Solving ill posed problems

Eduard Feireisl
feireisl@math.cas.cz

Institute of Mathematics, Academy of Sciences of the Czech Republic, Prague
Technische Universität Berlin

CRC 910 Symposium Berlin, February 2021

Einstein Stiftung Berlin
Einstein Foundation Berlin

Perfect fluids - Euler system

prefect = inviscid, non(heat) conducting

ϱ .... mass density
$\mathbf{m} = \varrho \mathbf{u}$ .... momentum
p .... pressure
E .... energy

Leonhard Paul Euler 1707–1783

Euler system of gas dynamics

$$\partial_t \varrho + \mathrm{div}_x \mathbf{m} = 0$$

$$\partial_t \mathbf{m} + \mathrm{div}_x \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right) + \nabla_x p = 0$$

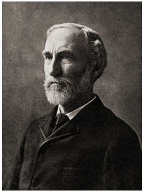
$$\partial_t E + \mathrm{div}_x \left[(E + p) \frac{\mathbf{m}}{\varrho} \right] = 0$$

$$E = \frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + \varrho e, \ e \text{ internal energy}$$

(Incomplete) equation of state (gases)

$$p = (\gamma - 1) \varrho e, \ \gamma \text{ - adiabatic coefficient}$$

Perfect fluids–entropy



Josiah Willard Gibbs 1839–1903

ϑ .... (absolute) temperature
e .... internal energy

Gibbs' relation

$$\vartheta Ds = De + pD\left(\frac{1}{\varrho}\right)$$

Entropy equation

$$\partial_t(\varrho s) + \mathrm{div}_x(s\mathbf{m}) = 0$$

"Nobody is perfect"

Entropy inequality - Second law of thermodynamics

$$\partial_t(\varrho s) + \mathrm{div}_x(s\mathbf{m}) \geq 0$$

(Mathematical) troubles with the Euler system

Well posedness

- Solution **exists** for any (physically) admissible data (initial, boundary)
- Solution is **unique** and depends continuously on the data

Known facts concerning well–posedness of the Euler system

- **Local well posedness.** Smooth solutions emenating from smooth data exist on a maximal time interval $T_{\max} > 0$. They are unique and depend continuously on the data
- **Finite time blow up.** There is a "generic" class of data for which $T_{\max} < \infty$. Solutions blow up in a finite time. Typically they become discontinuous shock waves
- **Weak solutions.** Shock waves represent *weak* solutions of the Euler system. Differential operators are understood in the distributional sense. Differential equations replaced by families of integral identities.

Weak solutions to Euler system

$\Omega \subset R^d, \ d = 1, 2, 3$ .... fluid domain

Mass conservation

$$\int_0^T \int_\Omega \left[\varrho \partial_t \varphi + \mathbf{m} \cdot \nabla_x \varphi \right] \, \mathrm{d}x = 0$$

Momentum balance

$$\int_0^T \int_\Omega \left[\mathbf{m} \cdot \partial \boldsymbol{\varphi} + 1_{\varrho > 0} \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} : \nabla_x \boldsymbol{\varphi} + p \mathrm{div}_x \boldsymbol{\varphi} \right] \, \mathrm{d}x \mathrm{d}t = 0$$

Energy balance

$$\int_0^T \int_\Omega \left[E \partial_t \varphi + (E + p) \frac{\mathbf{m}}{\varrho} \cdot \nabla_x \boldsymbol{\varphi} \right] \, \mathrm{d}x \mathrm{d}t = 0$$

$$\varphi \in C^1_c((0,T) \times \Omega), \ \boldsymbol{\varphi} \in C^1_c((0,T) \times \mathbb{T}^d; R^d)$$

Entropy admissibility condition

$$\int_0^T \int_\Omega \left[\varrho s \partial_t \varphi + s \mathbf{m} \cdot \nabla_x \varphi \right] \, \mathrm{d}x \mathrm{d}t \leq 0, \ \varphi \geq 0$$

Ill–posedness of Euler system

Theorem:
Let $\Omega \subset R^d$ be a bounded domain $d = 2, 3$. Let $\varrho_0 > 0$, $\vartheta_0 > 0$ be piecewise constant, arbitrary functions. Then there exists $\mathbf{u}_0 \in L^\infty$ such that the Euler system admits infinitely many *admissible* weak solutions in $(0, T) \times \Omega$ with the initial data $[\varrho_0, \vartheta_0, \mathbf{u}_0]$ satisfying the impermeability boundary condition

$$\mathbf{m} \cdot \mathbf{n}|_{\partial\Omega} = 0$$

Remarks:

