INSTITUTE OF MATHEMATICS

THE CZECH ACADEMY OF SCIENCES

Quantifying properties (K) and μ^s

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Preprint No. 8-2021

PRAHA 2021

QUANTIFYING PROPERTIES (K) AND (μ^s)

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Abstract. A Banach space X has *property* (K), whenever every weak* null sequence in the dual space admits a convex block subsequence $(f_n)_{n=1}^\infty$ so that $\langle f_n, x_n\rangle \to 0$ as $n\to\infty$ for every weakly null sequence $(x_n)_{n=1}^\infty$ in X; X has *property* (μ^s) if every weak* null sequence in X^* admits a subsequence so that all of its subsequences are Cesàro convergent to 0 with respect to the Mackey topology. Both property (μ^s) and reflexivity (or even the Grothendieck property) imply property (K). In the present paper we propose natural ways for quantifying the aforementioned properties in the spirit of recent results concerning other familiar properties of Banach spaces.

1. Introduction

The present paper is inspired by many recent results that quantify various familiar properties of Banach spaces such as weak sequential completeness [KPS], reciprocal Dunford–Pettis property [KS], Schur property [KS1], Dunford–Pettis property [KKS], Banach–Saks property [BKS], property (V) [Kr], Grothendieck property [Be], etc. We continue this line of research and investigate possible quantifications of related properties (K) and (μ^s) introduced by Kwapień and Rodríguez, respectively.

Mazur's lemma (see, *e.g.*, [D, p. 11]) states that every weakly convergent sequence in a Banach space has a convex block subsequence that is norm convergent to the same limit. (A sequence $(y_n)_{n=1}^\infty$ in a Banach space X is a *convex block subsequence* of a sequence $(x_n)_{n=1}^\infty$ provided that there exists a strictly increasing sequence of positive integers $(k_n)_{n=1}^\infty$ so that $y_n \in \mathrm{conv}(x_i)_{i=k_{n-1}+1}^{k_n}$ for every $n\in\mathbb{N}$, where we set $k_0=0$; we denote by $\mathrm{cbs}((x_n)_{n=1}^\infty)$ the collection of all convex block subsequences of $(x_n)_{n=1}^\infty$.) Kalton and Pełczyński [KP, Proposition 2.2] proved that if a Banach space X contains an isomorphic copy of c_0, then for every σ-finite measure μ the kernel of any surjection Q from $L_1(\mu)$ onto X is uncomplemented in its second dual. Consequently, $\ker Q$ is not isomorphic to a Banach lattice; the original argument relied on the Lindenstrauss Lifting Principle. Having read a preliminary version of [KP], Kwapień introduced property (K) to provide an alternative proof of [KP, Propositon 2.2] which did not appeal to the Lindenstrauss Lifting Principle (Kwapień's idea was incorporated in [KP], where it was presented with his permission). Property (K) is central to our considerations:

Date: Version: February 1, 2021.
2010 Mathematics Subject Classification: 46B26; 46B50.
Keywords: Property (K); Grothendieck property; Reflexivity; Banach–Saks property; Property (μ^s).
The first-named author was supported by the National Natural Science Foundation of China (Grant No.11971403) and the Natural Science Foundation of Fujian Province of China (Grant No. 2019J01024).

Definition 1.1. A Banach space X has *property* (K), whenever every weak* null sequence in X^* admits a convex block subsequence $(f_n)_{n=1}^\infty$ so that $\lim_{n\to\infty}\langle f_n, x_n\rangle = 0$ for every weakly null sequence $(x_n)_{n=1}^\infty$ in X.

Put equivalently, Definition 1.1 stipulates that the sequence $(f_n)_{n=1}^\infty$ converges to 0 with respect to the Mackey topology $\mu(X^*, X)$, which is the (locally convex) topology of uniform convergence in X^* on weakly compact subsets of X (see [DDS, Lemma 3.5]).

Property (K) may be thought of as a counterpart of Mazur's lemma with respect to the weak* topology. It was shown in [KP] that the space $L_1(\mu)$ for a σ-finite measure has property (K), yet c_0 fails to have this property. *Schur spaces*, *i.e.*, spaces in which weak convergence of sequences coincides with norm convergence have property (K) for trivial reasons. It follows from Mazur's lemma that Grothendieck spaces, in particular, reflexive spaces, have property (K). (A Banach space X is a *Grothendieck space*, whenever every weak* convergent sequence in X^* converges weakly.)

Figiel, Johnson, and Pełczyński [FJP] refined property (K) by introducing a weaker property that they call *property* (k); this property appeared implicitly also in [Jo]. Property (k) was used in [FJP] to show that the Separable Complementation Property need not pass to subspaces.

It was proved in [FJP] that property (k) is enjoyed by every separable subspace of a weakly sequentially complete Banach lattice, weakly sequentially complete Banach lattices with weak units, and every separable subspace of the predual of a von Neumann algebra. Oja [FJP] pointed out that the Radon–Nikodým property implies property (k). However, it was shown [FJP] that the ℓ_1-sum of continuum many copies of $L_1[0,1]$ as well as Banach spaces containing complemented subspaces isomorphic to c_0 fail property (k).

Property (K) admits a number of characterisations. More precisely, let X be a Banach space. Then the following assertions are equivalent:

(a) X has property (K).
(b) Every weak* null sequence in X^* admits a convex block subsequence $(f_n)_{n=1}^\infty$ so that $\lim_{n\to\infty}\langle f_n, x_n\rangle = 0$ for every weakly null sequence $(x_n)_{n=1}^\infty$ in X.
(c) Every weak* null sequence in X^* admits a convex block subsequence that is $\mu(X^*, X)$-null.
(d) Every weak* convergent sequence in X^* admits a convex block subsequence that is $\mu(X^*, X)$-Cauchy.

In Section 3 of the present paper, we prove quantitative versions of the aforestated characterisations. In order to do so, we introduce a quantity α that characterises $\mu(X^*, X)$-null sequences and subsequently we introduce a quantity K_1 that characterises property (K). This is a quantitative version of clause (c). In order to quantify (b), we introduce a quantity β that turns out to be equivalent to α for weak* null sequences. By using the quantity β, we introduce a further quantity K_2 that we then prove is then equivalent to the quantity K_1. Finally, to quantify (d), we introduce a quantity K_3 in terms of ca_{ρ^*} defined in [KKS] that measures $\mu(X^*, X)$-Cauchyness and prove that K_3 is equivalent to K_1. In summary, we quantify property (K) by means of the following estimates:

$$K_2(X) \leqslant K_1(X) \leqslant 2K_2(X)$$

and

$$K_1(X) \leqslant K_3(X) \leqslant 4K_1(X).$$

Furthermore, we investigate the values of the quantity K_2 in certain familiar Banach spaces failing property (K) and obtain that, in particular, $K_2(c_0) = 1$ and the K_2-value of the ℓ_1-sum of $\mathfrak{c}$ copies of $L_1[0,1]$ is equal to 1. (Curiously, Frankiewicz and Plebanek [FP] proved that under Martin's Axiom, the ℓ_1-sum of fewer than $\mathfrak{c}$ copies of $L_1[0,1]$ still has property (K).)

The purpose of Section 4 is to quantify the following widely known implications:

$$X \text{ is reflexive} \Rightarrow X \text{ is a Grothendieck space} \Rightarrow X \text{ has property } (K). \qquad (\star)$$

In order to quantify $(\star)$, we first make a slight improvement on characterisations of weak compactness due to Ülger [U] (see also [DRS]). Using this improvement, we establish a characterisation of the Grothendieck property, which is used to introduce a quantity G measuring the Grothendieck property. This quantification of the Grothendieck property is different from the quantitative Grothendieck property proposed by Bendová ([Be]). Again using the improvement, we introduce a new quantity R measuring reflexivity for Banach spaces. Meanwhile, the relationship between the quantity R and several classical equivalent quantities measuring weak non-compactness is discussed. We also investigate possible values of the quantity R of some classical Banach spaces. Having introduced G and R, we quantify the implications $(\star)$ as follows:

$$K_1(X) \leqslant G(X) \leqslant R(X^*).$$

Avilés and Rodríguez [AR] studied the implications $(\star)$ for Banach spaces not containing isomorphic copies of ℓ_1 and proved that for such space X:

$$X \text{ is reflexive} \Leftrightarrow X \text{ is a Grothendieck space} \Leftrightarrow X \text{ has property } (K). \qquad (\star\star)$$

Finally, we quantify $(\star\star)$ as follows:

$$K_1(X) \leqslant G(X) \leqslant R(X^*) \leqslant 8K_2(X).$$

A bounded subset A of a Banach space X is a *Banach–Saks set* if each sequence in A has a Cesàro convergent subsequence. A Banach space X is said to have the *Banach–Saks property* if its closed unit ball B_X is a Banach–Saks set. Banach and Saks proved in [BS] that the spaces $L_p[0,1]$ and ℓ_p $(1 < p < \infty)$ enjoy the Banach–Saks property, hence the name. Kakutani [Ka] later showed that uniformly convex spaces have the Banach–Saks property (hence so do superreflexive spaces). Any space with the Banach–Saks property is reflexive [NW], but there are reflexive spaces without the Banach–Saks property [Ba]. A localised version of the result of [NW] says that any Banach–Saks set is relatively weakly compact [LART].

