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THE CZECH ACADEMY OF SCIENCES

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Preprint No. 17-2021

PRAHA 2021

ASYMPTOTIC LIFTING FOR COMPLETELY POSITIVE MAPS

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Abstract. Let A and B be C^*-algebras, A separable and I an ideal in B. We show that for any completely positive contractive linear map $\psi\colon A \to B/I$ there is a continuous family $\Theta_t\colon A \to B$, for $t \in [1,\infty)$, of lifts of ψ that are asymptotically linear, asymptotically completely positive and asymptotically contractive. If ψ is orthogonality preserving, then Θ_t can be chosen to have this property asymptotically. If A and B carry continuous actions of a second countable locally compact group G such that I is G-invariant and ψ is equivariant, we show that the family Θ_t can be chosen to be asymptotically equivariant. If a linear completely positive lift for ψ exists, we can arrange that Θ_t is linear and completely positive for all $t \in [1,\infty)$. In the equivariant setting, if A, B and ψ are unital, we show that asymptotically linear unital lifts are only guaranteed to exist if G is amenable. This leads to a new characterization of amenability in terms of the existence of asymptotically equivariant unital sections for quotient maps.

1. Introduction

Lifts and sections for maps is a recurring theme in many mathematical fields. In the theory of C^*-algebras and its applications, one of the most prominent examples of this is the question about sections for a surjective $*$-homomorphism between C^*-algebras; for example in the work of Choi and Effros in [CE] and in the theory of extensions of C^*-algebras, initiated in [BDF] and [Ar]. In view of the recent increased activity in research on C^*-dynamical systems, it seems inevitable that questions about lifts and sections will become more important than they already are in this category also. Addressing questions of this nature is the main motivation for the present work, while concrete applications will be given elsewhere; see [FGT].

Let G be a second countable locally compact group. We shall work with pairs (A,α) where A is a C^*-algebra and $\alpha\colon G \to \mathrm{Aut}(A)$ is a

Version: March 16, 2021.

The first named author was supported by GAČR project 19-05271Y, RVO:67985840. The second named author was partially supported by the DFG through an *eigene Stelle* and under Germany's Excellence Strategy "Mathematics Münster: Dynamics–Geometry–Structure". The third named author was supported by the DFF-Research Project 2 'Automorphisms and Invariants of Operator Algebras', no. 7014-00145B.

homomorphism from G into the group $\mathrm{Aut}(A)$ of automorphisms of A (also called an action). When α is continuous in the sense that for all $a \in A$, the assignment $g \mapsto \alpha_g(a)$ is continuous as a map $G \to A$, we say that (A, α) is a *G-algebra*. In many cases it is clear from the context what α is and we shall then use the notation $g \cdot a = \alpha_g(a)$ for $g \in G$ and $a \in A$, since it often clarifies the statements.

We are concerned with short exact sequences of G-algebras

$$0 \longrightarrow (I, \gamma) \xrightarrow{\iota} (A, \alpha) \xrightarrow{q} (B, \beta) \longrightarrow 0,$$

where $\iota \colon I \to A$ and $q \colon A \to B$ are G-equivariant $*$-homomorphisms such that ι is injective, q is surjective and $\ker(q) = \mathrm{im}(\iota)$. Given another G-algebra (S, δ) and a map $\psi \colon S \to B$, which is not necessarily a $*$-homomorphism, a *lift* of ψ is a map $\psi' \colon S \to A$ such that the diagram

$$\begin{tikzcd} & A \arrow[d, "q"] \\ S \arrow[ur, dashed, "\psi'"] \arrow[r, "\psi"'] & B \end{tikzcd}$$

commutes. The most important case is when ψ is linear and completely positive, and this is also the case we shall consider. In a setting where the group actions are absent (or trivial), the question about existence of a lift which is also linear and completely positive was considered by Choi and Effros in [CE]. They showed that a completely positive linear lift exists when the map ψ is nuclear, which is in particular the case if either of A, B or S is nuclear. The main impetus for their work was the theory of extensions of C^*-algebra, which was also the main motivation for many subsequent examples, beginning with the example of J. Anderson, [A], showing that linear completely positive lifts do not always exist.

The present work was motivated by a study of G-algebras which made two of the authors suspect that if ψ is G-equivariant and if a completely positive linear lift exists, then there also exists an *almost equivariant*, completely positive linear lift ψ', in the sense that given $\varepsilon > 0$ and compact subsets $K \subseteq G$ and $F \subseteq S$, we have

$$\max_{g \in K} \|\psi'(g \cdot s) - g \cdot \psi'(s)\| \leq \varepsilon$$

for all $s \in F$. That this is indeed the case when S is separable is described with details, and the precise statement is given in Theorem 3.4 below. In the formulation of the result we are inspired by the E-theory of Connes and Higson, [CH], and this is also the case for the formulation of the main result of the paper, Theorem 3.3, which deals with the same setting except that we neither assume that ψ is equivariant nor that there is a linear completely positive contractive lift. We show that there is a continuous family $\Theta = (\Theta_t)_{t \in [1,\infty)} \colon S \to A$

of lifts of ψ, which is asymptotically linear, asymptotically completely positive and asymptotically (G, ψ)-equivariant in the sense that

$$\lim_{t \to \infty} \|g \cdot \Theta_t(s) - \Theta_t(h \cdot s)\| = \|g \cdot \psi(s) - \psi(h \cdot s)\|$$

for all $s \in S$, uniformly for g, h in compact subsets of G. In particular, Θ is asymptotically equivariant when ψ is equivariant. We refer to Section 3 for more precise statements.

Following its introduction by Winter and Zacharias in [WZ], the notion of completely positive linear maps of order zero has become important in the classification of simple C^*-algebras, and we make therefore the effort to show that if the given completely positive linear map ψ is of order zero, then the family Θ can be chosen to have this property asymptotically. We ensure this by arranging that the maps Θ_t can always be constructed such that

$$\lim_{t \to \infty} \|\Theta_t(s)\Theta_t(s')\| = \|\psi(s)\psi(s')\|$$

for all $s, s' \in S$.

When S, A and B are unital and ψ is also unital, it is natural to look for asymptotically (G, ψ)-equivariant lifts that are also unital. These, however, will generally not exist even with the relaxed algebraic requirements described above. We show that the existence of a unital asymptotically linear, asymptotically contractive and asymptotically (G, ψ)-equivariant lift is only guaranteed when G is amenable, leading to yet another characterization of amenability for second countable locally compact groups. See Theorem 4.1.

Even without the G-actions, our main result provides new information. As pointed out above, there are by now a wealth of examples where the quotient map in an extension of separable C^*-algebras does not admit a completely positive and linear section. It follows from Theorem 3.3 below that there always exists a family of sections which has these properties asymptotically, and at the same time is asymptotically orthogonality preserving. This result should be compared with the examples in [MT2] showing that there are separable C^*-algebras with extensions by the compact operators such that no other extension can be added to result in an extension for which the quotient map admits a family of sections that constitute an asymptotic $*$-homomorphism. Thus there are certainly limits to which properties of sections one can hope to get by relaxing algebraic conditions to asymptotic ones, and although our results may not be optimal, they appear to come close.

2. Lifting from Calkin algebras

In this section we fix a locally compact second countable group G, a G-algebra (U, δ) such that U is unital and separable, and a G-invariant state $\chi: U \to \mathbb{C}$; that is, a state on U such that $\chi \circ \delta_g = \chi$ for all $g \in G$. We also fix a G-algebra (I, γ) such that I is σ-unital.

We shall deal with group actions that are not necessarily continuous and for this we use the following terminology. Let $\delta\colon G \to \mathrm{Aut}(D)$ be a not necessarily continuous action of G on a C^*-algebra D. We let

$$D_\delta = \{d \in D\colon \text{the map } G \to D \text{ given by } g \mapsto \delta_g(d) \text{ is continuous}\}$$

denote the *continuous part* of D with respect to δ. As an example the given action γ on G extends to an action $\tilde{\gamma}$ of G on the multiplier algebra $M(I)$ of I defined such that $\tilde{\gamma}_g(m)x = \gamma_g(m\gamma_{g^{-1}}(x))$ for all $x \in I$. The continuous part of $M(I)$ with respect to this action is $M(I)_{\tilde{\gamma}}$. Denote by $Q(I) = M(I)/I$ the associated (generalized) Calkin algebra and by $q_I\colon M(I) \to Q(I)$ the quotient map. Since I is $\tilde{\gamma}$-invariant, there is an action $\overline{\gamma}\colon G \to \mathrm{Aut}(Q(I))$ such that $\overline{\gamma}_g \circ q_I = q_I \circ \tilde{\gamma}_g$ for all $g \in G$. Then $(M(I)_{\tilde{\gamma}}, \tilde{\gamma})$ and $(Q(I)_{\overline{\gamma}}, \overline{\gamma})$ are G-algebras, and

$$q_I\colon (M(I)_{\tilde{\gamma}}, \tilde{\gamma}) \to (Q(I)_{\overline{\gamma}}, \overline{\gamma})$$

is an equivariant $*$-homomorphism. As the main step towards more general cases, we consider first a unital linear completely positive map $\psi\colon U \to Q(I)_{\overline{\gamma}}$, and we seek to construct a lift ψ' in the diagram

$$\begin{tikzcd} & M(I)_{\tilde{\gamma}} \arrow[d, "q_I"] \\ U \arrow[ur, dashed, "\psi'"] \arrow[r, "\psi"'] & Q(I)_{\overline{\gamma}} \end{tikzcd}$$

such that ψ' has properties as close as possible to those of ψ.

While the focus in the following is on the general case, we want to simultaneously handle the case where there exists a unital completely positive linear lift ψ', in which case the problem reduces to that of making adjustments to ψ' so that it respects the G-actions as much as possible. As will become clear, the arguments necessary to handle the latter situation are much simpler than the ones we present to handle the general case, but they also lead to stronger conclusions. We treat both situations in parallel, pointing out the main differences in remarks.

Our starting point is the following variation of Lemma 3.1 in [K]. We include a proof because it is a key lemma and we shall rely on properties of the construction that are not explicit in [K].

Lemma 2.1. *(Kasparov,* [K]*) Let $0 \le d \le 1$ be a strictly positive element in I. For every separable C^*-subalgebra $M_0 \subseteq M(I)_{\tilde{\gamma}}$, there exists a countable approximate unit $(x_n)_{n=1}^\infty$ for I contained in $C^*(d)$ with the following properties:*

(a) $0 \le x_n \le 1$ *for all n;*
(b) $x_{n+1}x_n = x_n$ *for all n;*
(c) $\lim\limits_{n\to\infty} \|x_n b - b x_n\| = 0$ *for all $b \in M_0$;*
(d) $\lim\limits_{n\to\infty} \max\limits_{g\in K} \|\gamma_g(x_n) - x_n\| = 0$ *for all compact subsets $K \subseteq G$.*

Proof. For each $n \in \mathbb{N}$, let $g_n\colon [0,1] \to [0,1]$ be the continuous function which vanishes between 0 and $\frac{1}{2}\left(\frac{1}{n}+\frac{1}{n+1}\right)$; is linear between $\frac{1}{2}\left(\frac{1}{n}+\frac{1}{n+1}\right)$ and $\frac{1}{n}$, and is constant equal to 1 for $t \geq \frac{1}{n}$. Using functional calculus, we set $a_n = g_n(d) \in I$. Then $0 \leq a_n \leq 1, a_{n+1}a_n = a_n$, and $\lim_{n\to\infty} a_n d = d$. In particular, $(a_n)_{n\in\mathbb{N}}$ is an approximate unit in I since $\overline{dI} = I$. Since G is σ-compact, the set

$$\{\gamma_g(m)\colon g \in G, m \in M_0 \cup \{d\}\}$$

generates a separable γ-invariant C^*-subalgebra $M_{00} \subseteq M(I)_{\bar{\gamma}}$ which contains d and M_0. Let $(F_n)_{n=1}^{\infty}$ be an increasing sequence of finite subsets with dense union in M_{00} and let $(V_n)_{n\in\mathbb{N}}$ be an increasing sequence of open subsets of G with compact closures such that $\bigcup_{n\in\mathbb{N}} V_n = G$. For $N \in \mathbb{N}$, let X_N denote the convex hull of $\{a_k\colon k \geq N\}$.

