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COSIMPLICIAL RATIONAL FUNCTIONS COHOMOLOGY ON COMPLEX MANIFOLDS

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Abstract. Developing ideas of [13], we introduce canonical cosimplicial cohomology of rational functions for infinite-dimensional Lie algebra formal series with prescribed analytic behavior on domains of a complex manifold M. Differential graded cohomology of a sheaf of Lie algebras $\mathcal{G}$ via the cosimplicial cohomology of $\mathcal{G}$-formal series for any covering by Stein spaces on M is computed. A relation between cosimplicial cohomology (on a special set of open domains of M) of formal series of an infinite-dimensional Lie algebra $\mathcal{G}$ and singular cohomology of auxiliary manifold associated to a $\mathcal{G}$-module is found. Finally, multiple applications in conformal field theory, deformation theory, and in the theory of foliations are proposed.

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1. Introduction

The continuous cohomology of Lie algebras of C^∞-vector fields [6, 11, 12] has proven to be a subject of great geometrical interest. There exists the natural problem of calculating the continuous cohomology of holomorphic structures on complex manifolds [13, 35, 22, 23, 16, 6].

In [13] Feigin obtained various results concerning (co)homologies of certain Lie algebras associated to a complex curve M. For the Hodge decomposition of the tangent bundle complexification of M corresponding Lie bracket in the space of holomorphic vector fields extends to a differential Lie superalgebra structure on the Dolbeault complex. This is called the differential Lie superalgebra $\Gamma(Lie(M))$ of holomorphic vector fields on M. The Lie algebra of holomorphic vector fields $Lie_D(M)$ is defined as the cosimplicial object in the category of Lie algebras obtained from a covering of M by associating to any $i_1 < i_2 < \ldots < i_l$, the Lie algebra of holomorphic vector fields $Lie(U_{i_1} \cap \ldots \cap U_{i_l})$.

In [13] the author calculates the continuous (co)homologies with coefficients in certain one-dimensional representations $\tau_{c_{p,q}}$ of these Lie (super)algebras where $c_{p,q}$ denotes the value of the central charge for corresponding Virasoro algebra. The main result states that $H_0(Lie_D(M), \tau_{c_{p,q}})$ is isomorphic to $H(M, p, q)$, where the representation $\tau_{c_{p,q}}$ is derived from a vacuum representation of the Virasoro algebra, and $H(M, p, q)$ is the modular functor for the minimal conformal field theory [10]. The algebra $H^*_c(\Gamma(Lie(M)))$ of continuous cohomologies acts naturally on

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$H_*(Lie_D(M), \tau_{c_{p,q}})$, and the dual space is a free $H^*_c(\Gamma(Lie(M)))$-module with generators in degree zero.

The paper [35] continues work of Feigin [13] and Kawazumi [23] on the Gelfand-Fuks cohomology of the Lie algebra of holomorphic vector fields on complex manifolds. To enrich the cohomological structure, one has to involve cosimplicial and differential graded Lie algebras well known in Kodaira-Spencer deformation theory. The idea to use cosimplicial spaces to study the cohomology of mapping spaces goes back at least to Anderson [1], and it was further developed in [6]. In [35] they compute the corresponding cohomologies for arbitrary complex manifolds up to calculation of cohomology of sections spaces of complex bundles on extra manifolds. The results obtained are very similar to the results of Haefliger [16] and [6] in the case of C^∞ vector fields. Following constructions of [13] applications in conformal field theory (for Riemann surfaces), deformation theory, and foliation theory were proposed. In addition to that, in [35] the Quillen functor scheme was used for the sheaf of holomorphic vector fields on a complex manifold, and its fine resolution was given by the sheaf of $d\bar{z}$-forms with values in holomorphic vector fields, the sheaf of Kodaira-Spencer algebras.

Let M be a smooth compact manifold and $Vect(M)$ be the Lie algebra of vector fields on M. Bott and Segal [6] proved that the Gelfand-Fuks cohomology $H^*(Vect(M))$ is isomorphic to the singular cohomology $H^*(E)$ of the space E of continuous cross sections of a certain fibre bundle $\mathcal{E}$ over M. Authors of [27, 33] continued to use advanced topological methods for more general cosimplicial spaces of maps.