It follows from the Erdős–Magidor Theorem (Theorem 2.3) that a Banach space X has the Banach–Saks property if and only if every bounded sequence in X admits a subsequence such that all of its subsequences are Cesàro convergent. Property (μ^s), introduced by Rodríguez [Ro], is a statement refining the Banach–Saks property for weak* null sequences in the dual space.

Definition 1.2. A Banach space X has *property* (μ^s), whenever every weak* null sequence in X^* admits a subsequence so that all of its subsequences are Cesàro convergent to 0 with respect to $\mu(X^*, X)$.

Clearly, if X^* has the Banach–Saks property, then X has property (μ^s). The converse is true for reflexive spaces [Ro, Proposition 2.2]. Moreover, it was pointed out [Ro, Lemma 2.1, Remark 2.3] that property (μ^s) is strictly stronger than property (K).

The goal of Section 5 is to quantify the following implications ([Ro]):

$$X^* \text{ has the Banach–Saks property} \Rightarrow X \text{ has property } (\mu^s) \Rightarrow X \text{ has property } (K). \tag{$\star\star\star$}$$

To quantify property (μ^s), we first introduce a quantity $c\alpha$ by means of α that measures the rate of Cesàro convergence to 0 with respect to $\mu(X^*, X)$. By using the quantity $c\alpha$, we introduce a quantity μ^s that we then prove characterises property (μ^s). Furthermore, we introduce a quantity $\mathrm{bs}(X)$ that characterises the Banach–Saks property of a Banach space X. This quantity is stronger than the quantity introduced in [BKS] that measures how far a bounded set is from being Banach–Saks. By using the quantities μ^s and bs, we quantify $(\star\star\star)$ as follows:

$$\frac{1}{3}K_1(X) \leqslant \mu^s(X) \leqslant \mathrm{bs}(X^*).$$

Finally, we prove that, for a reflexive space X,

$$\mu^s(X) \leqslant \mathrm{bs}(X^*) \leqslant 4\mu^s(X),$$

which is a quantitative version of [Ro, Proposition 2.2].

2. Preliminaries

We use standard notation and terminology in-line with [AK] and [LT]. Throughout this paper, all Banach spaces are infinite-dimensional over the fixed field of real or complex numbers. By a *subspace* we mean a closed, linear subspace. An *operator* will always mean a bounded linear operator. If X is a Banach space, we denote by B_X the closed unit ball $\{x \in X : \|x\| \leqslant 1\}$ and by $\mathcal{F}_X$ the family of all weakly compact subsets in B_X. For a subset A of X, $\operatorname{conv}(A)$ stands for the convex hull of A. For brevity of notation, we denote by $\operatorname{ss}((x_n)_{n=1}^\infty)$ the family of all subsequences of a sequence $(x_n)_{n=1}^\infty$.

2.1. **Weak compactness.** Let us invoke the following characterisation of weak compactness due to Ülger [U].

Lemma 2.1. *A bounded subset A of a Banach space X is relatively weakly compact if and only if given any sequence $(x_n)_{n=1}^\infty$ in A, there exists a sequence $(z_n)_{n=1}^\infty$ with $z_n \in \operatorname{conv}(x_i : i \geqslant n)$ that converges weakly.*

Diestel, Ruess, and Schachermayer [DRS] improved Lemma 2.1 as follows.

Lemma 2.2. *For a bounded subset A of X the following statements are equivalent:*

(1) A is relatively weakly compact.
(2) For every sequence $(x_n)_{n=1}^\infty$ in A, there is a norm-convergent sequence $(z_n)_{n=1}^\infty$ such that $z_n \in \operatorname{conv}(x_i : i \geqslant n)$.
(3) For every sequence $(x_n)_{n=1}^\infty$ in A, there is a weakly convergent sequence $(z_n)_{n=1}^\infty$ such that $z_n \in \operatorname{conv}(x_i : i \geqslant n)$.

Let A and B are non-empty subsets of a Banach space X, we set

- $\mathrm{d}(A,B) = \inf\{\|a-b\| : a \in A, b \in B\}$,
- $\widehat{\mathrm{d}}(A,B) = \sup\{\mathrm{d}(a,B) : a \in A\}$.

$\mathrm{d}(A,B)$ is the ordinary distance between A and B, and $\widehat{\mathrm{d}}(A,B)$ is the (non-symmetrised) Hausdorff distance from A to B. When A is a bounded subset of a Banach space X, following [KKS], we set

- $\operatorname{wk}_X(A) = \widehat{\mathrm{d}}\big(\overline{A}^{\sigma(X^{**},X^*)}, X\big)$;
- $\operatorname{wck}_X(A) = \sup\{\mathrm{d}(\operatorname{clust}_{X^{**}}((x_n)_{n=1}^\infty), X) : (x_n)_{n=1}^\infty \text{ is a sequence in } A\}$, where $\operatorname{clust}_{X^{**}}((x_n)_{n=1}^\infty)$ is the set of all weak*-cluster points of $(x_n)_{n=1}^\infty$ in X^{**}.
- $\gamma_X(A) = \sup\{|\lim_n \lim_m \langle f_m, x_n\rangle - \lim_m \lim_n \langle f_m, x_n\rangle| : (x_n)_{n=1}^\infty$ is a sequence in A, $(f_m)_{m=1}^\infty$ is a sequence in B_{X^*} and all the involved limits exist$\}$.

It follows from [AC, Theorem 2.3] that

$$\operatorname{wck}_X(A) \leqslant \operatorname{wk}_X(A) \leqslant \gamma_X(A) \leqslant 2\operatorname{wck}_X(A).$$

2.2. **Mackey topology.** Let X be a Banach space. The Mackey topology, $\mu(X^*,X)$, is the strongest locally convex topology on X^* which is compatible with the dual pairing $\langle X^*, X\rangle$. In particular, $\overline{C}^{w^*} = \overline{C}^{\mu(X^*,X)}$ for every convex subset C of X^*. If the dual unit ball endowed with the relative Mackey topology, $(B_{X^*}, \mu(X^*,X))$, is metrisable, then X has property (K). Schlüchtermann and Wheeler [SW] term Banach spaces X for which

$(B_{X^*}, \mu(X^*, X))$ is metrisable as *strongly weakly compactly generated* (SWCG) spaces. As proved in [SW], a Banach space X is SWCG if and only if there exists a weakly compact subset K of X so that for every weakly compact subset L of X and $\varepsilon > 0$, there is a positive integer n with $L \subseteq nK + \varepsilon B_X$. Moreover, reflexive spaces, separable Schur spaces, the space of operators of trace-class on a separable Hilbert space, and $L_1(\mu)$ for a σ-finite measure μ are SWCG.

2.3. **Banach–Saks sets.** Let $(x_n)_{n=1}^\infty$ be a bounded sequence in a Banach space. We set

$$\mathrm{ca}((x_n)_{n=1}^\infty) = \inf_{n\in\mathbb{N}} \sup_{k,l\geqslant n} \|x_k - x_l\|.$$

Clearly $(x_n)_{n=1}^\infty$ is norm-Cauchy if and only if $\mathrm{ca}((x_n)_{n=1}^\infty) = 0$. Following [BKS], we define

$$\mathrm{cca}\big((x_n)_{n=1}^\infty\big) = \mathrm{ca}\,\big((\frac{1}{n}\sum_{i=1}^n x_i)_{n=1}^\infty\big).$$

Clearly, $\mathrm{cca}((x_n)_{n=1}^\infty) = 0$ if and only if $(x_n)_{n=1}^\infty$ is Cesàro convergent.