- There are examples of Lipschitz (and even C^∞) initial data for which the Euler system admits infinitely many admissible weak solutions
- The result can be extended to the Euler system driven by stochastic forcing

Numerics – consistent approximation

Mass conservation

$$\int_0^T \int_\Omega \left[\varrho_n \partial_t \varphi + \mathbf{m}_n \cdot \nabla_x \varphi\right] \, \mathrm{d}x = E_n^1(\varphi)$$

Momentum balance

$$\int_0^T \int_\Omega \left[\mathbf{m}_n \cdot \partial \boldsymbol{\varphi} + \frac{\mathbf{m}_n \otimes \mathbf{m}_n}{\varrho_n} : \nabla_x \boldsymbol{\varphi} + p_n \mathrm{div}_x \boldsymbol{\varphi}\right] \, \mathrm{d}x\mathrm{d}t = E_n^2(\boldsymbol{\varphi})$$

Energy balance

$$\int_0^T \int_\Omega \left[E_n \partial_t \varphi + (E_n + p_n) \frac{\mathbf{m}_n}{\varrho_n} \cdot \nabla_x \boldsymbol{\varphi}\right] \, \mathrm{d}x\mathrm{d}t = E_n^3(\varphi)$$

Entropy admissibility condition

$$\int_0^T \int_\Omega \left[\varrho_n s_n \partial_t \varphi + s_n \mathbf{m}_n \cdot \nabla_x \varphi\right] \, \mathrm{d}x\mathrm{d}t \leq E_n^4(\varphi), \ \varphi \geq 0$$

Consistency errors

$$E_n^1(\varphi), \ E_n^2(\boldsymbol{\varphi}), \ E_n^3(\varphi), \ E_n^4(\varphi)$$

Lax equivalence principle

Approximate solutions in conservative entropy variables

ϱ .... mass density
$\mathbf{m}$ .... momentum
$S = \varrho s$ .... total entropy

Formulation for LINEAR problems

Peter D. Lax

- **Stability** - uniform bounds on the sequence of approximate solutions $(\varrho_n, \mathbf{m}_n, S_n)$
- **Consistency** - vanishing approximation error

$$E_n^i(\varphi) \to 0 \text{ as } n \to \infty \text{ for } i = 1, \dots, 4 \text{ and any (smooth) } \varphi$$

$$\Longleftrightarrow$$

- **Convergence** - approximate solutions converge to an exact solution of the Euler system

Lax equivalence principle - nonlinear version ?

Thermodynamics stability

Total energy.

$$E = E(\varrho, \mathbf{m}, S) = \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + \varrho e(\varrho, S)$$

Hypothesis of thermodynamics stability.

$$(\varrho, \mathbf{m}, S) \mapsto E(\varrho, \mathbf{m}, S) \text{ (strictly) convex}$$

Stability estimates for the Euler system

$$\int_\Omega E(\varrho_n, \mathbf{m}_n, S_n)(t, \cdot)\ \mathrm{d}x \leq \overline{E} \text{ uniformly in } n \text{ and } t \in [0, T]$$

Consistency – vanishing “viscosity”

(Artificial) viscosity approximation

$$\partial_t \varrho_n + \mathrm{div}_x \mathbf{m}_n = \varepsilon_n \Delta_x \varrho_n$$

$$\partial_t \mathbf{m}_n + \mathrm{div}_x \left(\frac{\mathbf{m}_n \otimes \mathbf{m}_n}{\varrho_n} \right) + \nabla_x p_n = \varepsilon_n \Delta_x \mathbf{m}_n$$

$$\partial_t E_n + \mathrm{div}_x \left[(E_n + p_n) \frac{\mathbf{m}_n}{\varrho_n} \right] = \varepsilon_n \Delta_x E_n$$