The main purposes of this paper are: to compute the cosimplicial version of cohomology of rational functions with prescribed analytic behavior on domains of arbitrary complex manifolds, and to find relations with other types of cohomologies. We also propose applications in conformal field theory, deformation theory, cohomology and characteristic classes of foliations on smooth manifolds.

As it was demonstrated in [35], the ordinary cohomology of vector fields on complex manifolds turns to be not the most effective and general one. In order to avoid trivialization and reveal a richer cohomological structure of complex manifolds cohomology, one has to treat [13] holomorphic vector fields as a sheaf rather than taking global sections. Inspite results in previous approaches, it is desirable to find a way to enrich cohomological structure which motivates construction of more refined cohomology description for non-commutative algebraic structures. The idea of rational function cosimplician cohomology for complex manifolds was outlined in [13] in conformal field theory form (for Riemann surfaces) and is developing in this paper. In particular, we study relations of the sheaf of rational functions associated to certain Lie algebras to the sheaf $\mathfrak{g}$ of vector valued differential forms.

2. Rational functions with prescribed analytic behavior

In this section the space underlying cohomological complexes is defined in terms of rational functions with certain properties [20, 19]. Such rational functions depend implicitly on an infinite number of non-commutative parameters.

2.1. **Rational functions with non-commutative parameters.** In the whole body of the paper we use the notation $\mathbf{y} = (y_1, \ldots, y_n)$ for $n \geq 0$. Let M be an n-dimensional complex manifold. Denote by $\mathbf{p}_l$ be a set of l points on M. We denote by $\mathcal{U}_l$ a set of domains such that $\mathbf{p}_l \in \mathcal{U}_l$. Let $\mathbf{z}_l$ be l sets of n complex coordinates in $\mathcal{U}_l$ around origines $\mathbf{p}_l$. In this paper we consider meromorphic functions of several complex variables defined on sets of open domains of M with local coordinates $\mathbf{z}_l$ which are extandable to rational functions on larger domains on M. We denote such extensions by $R(f(\mathbf{z}_l))$.

Let $\mathcal{G}$ be an infinite-dimensional Lie algebra generated by g_i, $i \in \mathbb{Z}$. Then let G be a graded (with respect to a grading operator K_G) algebraic completion of a $\mathcal{G}$-module. Denote by $F_n\mathbb{C}$ the configuration space of $l \geq 1$ ordered coordinates in $\mathbb{C}^{ln}$,

$$F_{ln}\mathbb{C} = \{\mathbf{z}_l \in \mathbb{C}^{ln} \mid z_{i,l} \neq z_{j,l'}, i \neq j\}.$$

We assume that there exists a non-degenerate bilinear form $(.,.)$ on G, and denote by $\widetilde{G}$ the space dual with respect to this form. In order to work with objects having coordinate invariant formulation [3], for a set of G-elements $\mathbf{g}_l$ we consider converging rational functions $f(\mathbf{x}_l)$ of $\mathbf{z}_l \in F_{ln}\mathbb{C}$, with $\mathbf{x}_l = (\mathbf{g}_l, \mathbf{z}_l \mathbf{dz}_l)$, where $\mathbf{z}_l$ are multiplied by corresponding differentials $\mathbf{dz}_l$. Here we make use torsor notations [3] for $\mathbf{x}_l$.

Definition 1. For arbitrary $\vartheta \in \widetilde{G}$, we call a map linear in $\mathbf{g}_l$ and $\mathbf{z}_l$,

$$F : \mathbf{x}_l \mapsto R(\vartheta, f(\mathbf{x}_l)), \tag{2.1}$$

a rational function in $\mathbf{z}_l$ with the only possible poles at $z_{i,l} = z_{j,l'}$, $i \neq j$. Abusing notations, we denote $F(\mathbf{x}_l) = R(\vartheta, f(\mathbf{x}_l))$.

Definition 2. We define left action of the permutation group S_{ln} on $F(\mathbf{z}_l)$ by

$$\sigma(F)(\mathbf{x}_l) = F(\mathbf{g}_l, \mathbf{z}_{\sigma(i)}).$$

2.2. **Conditions on rational functions.** Let $\mathbf{z}_l \in F_{ln}\mathbb{C}$. Denote by T_G the translation operator [21]. We define now extra conditions on rational functions leading to the definition of restricted rational functions.