A subset A of a Banach space is *Banach–Saks*, whenever every sequence in A has a Cesàro convergent subsequence.

We shall require the well-known 0-1 law by Erdős–Magidor [EM].

Theorem 2.3. *(Erdős–Magidor) Every bounded sequence in a Banach space has a subsequence such that either all its further subsequences are Cesàro convergent, or none of them.*

For a bounded set A in a Banach space X, Bendová, Kalenda and Spurný [BKS] introduced the following quantity

$$\mathrm{bs}(A) = \sup_{(x_n)_{n=1}^\infty \subseteq A} \inf_{(y_n)_{n=1}^\infty \in \mathrm{ss}((x_n)_{n=1}^\infty)} \mathrm{cca}((y_n)_{n=1}^\infty)$$

measuring how far is A from being a Banach–Saks set. More precisely, they proved that A is a Banach–Saks set if and only if $\mathrm{bs}(A) = 0$.

3. Quantifications of property (K)

Let $(f_n)_{n=1}^\infty$ be a bounded sequence in X^*. Following [KKS], we set

$$\mathrm{ca}_{\rho^*}((f_n)_{n=1}^\infty) = \sup_{K\in\mathcal{F}_X} \inf_{n\in\mathbb{N}} \sup_{k,l\geqslant n} \sup_{x\in K} |\langle f_k - f_l, x\rangle|,$$

then $\mathrm{ca}_{\rho^*}((f_n)_{n=1}^\infty) = 0$ if and only if $(f_n)_{n=1}^\infty$ is $\mu(X^*, X)$-Cauchy (*i.e.*, $\mu(X^*, X)$-convergent as the Mackey topology $\mu(X^*, X)$ [Ja, Proposition 4 on p. 197] is complete). We set

$$\alpha((f_n)_{n=1}^\infty) = \sup_{K\in\mathcal{F}_X} \limsup_{n\to\infty} \sup_{x\in K} |\langle f_n, x\rangle|,$$

then $\alpha((f_n)_{n=1}^\infty) = 0$ if and only if $(f_n)_{n=1}^\infty$ is $\mu(X^*, X)$-null, and

$$\beta((f_n)_{n=1}^\infty) = \sup_{\substack{(x_n)_{n=1}^\infty \subseteq B_X \\ \text{weakly null}}} \limsup_{n\to\infty} |\langle f_n, x_n\rangle|.$$

The following result is a quantitative version of [DDS, Lemma 3.3].

Lemma 3.1. *Let $(f_n)_{n=1}^\infty$ be a weak* null sequence in X^*. Then*

$$\beta((f_n)_{n=1}^\infty) \leqslant \alpha((f_n)_{n=1}^\infty) \leqslant 2\beta((f_n)_{n=1}^\infty).$$

Proof. The former inequality is trivial. It remains to prove the latter one.

Let $0 < c < \alpha\big((f_n)_{n=1}^\infty\big)$. Then there exist a weakly compact subset $K \subseteq B_X$, a subsequence $(f_{k_n})_{n=1}^\infty$ of $(f_n)_{n=1}^\infty$, and a sequence $(x_n)_{n=1}^\infty$ in K so that $|\langle f_{k_n}, x_n\rangle| > c$ for all n. Since K is weakly compact, by the Eberlein–Šmulian theorem, $(x_n)_{n=1}^\infty$ admits a subsequence $(x_{n_m})_{m=1}^\infty$ that converges weakly to some $x \in K$. We define a sequence $(z_n)_{n=1}^\infty$ in X by

$$z_{k_{n_m}} = \frac{1}{2}(x_{n_m} - x) \quad (m = 1, 2, \ldots)$$

and $z_n = 0$ for $n \notin \{k_{n_m}\}_{m=1}^\infty$. Then $(z_n)_{n=1}^\infty$ is weakly null in B_X. For each m, we get

$$|\langle f_{k_{n_m}}, z_{k_{n_m}}\rangle| = \frac{1}{2}|\langle f_{k_{n_m}}, x_{n_m}\rangle - \langle f_{k_{n_m}}, x\rangle| \geqslant \frac{1}{2}(c - |\langle f_{k_{n_m}}, x\rangle|).$$

Since $(f_n)_{n=1}^\infty$ is $\sigma(X^*, X)$-null, we get

$$\limsup_{n\to\infty} |\langle f_n, z_n\rangle| \geqslant \limsup_{m\to\infty} |\langle f_{k_{n_m}}, z_{k_{n_m}}\rangle| \geqslant \frac{c}{2}.$$

Since c is arbitrary, we arrive at $\beta((f_n)_{n=1}^\infty) \geqslant \frac{1}{2}\alpha((f_n)_{n=1}^\infty)$. □

Lemma 3.2. *Let $(f_n)_{n=1}^\infty$ be a bounded sequence in X^* and $f \in X^*$. Then*

$$\mathrm{ca}_{\rho^*}((f_n)_{n=1}^\infty) \leqslant 2\alpha((f_n - f)_{n=1}^\infty).$$

Proof. Let $c > \alpha((f_n - f)_{n=1}^\infty)$ be arbitrary. Let $K \in \mathcal{F}_X$. Then there exists a positive integer n so that $\sup_{x\in K} |\langle f_k - f, x\rangle| < c$ for all $k \geqslant n$. Hence, for $k, l \geqslant n$, we get

$$\sup_{x\in K} |\langle f_k - f_l, x\rangle| = \sup_{x\in K} |\langle (f_k - f) - (f_l - f), x\rangle| \leqslant 2c.$$

This implies that $\mathrm{ca}_{\rho^*}((f_n)_{n=1}^\infty) \leqslant 2c$. As c was arbitrary, the proof is complete. □

Lemma 3.3. *Suppose that $(f_n)_{n=1}^\infty$ converges to $f \in X^*$ in the weak* topology. Then*

$$\alpha((f_n - f)_{n=1}^\infty) \leqslant \mathrm{ca}_{\rho^*}((f_n)_{n=1}^\infty).$$

Proof. Let $c > \mathrm{ca}_{\rho^*}((f_n)_{n=1}^\infty)$ be arbitrary. Let $K \in \mathcal{F}_X$. Then there exists a positive integer n so that $\sup_{x\in K} |\langle f_k - f_l, x\rangle| < c$ for all $k, l \geqslant n$. Hence, for each $x \in K$, we get $|\langle f_k - f_l, x\rangle| < c$ for all $k, l \geqslant n$. Letting $l \to \infty$, we get $|\langle f_k - f, x\rangle| \leqslant c$. This means that $\sup_{x\in K} |\langle f_k - f, x\rangle| \leqslant c$ for all $k \geqslant n$ and, consequently, $\alpha\big((f_n - f)_{n=1}^\infty\big) \leqslant c$. As c was arbitrary, the proof is finished. □

Definition 3.4. Let X be a Banach space. We set

$$K_1(X) = \sup_{\substack{(f_n)_{n=1}^\infty \subseteq B_{X^*} \\ \text{weak* null}}} \inf_{(g_n)_{n=1}^\infty \in \mathrm{cbs}((f_n)_{n=1}^\infty)} \alpha\big((g_n)_{n=1}^\infty\big),$$

$$K_2(X) = \sup_{\substack{(f_n)_{n=1}^\infty \subseteq B_{X^*} \\ \text{weak* null}}} \inf_{(g_n)_{n=1}^\infty \in \operatorname{cbs}((f_n)_{n=1}^\infty)} \beta\big((g_n)_{n=1}^\infty\big),$$

and

$$K_3(X) = \sup_{\substack{(f_n)_{n=1}^\infty \subseteq B_{X^*} \\ \text{weak* Cauchy}}} \inf_{(g_n)_{n=1}^\infty \in \operatorname{cbs}((f_n)_{n=1}^\infty)} \operatorname{ca}_{\rho^*}\big((g_n)_{n=1}^\infty\big).$$

The three quantities K_1, K_2, and K_3 are actually equivalent.