We claim that given $n, N \in \mathbb{N}$ and $\varepsilon > 0$, there is $x \in X_N$ such that $\|\gamma_g(x) - x\| \leq \varepsilon$ for all $g \in \overline{V_n}$ and $\|xb - bx\| \leq \varepsilon$ for all $b \in F_n$. To establish the claim, set $I_{00} = I \cap M_{00}$. For $x \in X_N$, let $f_x \in C(\overline{V_n}, I_{00})$ and $h_x\colon F_n \to I_{00}$ be given by

$$f_x(g) = \gamma_g(x) - x \quad \text{and} \quad h_x(b) = xb - bx$$

for $g \in \overline{V_n}$ and $b \in F_n$. Then $\Omega = \{(f_x, h_x)\colon x \in X_N\}$ is a convex subset of the C^*-algebra $C(\overline{V_n}, I_{00}) \oplus C(F_n, I_{00})$. Assuming that the claim is not true, we can find $n, N \in \mathbb{N}$ such that the norm-closure of Ω does not contain 0. By Hahn-Banach's separation theorem, there is then a $\delta > 0$ and elements $\Phi_1 \in C(\overline{V_n}, I_{00})^*$ and $\Phi_2 \in C(F_n, I_{00})^*$ such that

$$|\Phi_1(f_x) + \Phi_2(h_x)| \geq \delta$$

for all $x \in X_N$. Write $\Phi_1 = \omega_1 - \omega_2 + i(\omega_3 - \omega_4)$, where the ω_j's are positive linear functionals on $C(\overline{V_n}, I_{00})$. For each $k \geq N$ define $A_k, B_k \in C\left(\overline{V_n}, I_{00}\right)$ by $A_k(g) = a_k$ and $B_k(g) = \gamma_g(a_k)$ for $g \in \overline{V_n}$. Then $(A_k)_{k\in\mathbb{N}}$ and $(B_k)_{k\in\mathbb{N}}$ are both approximate units in $C(\overline{V_n}, I_{00})$ and hence $\|\omega_j\| = \lim_{k\to\infty} \omega_j(B_k) = \lim_{k\to\infty} \omega_j(A_k)$ for all $j \in \mathbb{N}$. This implies that $\lim_{k\to\infty} \Phi_1(B_k - A_k) = 0$. Since $B_k - A_k = f_{a_k}$ it follows that there is an $L \geq N$ such that

$$|\Phi_2(h_x)| \geq \frac{\delta}{2} \tag{2.1}$$

for all $x \in X_L$. Let $C_x \in C(F_n, I_{00})$ and $B \in C(F_n, M_{00})$ be the elements defined such that $C_x(b) = x$ and $B(b) = b$ for $b \in F_n$. Then $(C_x)_{x\in X_L}$ is a convex approximate unit in $C(F_n, I_{00})$ which is an ideal in $C(F_n, M_{00})$. Since $h_x = C_xB - BC_x$, the inequality (2.1) contradicts the lemma on page 330-331 in [Ar]. This proves the claim.

It is now straightforward to use the claim to construct an increasing sequence $(k_n)_{n\in\mathbb{N}}$ in $\mathbb{N}$, and elements x_n is the convex hull of

$$\{a_j\colon\ k_n \leq j \leq k_{n+1} - 1\}$$

such that

- $\|x_n b - b x_n\| \leq \frac{1}{n}$ for all $b \in F_n$;
- $\|\gamma_g(x_n) - x_n\| \leq \frac{1}{n}$ for all $g \in V_n$, and
- $\|x_n d - d\| \leq \frac{1}{n}$.

The sequence $\{x_n\}_{n\in\mathbb{N}}$ will then have the desired properties. $\square$

The following lemma is a direct consequence of work by L. Brown, [Br].

Lemma 2.2.

$$0 \longrightarrow (I,\gamma) \longrightarrow (M(I)_{\tilde{\gamma}}, \tilde{\gamma}) \xrightarrow{q_I} (Q(I)_{\overline{\gamma}}, \overline{\gamma}) \longrightarrow 0 \ ,$$

is a short exact sequence of G-algebras.

Proof. The only non-trivial fact is that $Q(I)_{\overline{\gamma}} \subseteq q_I\left(M(I)_{\tilde{\gamma}}\right)$, which follows from Theorem 2 in [Br] since that theorem implies that the restriction of $\tilde{\gamma}$ to $q_I^{-1}\left(Q(I)_{\overline{\gamma}}\right)$ is continuous. $\square$

Lemma 2.3. *Let $d \in I$ be a strictly positive element. Let $(Z_n)_{n\in\mathbb{N}}$ be an increasing sequence of compact subsets of $M(I)_{\tilde{\gamma}}$, and let $(V_n)_{n\in\mathbb{N}}$ be an increasing sequence of open subsets of G with compact closures. There is an approximate unit $(y_n)_{n=1}^{\infty}$ for I contained in $C^*(d)$, such that, when we set*

$$\Delta_0 = \sqrt{y_1} \text{ and } \Delta_n = \sqrt{y_{n+1} - y_n},$$

for $n \geq 1$, the following properties are satisfied:

(i) $0 \leq y_n = y_{n+1} y_n \leq y_{n+1} \leq 1$ *for all* $n \in \mathbb{N}$,
(ii) $\lim\limits_{n\to\infty} \max\limits_{g\in K} \|\gamma_g(y_n) - y_n\| = 0$ *for all compact subsets* $K \subseteq G$,
(iii) $\|z(1 - y_n)\| \leq \|q_I(z)\| + \frac{1}{n}$ *for all* $z \in Z_n$,
(iv) $\Delta_{n+1} y_n = 0$,
(v) $\|\Delta_n z - z\Delta_n\| \leq 2^{-n}$ *for all* $z \in Z_n$,
(vi) $\|\gamma_g(\Delta_n) - \Delta_n\| \leq 2^{-n}$ *for all* $g \in V_n$,

for all $n \geq 1$.

Proof. Let M_0 be the C^*-subalgebra of $M(I)_{\tilde{\gamma}}$ generated by $\bigcup_{n=1}^{\infty} Z_n$, and let $(x_n)_{n=1}^{\infty}$ be an approximate unit in I contained in $C^*(d)$ satisfying properties (a)-(d) in Lemma 2.1. It is easy to see, by approximating the square root function on $[0,1]$ by polynomials, that for every $\varepsilon > 0$ there is $\delta > 0$ such that when $a \in I$ is a positive contraction satisfying $\|\gamma_g(a) - a\| \leq \delta$, then $\|\gamma_g(\sqrt{a}) - \sqrt{a}\| \leq \varepsilon$. Combined with the lemma on page 332 in [Ar], it follows that for each $n \in \mathbb{N}$ there is a $\delta_n > 0$ such that whenever $a \in I$ is a positive contraction, then

$$\left.\begin{matrix} \|\gamma_g(a) - a\| \leq \delta_n \ \forall g \in V_n \\ \|az - za\| \leq \delta_n \ \forall z \in Z_n \end{matrix}\right\} \Rightarrow \begin{cases} \|\gamma_g(\sqrt{a}) - \sqrt{a}\| \leq 2^{-n} \ \forall g \in V_n \\ \|\sqrt{a}z - z\sqrt{a}\| \leq 2^{-n} \ \forall z \in Z_n \ . \end{cases}$$

We extract therefore a subsequence $(y_n)_{n\in\mathbb{N}}$ from $(x_n)_{n\in\mathbb{N}}$ such that

$$\|\gamma_g(y_{n+1} - y_n) - (y_{n+1} - y_n)\| \leq \delta_n$$

for all $g \in V_n$ and $\|(y_{n+1} - y_n)z - z(y_{n+1} - y_n)\| \leq \delta_n$ for all $z \in Z_n$. Since $\lim_{n\to\infty} \|m(1 - x_n)\| = \|q_I(m)\|$ for all $m \in M(I)$, we can also arrange that (iii) holds. Then $(y_n)_{n=1}^{\infty}$ will have all the stated properties. □

Lemma 2.4. *Adopt the assumptions from Lemma 2.3, and let $(y_n)_{n=1}^{\infty}$ and $(\Delta_n)_{n=0}^{\infty}$ be as in its conclusion. Let $(m_n)_{n=0}^{\infty}$ be a uniformly bounded sequence in $M(I)$. Then the sum*

$$\sum_{n=0}^{\infty} \Delta_n m_n \Delta_n$$

converges in the strict topology of $M(I)$ and for $k \in \mathbb{N}$ we have

$$\left\| \sum_{n=k}^{\infty} \Delta_n m_n \Delta_n \right\| \leq \sup_{n\geq k} \|m_n\|.$$

Proof. See Lemma 3.1 in [MT1]. □

We shall work with maps between C^*-algebras that may not respect anything of the algebraic structure. For this reason, we will be very explicit about the properties of the maps we consider.

Definition 2.5. Let $\theta\colon A \to B$ be a (not necessarily continuous or linear) map between C^*-algebras. We say that θ is

- *self-adjoint*, if $\theta(a^*) = \theta(a)^*$ for all $a \in A$.
- *unital*, if A and B are unital and $\theta(1) = 1$.

Let $n \in \mathbb{N}$. With a slight abuse of notation, we denote by $\theta \otimes \mathrm{id}_{M_n}$ the map $\theta \otimes \mathrm{id}_{M_n} : M_n(A) \to M_n(B)$ given by entry-wise application of θ.

Lemma 2.6. *Let D and E be C^*-algebras, D unital, and let $L\colon D \to Q(E)$ be a unital, continuous self-adjoint and linear map.*

(i) *For each finite set $F \subseteq D$ there is a unital continuous and self-adjoint map $L_F\colon D \to M(E)$ which is linear on* $\mathrm{Span}(F)$ *and satisfies $q_E \circ L_F = L$.*
(ii) *Let $F_1 \subseteq F_2$ be finite subsets of D, and let $L_1\colon D \to M(E)$ be a unital, continuous self-adjoint map which is linear on* $\mathrm{Span}(F_1)$ *and satisfies $q_E \circ L_1 = L$. There is a unital continuous and self-adjoint map $L_2\colon D \to M(E)$, which is linear on* $\mathrm{Span}(F_2)$ *and satisfies $q_E \circ L_2 = L$ and $L_2(x) = L_1(x)$ for $x \in$* $\mathrm{Span}(F_1)$.

