Compact spaces and their applications in Banach space theory

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Contents

Summary

We study classes of compact spaces which are useful in Banach space theory. Banach spaces serve as framework to both differential and integral calculus, including (among others) finding solutions of differential equations. The strength of this theory is in its abstraction – it enables us to consider complicated objects (for example sequences and functions) as points in a space with a geometrical structure. As it is usual in mathematics – once the theory was established, it became interesting in itself. It has its own inner structure, its own natural problems and deep theorems.

Banach spaces are closely related to compact spaces. Firstly, the space $C(K)$ of continuous functions on a compact space K equipped with the maximum norm is a Banach space. This is not only a natural example of a Banach space, but the spaces of this form are in a sense universal. More precisely, any Banach space is isometric to a subspace of a $C(K)$ space.

Secondly, if X is a Banach space, the unit ball of its dual space X^* is compact when equipped with the topology of pointwise convergence on X (i.e., with the weak* topology).

The main focus of the research presented in the thesis is the interplay of topological properties of compact spaces and properties of Banach spaces (including geometrical and topological ones). We address in particular the questions of the following type: Which topological properties of a compact space K ensure a given property of a Banach space $C(K)$ and vice versa? Which topological properties of the unit ball of the dual space X^* ensure a given property of a Banach space X and vice versa?

As this area is very large, we focus on two more narrow sets of properties. The first one is devoted to differentiability and the second one to decompositions of Banach spaces.

The basic idea of differentiation is to approximate complicated functions by affine ones. Therefore it is natural to set apart the classes of Banach spaces in which such an approximation is possible. It leads to the definition of the classes of Asplund spaces, weak Asplund spaces and Gâteaux differentiability spaces. They are defined using differentiability of convex continuous functions. But it turned out in such spaces also some non-convex functions can be differentiated.

The main problems addressed in Chapter 2 of the thesis concern distinguishing various classes of Banach spaces defined by differentiability, namely Gâteaux differentiability spaces, weak Asplund spaces and some subclasses of weak Asplund spaces.

As for the second area – one of the main tools in the investigation of nonseparable Banach spaces consists in decomposing the space to smaller subspaces. This is done by indexed families of projections – projectional resolutions of identity and, more recently, projectional skeletons. This research began in 1960s by results of J. Lindenstrauss and was continued by many authors proving the existence of such families of projections for larger and larger classes of Banach spaces.

The main topic of Chapter 3 of the thesis is the structure of almost the largest natural class of Banach spaces with such families of projections, and related classes of compact spaces. Namely, it deals with 1-Plichko and Plichko spaces and with Valdivia compact spaces and their continuous images.

Resumé

V disertaci studujeme třídy kompaktních prostorů, které jsou využívány v teorii Banachových prostorů. Banachovy prostory slouží jako rámec pro diferenciální i integrální počet, i jako prostředek pro hledání řešení diferenciálních rovnic. Účinnost této teorie spočívá v abstrakci, umožňuje totiž se složitými objekty (například posloupnostmi a funkcemi) jako s body v prostoru s geometrickou strukturou. A jak je v matematice obvyklé – jakmile byla tato teorie zformulována, stala se zajímavou i sama o sobě. Získala svou vnitřní strukturu, své přirozené otázky a též hluboké výsledky.

Banachovy prostory úzce souvisí s kompaktními prostory. Pokud prostor spojitých funkcí na kompaktním prostoru K, který se značí $C(K)$, vybavíme maximovou normou, dostaneme Banachův prostor. Tento prostor není jen přirozeným příkladem Banachova prostoru, ale je v jistém smyslu univerzální. Přesněji, každý Banachův prostor je izometrický podprostoru nějakého prostoru tvaru $C(K)$.

Na druhou stranu, je-li X Banachův prostor, pak uzavřená jednotková koule duálního prostoru X^* je kompaktní v topologii bodové konvergence na X (tj. ve slabé* topologii).

Výzkum, jehož výsledky jsou prezentovány v dizertaci, je zaměřen zejména na vztahy mezi topologickými vlastnostmi kompaktních prostorů a různými vlastnostmi Banachových prostorů (například geometrickými nebo topologickými). Zabývá se mimo jiné otázkami typu: Jaké topologické vlastnosti kompaktního prostoru K zaručí, že prostor $C(K)$ má danou vlastnost (a obráceně)? Jaké topologické vlastnosti uzavřené jednotkové koule duálního prostoru X^* zaručí, že prostor X má danou vlastnost (a obráceně)?

Jelikož jde o velmi širokou oblast výzkumu, pro dizertaci jsem zvolil dvě užší témata. První se týká diferencovatelnosti a druhé rozkladů neseparabilních prostorů na menší podprostory.

Základní myšlenka derivování je aproximace složitých funkcí afinními funkcemi. K tomu je přirozené vydělit třídy Banachových prostorů, kde taková aproximace je možná. To vede k zavedení Asplundových prostorů, slabě Asplundových prostorů a prostorů gâteauxovské diferencovatelnosti. Tyto třídy jsou definovány pomocí diferencovatelnosti

konvexních spojitých funkcí. Ukázalo se nicméně, že na těchto prostorech je možné derivovat i některé další, nekonvexní funkce.

Základní otázky, jimž se věnuje Kapitola 2, se týkají rozlišení některých tříd Banachových prostorů definovaných pomocí diferencovatelnosti. Konkrétně jde o prostory gâteauxovské diferencovatelnosti, slabě Asplundovy prostory a některé podtřídy slabě Asplundových prostorů.

Pokud jde o druhé téma – jedna z důležitých metod zkoumání neseparabilních Banachových prostorů spočívá v rozkládání prostoru na menší podprostory. K tomu se využívají indexované systémy projekcí – projekční rozklady identity a v posledních letech i projekční skeleteony. Na začátku výzkumu v tomto směru stály v 60. letech dvacátého století výsledky J. Lindenstrausse. Pokračováním byla úspěšná snaha mnoha matematiků původní výsledky rozšiřovat, a tak byla postupně ukazována existence příslušných systémů projekcí ve větších a větších třídách Banachových prostorů.

Kapitola 3 se věnuje struktuře téměř největší přirozené třídy Banachových prostorů s vhodnými systémy projekcí a též souvisejícím třídám kompaktních prostorů. Přesněji, je zaměřena na 1-Pličkovy a Pličkovy prostory, a také na Valdiviovy kompaktní prostory a jejich spojité obrazy.

1. Introduction

A *Banach space* is a (real or complex) normed linear space which is complete in the metric induced by the norm. In particular, $\mathbb{R}^n$ or $\mathbb{C}^n$ is a Banach space when equipped with the euclidean norm. The sequence spaces ℓ_p (for $p \in [1, \infty]$), the space c_0 of sequences converging to 0, Lebesgue function spaces $L_p[0, 1]$ (for $p \in [1, \infty]$) or the space $C[0, 1]$ of continuous functions on $[0, 1]$ are classical examples of infinite dimensional Banach spaces.

Banach spaces admit several structures including algebraical, geometrical and topological ones. One can view them as linear spaces, metric spaces or topological spaces. It is also possible to study the interplay of these points of view. There are several natural topologies on a Banach space. The first one is the *norm topology*, induced by the metric generated by the norm. Another very important one is the *weak topology*, which is the weakest topology having the same continuous linear functionals as the norm topology. On a dual space there is another topology – namely topology of pointwise convergence, which is called *weak* topology*.

A *compact space* is a topological space K such that each cover of K by open sets admits a finite subcover. For example, the unit interval $[0, 1]$ is compact. More generally, a subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.

Compact spaces are closely related to Banach spaces. The first result of this kind says that the closed unit ball B_X of a Banach space X is compact (in the norm topology) if and only if the space X has finite dimension. A deeper result is the Banach-Alaoglu theorem saying that the unit ball of the dual space X^* is compact in the weak* topology for any Banach space X.

One of the consequences of Banach-Alaoglu theorem is the characterization of reflexive spaces. A Banach space X is *reflexive* if the canonical embedding of X into the second dual X^{**} is onto, i.e. if each continuous linear functional on X^* is of the form $\xi \mapsto \xi(x)$ for some $x \in X$. And the promised characterization says that X is reflexive if and only if B_X is weakly compact.