Proposition 3.5. *Let X be a Banach space. Then*

(i) $K_2(X) \leqslant K_1(X) \leqslant 2K_2(X)$,
(ii) $K_1(X) \leqslant K_3(X) \leqslant 4K_1(X)$.

Proof. The statement (i) follows from Lemma 3.1. It suffices to prove (ii).

It follows from Lemma 3.3 that $K_1(X) \leqslant K_3(X)$. Let $0 < c < K_3(X)$. Then there exists a weak*-Cauchy sequence $(f_n)_{n=1}^\infty$ in B_{X^*} so that for every $(h_n)_{n=1}^\infty \in \operatorname{cbs}((f_n)_{n=1}^\infty)$ we have $\operatorname{ca}_{\rho^*}((h_n)_{n=1}^\infty) > c$. Clearly, $(f_n)_{n=1}^\infty$ converges to some $f \in B_{X^*}$ in the weak* topology. Take any $(g_n)_{n=1}^\infty \in \operatorname{cbs}((\frac{1}{2}(f_n - f))_{n=1}^\infty)$. Then $2g_n = h_n - f$ $(n \in \mathbb{N})$, where $(h_n)_{n=1}^\infty \in \operatorname{cbs}((f_n)_{n=1}^\infty)$. By Lemma 3.2, we get

$$c < \operatorname{ca}_{\rho^*}\big((h_n)_{n=1}^\infty\big) = 2\operatorname{ca}_{\rho^*}\big((g_n)_{n=1}^\infty\big) \leqslant 4\alpha\big((g_n)_{n=1}^\infty\big).$$

Hence $c \leqslant 4K_1(X)$. Since c is arbitrary, we get $K_3(X) \leqslant 4K_1(X)$. □

The subsequent result implies that the quantities K_1, K_2, and K_3 do characterise property (K).

Theorem 3.6. *A Banach space X has property (K) if and only if $K_1(X) = 0$.*

To prove Theorem 3.6, we require two elementary lemmata whose proofs are omitted.

Lemma 3.7. *If $(y_n)_{n=1}^\infty \in \operatorname{cbs}((x_n)_{n=1}^\infty)$, then $\operatorname{cbs}((y_n)_{n=1}^\infty) \subseteq \operatorname{cbs}((x_n)_{n=1}^\infty)$. More precisely, if $y_n \in \operatorname{conv}(x_i)_{i=k_{n-1}+1}^{k_n}$ and $z_n \in \operatorname{conv}(y_j)_{j=m_{n-1}+1}^{m_n}$, then $z_n \in \operatorname{conv}(x_i)_{i=k_{m_{n-1}}+1}^{k_{m_n}}$.*

Lemma 3.8. *Let $(f_n)_{n=1}^\infty$ be a bounded sequence in X^*. Then*

$$\alpha\big((g_n)_{n=1}^\infty\big) \leqslant \alpha\big((f_n)_{n=1}^\infty\big), \quad \big((g_n)_{n=1}^\infty \in \operatorname{cbs}((f_n)_{n=1}^\infty)\big).$$

Proof of Theorem 3.6. The necessity is trivial, so it suffices to prove the sufficiency.

Let $(f_n)_{n=1}^\infty$ be a weak*-null sequence in B_{X^*}. Since $K_1(X) = 0$, for each k we get inductively a sequence $(f_n^{(k)})_{n=1}^\infty$ in X^* so that for $k \in \mathbb{N}$.

- $(f_n^{(1)})_{n=1}^\infty \in \operatorname{cbs}((f_n)_{n=1}^\infty)$,
- $(f_n^{(k+1)})_{n=1}^\infty \in \operatorname{cbs}((f_n^{(k)})_{n=1}^\infty)$,
- $\alpha((f_n^{(k)})_{n=1}^\infty) < \frac{1}{k}$.

For $n \in \mathbb{N}$, we set $g_n = f_n^{(n)}$. By Lemma 3.7, we get $(g_n)_{n\geqslant k} \in \mathrm{cbs}((f_n^{(k)})_{n=1}^\infty)$ for each k. By Lemma 3.8, we get

$$\alpha\big((g_n)_{n=1}^\infty\big) = \alpha\big((g_n)_{n\geqslant k}\big) \leqslant \alpha\big((f_n^{(k)})_{n=1}^\infty\big) < \tfrac{1}{k} \quad (k = 1, 2, \ldots).$$

This means that $\alpha\big((g_n)_{n=1}^\infty\big) = 0$ and $(g_n)_{n=1}^\infty$ is $\mu(X^*, X)$-null. Consequently, X has property (K). □

Example 3.9.

(a) Let X be a Banach space so that B_{X^*} is $\sigma(X^*, X)$-sequentially compact or ℓ_1 does not embed into X. If X contains a subspace isomorphic to c_0, then $K_2(X) = 1$. In particular,

$$K_2(c_0) = K_2(c) = K_2(C[0,1]) = 1.$$

(b) $K_2\big(\ell_1(\mathbb{R}, L_1[0,1])\big) = 1$; here $\ell_1(\mathbb{R}, L_1[0,1])$ stands for the ℓ_1-sum of $\mathfrak{c}$ copies of $L_1[0,1]$.

Proof. (a). Let $\varepsilon > 0$. It follows from [DRT, Theorem 6] (respectively, [DF, Theorem 2.2]) that there exists a subspace Z of X so that Z is $(1+\varepsilon)$-isomorphic to c_0 and a projection P from X onto Z with $\|P\| \leqslant 1 + \varepsilon$. Let $T\colon c_0 \to Z$ be an operator so that

$$\frac{1}{1+\varepsilon}\|z\| \leqslant \|Tz\| \leqslant \|z\| \quad (z \in c_0).$$

Let $S = T^{-1}P$. Then $ST = I_{c_0}$ and $\|S\| \leqslant (1+\varepsilon)^2$. For each n, we set $f_n = \frac{S^* e_n^*}{(1+\varepsilon)^2}$, where $(e_n^*)_n$ is the unit vector basis of ℓ_1. Then $(f_n)_{n=1}^\infty$ is weak* null in B_{X^*}. Take any $(y_n^*)_{n=1}^\infty \in \mathrm{cbs}((f_n)_{n=1}^\infty)$ and write

$$y_n^* = \sum_{i=k_{n-1}+1}^{k_n} \lambda_i f_i,$$

where $\sum\limits_{i=k_{n-1}+1}^{k_n} \lambda_i = 1$, and $\lambda_i \geqslant 0$. For every n, let $z_n = \sum\limits_{i=k_{n-1}+1}^{k_n} e_i$, where $(e_n)_{n=1}^\infty$ is the unit vector unit basis of c_0. Clearly, $(Tz_n)_{n=1}^\infty$ is weakly null in B_X. Moreover, for every n, we get

$$|\langle y_n^*, Tz_n\rangle| = \frac{1}{(1+\varepsilon)^2}|\langle \sum_{i=k_{n-1}+1}^{k_n} \lambda_i e_i^*, \sum_{j=k_{n-1}+1}^{k_n} e_j\rangle| = \frac{1}{(1+\varepsilon)^2}.$$

This means that $\beta((g_n)_{n=1}^\infty) \geqslant \frac{1}{(1+\varepsilon)^2}$ and so $K_2(X) \geqslant \frac{1}{(1+\varepsilon)^2}$. Letting $\varepsilon \to 0$, we get $K_2(X) = 1$.

(b). Let Λ be the set of all strictly increasing sequences $(k_n)_{n=1}^\infty$ of positive integers with $k_1 = 1$. Set $X = \ell_1(\Lambda, L_1[0,1])$. Let $(r_j)_{j=1}^\infty$ be a sequence of Rademacher functions. Define $(g_n^*)_{n=1}^\infty \subseteq X^*$ by

$$g_n^*(t) = r_{j(n,t)},$$

where $t = (k_m)_{m=1}^\infty \in \Lambda$, $k_{j(n,t)} \leqslant n < k_{j(n,t)+1}$.

