Obstacle problem, Euler system and turbulence

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Obstacle problem

Fluid domain and obstacle

$$Q = R^d \setminus B, \ d = 2, 3$$

$$B \text{ compact, convex}$$

Navier–Stokes system

$$\boxed{\begin{gathered}\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0 \\ \partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}) \\ p(\varrho) \approx a\varrho^\gamma, \ \gamma > 1, \ \mathbb{S} = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{d} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) + \lambda \mathrm{div}_x \mathbf{u} \mathbb{I}, \ \mu > 0, \lambda \geq 0\end{gathered}}$$

Boundary and far field conditions

$$\mathbf{u}|_{\partial Q} = 0, \ \varrho \to \varrho_\infty, \ \mathbf{u} \to \mathbf{u}_\infty \text{ as } |x| \to \infty$$

High Reynolds number (vanishing viscosity) limit

Vanishing viscosity

$$\varepsilon_n \searrow 0,\ \ \mu_n = \varepsilon_n \mu, \mu > 0,\ \ \lambda_n = \varepsilon_n \lambda, \lambda \geq 0$$

Questions

- Identify the limit of the corresponding solutions $(\varrho_n, \mathbf{u}_n)$ as $n \to \infty$ in the fluid domain Q
- **Yakhot and Orszak [1986]:** *"The effect of the boundary in the turbulence regime can be modeled in a statistically equivalent way by fluid equations driven by stochastic forcing"*

  Clarify the meaning of "statistically equivalent way"

  Is the (compressible) Euler system driven by a general cylindrical white noise force adequate to describe the limit of $(\varrho_n, \mathbf{u}_n)$?

Bounded energy solutions

(Relative) energy

$$E\left(\varrho,\mathbf{u}\,\middle|\,\varrho_\infty,\mathbf{u}_\infty\right) = \frac{1}{2}\varrho|\mathbf{u}-\mathbf{u}_\infty|^2 + P(\varrho) - P'(\varrho_\infty)(\varrho-\varrho_\infty) - P(\varrho_\infty)$$

$$P(\varrho) = \frac{a}{\gamma-1}\varrho^\gamma,\ \mathbf{u}_\infty = 0 \text{ for } |x|<R_1,\ \mathbf{u}_\infty = \mathbf{u}_\infty \text{ for } |x|>R_2$$

Energy inequality

$$\boxed{\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_Q E\left(\varrho,\mathbf{u}\middle|\varrho_\infty,\mathbf{u}_\infty\right)\ \mathrm{d}x + \int_Q \mathbb{S}(\nabla_x\mathbf{u}):\nabla_x\mathbf{u}\ \mathrm{d}x\\ &\leq -\int_Q\Big(\varrho\mathbf{u}\otimes\mathbf{u}+p(\varrho)\mathbb{I}\Big):\nabla_x\mathbf{u}_\infty\ \mathrm{d}x + \frac{1}{2}\int_Q\varrho\mathbf{u}\cdot\nabla_x|\mathbf{u}_\infty|^2\ \mathrm{d}x\\ &+\int_Q\mathbb{S}(\nabla_x\mathbf{u}):\nabla_x\mathbf{u}_\infty\ \mathrm{d}x.\end{aligned}}$$

Statistical limit

Energy bounds

$$\mathbf{m} \equiv \varrho \mathbf{u}$$

$$\frac{1}{N}\sum_{n=1}^{N}\left[\sup_{0\leq\tau\leq T}\int_Q E\left(\varrho_n, \mathbf{m}_n \Big| \varrho_\infty, \mathbf{u}_\infty\right)(\tau,\cdot)\ \mathrm{d}x + \varepsilon_n \int_0^T\int_Q \mathbb{S}(\nabla_x \mathbf{u}_n) : \nabla_x \mathbf{u}_n\ \mathrm{d}x\mathrm{d}t\right] \leq \overline{\mathcal{E}}$$

uniformly for $N \to \infty$

Trajectory space

$$(\varrho_n, \mathbf{m}_n) \in \mathcal{T} \equiv C_{\mathrm{weak}}([0,T]; L^\gamma_{\mathrm{loc}}(Q) \times L^{\frac{2\gamma}{\gamma+1}}_{\mathrm{loc}}(Q; R^d))$$