Definition 3. Denote by $(T_G)_i$ the operator acting on the i-th entry. We then define the action of partial derivatives on an element $F(\mathbf{x}_l)$

$$\begin{aligned} \partial_{z_i} F(\mathbf{x}_l) &= F((T_G)_i\, \mathbf{g}_l, \mathbf{z}_l), \\ \sum_{i \geq 1} \partial_{z_i} F(\mathbf{x}_l) &= T_G F(\mathbf{x}_l), \end{aligned} \tag{2.2}$$

and call it T_G-derivative property.

Definition 4. For $z \in \mathbb{C}$, let

$$e^{zT_G} F(\mathbf{x}_l) = F(\mathbf{g}_l, \mathbf{z}_l + z). \tag{2.3}$$

Let $\mathrm{Ins}_i(A)$ denotes the operator of multiplication by $A \in \mathbb{C}$ at the i-th position. Then we define

$$F(\mathbf{g}_l, \mathrm{Ins}_i(z)\, \mathbf{z}_l) = F(\mathrm{Ins}_i(e^{zT_G})\, \mathbf{g}_l, \mathbf{z}_l), \tag{2.4}$$

are equel as power series expansions in z, in particular, absolutely convergent on the open disk $|z| < \min_{i \neq j}\{|z_{i,l} - z_{j,l'}|\}$.

Definition 5. A rational function has K_G-property if for $z \in \mathbb{C}^\times$ satisfies $z\mathbf{z}_l \in F_{ln}\mathbb{C}$,

$$z^{K_G} F(\mathbf{x}_l) = F(z^{K_G} \mathbf{g}_l, z\, \mathbf{z}_l). \tag{2.5}$$

2.3. **Rational functions with prescribed analytical behavior.** In this subsection we give the definition of rational functions with prescribed analytical behavior on a domain of complex manifold M of dimension n. We denote by $P_k : G \to G_{(k)}$, $k \in \mathbb{C}$, the projection of G on $G_{(k)}$. For each element $g_i \in G$, and $x_i = (g_i, z)$, $z \in \mathbb{C}$ let us associate a formal series $W_{g_i}(z) = W(x_i) = \sum_{k\in\mathbb{C}} g_i\, z^k\, dz$, $i \in \mathbb{Z}$. Following [**?**, 19], we formulate

Definition 6. We assume that there exist positive integers $\beta(g_{l',i}, g_{l",j})$ depending only on $g_{l',i}$, $g_{l'',j} \in G$ for i, $j = 1, \ldots, (l+k)n$, $k \geq 0$, $i \neq j$, $1 \leq l', l'' \leq n$. Let $\mathbf{l}_n$ be a partition of $(l+k)n = \sum_{i\geq 1} l_i$, and $k_i = l_1 + \cdots + l_{i-1}$. For $\zeta_i \in \mathbb{C}$, define $h_i = F(\mathbf{W}_{\mathbf{g}_{k_i+\mathbf{1}_i}}(\mathbf{z}_{k_i+\mathbf{l}_i} - \zeta_i))$, for $i = 1, \ldots, ln$. We then call a rational function F satisfying properties (2.2)–(2.5), a rational function with prescribed analytical behavior, if under the following conditions on domains,

$$|z_{k_i+p} - \zeta_i| + |z_{k_j+q} - \zeta_j| < |\zeta_i - \zeta_j|,$$

for i, $j = 1, \ldots, k$, $i \neq j$, and for $p = 1, \ldots, l_i$, $q = 1, \ldots, l_j$, the function $\sum_{\mathbf{r}_n\in\mathbb{Z}^n} F(\mathbf{P}_{\mathbf{r_i}} \mathbf{h_i}; (\zeta)_l)$, is absolutely convergent to an analytically extension in $\mathbf{z}_{l+k}$, independently of complex parameters $(\zeta)_l$, with the only possible poles on the diagonal of $\mathbf{z}_{l+k}$ of order less than or equal to $\beta(g_{l',i}, g_{l'',j})$. In addition to that, for $\mathbf{g}_{l+k} \in G$, the series $\sum_{q\in\mathbb{C}} F(\mathbf{W}(\mathbf{g_k}, \mathbf{P}_q(\mathbf{W}(\mathbf{g}_{l+k}, \mathbf{z}_k), \mathbf{z}_{k+1}))$, is absolutely convergent when $z_i \neq z_j$, $i \neq j$ $|z_i| > |z_s| > 0$, for $i = 1, \ldots, k$ and $s = k+1, \ldots, l+k$ and the sum can be analytically extended to a rational function in $\mathbf{z}_{l+k}$ with the only possible poles at $z_i = z_j$ of orders less than or equal to $\beta(g_{l',i}, g_{l'',j})$.

