When are surjective algebra homomorphisms of $\mathcal{B}(X)$ automatically injective?

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Some notation & motivation

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Let X and Y be Banach spaces, let $\psi \colon \mathscr{B}(X) \to \mathscr{B}(Y)$ be a surjective (continuous) algebra homomorphism. Is ψ automatically injective?

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Example

The following Banach spaces X are such that $\mathscr{B}(X)$ has a character:

- The James space J_p (where $1 < p < \infty$), the Semadeni space $C[0, \omega_1]$, any hereditarily indecomposable space (Gowers–Maurey, Argyros–Haydon, ...);

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- Mankiewicz's separable and superreflexive space X_M, Gowers' space $\mathcal{G}$, Tarbard's indecomposable but not H.I. space X_∞, the space $C(K_0)$ where K_0 is a connected "Koszmider" space, the Motakis–Puglisi–Zisimopoulou space X_K.

In examples of the second type the character is obtained from a commutative quotient of $\mathscr{B}(X)$.

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Remark

The same argument works if we replace $\mathcal{H}$ with c_0 or ℓ_p, where $1 \leqslant p < \infty$. Indeed if X is one of the above, then by the Gohberg–Markus–Feldman Theorem the ideal lattice of $\mathscr{B}(X)$ is given by

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Definition

A Banach space X has the *SHAI property* (Surjective Homomorphisms Are Injective) if for every non-zero Banach space Y every surjective algebra homomorphism $\psi \colon \mathscr{B}(X) \to \mathscr{B}(Y)$ is injective.

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This follows from a much more general result of B. E. Johnson.

Consequently, if X has the SHAI property, Y is non-zero and there is a surjective algebra homomorphism $\psi\colon \mathscr{B}(X) \to \mathscr{B}(Y)$, then

$$\mathscr{B}(X) \cong \mathscr{B}(Y) \iff X \cong Y.$$

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(The answer to the question is YES, but a different approach is needed.)

The method of large kernels I.

Recall that if X, Y are non-zero Banach spaces, and $\psi\colon \mathscr{B}(X) \to \mathscr{B}(Y)$ is a non-zero, continuous algebra homomorphism, then either

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$$\dim(\ker(I_X - ST)) < \infty, \quad \operatorname{codim}(\operatorname{Ran}(I_X - ST)) < \infty$$

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Fact

The set $\mathscr{E}(X)$ of inessential operators is a proper, closed, two-sided ideal of $\mathscr{B}(X)$ if X is infinite-dimensional.

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Proof.

(Sketch.) Under the hypothesis $\mathscr{B}(X)$ cannot have finite-codimensional proper two-sided ideals. □

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In both cases $\ker(\psi) = \mathscr{E}(X)$ thus $\mathscr{B}(X)/\mathscr{E}(X) \cong \mathscr{B}(Y)$. Note that LHS is simple because $\mathscr{E}(X)$ is maximal, but RHS is not simple as Y is infinite-dimensional. A contradiction. □

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Let X and W be Banach spaces. Define

$$\overline{\mathscr{G}}_W(X) := \overline{\operatorname{span}}\{ST\colon T \in \mathscr{B}(X, W), S \in \mathscr{B}(W, X)\}.$$

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$$0 = (P(x \otimes \xi)P)x = \langle Px, \xi \rangle Px = \langle x, \xi \rangle x = x,$$

a contradiction. Consequently $P \in \operatorname{Ker}(\psi)$ must hold.

Theorem (H.)

Let $X := (\bigoplus_{n\in\mathbb{N}} \ell_2^n)_Y$*, where* Y *is* c_0 *or* ℓ_1*. Then* X *has the SHAI property.*

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Let $X := (\bigoplus_{n\in\mathbb{N}} \ell_2^n)_Y$, where Y is c_0 or ℓ_1. Then X has the SHAI property.

Proof.

Main ingredient:

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Let $X = (\bigoplus_{n\in\mathbb{N}} \ell_2^n)_Y$ where Y is c_0 or ℓ_1. Then the lattice of closed, two-sided ideals in $\mathscr{B}(X)$ is given by

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Alternative proof: $\mathscr{B}(X)/\mathscr{K}(X)$ does not have minimal idempotents.

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The proof uses Spectral Theory to show that idempotents from $\mathscr{B}(\ell_2(\lambda))/\mathscr{J}$ can be lifted to idempotents in $\mathscr{B}(\ell_2(\lambda))$, where $\mathscr{J} \trianglelefteq \mathscr{B}(\ell_2(\lambda))$.

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Hence $\mathscr{B}(\ell_2(\lambda))/\mathscr{J}$ has no minimal idempotents.

Thus there is no Banach space Y with $\mathscr{B}(\ell_2(\lambda))/\mathscr{J} \cong \mathscr{B}(Y)$, as minimal idempotents in $\mathscr{B}(Y)$ are precisely the rank one idempotents.