Statistical limit

$$\mathcal{V}_N = \frac{1}{N}\sum_{n=1}^{N}\delta_{(\varrho_n,\mathbf{m}_n)},\ \mathbf{m}_n = \varrho_n\mathbf{u}_n$$

Prokhorov theorem $\Rightarrow$ $\mathcal{V}_N \to \mathcal{V}$ narrowly in $\mathfrak{P}[\mathcal{T}]$

$(\varrho, \mathbf{m}) \approx \mathcal{V}$ a random process with paths in $\mathcal{T}$

Limit problem

Statistical dissipative solutions to the Euler system

$$\partial_t \varrho + \mathrm{div}_x \mathbf{m} = 0$$

$$\partial_t \mathbf{m} + \mathrm{div}_x \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right) + \nabla_x p(\varrho) = -\mathrm{div}_x \mathfrak{R}$$

$\mathcal{V}$ a.s.

Reynolds stress

$$\mathfrak{R} \in L^\infty_{\mathrm{weak}-(*)}(0, T; \mathcal{M}^+(Q; R^{d \times d}_{\mathrm{sym}}))$$

$$\mathfrak{R} : (\xi \otimes \xi) \geq 0, \ \xi \in R^d$$

$$\mathbb{E} \left[\int_0^T \psi \int_Q \varphi \ \mathrm{d} \ \mathrm{trace}[\mathfrak{R}] \mathrm{d}t \right] \leq c \overline{\mathcal{E}} \| \psi \|_{L^1(0,T)} \| \varphi \|_{BC(Q)}$$

Reynolds stress

Skorokhod–Jakubowski representation theorem

$$\varrho_N \approx \widetilde{\varrho}_N,\ \mathbf{m}_N \approx \widetilde{\mathbf{m}}_N \text{ (equivalence in law)}$$

a.s. weak convergence

$$(\widetilde{\varrho}_N, \widetilde{\mathbf{m}}_N) \to (\varrho, \mathbf{m}) \text{ in } C_{\rm weak}([0,T]; L^\gamma_{\rm loc}(Q) \times L^{\frac{2\gamma}{\gamma+1}}_{\rm loc}(Q; R^d))$$

$$\frac{\widetilde{\mathbf{m}}_N \otimes \widetilde{\mathbf{m}}_N}{\widetilde{\varrho}_N} + p(\varrho_N)\mathbb{I} \to \overline{\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} + p(\varrho)\mathbb{I}}$$

$$\text{weakly-(*) in } L^\infty_{\rm weak-(*)}(0,T; \mathcal{M}(Q; R^{d\times d}_{\rm sym}))$$

Reynolds stress

$$\mathfrak{R} \equiv \overline{\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} + p(\varrho)} - \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} + p(\varrho)\mathbb{I} \right)$$

$$\text{convexity of } (\varrho, \mathbf{m}) \mapsto \left(\frac{|\mathbf{m}\cdot\xi|^2}{\varrho} + p(\varrho)|\xi|^2 \right) \ \Rightarrow \ \mathfrak{R} : (\xi \otimes \xi) \geq 0$$

Stochastic Euler system

Euler system with stochastic forcing

$$\mathrm{d}\widetilde{\varrho} + \mathrm{div}_x \widetilde{\mathbf{m}} \mathrm{d}t = 0$$

$$\mathrm{d}\widetilde{\mathbf{m}} + \mathrm{div}_x \left(\frac{\widetilde{\mathbf{m}} \otimes \widetilde{\mathbf{m}}}{\widetilde{\varrho}} \right) \mathrm{d}t + \nabla_x p(\widetilde{\varrho}) \mathrm{d}t = \mathbf{F} \mathrm{d}W$$