Proof. (i) Set $F' = \{1\} \cup F \cup F^*$. By the Bartle-Graves selection theorem, [BG], there is a continuous map $s_0\colon Q(E) \to M(E)$ such that $q_E \circ s_0 = \mathrm{id}_{Q(E)}$. By exchanging s_0 with $s_0 - s_0(0)$ we may assume that $s_0(0) = 0$. Consider the Banach space quotient $Q(E)/L(\mathrm{Span}(F'))$ and the corresponding quotient map $\pi\colon Q(E) \to Q(E)/L(\mathrm{Span}(F'))$. The Bartle-Graves selection theorem gives a continuous map

$$s_1\colon Q(E)/L(\mathrm{Span}(F')) \to Q(E)$$

such that $\pi \circ s_1$ is the identity on $Q(E)/L(\mathrm{Span}(F'))$, and again we may assume that $s_1(0) = 0$. Since $\mathrm{Span}\, L(F')$ is finite dimensional and $1 \in L(F')$, there is a continuous and linear map

$$s_F \colon L(\mathrm{Span}(F')) \to M(E)$$

with $s_F(1) = 1$ such that $q_E \circ s_F = \mathrm{id}_{L(\mathrm{Span}(F'))}$. Since $x - s_1(\pi(x)) \in L(\mathrm{Span}(F'))$ for all $x \in Q(E)$, we can define $\theta \colon Q(E) \to M(E)$ by

$$\theta(x) = s_0\left(s_1(\pi(x))\right) + s_F(x - s_1(\pi(x)))\ .$$

Then θ is continuous, linear on $L(\mathrm{Span}(F'))$, and satisfies $\theta(1) = 1$ and $q_E \circ \theta = \mathrm{id}_{Q(E)}$. By replacing $\theta(x)$ with $\frac{1}{2}(\theta(x) + \theta(x^*)^*)$ for $x \in Q(E)$, we may also assume that θ is self-adjoint. Then $L_F = \theta \circ L$ has the desired properties.

(ii) Apply first (i) to get a continuous and self-adjoint unital map $L_2' \colon D \to M(E)$ which is linear on $\mathrm{Span}(F_2 \cup F_2^*)$ and satisfies $q_E \circ L_2' = L$. Since $\mathrm{Span}(F_1 \cup F_1^*)$ is finite-dimensional, there is a continuous linear projection $P' \colon D \to \mathrm{Span}(F_1 \cup F_1^*)$. Set

$$P(d) = \frac{1}{2}\left(P'(d) + P'(d^*)^*\right)$$

Then the map $L_2 \colon D \to Q(E)$ given by

$$L_2(d) = L_1(P(d)) + L_2'(d - P(d))$$

has the desired properties. □

Let $(F_n')_{n\in\mathbb{N}}$ be an increasing sequence of finite subsets of U with dense union in U, and assume that F_1' contains $\{0,1\}$. Let $\mathbb{Q}[i]$ denote the countable set of complex numbers whose real and imaginary parts are rational numbers, and let $(Q_n)_{n\in\mathbb{N}}$ be an increasing sequence of finite subsets of $\mathbb{Q}[i]$ with $\bigcup_{n=1}^{\infty} Q_n = \mathbb{Q}[i]$, such that $\{0,1\} \subseteq Q_1$. Since we assume that G is second countable, there is an increasing sequence $(G_n)_{n\in\mathbb{N}}$ of finite subsets of G, all containing the unit, such that their union $G^{(0)} = \bigcup_n G_n$ is dense in G. Set $F_1 = F_1'$ and define F_n for $n \geq 2$ recursively such that $F_n = F_n'' \cup F_n''^* \cup F_n' \cup F_n'^*$, where

$$F_n'' = \left\{ \sum_{j=1}^{n} g_j \cdot (q_j u_j) \colon g_j \in G_n, q_j \in Q_n, u_j \in F_{n-1} \right\}$$
$$\bigcup \{u - \chi(u) :\ u \in F_{n-1}\}\ .$$

Then $F_{n-1} \subseteq F_n = F_n^*$ when $n \geq 2$. Set $U^{(0)} = \bigcup_{n\in\mathbb{N}} F_n$. Then $U^{(0)}$ is dense in U,

$$\mathbb{Q}[i] - \mathrm{Span}(U^{(0)}) \subseteq U^{(0)}, \quad G^{(0)} \cdot U^{(0)} \subseteq U^{(0)} \tag{2.2}$$

and $u - \chi(u) \in U^{(0)}$ for all $u \in U^{(0)}$.

Now we consider the unital linear completely positive map $\psi \colon U \to Q(I)_{\overline{\gamma}}$. By (i) in Lemma 2.6, there is a unital continuous and self-adjoint map $\psi_1 \colon U \to M(I)$ such that $q_I \circ \psi_1 = \psi$ and ψ_1 is linear on Span(F_1). By repeated use of (ii) in Lemma 2.6, we get unital continuous self-adjoint maps $\psi_n \colon U \to M(I)$ satisfying the following for all $n \in \mathbb{N}$:

(i) $q_I \circ \psi_n = \psi$
(ii) ψ_n is linear on Span(F_n), and
(iii) $\psi_{n+1}(x) = \psi_n(x)$ for all $x \in F_n$.

It follows from (i) and Lemma 2.2 that

(iv) $\psi_n(U) \subseteq M(I)_{\tilde{\gamma}}$ for all $n \in \mathbb{N}$.

Remark 2.7. In the case where a unital completely positive linear lift ψ' of ψ is given, we take all the ψ_n's above to be equal to ψ'.

Let $e \in V_1 \subseteq V_2 \subseteq V_3 \subseteq \cdots$ be open subsets of G with compact closures $\overline{V_n}$ such that $\bigcup_n V_n = G$. Fix a strictly positive contraction $d \in I$. For each $n \in \mathbb{N}$ we let $Z_n \subseteq M(I)_{\tilde{\gamma}}$ be a compact set containing all elements of the form $m\left(\tilde{\gamma}_g(\psi_j(u)) - \psi_j(\delta_h(v))\right)$ where $g, h \in \overline{V_n}$, $m \in \{1, d\}$, $u, v \in F_n$, and $1 \leq j \leq n+1$, as well as the set

$$\bigcup_{j=1}^{n} \left(\psi_j(F_n)\psi_j(F_n) \cup \psi_j(F_n)\right) .$$

We fix from now on an approximate unit $(y_n)_{n\in\mathbb{N}}$ for I contained in $C^*(d)$ satisfying the conclusion of Lemma 2.3 for the sets $(V_n)_{n\in\mathbb{N}}$ and $(Z_n)_{n\in\mathbb{N}}$, and set $\Delta_0 = \sqrt{y_1}$ and $\Delta_n = \sqrt{y_{n+1} - y_n}$ for $n \geq 1$.

For each $n \in \mathbb{N} \cup \{0\}$ and $t \geq 1$, set

$$\psi_n^t(u) = \begin{cases} (t-k+1)\psi_k(u) + (k-t)\psi_{k-1}(u), & t \in [k-1, k], k \leq n, \\ (t-n)\chi(u) + (n+1-t)\psi_n(u), & t \in [n, n+1], \\ \chi(u), & t \geq n+1. \end{cases}$$

For $t \in [k-1, k]$, the element $\psi_n^t(u)$ belongs to the convex hull of $\{\chi(u), \psi_{k-1}(u), \psi_k(u)\}$, and hence

$$\left\|\psi_n^t(u)\right\| \leq \max\left\{|\chi(u)|, \|\psi_{k-1}(u)\|, \|\psi_k(u)\|\right\}$$

for all $n \in \mathbb{N}$. By Lemma 2.4, we can therefore define $\psi^t \colon U \to M(I)$, for $t \geq 1$, such that

$$\psi^t(u) = \sum_{n=0}^{\infty} \Delta_n \psi_n^t(u) \Delta_n \tag{2.3}$$

for all $u \in U$. For $k \in \mathbb{N}$, the map ψ^k takes the form

$$\psi^k(u) = \chi(u)y_k + \sum_{n=k}^{\infty} \Delta_n \psi_k(u) \Delta_n \tag{2.4}$$

for all $u \in U$. Moreover, for $t \in [k, k+1]$, the map ψ^t is a convex combination of ψ^{k-1} and ψ^k; specifically we have

$$\psi^t = (t-k+1)\psi^k + (k-t)\psi^{k-1}. \tag{2.5}$$

Remark 2.8. Following Remark 2.7, note that when a unital completely positive linear lift ψ' of ψ is given, each ψ^t will be linear and a completely positive contraction.

Lemma 2.9. *The family of self-adjoint maps $\psi^t \colon U \to M(I), t \in [1,\infty)$, has the following properties.*

1. *$\psi^t(1) = 1$ for all $t \geq 1$;*
2. *The assignment $t \mapsto \psi^t(u)$ is continuous for all $u \in U$;*
3. *ψ^t is continuous for all $t \geq 1$;*
4. *ψ^t is linear on $\mathrm{Span}(F_n)$ when $t \geq n+1$;*
5. *$q_I \circ \psi^t = \psi$ for all $t \geq 1$;*
6. *$\psi^t(U) \subseteq M(I)_{\tilde{\gamma}}$ for all $t \geq 1$.*
7. *Let $u, v \in U^{(0)}$, $g, h \in G^{(0)}$, and let $m \in \{1, d\}$. Then*
$$\limsup_{t\to\infty} \left\| m\left(g \cdot \psi^t(u) - \psi^t(h\cdot v)\right) \right\| \leq \max\left\{ |\chi(u-v)|, \|q_I(m)\left(g\cdot\psi(u) - \psi(h\cdot v)\right)\| \right\}.$$
8. *Let $u, v \in U^{(0)}$. Then*
$$\limsup_{t\to\infty} \left\| \psi^t(u-\chi(u))\psi^t(v-\chi(v)) \right\| \leq \|\psi(u-\chi(u))\psi(v-\chi(v))\|.$$
9. *For $j \in \mathbb{N}$, write $M_j(U^{(0)})$ for the subset of $M_j(U)$ of matrices with entries in $U^{(0)}$. For $u \in M_j(U^{(0)})$, we have*

$$\limsup_{t\to\infty} \left\|(\psi^t \otimes \mathrm{id}_{M_j})(u)\right\| \leq \max\left\{ \left\|(\chi\otimes\mathrm{id}_{M_j})(u)\right\|, \left\|(\psi\otimes\mathrm{id}_{M_j})(u)\right\| \right\}.$$

Proof. (1) This follows immediately from (2.3) since χ and the ψ_n's are all unital.

(2) This follows directly from (2.5).

(3) By (2.5), it suffices to show that ψ^k is continuous for each fixed $k \in \mathbb{N}$. Moreover, by (2.4) it suffices to show that the assignment $u \mapsto \sum_{n=k}^{\infty} \Delta_n \psi_k(u)\Delta_n$ is continuous. Let $u \in U$ and $\varepsilon > 0$ be given. Using continuity of ψ_k, find $\delta > 0$ such that $\|\psi_k(u) - \psi_k(v)\| \leq \varepsilon$ when $\|u - v\| \leq \delta$. It thus follows from Lemma 2.4 that

$$\left\| \sum_{n=k}^{\infty} \Delta_n(\psi_k(u) - \psi_k(v))\Delta_n \right\| \leq \varepsilon ,$$

when $\|u - v\| \leq \delta$, as desired.

(4) This follows from (2.5), since ψ_k, and thus also ψ^k by (2.4), is linear on $\mathrm{Span}(F_n)$ when $k \geq n$.