For $m \in \mathbb{N}$ and $1 \leq p \leq m-1$, let $J_{m;p}$ be the set of elements of S_m which preserve the order of the first p numbers and the order of the last $m-p$ numbers, that is,

$$J_{m,p} = \{\sigma \in S_m \mid \sigma(1) < \cdots < \sigma(p),\ \sigma(p+1) < \cdots < \sigma(m)\}.$$

Let $J_{m;p}^{-1} = \{\sigma \mid \sigma \in J_{m;p}\}$. In addition to that, for some rational functions require the property:

$$\sum_{\sigma\in J_{ln;p}^{-1}} (-1)^{|\sigma|} \sigma(F(\mathbf{g}_{\sigma(i)}, \mathbf{z}_l)) = 0. \tag{2.6}$$

Finally, we formulate

Definition 7. We define the space $\Theta(ln, k, U)$ of l complex n-variable restricted rational functions with prescribed analytical behavior on a $F_{ln}\mathbb{C}$-domain $U \subset M$ and satisfying T_G- and K_G-properties (2.2)–(2.5), definition (6), and (2.6).

3. Properties of cosimplicial double complex spaces

In this section we define double complex spaces of restricted rational function cohomology on a complex manifold M of dimension n.

3.1. **Cosimplicial double complex spaces.** In [15] the original approach to cohomology of vector fields of manifolds was initiated. Another approach to cohomology of the Lie algebra of vector fields on a manifold in the cosimplicial setup we find in [13, 35]. Let $\mathcal{U}$ be a covering $\{U_j\}$ on M, and $\mathbf{z}_{l,j}$ be l sets of local complex coordinates on each domain U_j aroud l points $\mathbf{p}_{l,j}$. For a set of G-elements $\mathbf{g_l}$, and differentials $\mathbf{dz}_{l,j}$, we consider $\mathbf{x}_{l,j} = (\mathbf{g}_l, \mathbf{z}_{l,j}\ \mathbf{dz}_{l,j})$.

Definition 8. For a domain $U \subset M$, and l, $k \geq 0$, we denote by $C^l(\Theta(ln, 1), U)$ the space of all restricted rational functions $\Theta(ln, 1)$ with prescribed analytic behavior on U depending on l sets of n complex coordinates considered on U.

Remark 1. Note that according to our construction, M can be infinite-dimensional. Thus, in that case, we consider l infinite sets of complex coordinates. The set of ln G-elements $\mathbf{g_l}$ plays the role of non-commutative parameters in our cohomological construction.

Using the standard method of defining canonical (i.e., independent of the choice of covering $\mathcal{U}$) cosimplicial object [13, 35], we consider restricted rational functions $F(\mathbf{x}_{l,j})$, and give the following

Definition 9. Choose a covering $\mathcal{U} = \{U_i, i \in I\}$ of n-dimensional complex manifold M, where each U_i is affine. Let us associate to any subset $\{i_1 < \cdots < i_k\}$ of I, the space of restricted rational functions converging on the intersection $\{U_{i_1} \cap \ldots \cap U_{i_k}\}$. Let us introduce the space

$$C_k^l = C^l \left(\Theta(ln, kn), \bigcap_{i_1 \leq \ldots \leq i_k, k \geq 0} U_{i_k} \right). \tag{3.1}$$

We call this space a cosimplicial cohomology object in the category of algebras of restricted rational functions on M.

3.2. **Co-boundary operators.** Let us take $C_k^0 = G$. Then we have

$$C_k^l \subset C_{k-1}^l,$$

when lower index is zero the sequence terminates. We also define

$$C_\infty^n = \bigcap_{m \in \mathbb{N}} C_m^n.$$

For a set $J = (\mathbf{j}_k)$, and a set $\mathbf{x}_p$ of p elements denote

$$\widehat{\mathbf{x}}_J = (x_1, \ldots, \widehat{x}_{j_1}, \ldots, \widehat{x}_{j_k}, \ldots, x_p),$$

where $\widehat{.}$ means omission of corresponding element. For sets $J_1 = (1)$, $J_{l+1} = (l+1)$, and $F \in C_k^l$, we define the operator D_k^l by

$$D_k^l F = F(W(\mathbf{x}_1), \widehat{\mathbf{x}}_{J_1}) + \sum_{i=1}^{l} (-1)^i F(W(\mathbf{x}_i) W(\mathbf{x}_{i+1})) + (-1)^{l+1} F(W(\mathbf{x}_{l+1}), \widehat{\mathbf{x}}_{J_{l+1}}). \tag{3.2}$$