$W = (W_k)_{k \geq 1}$ cylindrical Wiener process

$\mathbf{F} = (\mathbf{F}_k)_{k \geq 1}$ − diffusion coefficient

$$\mathbb{E} \left[\int_0^T \sum_{k \geq 1} \| \mathbf{F}_k \|^2_{W^{-\ell,2}(Q;R^d)} \mathrm{d}t \right] < \infty$$

we allow $\mathbf{F} = \mathbf{F}(\varrho, \mathbf{m})$

Statistical equivalence

statistical equivalence $\Leftrightarrow$ identity in expectation of some quantities

$(\varrho, \mathbf{m})$ statistically equivalent to $(\widetilde{\varrho}, \widetilde{\mathbf{m}})$

$$\Leftrightarrow$$

- **density and momentum**

$$\mathbb{E}\left[\int_D \varrho\right] = \mathbb{E}\left[\int_D \widetilde{\varrho}\right],\ \mathbb{E}\left[\int_D \mathbf{m}\right] = \mathbb{E}\left[\int_D \widetilde{\mathbf{m}}\right]$$

- **kinetic and internal energy**

$$\mathbb{E}\left[\int_D \frac{|\mathbf{m}|^2}{\varrho}\right] = \mathbb{E}\left[\int_D \frac{|\widetilde{\mathbf{m}}|^2}{\widetilde{\varrho}}\right],\ \mathbb{E}\left[\int_D p(\varrho)\right] = \mathbb{E}\left[\int_D p(\widetilde{\varrho})\right]$$

- **angular energy**

$$\mathbb{E}\left[\int_D \frac{1}{\varrho}(\mathbb{J}_{x_0} \cdot \mathbf{m}) \cdot \mathbf{m}\right],\ \mathbb{E}\left[\int_D \frac{1}{\widetilde{\varrho}}(\mathbb{J}_{x_0} \cdot \widetilde{\mathbf{m}}) \cdot \widetilde{\mathbf{m}}\right]$$

$$D \subset (0,T) \times Q,\ x_0 \in R^d,\ \mathbb{J}_{x_0}(x) \equiv |x - x_0|^2 \mathbb{I} - (x - x_0) \otimes (x - x_0)$$

Results

Hypothesis:

$(\varrho, \mathbf{m})$ statistically equivalent to a solution of the stochastic Euler system $(\widetilde{\varrho}, \widetilde{\mathbf{m}})$

Conclusion:

- **Noise inactive**
  $\mathfrak{R} = 0$, $(\varrho, \mathbf{m})$ is a statistical solution to a deterministic Euler system
- **S-convergence (up to a subsequence) to the limit system**

$$\frac{1}{N}\sum_{n=1}^{N} b(\varrho_n, \mathbf{m}_n) \to \mathbb{E}\left[b(\varrho, \mathbf{m})\right] \text{ strongly in } L^1_{\rm loc}((0,T)\times Q)$$

  for any $b \in C_c(R^{d+1})$, $\varphi \in C_c^\infty((0,T)\times Q)$
- **Conditional statistical convergence**

$$\text{barycenter } (\overline{\varrho}, \overline{\mathbf{m}}) \equiv \mathbb{E}\left[(\varrho, \mathbf{m})\right] \text{ solves the Euler system}$$

$$\Rightarrow$$

$$\frac{1}{N}\#\left\{ n \leq N \Big| \|\varrho_n - \overline{\varrho}\|_{L^\gamma(K)} + \|\mathbf{m}_n - \overline{\mathbf{m}}\|_{L^{\frac{2\gamma}{\gamma+1}}(K;R^d)} > \varepsilon \right\} \to 0 \text{ as } N \to \infty$$

