5) we obtain

$$\ln b_1 - n\ln 2 - (s-1)\ln\ln n \leqslant \ln t \leqslant \ln b_2 - n\ln 2 - (s-1)\ln\ln n$$

which gives for some constants L_1, L_2

$$
\begin{aligned}
L_1 n \leqslant n\ln 2 + (s-1)\ln\ln n \leqslant \ln\frac{b_2}{t}, \\
\ln\frac{b_1}{t} \leqslant n\ln 2 + (s-1)\ln\ln n \leqslant L_2 n.
\end{aligned}
$$

So

$$\frac{1}{L_2}\ln\frac{b_1}{t} \leqslant n \leqslant \frac{1}{L_1}\ln\frac{b_2}{t}.$$

This implies

$$\left(\ln\Big(\frac{1}{L_2}\ln\frac{b_1}{t}\Big)\right)^{s-1} \leqslant \ln^{s-1} n \leqslant \left(\ln\Big(\frac{1}{L_1}\ln\frac{b_2}{t}\Big)\right)^{s-1}.$$

Finally we can find c_1, c_2 and b such that

$$c_1(\ln\ln(b/t))^{1-s} \leqslant |K(t)| \leqslant c_2(\ln\ln(b/t))^{1-s}.$$

□

All Cantor sets are constructed on an interval $[0,1]$ so far. But we can construct Cantor sets in $[0,1]^N$ as a cartesian product. But having $|K(t)| = \varphi(t)$ we have $|K^N(t)| \leqslant |K(t)|^N \leqslant \varphi^N(t)$. In Lemmas 4.3, 4.4 and 4.5 we essentially get nothing new, the behavior of $\varphi(t)$ stays qualitatively the same.

5. Examples of almost compact embeddings

Example 5.1. Let Ω be bounded, $p(\cdot), q(\cdot) \in \mathcal{E}(\Omega)$, and suppose that $q(x) \leqslant p(x) \leqslant p_+ < \infty$ for all $x \in \Omega$. Let $K \subset \Omega$ be compact and let φ be given by (3.11). Suppose that $\varphi(t) \leqslant Ct^s$ for some $C > 0$ and $s \in (0, N]$. Assume that $\psi : [\omega_0, \infty) \to (0, \infty)$ satisfies

(i) $\frac{\psi(t)}{\ln t}$ is decreasing;

(ii) $\lim\limits_{t\to\infty} \psi(t) = \infty$;

(iii) $s(x) := \frac{1}{p(x)-q(x)} \leqslant \frac{\ln(1/\mathrm{d}_K(x))}{\psi(1/\mathrm{d}_K(x))}$.

Then $L^{p(\cdot)}(\Omega) \overset{*}{\hookrightarrow} L^{q(\cdot)}(\Omega)$.

Proof. Let $a > 1$. Set $\omega(t) = \frac{\ln(1/t)}{\psi(1/t)}$. Clearly, $s(x) \leqslant \omega(\mathrm{d}_K(x))$ by (iii). Using (ii) we have

$$\frac{\omega(t)}{\ln(1/t)} = \frac{1}{\psi(1/t)} \to 0 \text{ for } \ t \to 0_+.$$

Since ω is decreasing by (i) on $(0, 1/\omega_0)$ we can take an inverse function and write $t = \omega^{-1}(y)$, $y \in [\omega(1/\omega_0), \infty)$. Then

$$\psi(1/\omega^{-1}(y)) = \frac{y}{\ln(1/\omega^{-1}(y))} \to 0 \text{ for } \ y \to \infty. \tag{5.1}$$

This gives us

$$\ln\frac{1}{\omega^{-1}(y)} = \frac{y}{\psi(1/\omega^{-1}(y))} \ \Rightarrow \ \omega^{-1}(y) = \mathrm{e}^{-\frac{y}{\psi(1/\omega^{-1}(y))}}$$

and consequently

$$\int_{\omega_0}^{\infty} \varphi(\omega^{-1}(y))a^y dy \leqslant C\int_{\omega_0}^{\infty} \mathrm{e}^{-\frac{sy}{\psi(1/\omega^{-1}(y))}}\mathrm{e}^{y\ln a}dy = C\int_{\omega_0}^{\infty} \mathrm{e}^{y(\ln a - \frac{s}{\psi(1/\omega^{-1}(y))})}dy = I.$$