(5) By (2.5), it suffices to show that $q_I \circ \psi^k(u) = \psi(u)$ for all $k \in \mathbb{N}$. Since $U^{(0)}$ is dense in U and ψ^k is continuous by part (3), it suffices to show that $q_I \circ \psi^k(u) = \psi(u)$ for $u \in F_m$, with $m \in \mathbb{N}$. For $N \geq$

$\max\{k, m\}$, we have

$$\psi^k(u) = \sum_{n=0}^{N-1} \Delta_n \psi_n^k(u) \Delta_n + \sum_{n=N}^{\infty} \Delta_n \psi_m(u) \Delta_n \ . \tag{2.6}$$

It follows from (v) in Lemma 2.3 and the fact that $\|\Delta_n\| \leq 1$ that

$$\left\| \Delta_n^2 \psi_m(u) - \Delta_n \psi_m(u) \Delta_n \right\| \leq \|\Delta_n \psi_m(u) - \psi_m(u) \Delta_n\| \leq 2^{-n}$$

when $n \geq m$. Using that the series $\sum_{n=N}^{\infty} \Delta_n^2$ converges in the strict topology of $M(I)$ to $1 - y_N$ at the first step, we therefore have

$$\begin{aligned}
(1 - y_N)\psi_m(u) - \sum_{n=N}^{\infty} \Delta_n \psi_m(u) \Delta_n &= \sum_{n=N}^{\infty} \Delta_n^2 \psi_m(u) - \sum_{n=N}^{\infty} \Delta_n \psi_m(u) \Delta_n \\
&= \sum_{n=N}^{\infty} \left[\Delta_n^2 \psi_m(u) - \Delta_n \psi_m(u) \Delta_n \right] \ ,
\end{aligned}$$

where the last sum converges in norm by the previous norm estimate. In particular, since $\Delta_n^2 \psi_m(u) - \Delta_n \psi_m(u) \Delta_n$ belongs to I (because $\Delta_n \in I$), it follows that

$$(1 - y_N)\psi_m(u) - \sum_{n=N}^{\infty} \Delta_n \psi_m(u) \Delta_n \in I \ .$$

We combine the above with (2.6), writing $=_I$ for equality modulo I:

$$\begin{aligned}
\psi^k(u) - \psi_m(u) &= \psi^k(u) - (\psi_m(u) - y_N \psi_m(u)) - y_N \psi_m(u) \\
&=_I \psi^k(u) - \sum_{n=N}^{\infty} \Delta_n \psi_m(u) \Delta_n \\
&= \sum_{n=0}^{N-1} \Delta_n \psi_n^k(u) \Delta_n =_I 0 \ .
\end{aligned}$$

It follows that $q_I(\psi^k(u)) = q_I(\psi_m(u)) = \psi(u)$, as desired.

(6) This follows from part (5) and Lemma 2.2, since $\psi(U) \subseteq Q(I)_{\overline{\gamma}}$ by assumption.

(7) By (2.5), it suffices to show that

$$\begin{aligned}
&\limsup_{k \to \infty} \|m(g \cdot \psi^k(u) - \psi^k(h \cdot v))\| \\
&\qquad \leq \max \left\{ |\chi(u - v)|, \|q_I(m)(g \cdot \psi(u) - \psi(h \cdot v))\| \right\} \ .
\end{aligned} \tag{2.7}$$

Let $n \in \mathbb{N}$ such that $u, v \in F_n$ and $g, h \in V_n$, and fix $k \geq n + 1$. We write

$$g \cdot \psi^k(u) - \psi^k(h \cdot v) = a + b,$$

where

$$a = \sum_{j=0}^{k-1} [\gamma_g(\Delta_j)\chi(u)\gamma_g(\Delta_j) - \Delta_j \chi(v) \Delta_j]$$

and

$$b = \sum_{j=k}^{\infty} \left[\gamma_g(\Delta_j)\tilde{\gamma}_g\left(\psi_k(u)\right)\gamma_g(\Delta_j) - \Delta_j\psi_k(\delta_h(v))\Delta_j\right] .$$

Note that $a = \chi(u)\gamma_g\left(y_k\right) - \chi(v)y_k$ and hence

$$\|a - a'\| \leq |\chi(u)| \left\|\gamma_g\left(y_k\right) - y_k\right\| \tag{2.8}$$

where

$$a' = \sum_{j=0}^{k-1} \Delta_j(\chi(u) - \chi(v))\Delta_j .$$

Set

$$b' = \sum_{j=k}^{\infty} \Delta_j\left(\tilde{\gamma}_g(\psi_n(u)) - \psi_k(\delta_h(v)\right)\Delta_j .$$

Since

$$\begin{aligned}&\left\|\gamma_g(\Delta_j)\tilde{\gamma}_g\left(\psi_k(u)\right)\gamma_g(\Delta_j) - \Delta_j\tilde{\gamma}_g\left(\psi_k(u)\right)\Delta_j\right\|\\ &\leq 2\left\|\psi_k(u)\right\|\left\|\gamma_g(\Delta_j) - \Delta_j\right\|\end{aligned}$$

and $\psi_k(u) = \psi_n(u)$, we find that

$$\|b - b'\| \leq 2\left\|\psi_n(u)\right\| \sum_{j=k}^{\infty} \left\|\gamma_g(\Delta_j) - \Delta_j\right\| \leq 2\left\|\psi_n(u)\right\| \sum_{j=k}^{\infty} 2^{-j} , \tag{2.9}$$

thanks to property (vi) in Lemma 2.3. Property (iv) in Lemma 2.3 implies that

$$b' = \sum_{j=k}^{\infty} \Delta_j\left(\tilde{\gamma}_g(\psi_n(u)) - \psi_k(\delta_h(v))\right)(1 - y_{j-1})\Delta_j .$$

Using the above, together with the fact that m commutes with Δ_j, we get

$$\begin{aligned}ma' + mb' = &\sum_{j=0}^{k-1} \Delta_j m(\chi(u) - \chi(v))\Delta_j\\ &+ \sum_{j=k}^{\infty} \Delta_j m\big(\tilde{\gamma}_g(\psi_n(u)) - \psi_k(\delta_h(v))\big)(1 - y_{j-1})\Delta_j\end{aligned}$$

when $m \in \{1, d\}$. By property (iii) in Lemma 2.3 we have that

$$\left\|m\big(\tilde{\gamma}_g(\psi_n(u)) - \psi_k(\delta_h(v))\big)(1 - y_{j-1})\right\| \leq \left\|q_I(m)\big(g\cdot\psi(u) - \psi(h\cdot v)\big)\right\| + \frac{1}{j-1}$$

for $j \geq k \geq n+1$, since $\tilde{\gamma}_g(\psi_n(u)) - \psi_k(\delta_h(v))$ belongs to Z_{j-1}. It follows therefore from Lemma 2.4 that

$$\|ma' + mb'\| \leq \max\{|\chi(u) - \chi(v)|, \|q_I(m)(g\cdot\psi(u) - \psi(h\cdot v))\|\} + \frac{1}{k-1}$$

for $m \in \{1, d\}$ when $k \geq n+1$. Combined with (2.8) and (2.9) we find that

$$\begin{aligned}
&\left\|m\left(g \cdot \psi^k(u) - \psi^k(h \cdot v)\right)\right\| \\
&= \|m(a-a') + m(b-b') + ma' + mb'\| \\
&\leq |\chi(u)|\,\|m\|\,\|\gamma_g(y_k) - y_k\| + 2\|m\|\,\|\psi_n(u)\| \sum_{j=k}^{\infty} 2^{-j} \\
&\qquad + \max\{|\chi(u) - \chi(v)|, \|q_I(m)(g \cdot \psi(u) - \psi(h \cdot v))\|\} + \frac{1}{k-1},
\end{aligned}$$

when $k \geq n+1$. This estimate and property (ii) in Lemma 2.3 show that (2.7) holds.

(8) Set $u' = u - \chi(u),\ v' = v - \chi(v)$, and find $n \in \mathbb{N}$ with $u', v' \in F_n$. Let $\epsilon > 0$. Since $\psi(u')\psi(v') = q_I(\psi_n(u')\psi_n(v'))$ there is an $x \in I$ such that

$$\|\psi_n(u')\psi_n(v') + x\| \leq \|\psi(u')\psi(v')\| + \epsilon \ . \tag{2.10}$$

Let $t \geq n+2$ and let $k \in \mathbb{N}$ satisfy $t \in [k, k+1]$. Define $s_k = t - k$ and $s_j = 1$ for $j \geq k+1$. Using that $\chi(u') = \chi(v') = 0$ at the first step, and using that $\Delta_l \Delta_j = 0$ when $|l-j| \geq 2$ at the second step, we get

$$\begin{aligned}
\psi^t(u')\psi^t(v') &= \left(\sum_{j=k}^{\infty} \Delta_j s_j \psi_n(u') \Delta_j\right)\left(\sum_{j=k}^{\infty} \Delta_j s_j \psi_n(v') \Delta_j\right) \\
&= \sum_{l=-1}^{1} \sum_{j=k+1}^{\infty} \Delta_j s_j \psi_n(u') \Delta_j \Delta_{j+l} s_{j+l} \psi_n(v') \Delta_{j+l} \\
&\qquad + \sum_{l=0}^{1} \Delta_k s_k \psi_n(u') \Delta_k \Delta_{k+l} s_{k+l} \psi_n(v') \Delta_{k+l} \ .
\end{aligned}$$

Using property (v) of Lemma 2.3, we get

$$\|\Delta_j \psi_n(v') - \psi_n(v')\Delta_j\| \leq 2^{-j} \tag{2.11}$$

when $j \geq n$. Set

$$\begin{aligned}
a = &\sum_{l=-1}^{1} \sum_{j=k+1}^{\infty} \Delta_j s_j s_{j+l} \psi_n(u') \Delta_{j+l} \psi_n(v') \Delta_{j+l} \Delta_j \\
&+ \sum_{l=0}^{1} \Delta_k s_k s_{k+l} \psi_n(u') \Delta_{k+l} \psi_n(v') \Delta_{k+l} \Delta_k \ .
\end{aligned}$$

Since $k \geq n+1$, we conclude from (2.11) that

$$\left\|\psi^t(u')\psi^t(v') - a\right\| \leq 5\|\psi_n(u')\| \sum_{j=k+1}^{\infty} 2^{-j+1} \ . \tag{2.12}$$

Similarly, by setting

$$\begin{aligned} b &= \sum_{l=-1}^{1} \sum_{j=k+1}^{\infty} \Delta_j s_j s_{j+l} \psi_n(u')\psi_n(v')\Delta_{j+l}^2 \Delta_j \\ &\quad + \sum_{l=0}^{1} \Delta_k s_k s_{k+l} \psi_n(u')\psi_n(v')\Delta_{k+l}^2 \Delta_k \\ &= \sum_{j=k+1}^{\infty} \Delta_j \left(\sum_{l=-1}^{1} s_j s_{j+l} \psi_n(u')\psi_n(v')\Delta_{j+l}^2 \right) \Delta_j \\ &\quad + \Delta_k \left(\sum_{l=0}^{1} s_k s_{k+l} \psi_n(u')\psi_n(v')\Delta_{k+l}^2 \right) \Delta_k \ , \end{aligned}$$

we have

$$\|a - b\| \leq 5\|\psi_n(u')\| \sum_{j=k+1}^{\infty} 2^{-j+1} \ . \tag{2.13}$$

Set

$$\begin{aligned} c = \sum_{j=k+1}^{\infty} \Delta_j &\left(\sum_{l=-1}^{1} s_j s_{j+l} (\psi_n(u')\psi_n(v') + x)\Delta_{j+l}^2 \right) \Delta_j \\ &+ \Delta_k \left(\sum_{l=0}^{1} s_k s_{k+l} (\psi_n(u')\psi_n(v') + x)\Delta_{k+l}^2 \right) \Delta_k \ . \end{aligned}$$