Zero viscosity – turbulent limit

$$\varepsilon_n \searrow 0$$

Weak convergence

- *Pointwise* (ideal) values of functions are replaced by their *integral averages*. This idea is close to the physical concept of *measurement*

$$U \approx \left[\varphi \mapsto \int U\varphi\right]$$

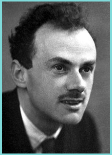
- **Weak convergence**

$$U_n \to U \text{ weakly} \;\Leftrightarrow\; \int U_n\varphi \to \int U\varphi$$

for any smooth φ

Example

Dirac distribution: $\delta_0 : \;\; \varphi \mapsto \varphi(0)$



Paul Adrien Maurice Dirac
[1902-1984]

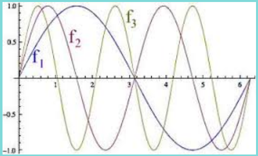
Troubles with weak convergence

Oscillatory solutions - velocity

$$U(x) \approx \sin(nx),\ \ U \to 0 \text{ in the sense of avarages (weakly)}$$

Oscillatory solutions - kinetic energy

$$\frac{1}{2}|U|^2(x) \approx \frac{1}{2}\sin^2(nx) \to \frac{1}{4} \neq \frac{1}{2}0^2 \text{ in the sense of avarages (weakly)}$$



Measure–valued solutions – limits of consistent approximations

$$(t,x) \mapsto \mathbf{U} = (\varrho, \mathbf{m}, S) \ \approx \left\langle \mathcal{V}_{t,x}; (\tilde{\varrho}, \widetilde{\mathbf{m}}, \widetilde{S}) \right\rangle$$

$$\mathcal{V}_{t,x} = \delta_{\mathbf{U}(t,x)} \ - \text{Dirac measure concentrated on } \mathbf{U}_{t,x}$$

Young measures

$$\mathbf{U}(t,x) \approx \mathcal{V}_{t,x}$$

$\mathcal{V}$ is a probability measure on the phase space related to the state variables $(\varrho, \mathbf{m}, S)$

Laurence Chisholm Young
[1905-2000]

$$\mathbf{U}_n \to \mathbf{U} \text{ weakly}, \ \int B(\mathbf{U}_n)\varphi \to \int \overline{B(\mathbf{U})}\varphi$$

$$\overline{B(\mathbf{U})}(t,x) \equiv \left\langle \mathcal{V}_{t,x}; B(\widetilde{\mathbf{U}}) \right\rangle$$

Lax equivalence principle – nonlinear version

- **Step 1 - stability.**
  $(\mathbf{U}_n)_{n=1}^{\infty}$ a stable approximation. Up to a subsequence, $\mathbf{U}_n$ converge (generate) a parametrized (Young) measure $\mathcal{V}_{t,x}$.
- **Step 2 - consistency.** $(\mathbf{U}_n)_{n=1}^{\infty}$ a stable and consistent approximation. Then $\mathcal{V}$ is a measure–valued solution of the Euler system
- **Step 3 - weak–strong uniqueness principle.**
  A measure valued and the strong solution emanating from the same data coincide on the life span of the latter. Here coincide means

$$\mathcal{V}_{t,x} = \delta_{\mathbf{U}(t,x)}$$

- **Step 4 - strong convergence**
  As the limit is a Dirac mass, the sequence does not oscillate and the convergence is (a.a.) pointwise

Convergence beyond the life span of strong solution

$\mathbf{U}_n = (\varrho_n, \mathbf{m}_n, S_n)$ ....stable consistent approximation of the Euler system

Convergence to a weak solution

- $$\mathbf{U}_n \to \mathbf{U} \text{ pointwise (a.a.)} \ \Rightarrow \ \text{the limit } \mathbf{U} = (\varrho, \mathbf{m}, S) \text{ is a weak solution}$$
- $$\text{the weak limit } \mathbf{U} = (\varrho, \mathbf{m}, S) \text{ is a weak solution}$$ $$\Rightarrow \ \mathbf{U}_n \to \mathbf{U} \text{ pointwise (a.a.)}$$

Conclusion:

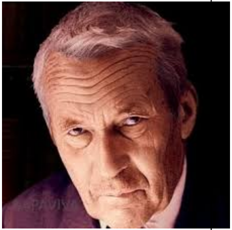
$\mathbf{U}_n \to \mathbf{U}$ (genuinely) weakly $\Rightarrow$ the limit is not a weak solution of the Euler system

Towards (compressible) turbulence

Scenario I. The viscous approximation $(\varrho_n, \mathbf{m}_n, S_n)$ converges weakly and generates a non–trivial (non monoatomic) Young measure. The weak limit – the barycenter of the Young measure – is not a weak solution of the Euler system

Scenario II - via Kolmogorov hypothesis

The viscous approximation $(\varrho_n, \mathbf{m}_n, S_n)$ is precompact in the strong Lebesgue type topology, in particular contains strongly convergent subsequence. The limit, however, is not unique.