Conversely, there is a natural way from compact spaces to Banach spaces. Namely, if K is a compact space, the space $C(K)$ of continuous functions on K equipped with the maximum norm, is a Banach space. More exactly – there are two such spaces – real space of real-valued functions and complex space of complex-valued functions. It should be clear from the context which one we have in mind (in most cases it does not matter).

These two ways relating Banach spaces and compact spaces result in a kind of duality. More exactly: Let us start with a Banach space X. Then the unit ball B_{X^*} of the dual space X^* is compact in the weak* topology. So, the space of continuous functions on this compact space is again a Banach space. Moreover, to each $x \in X$ we can associate a weak* continuous function f_x on B_{X^*} by setting $f_x(\xi) = \xi(x)$. Then $x \mapsto f_x$ is an isometric embedding of X into $C(B_{X^*}, w^*)$. So, in particular, X is isometric to a subspace of a space of the form $C(K)$.

Conversely, if we start with a compact space K, we get the Banach space $C(K)$. Further, the unit ball $B_{C(K)^*}$ is compact in the weak* topology. Again, to each $x \in K$ we can associate $\varepsilon_x \in B_{C(K)^*}$ by setting $\varepsilon_x(f) = f(x)$. Then the mapping $x \mapsto \varepsilon_x$ is a homeomorphic embedding of K into $(B_{C(K)^*}, w^*)$. We also remark that the unit ball $B_{C(K)^*}$ has an important subset $P(K)$ formed by those $\xi \in B_{C(K)^*}$ for which $\xi(\mathbf{1}) = 1$ (where $\mathbf{1}$ is the constant function with value 1). Note that $\varepsilon_x \in P(K)$ for each $x \in K$. If we use the identification of the dual space $C(K)^*$ with the space of (signed or complex) Radon measures on K (which is provided by Riesz representation theorem), then $P(K)$ is formed by Radon probability measures on K and ε_x is the Dirac measure supported at x.

Let us now name several results showing how the duality works for some concrete classes of Banach spaces and compact spaces. These results are due to a large number of mathematicians. A more detailed exposition, including the relevant references, can be found in the introductory chapter of the habilitation thesis of the author [37].

In the following tables X denotes a Banach space and K a compact space. The first table says that separable Banach spaces and metrizable compact spaces are in a complete duality.

X is separable	$C(K)$ is separable
$\Updownarrow$	$\Updownarrow$
(B_{X^*}, w^*) is metrizable	K is metrizable

The next table is devoted to weakly compactly generated Banach spaces and Eberlein compact spaces. We recall that a Banach space X is *weakly compactly generated* (shortly *WCG*) if there is a weakly compact subset $L \subset X$ such that the closed linear span of L is whole X (i.e., L generates X). A compact space K is called *Eberlein* if it is homeomorphic to a weakly compact subset of some Banach space. The table for these classes is a bit more complicated, as Eberlein compact spaces are preserved by continuous mappings but weakly compactly generated spaces are not preserved by subspaces. Therefore one more class enter there – the class of *subspaces of weakly compactly generated spaces*, i.e. of those spaces which are isomorphic to a subspace of a weakly compactly generated space.

X is weakly compactly generated	$C(K)$ is weakly compactly generated
$\Downarrow \quad \not\Uparrow$	$\Updownarrow$
X is a subspace of a WCG space	$C(K)$ is a subspace of a WCG space
$\Updownarrow$	$\Updownarrow$
(B_{X^*}, w^*) is Eberlein	K is Eberlein

We include one more table of this kind. It deals with Corson compact spaces and weakly Lindelöf determined Banach spaces. We recall that a compact space K is *Corson* if it is homeomorphic to a subset of the space

$$\Sigma(\Gamma) = \{x \in \mathbb{R}^\Gamma : \{\gamma \in \Gamma : x(\gamma) \neq 0\} \text{ is at most countable}\}$$

equipped with the topology of pointwise convergence inherited from $\mathbb{R}^\Gamma$. Further, a Banach space X is *weakly Lindelöf determined* (shortly *WLD*) if there is $M \subset X$ linearly dense (i.e., such that the closed linear span of M is equal to X) such that for each $\xi \in X^*$ there are at most countably many $x \in M$ with $\xi(x) \neq 0$. The duality of these classes is also not complete, we thus need one more property of compact spaces: A compact space K is said to have *property (M)* if each Radon probability measure on K has metrizable support. The table is then as follows:

X is weakly Lindelöf determined	$C(K)$ is weakly Lindelöf determined
$\Updownarrow$	$\Updownarrow$
(B_{X^*}, w^*) is Corson	K is Corson with property (M)
$\Uparrow \quad \not\Downarrow$	$\Downarrow \quad \not\Uparrow$
(B_{X^*}, w^*) is Corson with property (M)	K is Corson

The implication which are not valid are in fact independent of the standard axioms of the set theory. More exactly, under continuum hypothesis there are counterexamples and under Martin's axiom and negation of the continuum hypothesis all the implications are valid (as under these axioms any Corson compact space has property (M)).

There are much more such tables. We included the three ones for illustration. Some other ones will be included in the following section in the summary of the thesis.

2. Summary of the thesis

In this section we give the summary of the thesis. This section is divided into four subsections, the first two are devoted to Chapter 2 of the thesis, the remaining two to Chapter 3. In subsections 1 and 3 we explain the background of the respective chapters and we mention related results and problems. Subsections 2 and 4 then contain the summary of the results of the respective chapters.

2.1. Differentiability of convex functions and the respective classes of Banach spaces.

Let X be a Banach space, $a \in X$ and f be a real-valued function defined on a neighborhood of a.

A functional $L \in X^*$ is said to be the *Fréchet derivative* of f at a if

$$\lim_{h \to 0} \frac{f(a+h) - f(a) - L(h)}{\|h\|} = 0.$$

In this case the functional L is denoted by $f'_F(a)$. Note that the Fréchet derivative is a straightforward generalization of the notion of a differential of functions of several real variables.

Further, if $h \in X$ is arbitrary, the *directional derivative* of f at a in the direction h is defined by

$$\partial_h f(a) = \lim_{t \to 0} \frac{f(a+th) - f(a)}{t}$$

provided this limit exists and is finite. I.e., it is just the derivative of the function $t \mapsto f(a+th)$ at the point 0. This quantity has the real sense of a directional derivative if h is a unit vector. But it is useful to define it for any h. If the assignment $h \mapsto \partial_h f(a)$ defines a bounded linear functional on X, f is said to be *Gâteaux differentiable* at a and the respective functional is called *Gâteaux derivative* of f at a and is denoted by $f'_G(a)$.

It is easy to realize that a Fréchet derivative is automatically Gâteaux derivative as well. Moreover, f is Fréchet differentiable at a if and only if it is Gâteaux differentiable and the limit in the definition of the directional derivatives at a is uniform for directions $h \in S_X$.

For one-dimensional X (i.e., $X = \mathbb{R}$) both Fréchet differentiability and Gâteaux differentiability coincide with the ordinary differentiability. If the dimension of X is at least two, it is no longer the case. However, if X has finite dimension, Gâteaux and Fréchet differentiability coincide for locally Lipschitz functions, in particular for continuous convex functions. And differentiability of convex functions is the starting point for defining the classes of Banach spaces which we are dealing with in Chapter 2 of the thesis.

Let X be a Banach space. We say that the space X is

- an *Asplund space* if each real-valued convex continuous function defined on an open convex subset $G \subset X$ is Fréchet differentiable at all points of a dense G_δ subset of G;
- a *weak Asplund space* if each real-valued convex continuous function defined on an open convex subset $G \subset X$ is Gâteaux differentiable at all points of a dense G_δ subset of G;
- a *Gâteaux differentiability space* (shortly *GDS*) if each real-valued convex continuous function defined on an open convex subset $G \subset X$ is Gâteaux differentiable at all points of a dense subset of G.

Asplund and weak Asplund spaces were introduced by E. Asplund [7] who called them *strong differentiability spaces* and *weak differentiability spaces*. Gâteaux differentiability spaces were introduced eleven years later by D.G. Larman and R.R. Phelps [46]. Let us remark that the set of points of Fréchet differentiability of a convex continuous function is automatically G_δ, so it does not matter whether in the definition of an Asplund space we write "dense" or "dense G_δ". For Gâteaux differentiability it is not the case, we will comment it later in more detail.