Then

$$\begin{aligned} c - b = \sum_{j=k+1}^{\infty} \Delta_j &\left(\sum_{l=-1}^{1} s_j s_{j+l} x \Delta_{j+l}^2 \right) \Delta_j \\ &+ \Delta_k \left(\sum_{l=0}^{1} s_k s_{k+l} x \Delta_{k+l}^2 \right) \Delta_k \ . \end{aligned} \tag{2.14}$$

Note that

$$\left\| x\Delta_{j+l}^2 \right\| = \|x(y_{j+l+1} - y_{j+l})\| \ ,$$

which will be very small when j is big since $x \in I$. More precisely, there is a $K \in \mathbb{N}$ such that

$$\left\| \sum_{l=-1}^{1} s_j s_{j+l} x \Delta_{j+l}^2 \right\| \leq \epsilon \quad \text{and} \quad \left\| \sum_{l=0}^{1} s_k s_{k+l} x \Delta_{k+l}^2 \right\| \leq \epsilon$$

when $j \geq k+1$ and $k \geq K$. When we use these estimates in (2.14) and apply Lemma 2.4 it follows that

$$\|c - b\| \leq \epsilon \ , \tag{2.15}$$

provided $k \geq K$. Note that

$$\begin{aligned}
&\left\| \sum_{l=-1}^{1} s_j s_{j+l} (\psi_n(u')\psi_n(v') + x)\Delta_{j+l}^2 \right\| \\
&\leq \|\psi_n(u')\psi_n(v') + x\| \left\| \sum_{l=-1}^{1} s_j s_{j+l} \Delta_{j+l}^2 \right\| \\
&\leq \|\psi_n(u')\psi_n(v') + x\| \left\| \sum_{l=-1}^{1} \Delta_{j+l}^2 \right\| = \|\psi_n(u')\psi_n(v') + x\| \, \|y_{j+2} - y_{j-1}\| \\
&\leq \|\psi_n(u')\psi_n(v') + x\| \ ,
\end{aligned}$$

and similarly,

$$\left\| \sum_{l=0}^{1} s_k s_{k+l} (\psi_n(u')\psi_n(v') + x)\Delta_{k+l}^2 \right\| \leq \|\psi_n(u')\psi_n(v') + x\| \ .$$

It follows therefore from the definition of c and Lemma 2.4 that

$$\|c\| \leq \|\psi_n(u')\psi_n(v') + x\| \ .$$

Combining with (2.12), (2.13), (2.15) and (2.10) we find that

$$\left\|\psi^t(u')\psi^t(v')\right\| \leq \|\psi(u')\psi(v')\| + 10\|\psi_n(u')\| \sum_{j=k+1}^{\infty} 2^{-j+1} + 2\epsilon$$

when $t \in [k, k+1]$ for some $k \geq \max\{n+2, K\}$. This proves (8).

(9) Fix $j, n \in \mathbb{N}$ and fix $u \in M_j(F_n)$. We make the following abbreviations: $\chi' = \chi \otimes \mathrm{id}_{M_j}$, $\psi' = \psi \otimes \mathrm{id}_{M_j}$, $q_I' = q_I \otimes \mathrm{id}_{M_j}$, $\psi_k' = \psi_k \otimes \mathrm{id}_{M_j}$, $y_k' = y_k \otimes 1_{M_j(\mathbb{C})}$ and $\Delta_k' = \Delta_k \otimes 1_{M_j(\mathbb{C})}$ for $k \in \mathbb{N}$. Using property (iv) in Lemma 2.3, for $k \geq n$ we get

$$\begin{aligned}
(\psi^k \otimes \mathrm{id}_{M_j})(u) &= \sum_{\ell=0}^{k-1} \Delta_\ell' \chi'(u) \Delta_\ell' + \sum_{\ell=k}^{\infty} \Delta_\ell' \psi_k'(u) \Delta_\ell' \\
&= \sum_{\ell=0}^{k-1} \Delta_\ell' \chi'(u) \Delta_\ell' + \sum_{\ell=k}^{\infty} \Delta_\ell' \psi_n'(u)(1 - y_{\ell-1}') \Delta_\ell'.
\end{aligned}$$

It follows from Lemma 3.1 in [MT1] that

$$\left\|(\psi^k \otimes \mathrm{id}_{M_j})(u)\right\| \leq \max\left\{\|\chi'(u)\|, \ \sup_{\ell \geq k} \left\|\psi_n'(u)(1 - y_{\ell-1}')\right\|\right\} . \tag{2.16}$$

Since $(y_k')_{k\in\mathbb{N}}$ is an approximate unit in $M_j(I) = \ker(q_I \otimes \mathrm{id}_{M_j})$,

$$\lim_{\ell\to\infty} \left\|\psi_n'(u)(1 - y_{\ell-1}')\right\| = \|q_I'(\psi_n'(u))\| = \|\psi'(u)\| \ .$$

It follows therefore from (2.16) that

$$\limsup_{k\to\infty} \left\|(\psi^k \otimes \mathrm{id}_{M_j})(u)\right\| \leq \max\left\{ \|\chi'(u)\|, \|\psi'(u)\| \right\} .$$

Thanks to (2.5) this completes the proof. □

The cone of positive elements in a C^*-algebra A will be denoted by A_+. Given a self-adjoint element $a = a^* \in A$, there are unique elements $a^+, a^- \in A_+$ such that $a^+a^- = 0$ and $a = a^+ - a^-$. This notation is used in (e) of the following

Definition 2.10. Let A and B be C^*-algebras. We say that a collection $\Theta = (\Theta_t)_{t\in[1,\infty)}$ of maps $\Theta_t\colon A \to B$ is a *continuous path*, when $(\Theta_t)_{t\in[1,\infty)}$ is an equicontinuous family of maps and for every $a \in A$, the assignment $[1,\infty) \to A$ given by $t \mapsto \Theta_t(a)$ is continuous. Additionally, we say that Θ is

(a) *unital*, if $\Theta_t(1) = 1$ for all t;
(b) *self-adjoint*, if Θ_t is self-adjoint for all $t \in [1,\infty)$;
(c) *asymptotically linear*, if

$$\lim_{t\to\infty} \|\Theta_t(\lambda_1 a_1 + \lambda_2 a_2) - \lambda_1\Theta_t(a_1) - \lambda_2\Theta_t(a_2)\| = 0$$

for all $\lambda_1, \lambda_2 \in \mathbb{C}$ and all $a_1, a_2 \in A$;
(d) *asymptotically contractive*, if $\limsup_{t\to\infty} \|\Theta_t(a)\| \le \|a\|$ for $a \in A$;
(e) *asymptotically positive*, if Θ is self-adjoint and

$$\lim_{t\to\infty} \Theta_t(a)^- = 0$$

for all positive elements $a \in A_+$;
(f) *asymptotically completely positive*, if the continuous family

$$\Theta \otimes \mathrm{id}_{M_n} = (\Theta_t \otimes \mathrm{id}_{M_n})_{t\in[1,\infty)} : M_n(A) \to M_n(B)$$

is asymptotically linear and asymptotically positive for all n.

Lemma 2.11. *Let A and B be unital C^*-algebras, and let $\Theta\colon A \to B$ be a continuous family of maps which is self-adjoint, unital, asymptotically linear and an asymptotic contraction. Then Θ is asymptotically positive.*

Proof. Let $a \in A_+$. Assume for a contradiction that $\Theta_t(a)^-$ does not converge to 0 as $t \to \infty$. Then there are $\varepsilon > 0$ and a sequence $(t_n)_{n\in\mathbb{N}}$ in $[1,\infty)$ such that $\lim_{n\to\infty} t_n = \infty$ and $\|\Theta_{t_n}(a)^-\| \ge \varepsilon$ for all $n \in \mathbb{N}$. Find states ω_n on B, for $n \in \mathbb{N}$, such that $\omega_n(\Theta_{t_n}(a)^+) = 0$ and $\omega_n(\Theta_{t_n}(a)^-) \ge \varepsilon$ for all $n \in \mathbb{N}$. Using that Θ_t is asymptotically linear, we have.

$$\lim_{n\to\infty} \|\Theta_{t_n}(\|a\| - a) - (\|a\| - \Theta_{t_n}(a))\| = 0\ .$$

Similarly, using that Θ_t is an asymptotic contraction at the second step, we have

$$\limsup_{t\to\infty} \|\Theta_t(\|a\| - a)\| \le \|\|a\| - a\| \le \|a\|\ .$$

We deduce that there is $n \in \mathbb{N}$ such that

$$\|a\| + \frac{\varepsilon}{3} \geq \omega_n(\Theta_{t_n}(\|a\| - a)) \geq \omega_n(\|a\| - \Theta_{t_n}(a)) - \frac{\varepsilon}{3}$$
$$= \|a\| + \omega_n(\Theta_{t_n}(a)^-) - \frac{\varepsilon}{3} \ \geq \|a\| + \frac{2\varepsilon}{3} \ ,$$

which is a contradiction, and thus Θ is asymptotically positive. ☐

Definition 2.12. Let (A, α), (B, β) and (S, δ) be G-algebras and $q : (A, \alpha) \to (B, \beta)$ an equivariant $*$-homomorphism. Let $\psi\colon S \to B$ be a linear completely positive contraction, and let $\Theta = (\Theta_t)_{t\in[1,\infty)}\colon S \to A$ be a continuous path of maps.

(1) We say that Θ is an *asymptotically* (G, ψ)*-equivariant* lift of ψ when
- $q \circ \Theta_t = \psi$ for all $t \in [1, \infty)$;
- Θ is asymptotically completely positive;
- for all $s \in S$ we have
$$\lim_{t\to\infty} \|g \cdot \Theta_t(s) - \Theta_t(h \cdot s)\| = \|g \cdot \psi(s) - \psi(h \cdot s)\| \ ,$$
uniformly for g and h in compact subsets of G; and
- for all $s \in S$, $g \in G$ and $\varepsilon > 0$, there is an open neighborhood W of g such that for all $h \in W$ we have
$$\sup_{t\in[1,\infty)} \|h \cdot \Theta_t(s) - g \cdot \Theta_t(s)\| \leq \varepsilon \ .$$

(2) We say that Θ is a *completely positive asymptotically* (G, ψ)*-equivariant* lift of ψ when
- $q \circ \Theta_t = \psi$ for all $t \in [1, \infty)$;
- Θ_t is a linear completely positive contraction for all $t \in [1, \infty)$; and
- for all $s \in S$ we have
$$\lim_{t\to\infty} \|g \cdot \Theta_t(s) - \Theta_t(h \cdot s)\| = \|g \cdot \psi(s) - \psi(h \cdot s)\| \ ,$$
uniformly for g and h in compact subsets of G.