By (5.1) we have $\psi(1/\omega^{-1}(y)) \to 0$ for $y \to \infty$ and so, $\ln a - \frac{s}{\psi(1/\omega^{-1}(y))} \leqslant -1$ for large y which implies $I < \infty$. Now, Lemma 3.8 gives $L^{p(\cdot)}(\Omega) \overset{*}{\hookrightarrow} L^{q(\cdot)}(\Omega)$. □

Example 5.2. Let Ω be bounded, $p(\cdot), q(\cdot) \in \mathcal{E}(\Omega)$, and suppose that $q(x) \leqslant p(x) \leqslant p_+ < \infty$ for all $x \in \Omega$. Let $K \subset \Omega$ be compact and let φ be given by (3.11) and $\varphi(t) \leqslant C(\ln(\mathrm{e}/t))^{1-s}$ for some $C > 0$ and $s > 1$. Assume that $\psi : [\omega_0, \infty) \to (0, \infty)$ satisfies

(i) $\frac{\psi(t)}{\ln\ln t}$ is decreasing;

(ii) $\lim\limits_{t\to\infty} \psi(t) = \infty$;

(iii) $s(x) := \frac{1}{p(x)-q(x)} \leqslant \frac{\ln\ln(1/\mathrm{d}_K(x))}{\psi(1/\mathrm{d}_K(x))}$.

Then $L^{p(\cdot)}(\Omega) \overset{*}{\hookrightarrow} L^{q(\cdot)}(\Omega)$.

Proof. Let $a > 1$. Set $\omega(t) = \frac{\ln\ln(1/t)}{\psi(1/t)}$. Clearly, $s(x) \leqslant \omega(\mathrm{d}_K(x))$ by (iii). Using (ii) we have

$$\frac{\omega(t)}{\ln\ln(1/t)} = \frac{1}{\psi(1/t)} \to 0 \text{ for } \ t \to 0_+.$$

Since ω is decreasing by (i) on $(0, 1/\omega_0)$ we can take an inverse function and write $t = \omega^{-1}(y)$, $y \in [\omega(1/\omega_0), \infty)$. Then

$$\psi(1/\omega^{-1}(y)) = \frac{y}{\ln\ln(1/\omega^{-1}(y))} \to 0 \text{ for } y \to \infty. \tag{5.2}$$

It gives us

$$\ln\ln\frac{1}{\omega^{-1}(y)} = \frac{y}{\psi(1/\omega^{-1}(y))} \ \Rightarrow \ \frac{1}{\omega^{-1}(y)} = \exp(\exp(y/\psi(1/\omega^{-1}(y))))$$

and consequently

$$\int_{\omega_0}^{\infty} \varphi(\omega^{-1}(y))a^y dy = \int_{\omega_0}^{\infty} \Big(\ln\frac{1}{\omega^{-1}(y)}\Big)^{1-s} a^y dy = \int_{\omega_0}^{\infty} \mathrm{e}^{(1-s)\ln\ln\frac{1}{\omega^{-1}(y)}} \mathrm{e}^{y\ln a} dy$$
$$\leqslant c\int_{\omega_0}^{\infty} \mathrm{e}^{(1-s)\frac{y}{\psi(1/\omega^{-1}(y))}} \mathrm{e}^{y\ln a} dy = c\int_{\omega_0}^{\infty} \mathrm{e}^{y(\ln a + \frac{1-s}{\psi(1/\omega^{-1}(y))})} dy = I.$$

By (5.2) we have $\psi(1/\omega^{-1}(y)) \to 0$ for $y \to \infty$ and so, $\ln a + \frac{1-s}{\psi(1/\omega^{-1}(y))} \leqslant -1$ for large y which implies $I < \infty$. Now, Lemma 3.8 gives $L^{p(\cdot)}(\Omega) \overset{*}{\hookrightarrow} L^{q(\cdot)}(\Omega)$. □