Since $g_n^*(t) \overset{\text{weak}^*}{\longrightarrow} 0$ in $L_\infty[0,1]$ $(t \in \Lambda)$ and $\|g_n^*\| = 1$ $(n \in \mathbb{N})$, we get $g_n^* \overset{\text{weak}^*}{\longrightarrow} 0$. Given $(h_m^*)_{m=1}^\infty \in \operatorname{cbs}((g_n^*)_n)$, we write

$$h_m^* = \sum_{j=k_m^\circ}^{k_{m+1}^\circ - 1} \lambda_j g_j^* \quad (t_0 = (k_m^\circ)_m \in \Lambda).$$

For each m, define $h_m \in X$ by $h_m(t) = r_m$ if $t = t_0$ and $h_m(t) = 0$ otherwise. Then $(h_m)_{m=1}^\infty$ is weakly null in B_X. Moreover, $\langle h_m^*, h_m \rangle = 1$ for each m. This implies that $\beta((h_m^*)_{m=1}^\infty) = 1$. Consequently, $K_2(X) = 1$.

□

4. Quantifying the Grothendieck property and reflexivity

The following result is a slight improvement on Lemma 2.2. For the sake of completeness, we include the proof here.

Lemma 4.1. *For a bounded subset A of X the following are equivalent:*

(1) A is relatively weakly compact.
(2) Every sequence in A admits a convex block subsequence that is norm convergent.
(3) Every sequence in A admits a convex block subsequence that is weakly convergent.

Proof. (1) $\Rightarrow$ (2). Given a sequence $(x_n)_{n=1}^\infty$ in A. Then $(x_n)_{n=1}^\infty$ admits a subsequence $(y_n)_{n=1}^\infty$ that is weakly convergent. By Mazur's lemma, $(y_n)_{n=1}^\infty$ admits a convex block subsequence $(z_n)_{n=1}^\infty$ that is norm convergent. It follows from Lemma 3.7 that $(z_n)_{n=1}^\infty$ is a convex block subsequence of $(x_n)_{n=1}^\infty$.

(2) $\Rightarrow$ (3) is trivial. It remains to prove (3) $\Rightarrow$ (1).

Let $K = \overline{\operatorname{conv}}(A)$. Given any $f \in X^*$. We let $c = \sup_{x \in K} \langle f, x \rangle = \sup_{x \in A} \langle f, x \rangle$. Choose a sequence $(x_n)_{n=1}^\infty$ in A so that $\langle f, x_n \rangle \to c$. By the assumption, there exists a sequence $(z_n)_{n=1}^\infty \in \operatorname{cbs}((x_n)_{n=1}^\infty)$ so that $(z_n)_{n=1}^\infty$ converges weakly to some $x \in K$. It is easy to see that $\langle f, z_n \rangle \to c$. Hence $c = \langle f, x \rangle$. It follows from James' characterisation of weak compactness via norm-attaining functionals that K is weakly compact and so A is relatively weakly compact. □

Proposition 4.2. *A Banach space X has the Grothendieck property if and only if every weak* null sequence in X^* admits a convex block subsequence that is norm null.*

Proof. The necessity follows from Mazur's lemma. It remains to prove the sufficiency.

Given a weak* null sequence $(f_n)_{n=1}^\infty$ in X^* and any subsequence $(h_n)_{n=1}^\infty$ of $(f_n)_{n=1}^\infty$. By the hypothesis, $(h_n)_{n=1}^\infty$ admits a convex block subsequence $(g_n)_{n=1}^\infty$ that is norm null. By Lemma 4.1, the sequence $(f_n)_{n=1}^\infty$ is relatively weakly compact and hence is weakly null. Thus X has the Grothendieck property. □

Definition 4.3. Let X be a Banach space. We set

$$G(X) = \sup_{\substack{(f_n)_{n=1}^\infty \subseteq B_{X^*} \\ \text{weak}^* \text{ null}}} \inf_{(g_n)_{n=1}^\infty \in \operatorname{cbs}((f_n)_{n=1}^\infty)} \limsup_{n\to\infty} \|g_n\|.$$

The above-defined quantity measures, in a certain sense, how far is a given Banach space from being a Grothendieck space. This quantification of the Grothendieck property is very different from the one proposed by Bendová ([Be]) who introduced the so-called λ-*Grothendieck space* parametrised by $\lambda \geqslant 1$. Every λ-Grothendieck space is Grothendieck but not every Grothendieck space is λ-Grothendieck for some $\lambda \geqslant 1$ ([Be, Theorem 1.2]).

Example 4.4.

(1) $G(c_0) = 1$,
(2) $G(\ell_1) = 1$,
(3) $G(C[0,1]) = 1$.

Proof. (1) is clear.

For (2), let $(s_n)_{n=1}^\infty$ be the summing basis of c_0, that is, $s_n = \sum_{k=1}^{n} e_k$ $(n \in \mathbb{N})$. Then $(s_\omega - s_n)_{n=1}^\infty$ is a weak* null sequence in B_{ℓ_∞}, where s_ω is the sequence constantly equal to 1. It is easy to see that for any $(g_n)_{n=1}^\infty \in \operatorname{cbs}((s_\omega - s_n)_{n=1}^\infty)$ we have $\|g_n\| = 1$ $(n \in \mathbb{N})$. Consequently, $G(\ell_1) = 1$.

In order to prove (3), for the sake of convenience, we consider $C[-1,1]$ instead. For each n, we define

$$h_n(t) = \begin{cases} -\frac{n}{2}, & -\frac{1}{n} \leqslant t < 0 \\ \frac{n}{2}, & 0 \leqslant t \leqslant \frac{1}{n} \\ 0, & \text{otherwise} \end{cases}$$

and

$$\varphi(t) = \begin{cases} -1, & -1 \leqslant t < 0 \\ 1, & 0 \leqslant t \leqslant 1 \end{cases}$$

Let ν be the Lebesgue measure. A routine argument shows that $\lim_{n\to\infty} \int f h_n \,\mathrm{d}\nu = 0$ for all $f \in C[-1,1]$, which means that $(h_n)_{n=1}^\infty$ is a weak* null sequence in $B_{C[-1,1]^*}$ if we view each $h_n \in L_1[-1,1]$ as an element of $C[-1,1]^*$. Clearly, $\int \varphi \cdot h_n d\nu = 1$ for each n. Take any $(\nu_n)_{n=1}^\infty \in \operatorname{cbs}((h_n)_{n=1}^\infty)$ and write $\nu_n = \sum_{i=k_{n-1}+1}^{k_n} \lambda_i h_i$. Then

$$\langle \varphi, \nu_n \rangle = \sum_{i=k_{n-1}+1}^{k_n} \lambda_i \langle \varphi, h_i \rangle = 1 \quad (n \in \mathbb{N}),$$

which implies that $\|\nu_n\| = 1$ if we regard φ as an element of $B_{C[-1,1]^{**}}$. We have thus proved that $G(C[-1,1]) = 1$. □

We are going to use G to quantify how far is a given Banach space from being a Grothendieck space.

Theorem 4.5. *A Banach space X has the Grothendieck property if and only if $G(X) = 0$.*

Proof. The necessary implication follows from Proposition 4.2.

Suppose that $G(X) = 0$. Given a weak* null sequence $(f_n)_{n=1}^\infty$ in B_{X^*}, by induction, for each k, we get a sequence $(f_n^{(k)})_{n=1}^\infty$ so that for all $k = 1, 2, \dots,$

- $(f_n^{(1)})_{n=1}^\infty \in \operatorname{cbs}\big((f_n)_{n=1}^\infty\big)$,
- $(f_n^{(k+1)})_{n=1}^\infty \in \operatorname{cbs}\big((f_n^{(k)})_{n=1}^\infty\big)$,
- $\limsup\limits_{n\to\infty} \|f_n^{(k)}\| < \frac{1}{k}$.