Lemma 2.13. *Let $\psi\colon U \to Q(I)_{\overline{\gamma}}$ be a unital linear completely positive map. There is a unital asymptotically (G, ψ)-equivariant lift $\Theta = (\Theta_t)_{t\in[1,\infty)}\colon U \to M(I)_{\tilde{\gamma}}$ of ψ with the following additional properties:*

(i) *For all $u, v \in U$, every compact subset $K \subseteq G$ and every $\varepsilon > 0$, there is $T \geq 1$ such that*
$$\sup_{t\geq T} \|g \cdot \Theta_t(u) - \Theta_t(h \cdot v)\|$$
$$\leq \max \{|\chi(u - v)|, \|g \cdot \psi(u) - \psi(h \cdot v)\|\} + \varepsilon$$
for all $g, h \in K$,

(ii) *for all $u \in U$ and all $x \in I$, we have $\lim_{t\to\infty} \Theta_t(u)x = \chi(u)x$,*

(iii) *for all* $n \in \mathbb{N}$ *and all* $u \in M_n(U)$*, we have*

$$\lim_{t\to\infty} \|(\Theta_t \otimes \mathrm{id}_{M_n})(u)\| = \max\left\{ \|(\psi \otimes \mathrm{id}_{M_n})(u)\| , \|(\chi \otimes \mathrm{id}_{M_n})(u)\| \right\},$$

and

(iv) *for* $u, v \in \ker(\chi)$*, we have* $\lim_{t\to\infty} \|\Theta_t(u)\Theta_t(v)\| = \|\psi(u)\psi(v)\|$.

Proof. Consider the C^*-algebra

$$\mathcal{A} = \{f \in C_b([1,\infty), M(I)_{\tilde{\gamma}}) : f(1) - f(t) \in I \text{ for all } t \in [1,\infty)\},$$

and define an action $\mu\colon G \to \mathrm{Aut}(\mathcal{A})$ by $\mu_g(f)(t) = \tilde{\gamma}_g(f(t))$ for all $g \in G$, all $f \in \mathcal{A}$ and all $t \in [1,\infty)$. Set

$$\mathcal{A}_0 = \mathcal{A} \cap C_0([1,\infty), M(I)_{\tilde{\gamma}}),$$

which is an ideal in $\mathcal{A}$. Note that $\mathcal{A}_0 = C_0([1,\infty), I)$, $\mu_g(\mathcal{A}_0) = \mathcal{A}_0$ for all $g \in G$, and that $\mathcal{A}_0$ is contained in the continuous part $\mathcal{A}_\mu$ of $\mathcal{A}$. We denote by $\pi\colon \mathcal{A} \to \mathcal{A}/\mathcal{A}_0$ the quotient map and let $\overline{\mu}\colon G \to \mathrm{Aut}(\mathcal{A}/\mathcal{A}_0)$ be the action defined such that $\overline{\mu}_g \circ \pi = \pi \circ \mu_g$ for all $g \in G$.

Let $\psi^t\colon U \to M(I)_{\tilde{\gamma}}$ be the family of maps from Lemma 2.9. It follows from (7) in Lemma 2.9 (with $g = e$ and $v = 0$) that $\sup_{t\in[1,\infty)} \|\psi^t(u)\| < \infty$ when $u \in U^{(0)}$. In combination with (2) and (5) of Lemma 2.9, this gives us a map $\Phi_0\colon U^{(0)} \to \mathcal{A}$ defined by

$$\Phi_0(u)(t) = \psi^t(u)$$

for all $u \in U^{(0)}$ and all $t \in [1,\infty)$. Set $\Psi_0 = \pi \circ \Phi_0\colon U^{(0)} \to \mathcal{A}/\mathcal{A}_0$. It follows then from (7) in Lemma 2.9 that

$$\|\Psi_0(u) - \Psi_0(v)\| \leq \max\{|\chi(u-v)|, \|\psi(u-v)\|\}$$

for all $u, v \in U^{(0)}$. Since $U^{(0)}$ is dense in U, it follows from this estimate that Ψ_0 extends by continuity to a continuous map $\Psi\colon U \to \mathcal{A}/\mathcal{A}_0$ with the property that

$$\|\Psi(u) - \Psi(v)\| \leq \max\{|\chi(u-v)|, \|\psi(u-v)\|\}$$

for all $u, v \in U$. Note that Ψ is self-adjoint because each ψ^t is and $(U^{(0)})^* = U^{(0)}$.

Let $u_1, \ldots, u_n \in U^{(0)}$ and let $\lambda_1, \ldots, \lambda_n \in \mathbb{Q}[i]$, so that $\sum_{j=1}^n \lambda_j u_j$ belongs to $U^{(0)}$ by (2.2). Using (4) of Lemma 2.9 and linearity of π, we have

$$\Psi\Big(\sum_{j=1}^n \lambda_j u_j\Big) = \Psi_0\Big(\sum_{j=1}^n \lambda_j u_j\Big) = \sum_{j=1}^n \lambda_j \Psi_0(u_j) = \sum_{j=1}^n \lambda_j \Psi(u_j).$$

Hence Ψ is $\mathbb{Q}[i]$-linear on $U^{(0)}$, and by continuity it is linear on all of U. Note that Ψ is unital by (1) of Lemma 2.9.

Let $k \in \mathbb{N}$. Then $\Psi \otimes \mathrm{id}_{M_k}$ is self-adjoint because Ψ is and it follows from (9) of Lemma 2.9 that $\Psi\otimes\mathrm{id}_{M_k}$ is a contraction, since $\chi\otimes\mathrm{id}_{M_k}$ and $\psi\otimes\mathrm{id}_{M_k}$ are. Since $\Psi\otimes\mathrm{id}_{M_k}$ is also unital, we deduce from Lemma 2.11 that $\Psi \otimes \mathrm{id}_{M_k}$ is positive. We conclude that Ψ is completely positive.

Let $g, h \in G$ and $u, v \in U^{(0)}$. Since $G^{(0)}$ is dense in G, we can find a sequence $(h_n)_{n\in\mathbb{N}}$ in $G^{(0)}$ such that $\lim_{n\to\infty} h_n = h$. By (2.2), we have $h_n \cdot v \in U^{(0)}$ for all $n \in \mathbb{N}$. Taking $h_n \cdot v \in U^{(0)}$ in place of v and e in place of h in (7) of Lemma 2.9, we get

$$\begin{aligned}\|g\cdot\Psi(u) - \Psi(h_n\cdot v)\| &= \limsup_{t\to\infty}\left\|g\cdot\psi^t(u) - \psi^t(h_n\cdot v)\right\| \\ &\leq \max\left\{|\chi(u-v)|, \|g\cdot\psi(u) - \psi(h_n\cdot v)\|\right\}\ .\end{aligned}$$

By continuity of Ψ and ψ, we can take the limit on n to get

$$\|g\cdot\Psi(u) - \Psi(h\cdot v)\| \leq \max\left\{|\chi(u-v)|, \|g\cdot\psi(u) - \psi(h\cdot v)\|\right\}\ . \tag{2.17}$$

Since $U^{(0)}$ is dense in U and Ψ and ψ are continuous, we conclude that (2.17) above holds for all $u, v \in U$. It follows, in particular, that

$$\lim_{g\to e}\overline{\mu}_g(\Psi(u)) = \Psi(u)\ ,$$

showing that Ψ takes values in the continuous part $(\mathcal{A}/\mathcal{A}_0)_{\overline{\mu}}$ of $\mathcal{A}/\mathcal{A}_0$. It follows from [Br] that $\pi(\mathcal{A}_\mu) = (\mathcal{A}/\mathcal{A}_0)_{\overline{\mu}}$ and then from the Bartle-Graves selection theorem, [BG], that there is a continuous map $\Psi'\colon U \to \mathcal{A}_\mu$ such that $\pi\circ\Psi' = \Psi$. By substituting $\Psi'(u)$ with $\Psi'(u) + \chi(u)(1 - \Psi'(1))$, we may assume that $\Psi'(1) = 1$. By substituting $\Psi'(u)$ with $\frac{1}{2}\left(\Psi(u) + \Psi(u^*)^*\right)$, we may also assume that Ψ' is self-adjoint. For $t \in [1,\infty)$, let $\mathrm{ev}_t\colon \mathcal{A} \to M(I)_{\tilde{\gamma}}$ be evaluation at t and set

$$\Theta_t = \mathrm{ev}_t\circ\Psi'\colon U \to M(I)_{\tilde{\gamma}}\ .$$

Then $\Theta = (\Theta_t)_{t\in[1,\infty)}$ is a unital and self-adjoint continuous family of maps $U \to M(I)_{\tilde{\gamma}}$. In the remainder of the proof, we check that Θ has the properties required in the statement of the lemma.

Fix $t \in [1,\infty)$. To show that $q_I\circ\Theta_t = \psi$, it suffices to check the identity on $U^{(0)}$. Fix $u \in U^{(0)}$. Then $\Psi'(u) - \Phi_0(u) \in \mathcal{A}_0$ and hence $\Theta_t(u) - (\mathrm{ev}_t\circ\Phi_0)(u)$ belongs to I. Since $\mathrm{ev}_t(\Phi_0(u)) = \psi^t(u)$, it follows that $q_I(\Theta_t(u)) = q_I(\psi^t(u))$, which equals $\psi(u)$ by (5) of Lemma 2.9.

To check that Θ is asymptotically linear, let $\lambda_1, \lambda_2 \in \mathbb{C}$ and $u_1, u_2 \in U$. Using the linearity of Ψ we find that

$$\begin{aligned}&\limsup_{t\to\infty}\|\Theta_t(\lambda_1u_1 + \lambda_2u_2) - \lambda_1\Theta_t(u_1) - \lambda_2\Theta_t(u_2)\| \\ &\quad= \limsup_{t\to\infty}\left\|\mathrm{ev}_t\left(\Psi'(\lambda_1u_1 + \lambda_2u_2) - \lambda_1\Psi'(u_1) - \lambda_2\Psi'(u_2)\right)\right\| \\ &\quad= \left\|\pi\big(\Psi'(\lambda_1u_1 + \lambda_2u_2) - \lambda_1\Psi'(u_1) - \lambda_2\Psi'(u_2)\big)\right\| \\ &\quad= \|\Psi(\lambda_1u_1 + \lambda_2u_2) - \lambda_1\Psi(u_1) - \lambda_2\Psi(u_2)\| = 0\ ,\end{aligned}$$

as desired. To show that Θ is asymptotically completely positive, we fix $k \in \mathbb{N}$ and will prove that $\Theta\otimes\mathrm{id}_{M_k}$ is asymptotically positive. Note that $\Theta\otimes\mathrm{id}_{M_k}$ is self-adjoint, unital and asymptotically linear. By Lemma 2.11, it suffices to show that

(2.18)

$$\limsup_{t\to\infty}\|(\Theta_t\otimes\mathrm{id}_{M_k})(u)\| \leq \max\{\|(\psi\otimes\mathrm{id}_{M_k})(u)\|, \|(\chi\otimes\mathrm{id}_{M_k})(u)\|\}$$

for all $u \in M_k(U)$, since $\psi \otimes \mathrm{id}_{M_k}$ and $\chi \otimes \mathrm{id}_{M_k}$ are contractions. This estimate follows from equicontinuity of the continuous family $\Theta \otimes \mathrm{id}_{M_k}$ together with (8) of Lemma 2.9, since $M_k(U^{(0)})$ is dense in $M_k(U)$. We conclude that Θ is is asymptotically completely positive.