Example 5.3. Let $\Omega \in \mathcal{C}^{0,1}$, $p(\cdot)$, $q(\cdot) \in \mathcal{E}(\Omega)$, and suppose that $1 \leqslant p(x) \leqslant p_+ < N$, $1 \leqslant q(x) \leqslant p^{\#}(x)$ for all $x \in \Omega$. Let $K \subset \mathbb{R}^N$, φ be given by (3.11) and $\varphi(t) \leqslant C(\ln\ln\ln(\mathrm{e}^{\mathrm{e}}/t))^{1-s}$ for some $C > 0$ and $s > 1$. Assume that $\psi : [\omega_0, \infty) \to (0, \infty)$ satisfies

(i) $\frac{\psi(t)}{\ln\ln\ln t}$ is decreasing;

(ii) $\lim\limits_{t\to\infty} \psi(t) = \infty$;

(iii) $s(x) := \frac{1}{p(x)-q(x)} \leqslant \frac{\ln\ln\ln(1/\mathrm{d}_K(x))}{\psi(1/\mathrm{d}_K(x))}$.

Then $L^{p(\cdot)}(\Omega) \overset{*}{\hookrightarrow} L^{q(\cdot)}(\Omega)$.

Proof. Let $a > 1$. Set $\omega(t) = \frac{\ln\ln\ln(1/t)}{\psi(1/t)}$. Clearly, $s(x) \leqslant \omega(\mathrm{d}_K(x))$ by (iii). Using (ii) we have

$$\frac{\omega(t)}{\ln\ln\ln(1/t)} = \frac{1}{\psi(1/t)} \to 0 \text{ for } \ t \to 0_+.$$

Since ω is strictly monotone by (i) on $(0, 1/\omega_0)$ we can take an inverse function and write $t = \omega^{-1}(y)$, $y \in [\omega(1/\omega_0), \infty)$. Then

$$\psi(1/\omega^{-1}(y)) = \frac{y}{\ln\ln\ln(1/\omega^{-1}(y))} \to 0 \text{ for } y \to \infty. \tag{5.3}$$

Thus

$$\ln\ln\ln\frac{1}{\omega^{-1}(y)}=\frac{y}{\psi(1/\omega^{-1}(y))}\ \Rightarrow\ \frac{1}{\omega^{-1}(y)}=\exp(\exp(\exp(y/\psi(1/\omega^{-1}(y)))))$$

and consequently

$$\int_{\omega_0}^{\infty}\varphi(\omega^{-1}(y))a^y dy=\int_{\omega_0}^{\infty}\Big(\ln\ln\frac{1}{\omega^{-1}(y)}\Big)^{1-s}a^y dy=\int_{\omega_0}^{\infty}\mathrm{e}^{(1-s)\ln\ln\ln\frac{1}{\omega^{-1}(y)}}\mathrm{e}^{y\ln a}dy$$
$$\leqslant c\int_{\omega_0}^{\infty}\mathrm{e}^{(1-s)\frac{y}{\psi(1/\omega^{-1}(y))}}\mathrm{e}^{y\ln a}dy=c\int_{\omega_0}^{\infty}\mathrm{e}^{y(\ln a+\frac{1-s}{\psi(1/\omega^{-1}(y))})}dy=I.$$

By (5.3) we have $\psi(1/\omega^{-1}(y))\to 0$ for $y\to\infty$ and so, $\ln a+\frac{1-s}{\psi(1/\omega^{-1}(y))}\leqslant -1$ for large y which implies $I<\infty$. Now, Lemma 3.8 gives $L^{p(\cdot)}(\Omega)\overset{*}{\hookrightarrow}L^{q(\cdot)}(\Omega)$. □

6. Compact embeddings between variable Sobolev and variable Lebesgue spaces

First of all we establish a necessary condition for an embedding to be compact.