For each n, we set $h_n = f_n^{(n)}$. By Lemma 3.7, $(h_n)_{n\geqslant k} \in \operatorname{cbs}((f_n^{(k)})_{n=1}^\infty)$ for each k. Hence

$$\limsup_{n\to\infty} \|h_n\| \leqslant \limsup_{n\to\infty} \|f_n^{(k)}\| < \tfrac{1}{k} \qquad (k \in \mathbb{N}).$$

This implies that $(h_n)_{n=1}^\infty$ is a convex block subsequence of $(f_n)_{n=1}^\infty$ that converges to 0 in norm. Again by Proposition 4.2, X enjoys the Grothendieck property. □

Definition 4.6. Let X be a Banach space. We set

$$R(X) = \sup_{(x_n)_{n=1}^\infty \subseteq B_X} \ \inf_{(z_n)_{n=1}^\infty \in \operatorname{cbs}((x_n)_{n=1}^\infty)} \operatorname{ca}((z_n)_{n=1}^\infty).$$

Theorem 4.7. *A Banach space X is reflexive if and only if $R(X) = 0$.*

Proof. The necessity follows from Lemma 4.1. To prove the sufficiency, we need [BF, Fact 1]: an operator T from a Banach space X to a Banach space Y is weakly compact if and only if the image under T of every normalised basic sequence in X does not dominate the summing basis $(s_n)_{n=1}^\infty$ of c_0. In particular, a Banach space X is reflexive if and only if every normalised basic sequence in X does not dominate the summing basis $(s_n)_{n=1}^\infty$ of c_0.

Assume that X is non-reflexive. Then there exists a normalised basic sequence $(x_n)_{n=1}^\infty$ in X that dominates the summing basis $(s_n)_{n=1}^\infty$ in c_0. That is, for some constant $C > 0$, we get

$$\|\sum_{i=1}^n a_i x_i\| \geqslant C\|\sum_{i=1}^n a_i s_i\| = C \max_{1\leqslant k\leqslant n} |\sum_{i=k}^n a_i|,$$

for all n and all scalars $a_1, a_2, \dots, a_n$. By the hypothesis, there exists a sequence $(z_n)_{n=1}^\infty$ in $\operatorname{cbs}((x_n)_{n=1}^\infty)$, $z_n = \sum\limits_{i=k_{n-1}+1}^{k_n} \lambda_i x_i$, so that $\operatorname{ca}((z_n)_{n=1}^\infty) < C/2$. Thus, for $n \neq m$ we have $\|z_n - z_m\| < \frac{1}{2}C$, yet

$$\|z_n - z_m\| = \|\sum_{i=k_{n-1}+1}^{k_n} \lambda_i x_i - \sum_{i=k_{m-1}+1}^{k_m} \lambda_i x_i\| \geqslant C\|\sum_{i=k_{n-1}+1}^{k_n} \lambda_i s_i - \sum_{i=k_{m-1}+1}^{k_m} \lambda_i s_i\| \geqslant C.$$

This contradiction completes the proof. □

We discuss the relationship between the quantity R and several commonly used equivalent quantities measuring weak non-compactness.

Theorem 4.8. *Let X be a Banach space. Then*

$$\mathrm{wck}_X(B_X) \leqslant R(X).$$

Proof. Case 1. X is separable.

Let $0 < c < \mathrm{wck}_X(B_X)$ be arbitrary. Then there exists a sequence $(x_n)_{n=1}^{\infty}$ in B_X so that $\mathrm{d}(\mathrm{clust}_{X^{**}}((x_n)_{n=1}^{\infty}), X) > c$. Let $\varepsilon > 0$. Take any $x_0^{**} \in \mathrm{clust}_{X^{**}}((x_n)_{n=1}^{\infty})$ and let $d = \mathrm{d}(x_0^{**}, X)$. By the Hahn–Banach theorem, there exists $x_0^{***} \in S_{X^{***}}$ so that $\langle x_0^{***}, x_0^{**}\rangle = d$ and $\langle x_0^{***}, x\rangle = 0$ for all $x \in X$. We let

$$C = B_{X^*} \cap \{x^{***} \in X^{***} \colon |\langle x^{***}, x_0^{**}\rangle - d| < \varepsilon\}.$$

By Goldstine's theorem, $x_0^{***} \in \overline{C}^{\sigma(X^{***},X^{**})}$. Since $\langle x_0^{***}, x\rangle = 0$ for all $x \in X$, we get $0 \in \overline{C}^{\sigma(X^*,X)}$. Since X is separable, there exists a weak* null sequence $(f_m)_{m=1}^{\infty}$ in C. By passing to a subsequence, we may assume that the limit $\lim_m \langle x_0^{**}, f_m\rangle$ exists, which is denoted by a. By the definition of C, $|a - d| \leqslant \varepsilon$. Since $x_0^{**} \in \mathrm{clust}_{X^{**}}((x_n)_{n=1}^{\infty})$, we get a subsequence $(y_n)_{n=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$ so that $|\langle x_0^{**} - y_n, f_m\rangle| < \frac{1}{n}$ for $m = 1, 2, \ldots, n$. This implies that $\lim_{n\to\infty} \langle f_m, y_n\rangle = \langle x_0^{**}, f_m\rangle$ for each m and then $\lim_{m\to\infty}\lim_{n\to\infty} \langle f_m, y_n\rangle = a$. Given any $(z_n)_{n=1}^{\infty} \in \mathrm{cbs}((y_n)_{n=1}^{\infty})$. It is easy to see that $\lim_{m\to\infty}\lim_{n\to\infty} \langle f_m, z_n\rangle = a$.

We *claim* that $|a| \leqslant \mathrm{ca}((z_n)_{n=1}^{\infty})$. Indeed, for any $\delta > 0$, we may choose a $N \in \mathbb{N}$ so that $\|z_n - z_N\| < \mathrm{ca}((z_n)_{n=1}^{\infty}) + \delta$ for all $n \geqslant N$. Then for each m and $n \geqslant N$, we get

$$|\langle f_m, z_n\rangle| \leqslant \mathrm{ca}((z_n)_{n=1}^{\infty}) + \delta + |\langle f_m, z_N\rangle|.$$

Since $(f_m)_{m=1}^{\infty}$ is weak* null, we get, by letting $n \to \infty$ and $m \to \infty$, $|a| \leqslant \mathrm{ca}((z_n)_{n=1}^{\infty}) + \delta$. As δ was arbitrary, the proof of the claim is complete.

It follows that

$$c < d \leqslant |a| + \varepsilon \leqslant R(X) + \varepsilon.$$

As c and ε are arbitrary, we get $\mathrm{wck}_X(B_X) \leqslant R(X)$.
Case 2. X is possibly non-separable.

Let $0 < c < \mathrm{wck}_X(B_X)$ be arbitrary. Then there exists a sequence $(x_n)_{n=1}^{\infty}$ in B_X so that $\mathrm{d}(\mathrm{clust}_{X^{**}}((x_n)_{n=1}^{\infty}), X) > c$. Let $Y = \overline{\mathrm{span}}\{x_n \colon n = 1, 2, \ldots\}$ and $i_Y \colon Y \to X$ be the inclusion map. Since $i_Y^{**} \colon Y^{**} \to X^{**}$ is an isometric embedding, we get

$$\mathrm{d}(\mathrm{clust}_{Y^{**}}((x_n)_{n=1}^{\infty}), Y) \geqslant \mathrm{d}(\mathrm{clust}_{X^{**}}((x_n)_{n=1}^{\infty}), X) > c.$$

Indeed, let $y^{**} \in \mathrm{clust}_{Y^{**}}((x_n)_{n=1}^{\infty})$ and $y \in Y$ be arbitrary. Then $i_Y^{**}y^{**} \in \mathrm{clust}_{X^{**}}((x_n)_{n=1}^{\infty})$ and

$$\|y^{**} - y\| = \|i_Y^{**}y^{**} - y\| \geqslant \mathrm{d}(\mathrm{clust}_{X^{**}}((x_n)_{n=1}^{\infty}), X).$$

Finally, by Case 1, we get

$$c \leqslant \mathrm{wck}_Y(B_Y) \leqslant R(Y) \leqslant R(X).$$

As c was arbitrary, the proof is complete. □

Example 4.9.

(1) Let X be a Banach space containing a subspace isomorphic to ℓ_1. Then $R(X) = 2$. In particular, $R(\ell_1) = R(C[0,1]) = 2$.

(2) $R(c) = 2$, where c denotes the space of all convergent scalar sequences equipped with the supremum norm.

(3) $1 \leqslant R(c_0) \leqslant \frac{4}{3}$.