Then, using similar arguments as in the proof of proposition 4.1 of [19], we obtian

Proposition 1. *The operator* (3.2) *forms the double chain complex*

$$D_k^l : C_k^l \to C_{k-1}^{l+1}, \tag{3.3}$$
$$D_{k-1}^{l+1} \circ D_k^l = 0,$$

on the spaces (3.1)*, and one defines the l-th rational function cosimplicial cohomology H_k^l of M to be $H_k^l = \mathrm{Ker}\ D_k^l / \mathrm{Im}\ D_{k+1}^{l-1}$.*

4. Cosimplicial cohomology of restricted rational functions

4.1. **Differential graded (co-)homology.** In this subsection we recall certain facts [35] on construction of differential graded (co-)homology. Let $\mathcal{G}$ be an infinite dimensional topological Lie algebra. Its (co-)homology is calculated by associating to $\mathcal{G}$ a differential graded coalgebra $C_*(g) = (\Lambda^*(g), d)$, and a differential graded algebra $C^*(g) = (Hom(\Lambda^*(g), \mathbb{C}), d)$, the homological and the cohomological Chevalley-Eilenberg complex. In consideration of tensor products of infinite dimensional topological vector spaces, we take completions, in particular, G of $\mathcal{G}$. Let $(g = \bigoplus_{i=0}^n g^i, \partial)$, be a cohomological differential graded Lie algebra (which we will denote dgla in notations). There exist two functors, $C_{*,dg}$ and C_{dg}^*, associating to (g, ∂) differential graded coalgebras $C_{*,dg}(g)$ and $C_{dg}^*(g)$.

Definition 10. $C_{*,dg}(g)$ is called the Quillen functor, [28].

It was explicitly constructed in [17]. The cohomology version was used in [16] and [29]. Explicitly, it is given by

$$C_{k,dg}(g) = \bigoplus_{k=p+q} C_{dg}^p(g)^q = \bigoplus_{k=p+q} S^{-p}(g_{(q+1)}),$$

as graded vector spaces. Here $S^{-p}(g_{(q+1)})$ is the graded symmetric algebra S^* on the shifted by one graded vector space $g_{(q+1)}$. Note that for $g_{(0)} \neq 0$, we have in $g_{(1)}$ a component of degree (-1). $S^{-p}(g_{(q+1)})$ is bigraded by the tensor degree $-p$ and the internal degree q which is induced by the grading of $g_{(q+1)}$. The differential on $C_{*,dg}(g)$ is the direct sum of the graded homological Chevalley-Eilenberg [8] differential in the tensor direction (with degree reversed in order to have a cohomological differential) and the differential induced on $S^*(g_{(q+1)})^*$ by ∂.

4.2. **Cosimplicial sheaf formulation.** Let us generale the construction of [15, 13, 35] and replace the algebra of holomorphic vector field used in [35], with n-formal complex parameter series $\mathcal{G}(\mathbf{z})$ of an infinite-dimensional Lie algebra $\mathcal{G}$. Now let us transfer (as for the case of holomorphic vector fields in [35]) to the language of sheaves of restricted rational functions defined for infinite-dimensional graded Lie algebras via a non-degenerate form. Let M a complex manifold of dimension n. Denote by $\mathcal{F}_M$ the coherent sheaf of restricted rational functions on M, and consider $\mathcal{F}_M$-modules. Let us denote by $\mathcal{E}_M$ the sheaf of C^∞ functions on M. Denote by $\mathfrak{g}(\mathbf{z})$ a sheaf of Lie-algebraic n-parameter formal series represented (according definition (7) of restricted rational functions) by $\mathcal{F}_M$-module. In this case, the Lie bracket does not provide a morphism of $\mathcal{F}_M$-modules. An action of the elements of the Lie algebra on $f \in \mathcal{F}_M$ through $W_g(z)$-operators should be specified, i.e., this leads to the concept of twisted