To obtain (i), let $u, v \in U$ and let K be a compact subset of G. Let $\varepsilon > 0$. Since $\Psi'(u) \in \mathcal{A}_\mu$ and since Θ is equicontinuous, for every $g, h \in K$ there are open neighborhoods V_g of g and W_h of h such that

$$\|g' \cdot \Theta_t(u) - \Theta_t(h' \cdot v)\| \leq \|g \cdot \Theta_t(u) - \Theta_t(h \cdot v)\| + \frac{\varepsilon}{3}$$

for all $(g', h') \in V_g \times W_h$ and for all $t \in [1, \infty)$. By shrinking V_g and W_h we can also arrange that

$$\|g \cdot \psi(u) - \psi(h \cdot v)\| \leq \|g' \cdot \psi(u) - \psi(h' \cdot v)\| + \frac{\varepsilon}{3}$$

for all $(g', h') \in V_g \times W_h$. By compactness of $K \times K$, there is a finite set $K_0 \subseteq K \times K$ such that $\{V_g \times W_h \colon (g, h) \in K_0\}$ covers $K \times K$. Since K_0 is finite, it follows from (2.17) that there is a $T \geq 1$ such that

$$\|g \cdot \Theta_t(u) - \Theta_t(h \cdot v)\| \leq \max\{|\chi(u - v)|, \|g \cdot \psi(u) - \psi(h \cdot v)\|\} + \frac{\varepsilon}{3}$$

for all $(g, h) \in K_0$ when $t \geq T$. An easy application of the triangle inequality then yields

$$\|g \cdot \Theta_t(u) - \Theta_t(h \cdot v)\| \leq \max\{|\chi(u - v)|, \|g \cdot \psi(u) - \psi(h \cdot v)\|\} + \varepsilon$$

for all $g, h \in K$ when $t \geq T$, which establishes (i). On the other hand it follows from (2.17) that

$$\begin{aligned} &\limsup_{t \to \infty} \|g \cdot \Theta_t(u) - \Theta_t(h \cdot u)\| = \|g \cdot \Psi(u) - \Psi(h \cdot u)\| \\ &\leq \|g \cdot \psi(u) - \psi(h \cdot u)\| \ , \end{aligned}$$

for all $g, h \in G,\ u \in U$, which combined with (i) shows that Θ satisfies the condition in the third bullet of (1) in Definition 2.12. The fourth bullet of (1) in Definition 2.12 follows because $\Psi(U) \subseteq \mathcal{A}_\mu$.

It remains now only to establish the last three items, (ii)-(iv), in the statement. For (ii) consider first an element $u \in U^{(0)}$. Taking $m = d$, $g = h = e$ and $v = \chi(u) \in U^{(0)}$ in (7) of Lemma 2.9 gives

$$\lim_{t \to \infty} \left\| d(\psi^t(u) - \chi(u)) \right\| = 0 \ .$$

Hence, upon identifying d with the element of $\mathcal{A}$ which is constant with value d, we have $\pi(d\Psi'(u)) = \chi(u)\pi(d)$. This identity means that $d\Psi'(u) - \chi(u)d$ belongs to $\mathcal{A}_0$. It follows that

$$\lim_{t \to \infty} \Theta_t(u)d - \chi(u)d = \lim_{t \to \infty} (d\Theta_t(u^*) - \chi(u^*)d)^* = 0 \ .$$

Since $\overline{dI} = I$ and $\|\Theta_t(u)\|$ is uniformly bounded, it follows that

$$\lim_{t \to \infty} \Theta_t(u)x = \chi(u)x \tag{2.19}$$

for all $x \in I$. Since Θ is an equicontinuous family of maps this conclusion extends to all elements $u \in U$.

To obtain (iii) from (ii), fix $u \in M_n(U)$ and $\varepsilon > 0$. Since $\Theta_t \otimes \mathrm{id}_{M_n}$ is a lift of $\psi \otimes \mathrm{id}_{M_n}$, we have

$$\|(\psi \otimes \mathrm{id}_{M_n})(u)\| \leq \|(\Theta_t \otimes \mathrm{id}_{M_n})(u)\| \tag{2.20}$$

for all $t \in [1, \infty)$. Since $M_n(M(I)) \cong M(M_n(I))$, we can choose $x \in M_n(I)$ with $\|x\| \leq 1$ such that

$$\|(\chi \otimes \mathrm{id}_{M_n})(u)x\| > \|(\chi \otimes \mathrm{id}_{M_n})(u)\| - \varepsilon \ .$$

It follows from (2.19) that

$$\lim_{t\to\infty} (\Theta_t \otimes \mathrm{id}_{M_n}(u))x = (\chi \otimes \mathrm{id}_{M_n})(u)x \ ,$$

and hence

$$\|(\Theta_t \otimes \mathrm{id}_{M_n})(u)\| \geq \|(\Theta_t \otimes \mathrm{id}_{M_n})(u)x\| > \|(\chi \otimes \mathrm{id}_{M_n})(u)\| - \varepsilon$$

for all t big enough. In combination with (2.20), this shows that

$$\liminf_{t\to\infty} \|(\Theta_t \otimes \mathrm{id}_{M_n})(u)\| \geq \max \left\{ \|(\psi \otimes \mathrm{id}_{M_n})(u)\| , \|(\chi \otimes \mathrm{id}_{M_n})(u)\| \right\} .$$

In combination with (2.18) this gives us (iii).

Finally, from (8) of Lemma 2.9 we get the estimate

$$\limsup_{t\to\infty} \left\|\Theta_t(u - \chi(u))\Theta_t(v - \chi(v))\right\| \leq \|\psi(u - \chi(u))\psi(v - \chi(v))\|$$

for all $u, v \in U^{(0)}$. This estimate extends to all $u, v \in U$ by equicontinuity of Θ, implying that

$$\limsup_{t\to\infty} \|\Theta_t(u)\Theta_t(v)\| \leq \|\psi(u)\psi(v)\|$$

when $u, v \in \ker \chi$. Since $\|\psi(u)\psi(v)\| = \|q_I(\Theta_t(u)\Theta_t(v))\| \leq \|\Theta_t(u)\Theta_t(v)\|$ for all t, we have established (iv). □

Lemma 2.14. *Let $\psi\colon U \to Q(I)_{\overline{\gamma}}$ be a unital linear completely positive contraction and $\psi'\colon U \to M(I)$ a unital linear completely positive map such that $q_I \circ \psi' = \psi$. There is a unital completely positive (G, ψ)-equivariant lift of ψ such that i) through iv) in Lemma 2.13 hold.*

Proof. This follows from the arguments that proved Lemma 2.13, except that most of them can be greatly simplified thanks to the stronger assumption. In particular, we note that the reason we get completely positive and linear lifts is due to Remark 2.8, since it implies that the map Φ_0 in the proof of Lemma 2.13 in this case can be defined on all of U and gives a unital completely positive linear map $\Phi_0\colon U \to \mathcal{A}$ which can be used instead of the map Ψ' constructed above. We omit the details. □

3. Asymptotic lifts

Let

$$0 \longrightarrow (I,\gamma) \xrightarrow{\iota} (A,\alpha) \xrightarrow{q} (B,\beta) \longrightarrow 0 \tag{3.1}$$

be an extension of G-algebras. Let (S,δ) be a G-algebra and let $\psi\colon S \to B$ be a linear completely positive contraction. This is the general set-up we shall consider in this section, with varying additional assumptions.

3.1. **The unital case.** In this subsection, we assume that A is unital, I is σ-unital, S is unital and separable, ψ is unital, and that there is a G-invariant state χ on S.

Define $r\colon A \to M(I)$ by $\iota(r(a)x) = a\iota(x)$ for all $a \in A$ and all $x \in I$. Then $r\colon (A,\alpha) \to (M(I)_{\tilde{\gamma}},\tilde{\gamma})$ is a unital G-equivariant $*$-homomorphism, and thus $q_I\circ r\colon (A,\alpha) \to (Q(I)_{\overline{\gamma}},\overline{\gamma})$ is G-equivariant as well. Since $\iota(I) \subseteq \ker(q_I\circ r)$, we obtain an equivariant $*$-homomorphism $\phi\colon (B,\beta) \to (Q(I)_{\overline{\gamma}},\overline{\gamma})$ making the following diagram commute:

$$\begin{tikzcd} A \arrow[r, "q"] \arrow[d, "r"] & B \arrow[d, "\phi"] \\ M(I)_{\tilde{\gamma}} \arrow[r, "q_I"'] & Q(I)_{\overline{\gamma}} \end{tikzcd} .$$

The map ϕ is known as the Busby invariant of the extension (3.1). Set

$$E = \{(m,b) \in M(I)_{\tilde{\gamma}} \oplus B\colon\ q_I(m) = \phi(b)\}.$$

Define $\mu\colon G \to \mathrm{Aut}(E)$ by $\mu_g(m,b) = (\tilde{\gamma}_g(m), \beta_g(b))$ for all $g \in G$ and all $(m,b) \in E$. Then (E,μ) is a G-algebra. Define $*$-homomorphisms $\iota'\colon I \to E$, $p\colon E \to B$, and $\xi\colon A \to E$ by $\iota'(x) = (x,0)$, $p(m,b) = b$, and $\xi(a) = (r(a), q(a))$, for $x \in I$, $(m,b) \in E$ and $a \in A$. Then

$$\begin{tikzcd} 0 \arrow[r] & (I,\gamma) \arrow[r, "\iota"] \arrow[d, equal] & (A,\alpha) \arrow[r, "q"] \arrow[d, "\xi"] & (B,\beta) \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & (I,\gamma) \arrow[r, "\iota'"'] & (E,\mu) \arrow[r, "p"'] & (B,\beta) \arrow[r] & 0 \end{tikzcd}$$

is a commuting diagram of G-algebras with exact rows. In particular, ξ is an isomorphism of G-algebras. Let $\Theta' = (\Theta'_t)_{t\in[1,\infty)}\colon S \to M(I)_{\tilde{\gamma}}$ be a unital asymptotically $(G,\phi\circ\psi)$-equivariant lift of the unital completely positive contraction $\phi\circ\psi\colon S \to Q(I)_{\overline{\gamma}}$ satisfying the conclusion of Lemma 2.13. Define $\Theta = (\Theta_t)_{t\in[1,\infty)}\colon S \to A$ by

$$\Theta_t(s) = \xi^{-1}\left((\Theta'_t(s), \psi(s))\right)$$

for all $t \in [1,\infty)$ and all $s \in S$. Then Θ is a unital asymptotically (G,ψ)-equivariant lift of ψ to A. This leads to the following result. Note that most of the conclusions follow also when I is not assumed to be σ-unital.

Theorem 3.1. *Let G be a second countable locally compact group and consider the extension* (3.1) *of G-algebras with A unital. Let (S,δ) be a G-algebra with S separable and unital, and let $\psi\colon S\to B$ be a unital linear completely positive map. Assume that S has a G-invariant state χ. There is a unital asymptotically (G,ψ)-equivariant lift $\Theta=(\Theta_t)_{t\in[1,\infty)}\colon S\to A$ of ψ with the following additional properties:*

(a) *For all $s,s'\in S$, every compact subset $K\subseteq G$ and every $\varepsilon>0$, there is $T\geq 1$ such that*
$$\sup_{t\geq T}\|g\cdot\Theta_t(s)-\Theta_t(h\cdot s')\| \leq \max\{|\chi(s-s')|,\|g\cdot\psi(s)-\psi(h\cdot s')\|\}+\varepsilon$$
for all $g,h\in K$,

(b) *for all $n\in\mathbb{N}$ and all $s\in M_n(S)$, we have*
$$\lim_{t\to\infty}\|(\Theta_t\otimes\mathrm{id}_{M_n})(s)\|=\max\left\{\|(\psi\otimes\mathrm{id}_{M_n})(s)\|,\|(\chi\otimes\mathrm{id}_{M_n})(s)\|\right\},$$

(c) *for $s,s'\in\ker(\chi)$ we have $\lim_{t\to\infty}\|\Theta_t(s)\Theta_t(s')\|=\|\psi(s)\psi(s')\|$, and*

(d) *if I is σ-unital, we have $\lim\limits_{t\to\infty}\Theta_t(s)x=\chi(s)x$ for all $s\in S$ and all $x\in I$.*

Proof. Since S is separable and G is second countable, we can choose separable G-subalgebras $B_0\subseteq B$ and $A_0\subseteq A$ with $q(A_0)=B_0$ such that $\psi(S)\subseteq B_0$. When I is σ-unital, we may arrange that A_0 contains a strictly positive element of I. In either case, $I_0=I\cap A_0$ is separable, (I_0,γ) is a G-algebra, and the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & (I_0,\gamma) \arrow[r] \arrow[d, hook] & (A_0,\alpha) \arrow[r, "q"] \arrow[d, hook] & (B_0,\beta) \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & (I,\gamma) \arrow[r] & (A,\alpha) \arrow[r, "q"] & (B,\beta) \arrow[r] & 0\,.
\end{tikzcd}$$

Since ψ takes values in B_0 it follows from the preceding discussion that there is an asymptotically (G,ψ)-equivariant lift of ψ to A_0, and by the above diagram this is also a lift of ψ to A. Conditions (a), (b), (c) and (d) in the statement follow from conditions (i) through (iv) in Lemma 2.13. ☐

By using Lemma 2.14 instead of Lemma 2.13, the preceding proof works ad verbatim to prove the following.