Proof. (1). Let $\varepsilon > 0$. By James' distortion theorem, there is a sequence $(x_n)_{n=1}^{\infty}$ in B_X so that $\|\sum\limits_{i=1}^{n} a_i x_i\| \geqslant (1-\varepsilon)\sum\limits_{i=1}^{n} |a_i|$ for all n and all scalars $a_1, a_2, \ldots, a_n$. For each $(z_n)_{n=1}^{\infty} \in \text{cbs}((x_n)_{n=1}^{\infty})$ we write $z_n = \sum_{i=k_{n-1}+1}^{k_n} \lambda_i x_i$. Then, for $n < m$, we get

$$\|z_n - z_m\| = \|\sum_{i=k_{n-1}+1}^{k_n} \lambda_i x_i - \sum_{i=k_{m-1}+1}^{k_m} \lambda_i x_i\| \geqslant 2(1-\varepsilon).$$

This implies that $\text{ca}((z_n)_{n=1}^{\infty}) \geqslant 2(1-\varepsilon)$ and hence $R(X) \geqslant 2(1-\varepsilon)$. As ε was arbitrary, we proved (1).

(2). For each n, let

$$x_n(i) = \begin{cases} 1, & i \leqslant n \\ -1, & i > n \end{cases}$$

Given $(z_n)_{n=1}^{\infty} \in \text{cbs}((x_n)_{n=1}^{\infty})$, we write $z_n = \sum\limits_{i=k_{n-1}+1}^{k_n} \lambda_i x_i$. Then, for $n < m$

$$\sum_{i=k_{n-1}+1}^{k_n} \lambda_i x_i(k_{m-1}+1) = -1, \qquad \sum_{i=k_{m-1}+1}^{k_m} \lambda_i x_i(k_{m-1}+1) = 1.$$

This implies that $\|z_n - z_m\| = 2$ and so $\text{ca}((z_n)_{n=1}^{\infty}) = 2$. Thus, we obtain $R(c) = 2$.

(3). The inequality $R(c_0) \geqslant 1$ follows from Theorem 4.8 since for every non-reflexive space X one has $\text{wck}_X(B_X) = 1$, which follows for example from [GHP, Theorem 1] and [CKS, Proposition 2.2]. The inequality $R(c_0) \leqslant 4/3$ was pointed out by W. B. Johnson; we present it here with his permission.

Suppose that $(x_n)_{n=1}^{\infty}$ is a sequence in B_{c_0}. By passing to a subsequence, we may assume that $(x_n)_{n=1}^{\infty}$ converges coordinate-wise to some $x \in B_{\ell_\infty}$. By passing to further subsequence and making a small perturbation we may assume that there are $k_1 < k_2 < \ldots$ so that x_n is supported on $\{1, 2, \ldots, k_n\}$ and $x_{n+1}(i) = x(i), i = 1, 2, \ldots, k_n$. We define $z_n = \frac{2}{3}x_{2n} + \frac{1}{3}x_{2n+1}$ $(n \in \mathbb{N})$.

We *claim* that $\|z_n - z_m\| \leqslant \frac{4}{3}$ for all n, m, $m > n$. Indeed,

$$|z_n(i) - z_m(i)| = \begin{cases} |\frac{2}{3}x_{2n}(i) + \frac{1}{3}x(i) - \frac{2}{3}x_{2m}(i) - \frac{1}{3}x(i)| \leqslant \frac{4}{3}, & i \leqslant k_{2n} \\ |\frac{1}{3}x_{2n+1}(i) - \frac{2}{3}x_{2m}(i) - \frac{1}{3}x(i)| \leqslant \frac{4}{3}, & k_{2n} < i \leqslant k_{2n+1} \\ |-\frac{2}{3}x_{2m}(i) - \frac{1}{3}x(i)| \leqslant 1, & k_{2n+1} < i \leqslant k_{2m} \\ |-\frac{1}{3}x_{2m+1}(i)| \leqslant \frac{1}{3}, & k_{2m} < i \leqslant k_{2m+1} \end{cases}$$

Consequently, $\text{ca}((z_n)_{n=1}^{\infty}) \leqslant \frac{4}{3}$ and the proof is completed. $\square$

We require an elementary lemma whose proof is straightforward.

Lemma 4.10. *Suppose that* $(f_n)_{n=1}^\infty$ *is a weak* null sequence in* X^*. *Then*

$$\limsup_{n\to\infty} \|f_n\| \leqslant \operatorname{ca}((f_n)_{n=1}^\infty) \leqslant 2 \limsup_{n\to\infty} \|f_n\|.$$

An immediate consequence of Lemma 4.10 is the following quantification of implications $(\star)$.

Theorem 4.11. *Let* X *be a Banach space. Then*

$$K_1(X) \leqslant G(X) \leqslant R(X^*).$$

In order to quantify $(\star\star)$, we require a lemma.

Lemma 4.12. *Let* X *be a Banach space containing no subspaces isomorphic to* ℓ_1. *Suppose that* $f_n \stackrel{\text{weak}^*}{\longrightarrow} 0$ *in* X^*. *Then*

$$\limsup_{n\to\infty} \|f_n\| \leqslant 2\beta\big((f_n)_{n=1}^\infty\big).$$

Proof. Let $0 < c < \limsup_{n\to\infty} \|f_n\|$. By passing to a subsequence, we may assume that $\|f_n\| > c$ for all n. Choose $x_n \in B_X$ with $\langle f_n, x_n\rangle > c$ $(n \in \mathbb{N})$. Passing to a further subsequence, by Rosenthal's ℓ_1-theorem, we may assume that $(x_n)_{n=1}^\infty$ is weakly Cauchy. Let $\varepsilon > 0$. Since $f_n \stackrel{\text{weak}^*}{\longrightarrow} 0$, we obtain, by induction, a strictly increasing sequence $(k_n)_{n=1}^\infty$ of even integers so that $\langle f_{k_n}, x_{k_n} - x_{2n-1}\rangle > c - \varepsilon$ for all n. We set $y_n = \frac{1}{2}(x_{k_n} - x_{2n-1})$. Then $(y_n)_{n=1}^\infty$ is weakly null in B_X. Let us define a weakly null sequence $(z_n)_{n=1}^\infty$ in B_X by $z_{k_n} = y_n$ and 0 otherwise. Then

$$\beta\big((f_n)_{n=1}^\infty\big) \geqslant \limsup_{n\to\infty} |\langle f_n, z_n\rangle| \geqslant \limsup_{n\to\infty} |\langle f_{k_n}, z_{k_n}\rangle| \geqslant \frac{c-\varepsilon}{2}.$$

Letting $\varepsilon \to 0$, we get $\beta\big((f_n)_{n=1}^\infty\big) \geqslant \frac{c}{2}$. As c was arbitrary, the proof is complete. □

Theorem 4.13. *Let* X *be a Banach space containing no subspaces isomorphic to* ℓ_1. *Then*

$$K_1(X) \leqslant G(X) \leqslant R(X^*) \leqslant 8K_2(X).$$

Proof. By Theorem 4.11, it suffices to prove the inequality $R(X^*) \leqslant 8K_2(X)$.

Let $0 < c < R(X^*)$. Then there exists a sequence $(f_n)_{n=1}^\infty$ in B_{X^*} so that

$$\operatorname{ca}\left((g_n)_{n=1}^\infty\right) > c \qquad \Big((g_n)_{n=1}^\infty \in \operatorname{cbs}\left((f_n)_{n=1}^\infty\right)\Big).$$

Since X contains no isomorphic copy of ℓ_1, it follows from [Bo, Proposition 3.11] (*cf.* [Pf, Proposition 11]) that B_{X^*} is *weak* convex block compact*, that is, every sequence in B_{X^*} admits a weak* convergent convex block subsequence. By passing to a convex block subsequence, by Lemma 3.7 we may assume that $f_n \stackrel{\text{weak}^*}{\longrightarrow} f$ for some $f \in B_{X^*}$. Hence, we get

$$\operatorname{ca}((g_n)_{n=1}^\infty) > c \qquad \Big((g_n)_{n=1}^\infty \in \operatorname{cbs}\left((f_n - f)_{n=1}^\infty\right)\Big).$$

Rescaling if necessary, we may assume that $(f_n)_{n=1}^\infty$ is a weak* null sequence in B_{X^*} and

$$\operatorname{ca}\left((g_n)_{n=1}^\infty\right) > \frac{c}{2} \qquad \Big((g_n)_{n=1}^\infty \in \operatorname{cbs}\left((f_n)_{n=1}^\infty\right)\Big).$$