Asplund spaces are now quite well understood. They have many equivalent characterizations and nice stability properties. In particular, a Banach space X is Asplund if and only if each separable subspace of X has separable dual and if and only if the dual X^* has the Radon-Nikodým property (see e.g. [58, Chapters 2 and 5]). Further, Asplund spaces are stable to taking subspaces, quotients, finite products and moreover, to be Asplund is a *three-space property*. I.e., X is Asplund whenever there is an Asplund subspace $Y \subset X$ such that the quotient X/Y is also Asplund. These stability properties are collected in [14, Theorem 1.1.2]. The relationship of Asplund spaces to compact spaces is described by the following table.

X is Asplund	$C(K)$ is Asplund
$\Downarrow \quad \not\Uparrow$	$\Updownarrow$
(B_{X^*}, w^*) is Radon-Nikodým	K is scattered

The right-hand part of the table is proved for example in [14, Theorem 1.1.3]. Recall that a compact space K is *scattered* if each nonempty subset of K has an isolated point. As for the left-hand part – the valid implication is just a consequence of the definition. A compact space K is said to be a *Radon-Nikodým compact space* if it is homeomorphic to a subset of (X^*, w^*) for an Asplund space X. As for the converse implication – not only it is not valid, but in fact there is no topological property of the dual unit ball characterizing Asplund spaces. The reason is that for all infinite dimensional separable spaces the respective dual unit balls are weak* homeomorphic (it is a consequence of Keller's theorem [72, Theorem 8.2.4] that they are homeomorphic to the Hilbert cube $[0,1]^{\mathbb{N}}$), but some of these spaces are Asplund (for example, c_0 or ℓ_p for $p \in (1,\infty)$) and some of them are not Asplund (for example ℓ_1 or $C[0,1]$).

The structure of weak Asplund spaces is much less understood. It is known that this class is quite large, many important classes of Banach spaces are its subclasses, but the structure of the class itself is completely unclear. It is not known whether a subspace of a weak Asplund space is again weak Asplund, it is not known whether $X \times \mathbb{R}$ is weak Asplund whenever X is weak Asplund. The only known stability result says that weak Asplund spaces are preserved by quotients.

As we have already remarked, the set of points of Gâteaux differentiability of a convex continuous function on a Banach space need not be G_δ. More precisely, it is G_δ if the space is separable. In general it may be highly non-measurable, in particular non-Borel (see e.g. [22]). So, it is natural to introduce also Gâteaux differentiability spaces as it was done in [46]. One of the main questions asked in that paper was *whether any Gâteaux differentiability space is automatically weak Asplund.* This question was finally answered in the negative by W. Moors and S. Somasundaram in [52]. Later we will comment the way this counterexample as it is closely related to the content of Chapter 2 of the thesis.

Gâteaux differentiability spaces seem to have a bit better structure than weak Asplund spaces. There are some characterizations and more stability properties. For example, X is GDS if and only if each convex weak* compact subset of X^* is the weak* closed convex hull of its *weak* exposed points* (see [58, Chapter 6]). Moreover, a quotient of a GDS is again GDS and the product $X \times Y$ is GDS whenever X is GDS and Y is separable. The case of one-dimensional Y was proved by M. Fabian (his proof is reproduced in the book [58]). The general case is a rather recent result of L. Cheng and M. Fabian [9]. Anyway, it is not clear whether a subspace of a GDS is again GDS.

Let us now concentrate on subclasses of weak Asplund spaces. We will consider the following sequence of classes of Banach spaces ordered by inclusion:

$$\begin{aligned}\text{separable spaces} &\subset \text{weakly compactly generated spaces}\\ &\subset \text{subspaces of WCG spaces} \subset \text{weakly } K\text{-analytic space}\\ &\subset \text{weakly countably determined spaces}\\ &\subset \text{weakly Lindelöf determined spaces}\end{aligned}$$

Some of these classes were defined in the Introduction. *Weakly K-analytic* Banach spaces are those spaces which are K-analytic in the weak topology (see M. Talagrand's paper [67] or Section 4.1 of M. Fabian's book [14]). Weakly countably determined spaces were introduced by L. Vašák in [73]. They can be defined, similarly as weakly K-analytic spaces, by a topological property of the weak topology (see, e.g. [14, Section 7.1]).

Let us comment now the relationship of these classes to weak Asplund spaces. Separable Banach spaces are weak Asplund by a result of S. Mazur [49, Satz 2] published long time before weak Asplund spaces were introduced. E. Asplund proved in [7, Theorem 2] that a Banach space X is weak Asplund provided it admits an equivalent norm such that the respective norm on the dual space X^* is strictly convex (i.e. the unit sphere contains no segments). Any separable Banach space admits such a norm by M.M. Day [11, Theorem 4]. This result was extended to weakly compactly generated spaces by D. Amir and J. Lindendstrauss in [2, Theorem 3] and later to weakly countably determined spaces by S. Mercourakis in [50, Theorem 4.8]. It follows that weakly countably determined spaces are weak Asplund. Let us remark that one of the main ingredients of the method of the proof of these result consisted in decompositions of nonseparable spaces to smaller subspaces. This method will be discussed in more detail in Subsection 2.3.

As for weakly Lindelöf determined Banach spaces – they form a natural class with a simple definition and nice properties containing weakly countably determined spaces. We will discuss them more in Subsections 2.3 and 2.4. At this point we remark that the inclusion of weakly countably determined spaces in the class of WLD spaces is not trivial and follows from the results of S. Mercourakis [50, Section 4]. Further, WLD spaces do not have an easy relationship to weak Asplund spaces, since no inclusion between these two classes holds. This was proved by S. Argyros and S. Mercourakis in [4]. It is worth to mention that it is up to our knowledge still an open question whether each WLD space is a Gâteaux differentiability space, cf. Conjecture on page 410 of [4].

This list of subclasses of weak Asplund spaces was completed by a deep theorem of D. Preiss given in [61, Section 4.2]. This result says that a Banach space is weak Asplund whenever it admits an equivalent *smooth* norm (i.e. a norm which is Gâteaux differentiable at each point except for the origin). It is an extension of the above mentioned result of E. Asplund as it is easy to see that the norm on X is smooth as soon as the norm on the dual space X^* is strictly convex. (This was observed by V.L. Klee in [43, Appendix, (A1.1)].) Therefore we have the following list of classes of Banach spaces ordered by inclusion.

weakly countably determined spaces
$\subset$ spaces with an equivalent norm with strictly convex dual
$\subset$ spaces with an equivalent smooth norm $\subset$ weak Asplund spaces

All these inclusions are known to be strict. We focus now on the last inclusion. It was proved by R. Haydon in [21, Theorem 2.1] that there is even an Asplund space which admits no equivalent smooth norm. This not only shows that the last inclusion is strict but also suggests that there is another list of subclasses of weak Asplund spaces unrelated to the one discussed above. Indeed, it is the following one:

$$\begin{aligned}\text{Asplund spaces} &\subset \text{Asplund generated spaces}\\ &\subset \text{subspaces of Asplund generated spaces}\\ &\subset \sigma\text{-Asplund generated spaces} \subset \text{weak Asplund spaces}\end{aligned}$$

We recall some of the definitions. A space X is *Asplund generated* if there is an Asplund space Y and a bounded linear operator $T : Y \to X$ with dense range. *Subspaces of Asplund generated spaces* are those spaces which are isomorphic to a subspace of an Asplund generated space. The definition of a σ-Asplund generated space is a bit more complicated, we refer to [16], where also important properties of this class are studied.

The relationship between these classes and the related classes of compact spaces is described by the following table (for proofs see [14, Section 1.5] and [16]).

X is Asplund generated	$C(K)$ is Asplund generated
$\Downarrow \quad \not\Uparrow$	$\Updownarrow$
(B_{X^*}, w^*) is Radon-Nikodým	K is Radon-Nikodým
$\Downarrow$	$\Downarrow$
(B_{X^*}, w^*) is the image of a RN compact	K is the image of a RN compact
$\Updownarrow$	$\Updownarrow$
X is a subspace of an AG space	$C(K)$ is a subspace of an AG space
$\Downarrow$	$\Downarrow$
X is σ-Asplund generated	$C(K)$ is σ- Asplund generated
$\Updownarrow$	$\Updownarrow$
(B_{X^*}, w^*) is quasi-Radon-Nikodým	K is quasi-Radon-Nikodým

As for *quasi-Radon-Nikodým compact* spaces, the above table may serve as a definition (although the original one was different, see [6]). We remark that all the missing implications are open questions which are related to the long-standing problem whether Radon-Nikodým compact spaces are preserved by continuous mappings.