Theorem 3.2. *In the setting of Theorem 3.1, assume that there is a unital linear completely positive lift $\psi'\colon S\to A$ for ψ. There is a unital completely positive asymptotically (G,ψ)-equivariant lift $\Theta=(\Theta_t)_{t\in[1,\infty)}\colon S\to A$ of ψ satisfying conditions (a), (b), (c) and (d) of Theorem 3.1.*

3.2. **The non-unital case.** Let A be a C^*-algebra. We denote by $A^\dagger$ the "forced" unitization of A, namely $A^\dagger$ is the (minimal) unitization of A if A is not unital, and $A^\dagger = A \oplus \mathbb{C}$ if A is unital. Note that A is an ideal in $A^\dagger$. We denote by $\chi_A\colon A^\dagger \to \mathbb{C}$ the unique state satisfying $\ker(\chi_A) = A$. This construction is functorial, and for a linear map $\phi\colon A \to B$ between C^*-algebras, we write $\phi^\dagger\colon A^\dagger \to B^\dagger$ for the unique linear extension of ϕ which satisfies $\chi_B \circ \phi^\dagger = \chi_A$. It is easy to see that $\phi^\dagger$ is a $*$-homomorphism if ϕ is. When ϕ is a completely positive contractive linear map, then the same is true for $\phi^\dagger$ by Lemma 3.9 in [CE], or by A.4 on page 266 in [NS]. In particular, if (A, α) is a G-algebra, then α extends uniquely to an action $\alpha^\dagger\colon G \to \mathrm{Aut}(A^\dagger)$ satisfying $\chi_A \circ \alpha_g^\dagger = \chi_A$ for all $g \in G$. Finally, if $\phi\colon (A, \alpha) \to (B, \beta)$ is an equivariant linear map between G-algebras, then $\phi^\dagger\colon (A^\dagger, \alpha^\dagger) \to (B^\dagger, \beta^\dagger)$ is also equivariant.

Theorem 3.3. *Let G be a second countable locally compact group and consider the extension* (3.1) *of G-algebras. Let (S, δ) be a G-algebra with S separable, and let $\psi\colon S \to B$ be a completely positive linear contraction. There is an asymptotically (G, ψ)-equivariant lift $\Theta = (\Theta_t)_{t\in[1,\infty)}\colon S \to A$ of ψ with the following additional properties:*

(a) *For all $s, s' \in S$, we have*
$$\lim_{t\to\infty} \|g \cdot \Theta_t(s) - \Theta_t(h \cdot s')\| = \|g \cdot \psi(s) - \psi(h \cdot s')\| \ ,$$
uniformly for g, h in compact subsets of G;

(b) *for all $n \in \mathbb{N}$ and all $s \in M_n(S)$, we have*
$$\lim_{t\to\infty} \|(\Theta_t \otimes \mathrm{id}_{M_n})(s)\| = \|(\psi \otimes \mathrm{id}_{M_n})(s)\| \ .$$

(c) *for $s, s' \in S$, we have $\lim_{t\to\infty} \|\Theta_t(s)\Theta_t(s')\| = \|\psi(s)\psi(s')\|$;*

(d) *if I is σ-unital, then $\lim\limits_{t\to\infty} \Theta_t(s)x = 0$ for all $s \in S$ and $x \in I$.*

Proof. From (3.1) we get the extension
$$0 \longrightarrow (I, \gamma) \xrightarrow{\iota} (A^\dagger, \alpha^\dagger) \xrightarrow{q^\dagger} (B^\dagger, \beta^\dagger) \longrightarrow 0$$
of G-algebras. Moreover, $\psi^\dagger\colon S^\dagger \to B^\dagger$ is a unital linear completely positive contraction satisfying $\chi_S = \chi_B \circ \psi^\dagger$. Using that χ_S is G-invariant, let $\Theta' = (\Theta'_t)_{t\in[1,\infty)}\colon S^\dagger \to A^\dagger$ be a unital asymptotically $(G, \psi^\dagger)$-equivariant lift of $\psi^\dagger$ to $A^\dagger$ satisfying the conclusions from Theorem 3.1. Given $s \in S$, we have
$$\chi_A \circ \Theta'_t(s) = \chi_B \circ q^\dagger \circ \Theta'_t(s) = \chi_B \circ \psi^\dagger(s) = \chi_S(s) = 0 \ .$$
Hence, the restriction of Θ' to S, which we denote by Θ, is an asymptotically (G, ψ)-equivariant lift of ψ to A. Parts (b), (c) and (d) of this theorem follow, respectively, from (b), (c) and (d) in Theorem 3.1, since $S = \ker(\chi_S)$. Part (a), which is a strengthening of asymptotic

(G,ψ)-equivariance, follows from (a) in Theorem 3.1 together with the fact that

$$\begin{aligned}\|g\cdot\psi(s)-\psi(h\cdot s')\| &= \|q(g\cdot\Theta_t(s)-\Theta_t(h\cdot s'))\|\\ &\le \|g\cdot\Theta_t(s)-\Theta_t(h\cdot s')\|\end{aligned}$$

for all $g,h\in G$, all $s,s'\in S$, and all $t\in[1,\infty)$. □

In the same way, Theorem 3.2 gives the following:

Theorem 3.4. *In the setting of Theorem 3.3, assume that there is a linear completely positive contractive lift $\psi'\colon S\to A$ for ψ. There is a completely positive asymptotically (G,ψ)-equivariant lift*

$$\Theta=(\Theta_t)_{t\in[1,\infty)}\colon S\to A$$

of ψ with the additional properties (a), (b), (c) and (d) of Theorem 3.3.

4. Unital asymptotic sections and amenability

Let $q\colon(A,\alpha)\to(B,\beta)$ be a surjective $*$-homomorphism between G-algebras. An *asymptotically equivariant linear section* for q is an asymptotically linear asymptotic contraction $\Theta=(\Theta_t)_{t\in[1,\infty)}\colon B\to A$ such that $q\circ\Theta_t=\mathrm{id}_B$ for all t and such that

$$\lim_{t\to\infty}\Theta_t(g\cdot b)-g\cdot\Theta_t(b)=0$$

for all $g\in G$ and all $b\in B$.

Theorem 3.3 guarantees the existence of an asymptotically equivariant linear section for any extension of G-algebras (3.1) with B separable, and in fact one which is also asymptotically completely positive. However, when A is unital so that B is also unital, it is natural to look for an asymptotically equivariant linear section Θ which is also unital, or at least *asymptotically unital* in the sense that

$$\lim_{t\to\infty}\Theta_t(1)=1\ .$$

The section provided by Theorem 3.3 is never unital, as item (d) shows. Theorem 3.1, on the other hand, does provide a unital asymptotically equivariant linear section and in fact one which is also asymptotically completely positive, provided there is a G-invariant state on B. When G is amenable, it is a consequence of Day's fixed point theorem that every unital G-algebra has an invariant state. Therefore, when G is amenable, Theorem 3.1 guarantees the existence of a unital asymptotically equivariant linear section for any equivariant surjective $*$-homomorphism $q\colon(A,\alpha)\to(B,\beta)$ whenever A is unital and B separable. In fact, said theorem provides a unital asymptotically (G,id_B)-equivariant lift of id_B. We show next that this property characterises amenability within the class of second countable locally compact groups.

Theorem 4.1. *Let G be a second countable locally compact group. The following are equivalent:*

(1) *For every compact metrizable G-space X, the natural extension*

$$0 \to \mathbb{C} \to C(X) \oplus \mathbb{C} \to C(X) \to 0$$

admits an asymptotically equivariant linear section which is also aymptotically unital.

(2) *For every extension* (3.1) *of G-algebras with B separable and A unital there is a unital asymptotically (G, id_B)-equivariant lift of id_B.*

(3) *Every G-algebra (B, β) with B unital and separable has a G-invariant state.*

(4) *G is amenable.*

Proof. The implication (4) $\Rightarrow$ (3) follows from the characterization of amenability mentioned above and (3) $\Rightarrow$ (2) follows from Theorem 3.1. Since (2) $\Rightarrow$ (1) is trivial it suffices to show that (1) $\Rightarrow$ (4). Assume therefore that G has the property stipulated in (1), and let X be a compact, metrizable G-space. Then $C(X)$ is a G-algebra in the natural way and $C(X)$ is separable. Let $p_1 \colon C(X) \oplus \mathbb{C} \to C(X)$ and $p_2 \colon C(X) \oplus \mathbb{C} \to \mathbb{C}$ be the canonical projections. By assumption, there is an asymptotically equivariant linear section $\Theta \colon C(X) \to C(X) \oplus \mathbb{C}$ for p_1 which is also asymptotically unital. By exchanging $\Theta_t(f)$ with

$$\frac{1}{2}\left(\Theta_t(f) + \Theta_t(f^*)^*\right)$$

for $f \in C(X)$, we may assume that Θ is self-adjoint. Let $C_b[1, \infty)$ be the C^*-algebra of continuous bounded functions on $[1, \infty)$ and denote by $C_0[1, \infty)$ the ideal in $C_b[1, \infty)$ consisting of the functions vanishing at infinity. Define $\Phi \colon C(X) \to C_b[1, \infty)$ by $\Phi(f)(t) = p_2(\Theta_t(f))$ for $f \in C(X)$, and let

$$\pi \colon C_b[1, \infty) \to C_b[1, \infty)/C_0[1, \infty)$$

be the quotient map. Then

$$\pi \circ \Phi \colon C(X) \to C_b[1, \infty)/C_0([1, \infty)$$

is a linear unital self-adjoint contraction satisfying that

$$(\pi \circ \Phi)(g \cdot f) = (\pi \circ \Phi)(f)$$

for all $g \in G$ and all $f \in C(X)$. Let ω be a state of $C_b[1, \infty)/C_0[1, \infty)$. The composition $\omega \circ \pi \circ \Phi$ is then a self-adjoint G-invariant linear contraction into $\mathbb{C}$ which takes 1 to 1, and it is therefore a state on $C(X)$. It follows that X has a G-invariant Borel probability measure. We have shown that all compact metrizable G-spaces have a G-invariant Borel probability measure which is one of the many equivalent conditions for amenability of G. □

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