Andrey Nikolaevich Kolmogorov 1903–1987

Visualizing (computing) the limit object

Scenario I

$$\mathbf{U}_n(t,x) \approx \delta_{\mathbf{U}_n(t,x)}$$

$$\text{Young measure } \left\langle \mathcal{V}_{t,x}; b(\widetilde{\mathbf{U}}) \right\rangle = \overline{b(\mathbf{U})}(t,x)$$

$$\overline{b(\mathbf{U})} \ - \text{weak limit of } b(\mathbf{U}_n)$$

S–convergence

$$\frac{1}{N}\sum_{n=1}^{N} b(\mathbf{U}_n) \to \overline{b(\mathbf{U})} \text{ strongly}$$

Scenario II - statistical limit

$$\mathbf{U}_n \in C([0,T];X),\ X \text{ a suitable Hilbert space}$$

S–convergence

$$\frac{1}{N}\sum_{n=1}^{N} \delta_{\mathbf{U}_n} \to \mathbf{U} \text{ strongly}$$

The limit is a statistical solution of the Euler system – a probability measure on the trajectory space

Obstacle problem

Leonhard Paul Euler 1707–1783

Isentropic Euler system driven by temporal white noise

$$\mathrm{d}\varrho + \mathrm{div}_x \mathbf{m} \mathrm{d}t = 0$$

$$\mathrm{d}\mathbf{m} + \mathrm{div}_x \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right) \mathrm{d}t + \nabla_x p(\varrho) \mathrm{d}t = \sigma \mathrm{d}W, \ p(\varrho) = a\varrho^\gamma$$

$W = W(t)$ cylindrical Wiener process

$$\Omega = R^d \setminus B, \ B \text{ compact convex obstacle}$$

Far field conditions

$$\varrho \to \varrho_\infty, \ \mathbf{m} \to \mathbf{m}_\infty \text{ as } |x| \to \infty.$$

Statistical equivalence

$(\varrho^1, \mathbf{m}^1)$, $(\varrho^2, \mathbf{m}^2)$ .... fluid motion

$(\varrho^1, \mathbf{m}^1)$ is statistically equivalent to $(\varrho^2, \mathbf{m}^2)$

$$\Leftrightarrow$$

- **Equality of expected values**

$$E[\varrho^1] = E[\varrho^2], \ E[\mathbf{m}^1] = E[\mathbf{m}^2] \text{ in } (0, T) \times \Omega$$

- **Equality of expected values of kinetic and internal energy (pressure)**

$$E\left[\frac{|\mathbf{m}^1|^2}{\varrho^1}\right] = E\left[\frac{|\mathbf{m}^2|^2}{\varrho^2}\right], \ E[p(\varrho^1)] = E[p(\varrho^2)] \text{ in } (0, T) \times \Omega$$

- **Equality of expected values of angular energy**

$$E\left[\frac{\mathbf{m}^1 \otimes \mathbf{m}^1}{\varrho^1} : (\xi \otimes \xi)\right] = E\left[\frac{\mathbf{m}^2 \otimes \mathbf{m}^2}{\varrho^2} : (\xi \otimes \xi)\right] \text{ in } (0, T) \times \Omega$$

Modeling turbulence by noise ?

Absence of noise in the high Reynolds (low viscosity) limit

- $$(\varrho, \mathbf{m}) - \text{statistical limit in the vanishing viscosity regime}$$
- $$(\varrho, \mathbf{m}) \text{ statistically equivalent}$$

  to a solution of the Euler system driven by the temporal white noise

  $$\Rightarrow$$

  $(\varrho, \mathbf{m})$ is a statistical solution of the *deterministic* Euler system

  $$\Rightarrow$$

  the noise is not active

Collaborators

D.Breit (Edinburgh)

M. Hofmanová (Bielefeld)

C.Klingenberg (Wuerzburg)

O.Kreml (Praha)

M.Lukáčová (Mainz)

S. Markfelder (Wuerzburg)

H.Mizerová (Bratislava)

B.She (Praha)