We have presented two sequences of subclasses of weak Asplund spaces. They are essentially unrelated. This is witnessed by the following results (see [14, Section 8.3] and [6]).

$$X \text{ is WLD and Asplund generated} \Leftrightarrow X \text{ is WCG}$$

$$X \text{ is WLD and a subspace of an AG space} \Leftrightarrow X \text{ is a subspace of a WCG space}$$

$$K \text{ is Corson and quasi-Radon-Nikodým} \Leftrightarrow K \text{ is Eberlein}$$

Anyway, these two unrelated sequences of subclasses of weak Asplund spaces do have a common roof. It is the class of Banach spaces with weak* fragmentable dual. To introduce it we need to recall the definition of fragmentability.

Let (X, τ) be a topological space and ρ be a metric on the set X (a priori unrelated with the topology τ). We say that (X, τ) is *fragmented by* ρ if for each nonempty set $A \subset X$ and each $\varepsilon > 0$ there is a nonempty relatively τ-open subset $U \subset A$ with ρ-diameter less than ε. Further, a topological space (X, τ) is called *fragmentable* provided it is fragmented by some metric.

Fragmentability is closely related to differentiability. It is witnessed for example by the fact that a Banach space X is Asplund if and only if the dual unit ball (B_{X^*}, w^*) is fragmented by the norm metric (see [14, Theorem 5.2.3]). It follows that each Radon-Nikodým compact space is fragmentable. More exactly, it is fragmented by some lower-semicontinuous metric. This is in fact a characterization of Radon-Nikodým compact spaces by [53]. Further, quasi-Radon-Nikodým compact spaces are easily seen to be fragmentable and they can be characterized using a stronger variant of fragmentability.

Further, it is not hard to show that a Banach space X is weak Asplund provided (X^*, w^*) is fragmentable. This class of Banach spaces we denote (following [14]) by $\widetilde{\mathcal{F}}$. The class $\widetilde{\mathcal{F}}$ is quite stable, in particular it it stable to subspaces, quotients and finite products. Moreover, we have the following table:

$$\begin{array}{ccc} X \in \widetilde{\mathcal{F}} & \quad & C(K) \in \widetilde{\mathcal{F}} \\ \Updownarrow & & \Updownarrow \\ (B_{X^*}, w^*) \text{ is fragmentable} & & K \text{ is fragmentable} \end{array}$$

The only non-easy implication is the implication $\Uparrow$ from the right-hand part and it is due to N.K. Ribarska [62]. It follows from the above remarks that σ-Asplund generated spaces belong to the class $\widetilde{\mathcal{F}}$. Moreover, Banach spaces admitting an equivalent smooth norm belong to the class $\widetilde{\mathcal{F}}$ as well. This was proved by N.K. Ribarska in [63] by refining the above mentioned result of D. Preiss from [61]. This was further extended to Banach spaces having a Lipschitz Gâteaux differentiable bump function (which is a function with nonempty bounded support) by M. Fosgerau [17]. Thus we have the following inclusions:

$$\begin{array}{ccc} \text{spaces with a smooth norm} & \substack{\not\subset \\ \not\supset} & \sigma\text{-Asplund generated spaces} \\ \cap & & \cap \\ \begin{array}{c}\text{spaces with a Lipschitz} \\ \text{Gâteaux smooth bump}\end{array} & \subset & \widetilde{\mathcal{F}} \\ & & \cap \\ & & \text{weak Asplund spaces} \end{array}$$

As all the known subclasses of weak Asplund spaces were in fact subclasses of $\widetilde{\mathcal{F}}$, it was natural to ask whether all weak Asplund spaces belong to $\widetilde{\mathcal{F}}$.

Up to now we discussed which concrete classes of Banach spaces are subclasses of weak Asplund spaces and finally we described a common roof of these classes – the class $\widetilde{\mathcal{F}}$. But one can proceed in the opposite direction – start from weak Asplund spaces and try to find a reasonable characterization, or at least a nice subclass which is as large as possible. The result of such search is Stegall's class. Before defining it we recall some facts on Gâteaux differentiability of convex functions.

Let X be a Banach space, D a nonempty open subset of X, $f : D \to \mathbb{R}$ a continuous convex function and $a \in D$. By a *subdifferential of f at a* we mean the set

$$\partial f(a) = \{x^* \in X^* : f(a+h) \leq f(a) + x^*(h) \text{ for } h \in X \text{ small enough}\}.$$

This set is automatically a nonempty convex weak* compact set. Moreover, f is Gâteaux differentiable at a if and only if $\partial f(a)$ is a singleton. The set-valued mapping $a \mapsto \partial f(a)$ has many nice properties, it is in particular upper-semicontinuous from $(D, \|\cdot\|)$ to (X^*, w^*), i.e. the set $\{a \in D : \partial f(a) \subset U\}$ is norm-open whenever U is a weak*-open subset of X^*. It is also minimal with respect to inclusion among upper-semicontinuous mappings with nonempty convex weak* compact values. These properties and also other ones can be found for example in [58, Chapter 2].

In view of these facts the following definitions are natural: Let T and X be topological spaces and φ a set-valued mapping defined on T whose values are subsets of X. The mapping φ is called *usco* if it is upper semicontinuous and has nonempty compact values. The mapping

φ is called *minimal usco* if it is usco and, moreover, it is minimal with respect to inclusion among usco maps.

Finally, a topological space X is said to be in *Stegall's class* if for each Baire topological space T and each minimal usco map $\varphi : T \to X$ there is at least one point $t \in T$ such that $\varphi(t)$ is a singleton. The set of singlevaluedness is then automatically residual (i.e., its complement is of first category) by the Banach localization principle. In particular, it easily follows that X is weak Asplund as soon as (X^*, w^*) belongs to Stegall's class. This class of Banach spaces will be denoted by $\widetilde{\mathcal{S}}$. The class $\widetilde{\mathcal{S}}$ has also nice stability properties – it is preserved by subspaces, quotients, finite products and even more. However, the respective duality table is not satisfactory, as only the easy implications are known:

$$\begin{array}{ccc} X \in \widetilde{\mathcal{S}} & \qquad & C(K) \in \widetilde{\mathcal{S}} \\ \Updownarrow & & \Downarrow \\ (B_{X^*}, w^*) \text{ is in Stegall's class} & & K \text{ is in Stegall's class} \end{array}$$

It is not hard to show that each fragmentable topological space belong to Stegall's class, in particular $\widetilde{\mathcal{F}} \subset \widetilde{\mathcal{S}}$. So, we get:

$$\widetilde{\mathcal{F}} \subset \widetilde{\mathcal{S}} \subset \text{weak Asplund spaces} \subset \text{GDS}$$

It was a longstanding problem whether these inclusions are proper. The way to distinguish these classes will be described in the next subsection.

2.2. Summary of the results of Chapter 2.

Section 2.1: Stegall compact spaces which are not fragmentable. This section contains the paper [26]. The main result is distinguishing the class of fragmentable spaces and Stegall's class in the framework of compact spaces. Let us describe the results and methods in more detail. One of the key ingredients is a proper choice of a smaller class of compact spaces. It is the following class:

Let $I = [0, 1]$ and $A \subset I$ be an arbitrary subset. Set

$$I_A = ((0, 1] \times \{0\}) \cup ((\{0\} \cup A) \times \{1\})$$

and equip this set with lexicographic order (i.e., $(x, s) < (y, t)$ if and only if either $x < y$ or $x = y$ and $s < t$). We then equip I_A with the order topology. (Such a space I_A is a special case of the spaces K_A defined in the paper.) These spaces are a generalization of the well-known “double arrow space”, which is the space I_A for $A = (0, 1)$. Each K_A is a Hausdorff compact space, it is first countable, hereditarily separable and hereditarily Lindelöf. (In fact, spaces K_A are known in topology, as any perfect separable linearly ordered compact space is of that form, see [57].)

The first important part of the results is a characterization of fragmentable spaces and spaces from Stegall's class within spaces K_A. The following proposition is [26, Proposition 3].