Lemma 6.1. *Let $B_r=B(0,r)$ denote the ball in $\mathbb{R}^N$ centered at 0 with radius r. Assume $M\subset B_r$ and $s\in\mathbb{R}$ are such that $|B_r\setminus B_s|\leqslant|M|$. Suppose that $\varphi:(0,r]\to\mathbb{R}$ is non-negative and non-increasing and set $\psi(x)=\varphi(|x|)$, $x\in B_r$. Then*

$$\int_M\psi(x)dx\geqslant\int_{B_r\setminus B_s}\psi(x)dx.$$

Proof. By the assumption $|B_r\setminus B_s|\leqslant|M|$ we have

$$\begin{aligned}|(B_r\setminus B_s)\setminus M|+|(B_r\setminus B_s)\cap M|&=|(B_r\setminus B_s)|\leqslant|M|\\&=|(B_r\setminus B_s)\cap M|+|M\cap B_s|.\end{aligned}$$

and consequently

$$|M\cap B_s|\geqslant|(B_r\setminus B_s)\setminus M|.$$

By the assumptions on ψ we have $\psi(x)\geqslant\psi(y)$ for every $x\in B_s$ and every $y\in B_r\setminus B_s$. This implies

$$\begin{aligned}\int_M\psi(x)dx&=\int_{M\cap B_s}\psi(x)dx+\int_{(B_r\setminus B_s)\cap M}\psi(x)dx\\&\geqslant\frac{|M\cap B_s|}{|(B_r\setminus B_s)\setminus M|}\int_{(B_r\setminus B_s)\setminus M}\psi(x)dx+\int_{(B_r\setminus B_s)\cap M}\psi(x)dx\\&\geqslant\int_{(B_r\setminus B_s)\setminus M}\psi(x)dx+\int_{(B_r\setminus B_s)\cap M}\psi(x)dx\\&=\int_{B_r\setminus B_s}\psi(x)dx.\end{aligned}$$

which finishes the proof. □

Theorem 6.2. *Let $p,\,q\in\mathcal{E}(\Omega)$, $1\leqslant p(x)\leqslant p_+<N$ on Ω and let $p(\cdot)$ satisfy (2.2), $1\leqslant q(x)\leqslant p^{\#}(x)$ and let $M=\{x\in\Omega;\ p(x)=q(x)\}$. Assume*

$$W^{1,p(\cdot)}(\Omega)\hookrightarrow\hookrightarrow L^{q(\cdot)}(\Omega).$$

Then $|M|=0$.

Proof. Suppose $|M|>0$. Let $x_0\in\Omega$ be a point of Lebesgue density of M. Given $\varepsilon>0$ denote

$$p_-^\varepsilon=\inf\{p(x);x\in B(x_0,\varepsilon)\},\quad p_+^\varepsilon=\sup\{p(x);x\in B(x_0,\varepsilon)\}$$

and define a function u_ε by

$$u_\varepsilon(x)=\varepsilon^{\frac{p_-^\varepsilon-N}{p_-^\varepsilon}}(1-|x|/\varepsilon)\chi_{B(x_0,\varepsilon)}(x).$$

Clearly,

$$|\nabla u_\varepsilon(x)|=\varepsilon^{\frac{p_-^\varepsilon-N}{p_-^\varepsilon}}\ 1/\varepsilon\ \chi_{B(x_0,\varepsilon)}(x)=\varepsilon^{-\frac{N}{p_-^\varepsilon}}\ \chi_{B(x_0,\varepsilon)}(x).$$

First we prove that the set $\{u_\varepsilon\}$ is bounded in $W^{1,p(\cdot)}(\Omega)$ for $\varepsilon\leqslant 1$. Plainly,

$$\begin{aligned}\int_\Omega |u_\varepsilon(x)|^{p(x)}dx&=\int_{B(x_0,\varepsilon)}\varepsilon^{\frac{p_-^\varepsilon-N}{p_-^\varepsilon}p(x)}(1-|x|/\varepsilon)^{p(x)}dx\\ &\leqslant\int_{B(x_0,\varepsilon)}\varepsilon^{p(x)}\varepsilon^{-\frac{N}{p_-^\varepsilon}p(x)}dx\leqslant\int_{B(x_0,\varepsilon)}\varepsilon^{-\frac{N}{p_-^\varepsilon}p(x)}dx:=I_\varepsilon.\end{aligned}$$