By Lemma 4.10 and Lemma 4.12, we arrive at

$$\frac{c}{2} < \operatorname{ca}\left((g_n)_{n=1}^\infty\right) \leqslant 2\limsup_n \|g_n\| \leqslant 4\beta\big((g_n)_{n=1}^\infty\big) \qquad \Big((g_n)_{n=1}^\infty \in \operatorname{cbs}\left((f_n)_{n=1}^\infty\right)\Big).$$

This implies that $K_2(X) \geqslant \frac{c}{8}$. Since c was arbitrary, the proof is complete. □

5. Quantifying property (μ^s)

For a bounded sequence $(f_n)_{n=1}^\infty$ in X^*, we define

$$c\alpha\big((f_n)_{n=1}^\infty\big) = \alpha\big((\frac{1}{n}\sum_{i=1}^n f_i)_{n=1}^\infty\big).$$

Then $c\alpha((f_n)_{n=1}^\infty) = 0$ if and only if $(f_n)_{n=1}^\infty$ is Cesàro convergent to 0 with respect to $\mu(X^*, X)$. A direct argument shows that $c\alpha((f_n)_{n=1}^\infty) = c\alpha((f_n)_{n\geqslant N+1})$ for every positive integer N.

Definition 5.1. Let X be a Banach space. We set

$$\mu^s(X) = \sup_{\substack{(f_n)_{n=1}^\infty \subseteq B_{X^*} \\ \text{weak* null}}} \inf_{(g_n)_{n=1}^\infty \in \operatorname{ss}((f_n)_{n=1}^\infty)} \sup_{(h_n)_{n=1}^\infty \in \operatorname{ss}((g_n)_{n=1}^\infty)} c\alpha\big((h_n)_{n=1}^\infty\big).$$

Theorem 5.2. *A Banach space X has property (μ^s) if and only if $\mu^s(X) = 0$.*

Proof. The sufficient part is trivial. We only prove the necessary part.

Given a weak* null sequence $(f_n)_{n=1}^\infty$ in B_{X^*}, by induction, for each k we may find a sequence $((g_n)^{(k)})_{n=1}^\infty$ in X^* such that

- $((g_n)^{(1)})_{n=1}^\infty \in \operatorname{ss}((f_n)_{n=1}^\infty)$,
- $((g_n)^{(k+1)})_n \in \operatorname{ss}(((g_n)^{(k)})_n)$,
- $c\alpha\big((g_n)_{n=1}^\infty\big) < \frac{1}{k} \quad \big((g_n)_{n=1}^\infty \in \operatorname{ss}(((g_n)^{(k)})_n)\big)$.

Let $g_n = (g_n)^{(n)}$ $(n = 1, 2, \ldots)$. Then $(g_n)_{n=1}^\infty$ is a subsequence of $(f_n)_{n=1}^\infty$. Take any subsequence $(h_n)_{n=1}^\infty$ of $(g_n)_{n=1}^\infty$. By construction, for each k, there exists $N_k \in \mathbb{N}$ so that $(h_n)_{n\geqslant N_k+1} \in \operatorname{ss}(((g_n)^{(k)})_n)$. Consequently,

$$c\alpha\big((h_n)_{n=1}^\infty\big) = c\alpha((h_n)_{n\geqslant N_k+1}) < \frac{1}{k}.$$

As k was arbitrary, $c\alpha\big((h_n)_{n=1}^\infty\big) = 0$. Thus the sequence $(h_n)_{n=1}^\infty$ is Cesàro convergent to 0 with respect to $\mu(X^*, X)$, which completes the proof. □

Definition 5.3. For a Banach space X, we set

$$\operatorname{bs}(X) = \sup_{(x_n)_{n=1}^\infty \subseteq B_X} \inf_{(y_n)_{n=1}^\infty \in \operatorname{ss}((x_n)_{n=1}^\infty)} \sup_{(z_n)_{n=1}^\infty \in \operatorname{ss}((y_n)_{n=1}^\infty)} \operatorname{cca}((z_n)_{n=1}^\infty).$$

Clearly, $\operatorname{bs}(B_X) \leqslant \operatorname{bs}(X)$. Combining Theorem 2.3 with [BKS, Corollary 4.3], we see that $\operatorname{bs}(X) = 0$ if and only if X has the Banach–Saks property.

Theorem 5.4. *Let X be a Banach space. Then*

$$\frac{1}{3}K_1(X) \leqslant \mu^s(X) \leqslant \mathrm{bs}(X^*).$$

Proof. The latter inequality follows from Lemma 3.3, so it remains to prove only the former one.

Let $0 < c < K_1(X)$ and let $\varepsilon > 0$. Then there exist a weak* null sequence $(f_n)_{n=1}^\infty$ in B_{X^*} and a subsequence $(g_n)_{n=1}^\infty$ of $(f_n)_{n=1}^\infty$ such that

- $\alpha\big((h_n)_{n=1}^\infty\big) > c \quad \big((h_n)_{n=1}^\infty \in \mathrm{cbs}((f_n)_{n=1}^\infty)\big)$,
- $c\alpha((g_n)_{n=1}^\infty) < \mu^s(X) + \varepsilon$.

It follows from Lemma 3.7 that

$$\begin{aligned} c &< \alpha\Big(\big(\frac{1}{2^{n-1}}\sum_{i=2^{n-1}+1}^{2^n} g_i\big)_{n=1}^\infty\Big) \\ &\leqslant 2\alpha\Big(\big(\frac{1}{2^n}\sum_{i=1}^{2^n} g_i\big)_{n=1}^\infty\Big) + \alpha\Big(\big(\frac{1}{2^{n-1}}\sum_{i=1}^{2^{n-1}} g_i\big)_{n=1}^\infty\Big) \\ &\leqslant 3\alpha\Big(\big(\frac{1}{n}\sum_{i=1}^{n} g_i\big)_{n=1}^\infty\Big) \\ &\leqslant 3\mu^s(X) + 3\varepsilon. \end{aligned}$$

As c and ε are arbitrary, we arrive at $K_1(X) \leqslant 3\mu^s(X)$, which completes the proof. ☐

Finally, we present a result that directly quantifies [Ro, Proposition 2.2].

Theorem 5.5. *Let X be a reflexive space. Then*

$$\mu^s(X) \leqslant \mathrm{bs}(X^*) \leqslant 4\mu^s(X).$$

Proof. The former inequality follows from Lemma 3.3, so we need to prove the latter one.

Let $0 < c < \mathrm{bs}(X^*)$. Then there exists a sequence $(f_n)_{n=1}^\infty$ in B_{X^*} so that

$$\sup_{(h_n)_{n=1}^\infty \in \mathrm{ss}((g_n)_{n=1}^\infty)} \mathrm{cca}((h_n)_{n=1}^\infty) > c \quad \big((g_n)_{n=1}^\infty \in \mathrm{ss}((f_n)_{n=1}^\infty)\big). \tag{5.1}$$

Due to reflexivity, we may assume that $f_n \overset{\text{weak}^*}{\longrightarrow} f$ for some $f \in B_{X^*}$.

Given any $(g_n)_{n=1}^\infty \in \mathrm{ss}((\frac{f_n - f}{2})_{n=1}^\infty)$, by (5.1), there exists a subsequence $(h_n)_{n=1}^\infty$ of $(2g_n + f)_{n=1}^\infty$ such that $\mathrm{cca}((h_n)_{n=1}^\infty) > c$. Again, by reflexivity of X, we get

$$2c\alpha\big((h_n - f)_{n=1}^\infty\big) \geqslant \mathrm{cca}((h_n - f)_{n=1}^\infty) = \mathrm{cca}((g_n)_{n=1}^\infty) > c.$$

As $(\frac{h_n - f}{2})_{n=1}^\infty$ is a subsequence of $(g_n)_{n=1}^\infty$, $\mu^s(X) \geqslant \frac{c}{4}$. Since c was arbitrary, the proof is complete. ☐

Acknowledgements. The first-named author would like to thank W. B. Johnson and B. Wallis for helpful discussions and comments. The second-named author is indebted to G. Plebanek for making available [FP] to us.

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