Proposition 1. *Consider a space of the form K_A. Then*

$$A \text{ is countable } \Leftrightarrow K_A \text{ is metrizable } \Leftrightarrow K_A \text{ is fragmentable.}$$

This proposition follows essentially from the fact that K_A is hereditarily Lindelöf. Further, let us turn to Stegall's class. The characterization is formulated in a more general setting, which proved to be useful later. So, suppose that $\mathcal{C}$ is a class of Baire topological spaces which is closed with respect to open subspaces and dense G_δ-subspaces. We say that a topological space X is in *Stegall's class with respect to* $\mathcal{C}$ if for each nonempty $T \in \mathcal{C}$ and each minimal usco map $\varphi : T \to X$ there is at least one point $t \in T$ such that $\varphi(t)$ is a singleton.

If $\mathcal{C}$ is the class of all Baire spaces, we get the original Stegall's class. But there are other reasonable choices of $\mathcal{C}$ – for example Baire spaces of weight at most κ for a cardinal number κ, or complete metric spaces.

Now we formulate the characterization given in [26, Proposition 4]:

Proposition 2. *Consider a space of the form K_A and a class $\mathcal{C}$ of Baire spaces stable with respect to open subspaces and dense G_δ subspaces. The following assertions are equivalent.*

(1) *K_A is in Stegall's class with respect to $\mathcal{C}$.*
(2) *Every continuous function $f : T \to A$ for any nonempty $T \in \mathcal{C}$ has a local extreme (i.e., a local maximum or a local minimum).*

The second ingredient of the paper are examples of uncountable sets $A \subset \mathbb{R}$ satisfying the condition (2) of Proposition 2. It is done using a formally stronger condition:

($*$) For any nonempty $T \in \mathcal{C}$ and any continuous $f : T \to A$ there is a nonempty open subset $U \subset T$ such that f is constant on U.

Then we have the following results:

Proposition 3. (a) *Under Martin's axiom and negation of continuum hypothesis each subset of $\mathbb{R}$ of cardinality $\aleph_1$ satisfies the condition ($*$) with respect to Baire spaces of weight at most $\aleph_1$.*

(b) *Assume Martin's axiom, negation of continuum hypothesis and* $\aleph_1 = \aleph_1^L$. *Then each subset of* $\mathbb{R}$ *of cardinality* $\aleph_1$ *satisfies the condition* $(*)$ *with respect to all Baire spaces.*

(c) *If* A *is a coanalytic set with no perfect subset, then* A *satisfies the condition* $(*)$ *with respect to all completely regular Baire spaces.*

This is a part of [26, Proposition 7] (which contains three more cases which are not so important for our purpose). Let us recall that L denotes the constructible universe and that $\aleph_1^L$ is the ordinal number, which in L plays the role of $\aleph_1$. So, in general we have $\aleph_1^L \leq \aleph_1$. Thus the assumption $\aleph_1 = \aleph_1^L$ says that, in a sense, the whole universe is not so far from the constructible one. The proofs of (a) and (b) use a result from the author's diploma thesis, in case (b) completed by a result of R. Frankiewicz and K. Kunen [18]. The case (c) was proved already by I. Namioka and R. Pol [54] for another purpose. Let us remark that the existence of an uncountable set satisfying the assumption of (c) follows from the axiom of constructibility $V = L$.

So, we get the following result [26, Theorem].

Theorem 1. (1) *Assume Martin's axiom and negation of continuum hypothesis. Then there is a non-fragmentable compact space which is in Stegall's class with respect to Baire spaces of weight at most* $\aleph_1$.

(2) *It is consistent with the usual axioms of the set theory that there is a non-fragmentable compact space which is in Stegall's class.*

The assertion (2) of this theorem yields a distinction of fragmentable spaces and Stegall's class within compact spaces. One disadvantage is that it is only a consistent result, depending on some additional axioms of the set theory. This difficulty cannot be easily overcome – it is witnessed by [26, Proposition 8] which we quote here:

Proposition 4. *Suppose that there is a precipitous ideal over* ω_1. *Then no uncountable subset of* $\mathbb{R}$ *satisfies the condition* $(*)$ *with respect to Baire metric spaces of weight at most* $2^{\aleph_1}$.

The definition of a precipitous ideal can be found in [18]. For a more detailed study we refer to [23, Chapter 22], where it is also proved that the existence of a precipitous ideal over ω_1 is equiconsistent with the existence of a measurable cardinal.

There is one more disadvantage of the above theorem – it does not solve the more interesting problem whether $\widetilde{\mathcal{F}} = \widetilde{\mathcal{S}}$. This problem was left open by the paper [26].

Section 2.2: A weak Asplund space whose dual is not in Stegall's class. This section contains the paper [33]. The main result is the example mentioned in the title. Let us comment it in more detail.

As we have remarked above, in the paper [26] which form Section 2.1 it is proved that there are (under some additional axioms of the set theory) non-fragmentable compact spaces which belong to Stegall's class. The question whether there are such examples in the framework of Banach spaces, i.e. whether there are Banach spaces belonging to $\widetilde{\mathcal{S}}$ but not to $\widetilde{\mathcal{F}}$ was left open. This question was solved by P. Kenderov, W. Moors and S. Sciffer in [41]. They proved that the space $C(I_A)$ belongs to $\widetilde{\mathcal{S}}$ whenever A satisfies the condition $(*)$. In the proof they used a suitable representation of the dual space $C(I_A)^*$. In [33] we used this result to prove the following proposition:

Proposition 5. *Let $\mathcal{C}$ be a class of Baire metric spaces closed to taking open subspaces and dense Baire subspaces. Consider a compact space of the form I_A. Then the following assertions are equivalent.*

(a) $(C(I_A)^*, w^*)$ *is in Stegall's class with respect to* $\mathcal{C}$.
(b) I_A *is in Stegall's class with respect to* $\mathcal{C}$.
(c) A *satisfies the condition (2) of Proposition 2.*
(d) A *satisfies the condition* $(*)$.

The implication (c)⇒(d) is a new result of this section. The implication (d)⇒(a) is in fact the mentioned result of [41]. Further (a)⇒(b) is trivial and (b)⇔(c) follows from Proposition 2 above.

As a consequence of the above proposition one gets the following theorem.

Theorem 2. (1) *If $\aleph_1 = \aleph_1^L$, then there is a Banach space belonging to $\widetilde{\mathcal{S}} \setminus \widetilde{\mathcal{F}}$.*

(2) *If Martin's axiom and the negation of continuum hypothesis hold, then there is a weak Asplund space which does not belong to $\widetilde{\mathcal{F}}$.*

(3) *If there is a precipitous ideal on ω_1 and Martin's axiom and the negation of continuum hypothesis hold, then there is a weak Asplund space which does not belong to $\widetilde{\mathcal{S}}$.*

The only really new result in this section is the assertion (3). The respective space is the space $C(I_A)$ for any $A \subset (0,1)$ of cardinality $\aleph_1$. This follows by combining previous proposition with results of the previous section. One more thing required a proof – consistency of the set-theoretical assumptions (assuming consistency of a measurable cardinal). This is also shown in the paper using a result of Y. Kakuda [24] on forcing extensions.

The assertion (1) is just a minor improvement of the result of [41] done with help of the results of [64] and [42]. As for the assertion (2) – it follows immediately from the results of the previous section and [41]. But it is pointed out as the assumptions are commonly used axioms. We stress that in this case one cannot determined whether the respective example belongs to the class $\widetilde{\mathcal{S}}$ of not.

Section 2.3: On subclasses of weak Asplund spaces. This section contains the paper [40], co-authored by K. Kunen. The results of [41] and of the previous section, summed up in Theorem 2, say that under suitable additional axioms of the set theory there is a Banach space from $\widetilde{\mathcal{S}} \setminus \widetilde{\mathcal{F}}$ and under another additional axioms there is a weak Asplund space which does not belong to $\widetilde{\mathcal{S}}$. However, the respective two sets of axioms are incompatible. Therefore it is natural to ask whether it is consistent to have simultaneously both counterexample. The answer to this question is the content of the paper [40].

The right set of axioms is the following one:

Axioms $\mathcal{A}$.

(i) Martin's axiom and $2^{\aleph_0} = \aleph_3$ hold.
(ii) There is a precipitous ideal over ω_2.
(iii) The cardinal $\aleph_1$ is not measurable in any transitive model of ZFC containing all the ordinals.

It is first proved that this set of axioms is consistent with ZFC provided the existence of a measurable cardinal is consistent. This part is due to K. Kunen who is an expert in set theory and in particular in forcing. Further, the following result is proved.