Lie algebras action., in particular, in the case when considering tensor products of $\mathcal{G}(\mathbf{z})$ elements for $\mathcal{F}_M$. In the same way, let $(\mathfrak{g}, \partial)$ be a sheaf of differential graded Lie algebras which are $\mathcal{E}_M$-modules. We denote by $\Gamma(\mathfrak{g}, M)$ the differential graded Lie algebra of global sections of the sheaf $\mathfrak{g}$. As in [35], we can associate to $\mathfrak{g}$ respectively to $(\mathfrak{g}, \partial)$ sheaves of differential graded coalgebras $C_*(\mathfrak{g})$ and $C_{*,dg}(\mathfrak{g})$. We obtain sheaves of differential graded algebras $C^*_{cont}(\mathfrak{g})$, $C^*_{dg}(\mathfrak{g})$, and differential graded coalgebras $C_*(\Gamma(\mathfrak{g}))$, $C_{*,dg}(\Gamma(\mathfrak{g}))$,

4.3. **Computation of sheaf cohomology via cosimplicial rational function cohomology.** The main idea of this subsection is that we are able to compute the graded differential algebra cohomology associated to a Lie algebra on a n-dimensional complex manifold M via the restricted rational function cosimplicial cohomology for this Lie algebra considered on special type of open domains on M. Since, in particular, the Čern-de Rham cohomological complex for a Lie algebra $\mathcal{G}$ formal series is a subcomplex of the double complex of restricted rational functions for $\mathcal{G}$, we find (following the lines of [17]) a relation between cohomologies of the sheaf $\mathcal{F}_M$ of restricted rational functions for a Lie algebra $\mathcal{G}$ formal series and the sheaf of graded differential algebras associated to $\mathcal{G}$. Developing ideas of [35, 8, 30], we now formulate

Definition 11. The cosimplicial cohomology of the complex $\check{C}(\mathcal{G}(\mathbf{z}),\mathcal{U})$ of formal series Lie algebra $\mathcal{G}(\mathbf{z})$ for a covering by Stein open sets $\mathcal{U}$ is the cohomology of the realization of the simplicial cochain complex obtained from applying the continuous Chevalley-Eilenberg complex as a functor C^*_{cont} to the cosimplicial Lie algebra. We denote cosimplicial cohomology by H^*_{cos}.

In the case when a sheaf $\mathfrak{g}$ is given by differential algebra of [9], we construct, as in [35], a morphism of simplicial cochain complexes

$$\tilde{f} : C^*_{dg}(\mathfrak{g}, N_*) \to C^*_{cont}(\mathcal{G}(\mathbf{z}), N_*),$$

induced by applying the modification of the Quillen functor C^*_{dg} to the inclusion

$$f : \mathcal{G}(\mathbf{z})(N_{M,q}) \hookrightarrow \mathfrak{g}(N_{M,q}),$$

Here, as in [35], N_* denotes the thickened nerve of the covering $\mathcal{U}$, i.e., the simplicial complex manifold associated to the covering $\mathcal{U}$. By lemma 5.9 in [6], the morphism $\tilde{f}$ induces a cohomology equivalence between the realizations of the two simplicial cochain complexes. The conditions of the lemma are fulfilled because of the isomorphism of the cohomologies on a Stein open set of the covering and the Künneth theorem [35]. Using Proposition 6.2 of [6], and involving partitions of unity, one shows that the cohomology of the realization of the simplicial cochain complex on the left hand side gives the differential graded cohomology of $\Gamma(X, \mathfrak{g})$. We finally obtain the following

Proposition 2. *On a complex manifold M of dimension n, one has*

$$H^*_{*\ dg}(\Gamma(M, \mathfrak{g})) \cong H^*_{*\ cos}(\check{C}(\mathcal{G}(\mathbf{z}),\mathcal{U})),$$

for any covering of M by Stein open sets $\mathcal{U}$.

4.4. **Relation of cosimplicial and singular cohomology.** Gelfand and Fuks [11] calculated the cohomology of the Lie algebra of formal vector fields in n variables W_n, i.e., is the Lie algebra of formal vector fields of n complex variables. In particular, they proved [11, 12]

Theorem 1. *There exists a manifold $X(n)$ such that the continuous cohomology of W_n is equivalent to singular cohomology of $X(n)$*

$$H^*_{cont}(W_n) \cong H^*_{sing}(X(n)).$$

In [6] they showed that for R^n or more generally a starshaped open set U of an n-dimensional manifold M, the Lie algebra of C^∞-vector fields $Vect(U)$ has the same cohomology as W_n. In [35] it was proven that the same is true for the Lie algebra of holomorphic vector fields on a disk of radius R in $\mathbb{C}^n$.