Moreover,

$$\int_\Omega|\nabla u_\varepsilon(x)|^{p(x)}dx\leqslant\int_{B(x_0,\varepsilon)}\varepsilon^{-\frac{N}{p_-^\varepsilon}p(x)}dx:=I_\varepsilon.$$

Now,

$$\begin{aligned}I_\varepsilon&=\int_{B(x_0,\varepsilon)}\varepsilon^{-\frac{N}{p_-^\varepsilon}(p(x)-p_-^\varepsilon)}\ \varepsilon^{-N}dx=\int_{B(x_0,\varepsilon)}\mathrm{e}^{-\frac{N}{p_-^\varepsilon}(p(x)-p_-^\varepsilon)\ln\varepsilon}\ \varepsilon^{-N}dx\\ &=\int_{B(x_0,\varepsilon)}\mathrm{e}^{\frac{N}{p_-^\varepsilon}(p(x)-p_-^\varepsilon)\ln(1/\varepsilon)}\ \varepsilon^{-N}dx.\end{aligned}$$

From the log-Lipschitz condition (2.2) we have

$$(p(x)-p_-^\varepsilon)\ln(1/\varepsilon)\leqslant C$$

and so

$$I_\varepsilon\leqslant\int_{B(x_0,\varepsilon)}\mathrm{e}^{\frac{CN}{p_-^\varepsilon}}\varepsilon^{-N}dx\leqslant\mathrm{e}^{CN}\int_{B(x_0,\varepsilon)}\varepsilon^{-N}dx:=A.$$

This immediately implies that $\|u_\varepsilon\|_{W^{1,p(\cdot)}(\Omega)}$ is bounded for $\varepsilon\leqslant 1$.

Now fix ε_0 such that for all $\varepsilon\leqslant\varepsilon_0$ we have

$$|B(x_0,\varepsilon)\cap M|\geqslant|B(x_0,7\varepsilon/8)|.$$

Fix for a moment $\varepsilon \leqslant \varepsilon_0$. Then

$$
\begin{aligned}
A_\varepsilon &:= \int_{(B(x_0,\varepsilon)\setminus B(x_0,3/4\ \varepsilon)\cap M} (1-|x|/\varepsilon)^{p^\#(x)} \Big(\varepsilon^{\frac{p_-^\varepsilon - N}{p_-^\varepsilon}}\Big)^{p^\#(x)} dx \\
&\geqslant \int_{(B(x_0,\varepsilon)\setminus B(x_0,3/4\ \varepsilon)\cap M} (1-|x|/\varepsilon)^{p^\#(x)} \varepsilon^{\frac{p_-^\varepsilon - N}{p_-^\varepsilon}(p_-^\varepsilon)^\#} dx \\
&= \int_{(B(x_0,\varepsilon)\setminus B(x_0,3/4\ \varepsilon)\cap M} (1-|x|/\varepsilon)^{p^\#(x)} \varepsilon^{\frac{p_-^\varepsilon - N}{p_-^\varepsilon}\frac{Np_-^\varepsilon}{N-p_-^\varepsilon}} dx \\
&= \int_{(B(x_0,\varepsilon)\setminus B(x_0,3/4\ \varepsilon)\cap M} (1-|x|/\varepsilon)^{p^\#(x)} \varepsilon^{-N} dx \\
&\geqslant \int_{(B(x_0,\varepsilon)\setminus B(x_0,3/4\ \varepsilon)\cap M} (1-|x|/\varepsilon)^{(p_+^\varepsilon)^\#} \varepsilon^{-N} dx := B_\varepsilon.
\end{aligned}
$$