Theorem 3. *Suppose ZFC + $\mathcal{A}$ holds.*

(a) *If $A \subset (0,1)$ has cardinality $\aleph_1$, then $C(I_A) \in \widetilde{\mathcal{S}} \setminus \widetilde{\mathcal{F}}$.*

(b) *If $A \subset (0,1)$ has cardinality $\aleph_2$, then $C(I_A)$ is weak Asplund but does not belong to $\widetilde{\mathcal{S}}$.*

In view of the described results the most natural question in this area is whether some additional axioms beyond ZFC are needed. In other words:

Question. *Is it consistent with ZFC that each weak Asplund space belongs to $\widetilde{\mathcal{F}}$?*

Sec. 2.4: Weakly Stegall spaces. This section contains the manuscript [25]. This manuscript was not published and in fact contains no deep results, but it is included as it started the way to a solution of a longstanding open problem.

The original motivation which began the research described in the previous sections was to find a Banach space which is GDS but not weak Asplund. This question was asked in [46] and remained open for a quite long time.

The idea of the manuscript [25] was simple. Weakly Stegall spaces are those spaces which are in Stegall's class with respect to the class of all complete metric spaces. And similarly as any Banach space from $\widetilde{\mathcal{S}}$ is weak Asplund, one can easily prove that any Banach space whose dual in its weak* topology is weakly Stegall is necessarily GDS.

In the manuscript weakly Stegall spaces are introduced. Some basic properties are proved. Further, it is proved that the compact space of the form K_A is weakly Stegall if and only if the set A contains no perfect compact subset. Moreover, it is observed that the class of weakly Stegall compact spaces is not preserved by finite products, so it is not clear how to get a compact space such that $(C(K)^*, w^*)$ is weakly Stegall. Therefore the author did not continue the research.

However, this research was continued by W. Moors and S. Somasundaram. In their paper [51] they characterized weakly Stegall compact space in terms of an infinite game. They applied this characterization to show that, unlike for Stegall's class, to prove that a compact space K is weakly Stegall it is sufficient to test the definition for complete metric spaces of the weight at most equal to the weight of K. In particular, they showed that $(C(I_A), w^*)$ is weakly Stegall if $C(I_A)$ is the space constructed in [33] (see Theorem 2(3) above). In the following paper

[52] they used the results on weakly Stegall spaces to find a GDS which is not weak Asplund. It is again a space of the form $C(I_A)$.

2.3. Decompositions of nonseparable Banach spaces.

Separable Banach spaces have many nice properties. In particular:

- Any separable Banach space admits an equivalent norm which is locally uniformly convex (in particular strictly convex) and whose dual norm is strictly convex (see e.g. [13, Corollary II.4.3]).
- Any separable Banach space X admits a Markushevich basis (see e.g. [20, Theorem 272]), i.e. there is a sequence $(x_n, f_n)_{n\in\mathbb{N}}$ in $X \times X^*$ satisfying the following properties:
  - $f_n(x_n) = 1$ and $f_n(x_m) = 0$ for $m, n \in \mathbb{N}$, $m \neq n$.
  - span $\{x_n : n \in \mathbb{N}\}$ is dense in X.
  - For each $x \in X \setminus \{0\}$ there is $n \in \mathbb{N}$ with $f_n(x) \neq 0$.
- In any separable Banach space any norm-open set is weakly F_σ. In particular, Borel sets with respect to norm and weak topologies coincide. (This is an easy exercise.)

Nonseparable Banach spaces need not have these properties. For example, if $X = \ell_\infty$, then:

- X does not admit any equivalent locally uniformly convex norm (see [13, Theorem II.7.10]).
- X does not admit any equivalent smooth norm. (This follows from [13, Proposition II.5.5]).
- X does not admit any Markushevich basis (see e.g. [20, Theorem 306]; the definition of Markushevich basis of a nonseparable space is the same as in the separable case, only instead of natural numbers we use an arbitrary index set).
- There is a norm-open subset of X which is not weakly Borel (see [66]).

If $X = \ell_\infty(\Gamma)$ for an uncountable set Γ, then even X does not admit any equivalent strictly convex norm (see [13, Corollary II.7.13]).

But on the other hand, some nonseparable spaces share the properties of separable ones. Consider, for example, a (possibly nonseparable) Hilbert space H. Then the canonical norm on H is uniformly convex and uniformly Fréchet differentiable on the unit sphere. Further, H admits an orthonormal basis, which is much stronger than a Markushevich basis. Finally, any norm-open set in H is weakly Borel.

The nice properties of a Hilbert space are closely related with the existence of a nice basis (the orthonormal one). Other spaces with a nice basis – like $c_0(\Gamma)$ or $\ell_p(\Gamma)$ for $p \in [1, \infty)$ – have also nice properties (even though not so nice as a Hilbert space). The reason of this phenomenon is that a basis essentially provides a decomposition of the space to one-dimensional pieces.

Let us name a result due to V. Zizler [74] which illustrates the use of decompositions to smaller subspaces.

Theorem 4. *Let X be a Banach space and $\{P_\alpha : \alpha \in \Lambda\}$ be a family of bounded linear operators on X such that:*

(i) $(\|P_\alpha x\|)_{\alpha\in\Lambda}$ *belongs to* $c_0(\Lambda)$ *for each* $x \in X$.

(ii) *Each* $x \in X$ *belongs to the closed linear span of* $\{P_\alpha x : \alpha \in \Lambda\}$.

(iii) *The space* $P_\alpha X$ *admits an equivalent locally uniformly convex norm for each* $\alpha \in \Lambda$.

Then X admits an equivalent locally uniformly convex norm.

One possible kind of such a family of operators is derived from projectional resolutions of the identity. Let us give a definition of this important notion. Let X be a nonseparable Banach space with density κ (i.e., κ is the smallest possible cardinality of a dense subset of X). By a *projectional resolution of identity* (shortly *PRI*) we mean an indexed family $(P_\alpha : \omega \leq \alpha \leq \kappa)$ of linear projections on X satisfying the following conditions:

(1) $P_\omega = 0$, $P_\kappa = \mathrm{Id}_X$;

(2) $\|P_\alpha\| = 1$ for $\alpha \in (\omega, \kappa]$;

(3) $P_\alpha P_\beta = P_\beta P_\alpha = P_\alpha$ whenever $\omega \leq \alpha \leq \beta \leq \kappa$;

(4) $\operatorname{dens} P_\alpha X \leq \operatorname{card} \alpha$;

(5) $P_\mu X = \overline{\bigcup_{\alpha<\mu} P_\alpha X}$ for $\mu \in (\omega, \kappa]$ limit.

If $(P_\alpha : \omega \leq \alpha \leq \kappa)$ is a PRI on X, then the operators $(P_{\alpha+1} - P_\alpha : \omega \leq \alpha < \kappa)$ satisfy the properties (i) and (ii) from the above Zizler's theorem. So we get immediately that any Banach space with density $\aleph_1$ which admits a PRI has an equivalent locally uniformly convex norm. For larger densities one can use transfinite induction. As a consequence we get the following:

Suppose that $\mathcal{C}$ is a class of Banach spaces such that each nonseparable $X \in \mathcal{C}$ admits a PRI $(P_\alpha : \omega \le \alpha \le \kappa)$ such that $(P_{\alpha+1} - P_\alpha)X \in \mathcal{C}$ for each $\alpha \in [\omega, \kappa)$. Then each space from $\mathcal{C}$ admits an equivalent locally uniformly convex norm.

The proof is done by obvious transfinite induction using the fact that any separable space admits an equivalent locally convex norm and Zizler's theorem. This illustrates the main type of applications of PRIs. Although there are other methods of renorming, there are other properties which can be proved by transfinite induction using PRIs – for example the existence of a Markushevich basis or the existence of a bounded linear injection to some $c_0(\Gamma)$ (see [14, Chapter 6]). Moreover, the existence of a PRI provides an insight into the structure of the space.

First projectional resolutions of the identity were constructed by J. Lindenstrauss. In [47] he constructed a PRI in any nonseparable reflexive Banach space having the metric approximation property. The existence of a PRI is not stated there as a theorem, but it is just a step in proving the main result. In [48] he dropped the assumption of metric approximation property. In fact, in that paper PRI is not explicitly mentioned. But combining it with results of [47] it follows that any nonseparable reflexive Banach space admits a PRI.