In this paper, we consider cohomology of restricted rational functions provided by bilinear forms for arbitrary n-formal parameter Lie algebra $\mathcal{G}(\mathbf{z})$ series localized on complex n-dimensional manifold M. We prove

Proposition 3. *There exists a manifold $X(n)$ such that the cosimplicial cohomology of restricted rational functions on a complex manifold for $\mathcal{G}(\mathbf{z})$ is equivalent to singular cohomology of $X(n)$, i.e.,*

$$H^*_{*\ cos}(\mathcal{G}(\mathbf{z}),\, \mathcal{U}) \cong H^*_{*\ sing}(X(G,n)).$$

Proof. Let us construct the special manifold $X(n)$ present in (3). According to the construction of Section 2.1, non-commutative coefficients of a formal Lie-algebraic series are elements of a $\mathcal{G}$-module G. For each element $g_{j,k}$, $k \in \mathbb{C}$, $j \ge \mathbb{Z}$, associate a diagram [26] representing it as action of generators on the union element. The properties of non-degenerate bilinear form $(.,.)$ allow us to find appropriate diagram for the element $\widetilde{g}_{j,k}$ of $\widetilde{G}$ dual to $g_{j,k}$. Recall the chequered cycle construction for elements $g_{j,k}$ used in [26]. In this paper we assume that chequered cycle is in one to one correspondence with the formation of an G-element, and all possible singularities are missing. Associate a knot of such diagram to a point of $X(n)$. Each point of the G-cycle is endowed with a power of z_j, $j \in \mathbb{Z}$. Let us associate to a point on the G-cycle the zero power of z_j the zero point of a local domain U_j ob $X(n)$. The union $V(G,n)$ of all chequered cycles together with local domains U_j present in the definition of the double complex constitutes the cells of a skeleton for G. Thus, we obtain an analog of a $2n$-skeleton for W_n of formal vector fields. In contrast to [11, 12] it is endowed with a power of j-th formal parameter z_j. We define a map

$$\pi : V(G,n) \to G(\infty,n),$$

from the the G-skeleton to an infinite Grassmanian $G(\infty,n)$. Since the inverse image of the union of the cells is not a manifold, we consider an open neighborhood of the inverse image under π of the G-skeleton of the Grassmannian $G(\infty,n)$. The union of such open neighborhoods constitutes the manifold $X(p)$. We now describe the singular cohomology of $H^*_{*\ sing}(X(G,n))$ of $X(n)$. This is the cohomology of double Lie-algebraic complexes $C^l_k(\Theta(n,k))$ which is the union of complexes $C^l_k(\Theta(1,k))$ for each local coordinate and g_{jk}-generators. It coincides with the cosimplicial cohomology $H^*_{*\ cos}(\mathcal{G}(\mathbf{z}),\mathcal{U})$ of W_n of n complex variables. □

Note that another way to prove Proposition 3 is to use the same technique as in [11, 12] since for the complex $C^l_k(\mathcal{G}(\mathbf{z}), \mathcal{U})$ there exist converging spectral sequences. i.e., to show that an isomorphism of the Hochschild-Serre spectral sequence [18] for the subalgebra $gl(n)$ with the Leray spectral of the restriction to the $2n$ skeleton of the universal $U(n)$ principal bundle.

5. Conclusions

In this section we list multiple applications of the research of this paper are in conformal field theory [13, 35, 3, 5, 10, 34], in deformation theory [2, 17], and in the theory of foliations [7].