By Lemma 6.1 we have

$$
\begin{aligned}
B_\varepsilon &\geqslant \int_{(B(x_0,\varepsilon)\setminus B(x_0,7/8\ \varepsilon)} (1-|x|/\varepsilon)^{(p_+^\varepsilon)^\#} \varepsilon^{-N} dx \\
&= \sigma_N \varepsilon^{-N} \int_{7/8\ \varepsilon}^{\varepsilon} (1-r/\varepsilon)^{(p_+^\varepsilon)^\#} r^{N-1} dr \\
&\geqslant \sigma_N \varepsilon^{-N} \int_{14/16\ \varepsilon}^{15/16\ \varepsilon} (1-(15\varepsilon/16)/\varepsilon)^{(p_+^\varepsilon)^\#} r^{N-1} dr \\
&= \sigma_N (1/16)^{(p_+^\varepsilon)^\#}\ \varepsilon^{-N} \int_{14/16\ \varepsilon}^{15/16\varepsilon} r^{N-1} dr := K.
\end{aligned}
$$

σ_N denotes the area of N-dimensional unit sphere S^N.

Denote $\varepsilon_n = (3/4)^n \varepsilon_0$ and consider the corresponding sequence $u_{\varepsilon_n}(x)$. Let $m > n$. Then $u_m(x) = 0$ for $x \in B(x_0, \varepsilon_n) \setminus B(x_0, \varepsilon_m)$ and so,

$$
\begin{aligned}
&\int_\Omega |u_m(x) - u_n(x)|^{q(x)} dx = \int_{B(x_0,\varepsilon_n)} |u_m(x) - u_n(x)|^{q(x)} dx \\
&\geqslant \int_{B(x_0,\varepsilon_n)\setminus B(x_0,\varepsilon_m)} |u_n(x)|^{q(x)} dx = \int_{(B(x_0,\varepsilon_n)\setminus B(x_0,3/4\ \varepsilon_n))\cap M} |u_n(x)|^{q(x)} dx \\
&= \int_{(B(x_0,\varepsilon_n)\setminus B(x_0,3/4\ \varepsilon_n))\cap M} |u_n(x)|^{p^\#(x)} dx \geqslant K.
\end{aligned}
$$

Hence, there is a constant $L > 0$ such that

$$\|u_m - u_n\|_{L^{q(\cdot)}(\Omega)} \geqslant L$$

and the embedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^{q(\cdot)}(\Omega)$ is not compact. □

The next lemma is proved in [2] (see Corollary 8.3.2.).

Lemma 6.3. *Let $\Omega \in \mathcal{C}^{0,1}$, p, $q \in \mathcal{E}(\Omega)$ and $p(\cdot)$ satisfies the log-Hölder condition* (2.2). *Assume that for all $x \in \Omega$*

$$1 \leqslant p(x) \leqslant p_+ < N.$$

Then $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{p^\#(\cdot)}(\Omega)$ where $p^\#(x)$ is given in (1.6).

Theorem 6.4. *Let $\Omega \in \mathcal{C}^{0,1}$, p, $q \in \mathcal{E}(\Omega)$ and let $p(\cdot)$ satisfy the log-Hölder condition* (2.2)*. Assume that for all $x \in \Omega$,*

$$1 \leqslant p(x) \leqslant p_+ < N, \quad 1 \leqslant q(x) \leqslant p^{\#}(x)$$

where $p^{\#}(x)$ is given in (1.6)*. Let $K \subset \overline{\Omega}$ be compact, $|K| = 0$ and denote $\varphi(t) = |K(t)|$. Let $\omega : [0, \operatorname{diam}(\Omega)] \to \mathbb{R}$ be a decreasing continuous non-negative function, $\omega_0 := \omega(\operatorname{diam}(\Omega))$. Suppose that $\omega(\cdot)$ satisfies*

$$\frac{1}{p^{\#}(x) - q(x)} \leqslant c\, \omega(\mathrm{d}_K(x)), \ x \in \Omega,$$

$$\int_{\omega_0}^{\infty} \varphi(\omega^{-1}(y)) a^y dy < \infty \text{ for all } a > 1.$$

Then $W^{1,p(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^{q(\cdot)}(\Omega)$.