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Let λ be an infinite cardinal. Then $c_0(\lambda)$, $\ell_\infty^c(\lambda)$ and $\ell_p(\lambda)$ (for $1 \leqslant p < \infty$) have the SHAI property.

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Ingredients of the proof.

Definition

Let X and Y be Banach spaces. Let $\mathscr{S}_Y(X)$ be a subset of $\mathscr{B}(X)$ defined by

$$T \notin \mathscr{S}_Y(X) \Longleftrightarrow \exists W \subseteq X \text{ subspace with } W \cong Y \text{ such that } T|_W \text{ is bounded below.}$$

The long sequence spaces

Theorem (H.–Kania)

Let λ be an infinite cardinal. Then $c_0(\lambda)$, $\ell_\infty^c(\lambda)$ and $\ell_p(\lambda)$ (for $1 \leqslant p < \infty$) have the SHAI property.

Ingredients of the proof.

Definition

Let X and Y be Banach spaces. Let $\mathscr{S}_Y(X)$ be a subset of $\mathscr{B}(X)$ defined by

$$T \notin \mathscr{S}_Y(X) \iff \exists W \subseteq X \text{ subspace with } W \cong Y \text{ such that } T|_W \text{ is bounded below.}$$

$\mathscr{S}_Y(X)$ is called the set of *Y-singular operators on X*.

Ingredients of the proof (con't).

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(X is complementably homogeneous if whenever Y is a subspace of X with $Y \cong X$ then there is $Z \subseteq Y$ subspace which is complemented in X and $Z \cong X$.)
The spaces $c_0(\lambda)$, $\ell_\infty^c(\lambda)$ and $\ell_p(\lambda)$ (where $1 \leqslant p < \infty$) are complementably homogeneous.

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Theorem (Johnson – Kania – Schechtman)

The set $\mathscr{S}_{E_\kappa}(E_\lambda)$ is a closed, non-zero, proper two-sided ideal in $\mathscr{B}(E_\lambda)$ for every infinite cardinal $\kappa \leq \lambda$. In particular $\mathscr{S}_{E_\lambda}(E_\lambda)$ is maximal.

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Theorem (Johnson – Kania – Schechtman)

Let λ and κ be uncountable cardinals with $\lambda \geqslant \kappa$, and suppose that κ is not a successor of any cardinal number. Then

$$\mathscr{S}_{E_\kappa}(E_\lambda) = \overline{\bigcup_{\alpha<\kappa} \mathscr{S}_{E_\alpha}(E_\lambda)}.$$

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The proof that E_λ has SHAI uses:

- Transfinite induction on the cardinals $\kappa \leqslant \lambda$;
- the above 3 theorems;
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The proof that E_λ has SHAI uses:

- Transfinite induction on the cardinals $\kappa \leqslant \lambda$;
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- and the Dichotomy Result II. In this context, $\overline{\mathscr{G}}_{E_\kappa}(E_\lambda) \subseteq \ker(\psi)$, where $\psi \colon \mathscr{B}(E_\lambda) \to \mathscr{B}(Y)$ is some surjective, non-injective algebra hom.

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We *claim* that $\mathscr{S}_{E_\lambda}(E_\lambda) \subseteq \mathrm{Ker}(\psi)$. We consider three cases:

1. $\lambda = \omega$;
2. λ is a successor cardinal;
3. λ is uncountable and not a successor cardinal.

Proof (con't.)

(1) If $\lambda = \omega$ then $E_\lambda = c_0$ or $E_\lambda = \ell_p$, where $p \in [1, \infty]$. Then Dichotomy Result I yields

$$\mathscr{S}_{E_\lambda}(E_\lambda) = \mathscr{E}(E_\lambda) \subseteq \mathrm{Ker}(\psi).$$

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Further results, remarks

- $L_p[0,1]$ has the SHAI property for $1 < p < \infty$ [Johnson – Phillips – Schechtman, 2020+].

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  - Both c_0 and $c_0(\mathfrak{c})$ have SHAI but $C_0(K_\mathcal{A})$ does not.

Further results, remarks

Recall that so far that all examples of Banach spaces X which lack SHAI have the property that there exists a character $\varphi \colon \mathscr{B}(X) \to \mathbb{C}$. (Or finite sums thereof, we can quotient to $M_n(\mathbb{C})$.)

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We can have infinite-dimensional targets for surjective, non-injective algebra homomorphisms:

Theorem (H.)

Let Y be a separable, reflexive Banach space. Let

$$X_Y := C_0[0,\omega_1)\hat{\otimes}_\varepsilon Y \overset{(1)}{\cong} \left\{ f \in C\big([0,\omega_1]; Y\big) \colon f(\omega_1) = 0_Y \right\}.$$

There exists a surjective, non-injective algebra homomorphism

$$\psi\colon \mathscr{B}(X_Y) \to \mathscr{B}(Y).$$

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Sources

- B. Horváth, "When are full representations of algebras of operators on Banach spaces automatically faithful?", Studia Mathematica (2020), available on the arXiv;
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