5.1. **Applications in conformal field theory and moduli spaces.** In [13, 4] applications of cosimplicial computations on compact Riemann surfaces in conformal field [10, 3, 5] theory were treated. As we deal with special homology, we replace the sheaf of holomorphic vector fields by the sheaf of rational functions associated to corresponding Lie algebra. In [13], for Riemann surface $\Sigma^{(g)}$, Feigin calculated the cosimplicial homology of $Lie(\Sigma^{(g)})$ with values in the representations mentioned in Introduction. It is possible to compute cosimplicial homology of a space of rational function complexes associated to various Lie algebras. The space of coinvariants on the right hand side defining so-called modular functor is usually associated to locally defined objects. We will obtain its homological description in terms of globally defined objects. The space of coinvariants supposed to be the continuous dual to the local ring completion of the moduli space of compact Riemann surfaces of genus $g \geq 2$ at the point $\Sigma^{(g)}$, provided that $\Sigma^{(g)}$ is a smooth point. This gives an important link between Lie algebra homology and the geometry of the moduli space.

5.2. **Applications in deformation theory.** *Deformations of complex manifolds.* Cosimplician considerations above are applicable to cohomology computations in the deformation theory of complex manifolds [25, 13, 17, 14]. The completion of a local ring of moduli space at a given point M is isomorphic to the dual of the Lie algebra of M-infinitesimal automorphisms zero-th homology group. This links Lie algebra homology and geometry of the moduli space in a formal neighborhood of a point. We expect results in this direction for higher dimensional complex manifolds. In [35] we find the condition for the first cohomology in the case of higher dimensional complex manifolds M. For restricted function cohomology one can consider also related the deformation theory following Kodaira and Spencer [24].

Deformations of Lie algebras. It is well known that the Lie algebra cohomology with values in the adjoint representation $H^*(L, L)$ of a Lie algebra L answers questions about deformations of L as an algebraic object. For example, $H^2(L, L)$ can be interpreted as the space of equivalence classes of infinitesimal deformations of L, see [11, 12]. There arise natural questions of this type for differential bi-graded Lie algebras resulting from chain complex constructions. For a disk $D \subset \mathbb{C}^n$, holomorphic vector fields are rigid, i.e., [35] $H^*_{cont}(Hol(D), Hol(D)) = 0$. Using cosimplicial cohomology results, we will study rigidity of differential bi-graded Lie algebras resulting from chain complex constructions. For a compact Riemann surface $\Sigma^{(g)}$ of genus

$g \geq 2$, we expect to find a relations for cohomologies in terms of elements of Fréchet spaces given by the polynomials on $T_{\Sigma(g)}\mathcal{M}(g,0)$. It's the space of formal power series on $T_{\Sigma(g)}\mathcal{M}(g,0)^*$. This could be interpreted as a relation between cohomology with adjoint coefficients of $\mathfrak{g}$, i.e., differential graded deformations of global sections of $\mathfrak{g}$, and deformations of the underlying manifold. As it is explained in [13, 35], the choice of the coefficients in the Lie algebra cohomology determines a geometric object on the moduli space in a formal neighborhood of a point. Namely, trivial coefficients correspond to the structure sheaf, adjoint coefficients correspond to vector fields, adjoint coefficients in the universal enveloping algebra correspond to differential operators.

5.3. **Applications in foliation theory.** Applications in foliation theory are inspired by the link between cohomology of Lie algebras and characteristic classes of foliations [11, 12]. In [35] the author considered the case of characteristic classes of g-structures. For a complex manifold M, and $\mathcal{U} = \{U_i\}_{i\in I}$ a covering of M by open sets such that I is a countable directed index set, consider the sheave of rational functions in cosimplicial setup given above. Denote by $W_{2n}|_{hol}$ the Lie subalgebra of W_{2n} generated by the $\frac{\partial}{\partial z_i}$ for $i = 1,\ldots,n$. Given a structure of rational functions associated to such a covering, we obtain that the space of moduli of $\{\omega_U\}_{U\in\mathcal{U}}$ is isomorphic to the moduli space of $W_{2n}|_{hol}$-valued differential forms ω. To such a structure we may assign as in [35] characteristic classes by considering $H^*(|C^*_{cont}(C(\mathcal{U},\mathcal{F}))|)$. The cosimplicial rational function structure is defined such that by inserting p-times $\chi_{U_{i_0}\cap\ldots\cap U_{i_q}}$ into each $c \in C^p_{cont}(\prod_{i_0<\ldots<i_q}\mathcal{F}(U_{i_0}\cap\ldots\cap U_{i_q}))$, one associates an element χ of the generalized Cech-de Rham complex associated to the covering $\mathcal{U}$ on M. By the standard reasoning this χ provide a well-defined cohomology class $[\chi]$, the characteristic class associated to the cosimplicial $\mathcal{F}$-structure.

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