Proof. Lemmas 6.3 and 3.8 give

$$W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{p^{\#}}(\Omega) \overset{*}{\hookrightarrow} L^{q(\cdot)}(\Omega).$$

Now Proposition 2.2 finishes the proof. □

As an application we introduce the following several examples. The first one is in fact proved in [9] (see Theorem 3.4) but we obtain it as an easy consequence of the previous theorem. Let φ and q denote in following three examples the same as in Theorem 6.4.

Example 6.5. Let $\Omega \in \mathcal{C}^{0,1}$ and $p(\cdot) : \Omega \to \mathbb{R}$ satisfy (2.2). Assume

$$1 \leqslant p(x) \leqslant p_+ < N.$$

Let $K \subset \overline{\Omega}$ and $\varphi(t) \leqslant Ct^s$ for some $C > 0$ and $s \in (0, N]$. Assume that $\psi : [\omega_0, \infty) \to (0, \infty)$ satisfies

(i) $\frac{\psi(t)}{\ln t}$ is decreasing;

(ii) $\lim\limits_{t\to\infty} \psi(t) = \infty$;

(iii) $s(x) := \frac{1}{p^{\#}(x) - q(x)} \leqslant \frac{\ln(1/\mathrm{d}_K(x))}{\psi(1/\mathrm{d}_K(x))}$.

Then $W^{1,p(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^{q(\cdot)}(\Omega)$.

Proof. It suffices to choose

$$\omega(t) = \frac{\ln(1/t)}{\psi(1/t)}$$

and then to use Theorem 6.4. □

Example 6.6. Let $\Omega \in \mathcal{C}^{0,1}$ and $p(\cdot) : \Omega \to \mathbb{R}$ satisfy (2.2). Assume

$$1 \leqslant p(x) \leqslant p_+ < N.$$

Let $K \subset \overline{\Omega}$ and $\varphi(t) \leqslant C(\ln(\mathrm{e}/t))^{1-s}$ for some $C > 0$ and $s > 1$. Assume that $\psi : [\omega_0, \infty) \to (0, \infty)$ satisfies

(i) $\frac{\psi(t)}{\ln \ln t}$ is decreasing;

(ii) $\lim\limits_{t\to\infty} \psi(t) = \infty$;

(iii) $s(x) := \frac{1}{p^{\#}(x)-q(x)} \leqslant \frac{\ln\ln(1/\mathrm{d}_K(x))}{\psi(1/\mathrm{d}_K(x))}$.

Then $W^{1,p(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^{q(\cdot)}(\Omega)$.

Proof. Take

$$\omega(t) = \frac{\ln\ln(1/t)}{\psi(1/t)}$$

and use Theorem 6.4. $\square$

Example 6.7. Let $\Omega \in \mathcal{C}^{0,1}$ and $p(\cdot) : \Omega \to \mathbb{R}$ satisfy (2.2). Assume

$$1 \leqslant p(x) \leqslant p_+ < N.$$

Let $K \subset \overline{\Omega}$ and $\varphi(t) \leqslant C(\ln\ln(\mathrm{e}/t))^{1-s}$ for some $C > 0$ and $s > 1$. Assume that $\psi : [\omega_0, \infty) \to (0, \infty)$ satisfies

(i) $\frac{\psi(t)}{\ln\ln\ln t}$ is decreasing;

(ii) $\lim\limits_{t\to\infty} \psi(t) = \infty$;

(iii) $s(x) := \frac{1}{p^{\#}(x)-q(x)} \leqslant \frac{\ln\ln\ln(1/\mathrm{d}_K(x))}{\psi(1/\mathrm{d}_K(x))}$.

Then $W^{1,p(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^{q(\cdot)}(\Omega)$.

Proof. Take

$$\omega(t) = \frac{\ln\ln\ln(1/t)}{\psi(1/t)}$$

and use Theorem 6.4. $\square$

To conclude we remark that the construction of Cantor sets could be refined, adding some more logarithms to give additional examples.

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