A substantial progress was made by D. Amir and J. Lindenstrauss in [2]. They proved the existence of a PRI in every nonseparable weakly compactly generated Banach space. And again, they did not formulated it explicitly, the formulation is rather hidden in the proof of the main result.

This result was further extended to larger classes of Banach spaces. L. Vašák [73] extended it to weakly countably determined Banach spaces. M. Valdivia [68] extended it to weakly Lindelöf determined spaces (he used another definition and terminology). The fact that it is really an extension is not trivial, it follows from results of S. Mercourakis [50].

There is also another line of results on the existence of a PRI, namely PRIs in dual spaces. D.G. Tacon in [65] proved that X^* admits a PRI whenever X is smooth and the mapping $x \mapsto \|x\| \cdot \|\cdot\|'_G(x)$ is norm-to-weak continuous. This result was generalized by M. Fabian and G. Godefroy who proved in [15] that X^* admits a PRI whenever X is an Asplund space. Let us remark that the PRI is constructed in such a way that each $P_\alpha X^*$ is isometric to a dual of a subspace of X, however the range need not be weak* closed. In particular, the projections need not be dual mappings.

A connection of the mentioned two lines is the notion of a *shrinking PRI*. It is a PRI on X such that the dual mappings form a PRI on X^*. A shrinking PRI can be constructed on any Asplund WCG space. It is also used as a tool of proving that any Asplund WLD space is already WCG in [55].

Further extensions of the first line of results are related to Valdivia compact spaces and associated Banach spaces. Therefore we give the respective definitions:

Let K be a compact space.

- A subset $A \subset K$ is called a Σ-*subset* of K if there is a set Γ and a homeomorphic injection $h : K \to \mathbb{R}^\Gamma$ such that $A = h^{-1}(\Sigma(\Gamma))$.
- K is said to be a *Valdivia compact space* if it admits a dense Σ-subset.

So, Valdivia compact spaces are a generalization of Corson compact spaces. In this terminology a compact space is Corson if and only if it is a Σ-subset of itself. We continue by the associated classes of Banach spaces.

Let X be a Banach space.

- A subspace $S \subset X^*$ is called a Σ-*subspace* of X^* if there is a linearly dense set $M \subset X$ such that

$$S = \{x^* \in X^* : \{x \in M : x^*(x) \neq 0\} \text{ is countable}\}.$$

- X is said to be a 1-*Plichko space* if X^* admits a 1-norming Σ-subspace.
- X is said to be a *Plichko space* if X^* admits a norming Σ-subspace.

The notion of Valdivia compact space appeared (without a name) in a paper by S. Argyros, S. Mercourakis and S. Negrepontis [5]. In their Lemma 1.3 it is proved that any Valdivia compact space admits a *retractional resolution of the identity* (which is an indexed family of retractions with properties similar to those of a PRI). Using Stone-Weierstrass theorem it is easy to check that any retractional resolution of the identity on a compact space K induces a PRI on $C(K)$. This was done by M. Valdivia [69] who explicitly formulated and proved that $C(K)$ admits a PRI whenever K is Valdivia (he called these compact spaces to be in the class $\mathcal{A}$). He proved even more – the spaces $P_\alpha(C(K))$ are again of the form

$C(L)$ where L is Valdivia. In fact, his PRI is exactly the PRI induced by the retractions from [5]. This topic was elaborated by M. Valdivia in [70]. This is a long paper with a number of results. Let us point out a result given in Note 1 on page 274: Let X be a Banach space such that (B_{X^*}, w^*) admits a Σ-subset containing $Y \cap B_{X^*}$ for a 1-norming subspace $Y \subset X^*$. Then X admits a PRI. One of the important tools is the topology of pointwise convergence on a dense Σ-subset.

The name *Valdivia compact space* was introduced by R. Deville and G. Godefroy in [12]. As for Plichko and 1-Plichko spaces – the terminology is inspired by older results of A. Plichko [59]. He proved that a Banach space with a countably norming Markushevich basis admits a *bounded projectional resolution* (it is the same thing as PRI, only the condition (1) is replaced by the requirements that the norms are uniformly bounded). It turned out that Plichko spaces as defined above are exactly the spaces with a countably norming Markushevich basis.

Valdivia compact spaces and Plichko and 1-Plichko Banach spaces are the main topic of the Chapter 3 of the thesis. Its content will be described in more detail in the following section.

As remarked above, an important application of PRIs is the possibility to prove results by transfinite induction. But to use transfinite induction we also need the induction hypothesis to be satisfied. So, we need an assumption on the ranges of projections P_α (or, in some cases, $P_{\alpha+1} - P_\alpha$). If the density of X is $\aleph_1$, these ranges are separable. But for spaces of a larger density mere existence of a PRI provides very few information. There are some ways to solve this problem. One of them is the use of a *projectional generator*. This notion was introduced by J. Orihuela and M. Valdivia in [56] as a technical tool to construct a PRI. A simplified version with equivalent applications is given in M. Fabian's book [14]. The existence of a projectional generator not only implies the existence of a PRI, but has consequences also for the structure of the space. More exactly, the ranges of the constructed projections again admit a projectional generator. Therefore the transfinite induction can be applied. As a consequence, one obtains characterizations of Asplund spaces and of WLD spaces using projectional generators.

Nonetheless, the notion of a projectional generator is quite technical. The right notion is that of a *projectional skeleton* introduced by

W. Kubiś in [45]. The results mentioned below come from this paper. A projectional skeleton on a Banach space X is an indexed family $(P_s : s \in \Sigma)$ of projections on X such that the following conditions are satisfied:

(i) $\sup_{s\in\Sigma} \|P_s\| < +\infty$
(ii) P_sX is separable for each $s \in \Sigma$.
(iii) $X = \bigcup_{s\in\Sigma} P_sX$
(iv) Σ is a directed set.
(v) If $s, t \in \Sigma$ are such that $s \leq t$, then $P_sP_t = P_tP_s = P_s$.
(vi) If $(s_n)_{n=1}^{\infty}$ is an increasing sequence in Σ, then it has a supremum $s \in \Sigma$ and $P_sX = \overline{\bigcup_{n=1}^{\infty} P_{s_n}X}$.

If all the respective projections have norm one, the skeleton is called 1-*projectional skeleton*.

If a Banach space admits a projectional generator, it admits a 1-projectional skeleton as well. Further, the existence of a 1-projectional skeleton implies the existence of a PRI such that the ranges of projections admit 1-projectional skeleton as well. It is worth to mention that 1-Plichko spaces are exactly the spaces admitting a commutative 1-projectional skeleton. Here, the word commutative mean that any two projections from the skeleton commute, even if their indices are incomparable. There are also spaces with a noncommutative 1-projectional skeleton which are not Plichko.

2.4. Summary of the results of Chapter 3.

Section 3.1: Valdivia compact spaces in topology and Banach space theory. This section contains the paper [32]. It is a survey paper written on the request of J. Castillo, the Editor in chief of Extracta Mathematica. It surveys the knowledge on Valdivia compact spaces and their role in topology and Banach space theory at that time. It contains in particular previous results by the author from [27, 28, 29, 30, 31] and also several new results.

The paper is divided to six chapters. The first one is an introductory one and contains basic notions and tools. The second one contains a characterization of Valdivia compact spaces and 1-Plichko spaces (or, more exactly, of dense Σ-subsets and of 1-norming Σ-subspaces) in terms of a suitable weak topology. The third chapter is devoted to topological properties of Valdivia compacta, the fourth one to Plichko and 1-Plichko Banach spaces. The fifth chapter focuses on Plichko and 1-Plichko $C(K)$ spaces. The last chapter collects illustrative examples of Valdivia compact spaces.

Let us name some of the important results.

Theorem 5. *Let K be a compact space and $A \subset K$ be a dense subset. The following assertions are equivalent:*

- *A is a Σ-subset of K.*
- *A is countably compact and $C(K)$ is primarily Lindelöf in the topology $\tau_p(A)$ of the pointwise convergence on A.*

Theorem 6. *Let X be a Banach space and $A \subset B_{X^*}$ be a weak* dense subset. The following assertions are equivalent:*

- *There is a Σ-subspace $S \subset X^*$ with $A = S \cap B_{X^*}$.*
- *A is an absolutely convex Σ-subset of (B_{X^*}, w^*).*
- *A is weak* countably compact and X is primarily Lindelöf in the topology $\sigma(X, A)$.*

A topological space T is called *primarily Lindelöf* if it is a continuous image of a closed subspace of the space $(L_\Gamma)^{\mathbb{N}}$ for some Γ, where L_Γ is the *one-point lindelöfication* of the discrete space Γ. (I.e., $L_\Gamma = \Gamma \cup \{\infty\}$, where points of Γ are isolated and neighborhoods of ∞ are complements of countable subsets of Γ.) The two above theorems generalize results of R. Pol [60] and serve as an important tool in the study of structure of Valdivia compact spaces.