  for any $\varepsilon > 0$, and any compact $K \subset [0,T]\times Q$

Main ideas

- Use statistical equivalence of $(\varrho, \mathbf{m})$ to $(\widetilde{\varrho}, \widetilde{\mathbf{m}})$ and the fact that the Itô integral is a martingale to obtain the identity

$$\mathbb{E}\left[\mathrm{div}_x \mathfrak{R}\right] = \mathbb{E}\left[\mathrm{div}_x \left(\frac{\widetilde{\mathbf{m}} \otimes \widetilde{\mathbf{m}}}{\widetilde{\varrho}} - \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right)\right] \tag{1}$$

  in $\mathcal{D}'((0,T) \times Q)$

- Show that if Q is exterior to a ball and $(\varrho, \mathbf{m})$ statistically equivalent to $(\widetilde{\varrho}, \widetilde{\mathbf{m}})$, then

$$\mathfrak{R} = 0 \text{ a.s.}$$

  Hint: Use test functions of the form

$$\phi_L(x) = \chi\left(\frac{|x|}{L}\right) \nabla_x F(|x|^2),\ \phi \in C^1_c(Q),\ L \geq 1$$

$$\chi \in C^\infty_c[0,\infty),\ \chi(Z) = 1 \text{ for } Z \leq 1,\ \chi(Z) = 0 \text{ for } Z \geq 2$$

$$F \text{ convex},\ F(Z) = 0 \text{ for } 0 \leq Z \leq R^2,\ 0 < F'(Z) \leq \overline{F} \text{ for } R^2 < Z < R^2 + 1$$

$$F'(Z) = \overline{F} \text{ if } Z \geq R^2 + 1,$$

  and let $L \to \infty$ to conclude $\mathbb{E}\left[\int_0^T \int_Q \mathrm{tr}[\mathfrak{R}]\right] = 0$

- Extend the result to $Q = R^d \setminus B$, B compact, convex.

Stratonovich drift

Stochastic Euler system

$$\mathrm{d}\widetilde{\varrho} + \mathrm{div}_x \widetilde{\mathbf{m}} \mathrm{d}t = 0$$

$$\mathrm{d}\widetilde{\mathbf{m}} + \mathrm{div}_x \left(\frac{\widetilde{\mathbf{m}} \otimes \widetilde{\mathbf{m}}}{\widetilde{\varrho}} \right) \mathrm{d}t + \nabla_x p(\widetilde{\varrho}) \mathrm{d}t = \boxed{(\sigma \cdot \nabla_x) \widetilde{\mathbf{m}} \circ \mathrm{d}W_1} + \mathbf{F}\, \mathrm{d}W_2$$

Additional hypotheses

- $Q = R^d$
- If $d = 2$, we need $\varrho_\infty = 0$; if $d = 3$, we need $\varrho_\infty = 0$, $\mathbf{u}_\infty = 0$, and $1 < \gamma \leq 3$

Similar type of noise used recently by Flandoli et al to produce a regularizig effect in the incompressible Navier–Stokes system

Conclusion

- Stochastically driven Euler system irrelevant in the description of compressible turbulence (slightly extrapolated statement)

Possible scenarios:

- **Oscillatory limit.** The sequence $(\varrho_n, \mathbf{m}_n)$ generates a Young measure. Its barycenter (weak limit of $(\varrho_n, \mathbf{m}_n)$) is not a weak solution of the Euler system. Statistically, however, the limit is a single object. This scenario is compatible with the hypothesis that the limit is independent of the choice of $\varepsilon_n \searrow 0 \Rightarrow$ computable numerically.
- **Statistical limit.** The limit is a statistical solution of the Euler system. In agreement with Kolmogorov hypothesis concerning turbulent flow advocated in the compressible setting by Chen and Glimm. This scenario is not compatible with the hypothesis that the limit is independent of $\varepsilon_n \searrow 0$ ($\Rightarrow$ numerically problematic) unless the limit is a monoatomic measure in which case the convergence must be strong.