Theorem 7. *Let K be a compact space. The following assertions are equivalent.*

- *K is a Corson compact space.*
- *Every continuous image of K is a Valdivia compact space.*
- *Every at most two-to-one continuous image of K is a Valdivia compact space.*

It is well know that Corson compact spaces are preserved by continuous images (see e.g. [3, Section IV.3]). M. Valdivia in [71] showed that it is not the case for Valdivia compact spaces, answering a question raised in [13, Problem VII.2]. The above theorem provides an answer to a question asked in [71].

Theorem 8. *Let K be a continuous image of a Valdivia compact space. If K is not Corson, then K contains a homeomorphic copy of the ordinal interval* $[0, \omega_1]$.

This theorem is a generalization of a result of R. Deville and G. Godefroy [12], who proved the same under the assumption that K is Valdivia. This generalization was then applied in particular in [29].

Theorem 9. *Let X be a Banach space. The following are equivalent.*

- *X is weakly Lindelöf determined.*
- *The dual unit ball $(B_{(X,|\cdot|)^*}, w^*)$ is a Valdivia compact space for each equivalent norm $|\cdot|$ on X.*

This theorem was applied to show the limits of the existence of a PRI. In particular, in [35] it is proved (using the above result) that a Banach space X is WLD if an only if each nonseparable Banach space isomorphic to a complemented subspace of X admits a PRI.

Theorem 10. *Let K be a compact space. Consider the following assertions:*

(1) *K is Valdivia.*
(2) *$C(K)$ is 1-Plichko.*
(3) *The space of Radon probability measures $P(K)$ admits a dense convex Σ-subset.*
(4) *$(B_{C(K)^*}, w^*)$ is Valdivia.*
(5) *$P(K)$ is Valdivia.*

Then the following implications hold true:

$$1 \Rightarrow 2 \Leftrightarrow 3 \Rightarrow 4 \Rightarrow 5$$

If K has a dense set of G_δ-points, then all these assertions are equivalent.

This theorem shows relationship between the Valdivia property of K and the 1-Plichko property of $C(K)$. An example showing the failure of $2 \Rightarrow 1$ was found by T. Banakh and W. Kubiś in [8].

Section 3.2: M-bases in spaces of continuous functions on ordinals. This section contains the paper [34]. The main result of this paper says that the space $C([0, \omega_2])$ of the continuous functions on the ordinal interval $[0, \omega_2]$ is not Plichko.

This result is inspired by basic examples of Valdivia and non-Valdivia compact spaces. The ordinal interval $[0, \omega_1]$ is a basic example of a Valdivia compact space which is not Corson. Further, the ordinal interval $[0, \omega_2]$ is not Valdivia. It follows that the space $C([0, \omega_2])$ is not 1-Plichko (in fact, the dual unit ball is not Valdivia). All these results are quite easy. But the question whether $C([0, \omega_2])$ is Plichko appeared to be more difficult.

The answer is provided in [34]. The proof uses ideas of G. Alexandrov and A. Plichko [1] who proved that the space $C([0, \omega_1])$ has no norming Markushevich basis. This idea had to be completed by one more idea which consists, roughly speaking, in "adjusting" a norming Σ-subspace.

Section 3.3: On the class of continuous images of Valdivia compacta. This section contains the paper [36]. It is devoted to the study of the class of compact spaces which are continuous images of Valdivia compact spaces and of related classes of Banach spaces.

The definition of a Corson compact spaces is extended in the obvious way to countably compact setting. Hence, a countably compact space is called *Corson* if it is homeomorphic to a subset of $\Sigma(\Gamma)$ for a set Γ. Further, continuous images of Corson countably compact spaces are said to be *weakly Corson countably compact spaces*. It is well known that Corson compact spaces are preserved by continuous images, and, more generally, Corson countably compact spaces are preserved by quotient images. But they are not preserved by continuous images. For example, the ordinal interval $[0, \omega_1]$ is not Corson, but it is weakly Corson as it is a continuous image of the Corson countably compact space $[0, \omega_1)$.

An interesting class is that of weakly Corson compact spaces. It is proved there that weakly Corson compact spaces share some stability properties of Corson ones. It is also observed that continuous images of Valdivia compact spaces are exactly those compact spaces which contain a dense weakly Corson countably compact subset. This easy observation simplifies the work with continuous images of Valdivia compact spaces. In particular we get the following theorem on ordinal intervals:

Theorem 11. *Let η be an ordinal.*

- *If $\eta < \omega_2$, then $[0, \eta]$ is Valdivia and weakly Corson.*
- *If $\eta \geq \omega_2$, then $[0, \eta]$ is not a continuous image of a Valdivia compact space.*

In the paper we further study the related classes of Banach spaces – *weakly Plichko spaces* (which are subspaces of Plichko spaces), *weakly* 1-*Plichko spaces* (which are isometric subspaces of 1-Plichko spaces) and *weakly WLD spaces* (which are spaces whose dual unit ball is weakly Corson in the weak* topology).

Section 3.4: Natural examples of Valdivia compact spaces. This section contains the paper [38] which collects examples of Valdivia compact spaces naturally appearing in various branches of mathematics. Let us name some of the included results:

Theorem 12. *Let K be a linearly ordered compact space.*

- *If K is Valdivia, then so is each closed subset of K.*
- *If K is scattered, then K is Valdivia if and only if K has cardinality at most $\aleph_1$ and each point of uncountable character is isolated from one side.*

This theorem sums up some special properties of linearly ordered Valdivia compact spaces. Let us remark that general Valdivia compact spaces are not preserved by closed subsets. Indeed, the space $[0,1]^\Gamma$ is Valdivia for any set Γ and any compact space can be embedded into such a space. The study of linearly ordered Valdivia compact lines was continued in a joint paper with W. Kubiś [39].

Theorem 13. *Let G be a compact group.*

- *G is an open continuous image of a Valdivia compact space.*
- *If G is abelian, then $C(G)$ is 1-Plichko.*

We note that a compact group is Valdivia if and only if it is homeomorphic to a product of compact metric spaces. This was proved for abelian groups by W. Kubiś [44] and in general by A. Chigogidze [10]. The second assertion is valid also for noncommutative groups. In case G has weight $\aleph_1$ it follows from [8], the general case is due to A. Plichko (unpublished).

Theorem 14. *The following Banach spaces are 1-Plichko:*

- *The space $L^1(\mu)$ for an arbitrary σ-additive non-negative measure.*
- *The dual spaces $C_0(T)^*$ for any locally compact space T. In particular, the dual spaces $C(K)^*$ for a compact space K.*
- *Banach lattices with order-continuous norm.*

- *Preduals to semifinite von Neumann algebras.*

We remark that the first assertion is essentially well-known and the second one is a consequence of the first one. The third assertion is an unpublished result of A. Plichko. The last one is new. It is not clear whether the semifiniteness assumption may be dropped. This is also related to a result of U. Haagerup [19, Theorem IX.1] showing that preduals to von Neumann algebras have separable complementation property.

3. List of articles (sections)

The thesis is formed by the eight articles which are listed below. Seven of them were published in international journals. One of them is an unpublished manuscript included here for the sake of completeness, as it was later used and quoted as explained above. The symbol IF denotes the value of the impact factor of the corresponding journal in the year preceding the year of the publication (according to the common convention). The list of citations of each article is typed in a small font.

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[Q6] MOORS, W.B. and SOMASUNDARAM, S. Some recent results concerning weak Asplund spaces. *Acta Univ. Carolin. Math. Phys.* 2002, vol. 43, no. 2, 67-86.

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Sec. 2.2: O.Kalenda, *A weak Asplund space whose dual is not Stegall,* Proc. Am. Math. Soc. **130** (2002), no. 7, 2139–2143. IF=0.369

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Sec. 2.3: O.Kalenda and K.Kunen, *On subclasses of weak Asplund spaces,* Proc. Amer. Math. Soc. **133** (2005), no. 2, 425–429. IF=0.508

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