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Efimov spaces and the separable quotient problem for spaces $C_p(K)$

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EFIMOV SPACES AND THE SEPARABLE QUOTIENT PROBLEM FOR SPACES $C_p(K)$

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Abstract. The classic Rosenthal-Lacey theorem asserts that the Banach space $C(K)$ of continuous real-valued maps on an infinite compact space K has a quotient isomorphic to c or ℓ_2. In [22] we proved that the space $C_p(K)$ endowed with the pointwise topology has an infinite-dimensional separable quotient algebra iff K has an infinite countable closed subset. Hence $C_p(\beta\mathbb{N})$ lacks infinite-dimensional separable quotient algebras. This motivates the following question: $(*)$ *Does $C_p(K)$ admit an infinite-dimensional separable quotient* (shortly *SQ*) *for any infinite compact space K?* Particularly, does $C_p(\beta\mathbb{N})$ admit *SQ*? Our main theorem implies that $C_p(K)$ has *SQ* for any compact space K containing a copy of $\beta\mathbb{N}$. Consequently, this result reduces problem $(*)$ to the case when K is an *Efimov space* (i.e. K is an infinite compact space that contains neither a non-trivial convergent sequence nor a copy of $\beta\mathbb{N}$). Although, it is unknown if Efimov spaces exist in ZFC, we show, making use of some result of R. de la Vega (2008) (under $\diamondsuit$), that for some Efimov space K the space $C_p(K)$ has *SQ*. Some applications of the main result are provided.

1. Preliminaries

One of famous unsolved problems of Functional Analysis (posed by S. Mazur 1932) asks (*) *whether any infinite-dimensional Banach space has an infinite-dimensional separable (Hausdorff) quotient* (in short SQ)?

We refer to [31] and [29], [21] (and references there) concerning several aspects related with problem (*) for Banach spaces. Clearly:

(**) *A Banach space X has SQ if and only if X is mapped on an infinite-dimensional separable Banach space under a continuous linear map.*

While this problem still remains open, several concrete classes of Banach spaces admit infinite-dimensional separable quotients. For example, all infinite-dimensional reflexive (or WCG) Banach spaces are of that type. With Mazur's problem yet unsolved, analysts since Eidelheit (1936) have studied the separable quotient problem for non-Banach spaces; see [20], [22] for more details.

Rosenthal [30] and Lacey [24] proved that for any infinite compact space K the Banach space $C(K)$ of real-valued continuous functions has a quotient isomorphic to c or ℓ_2; see also [25]. One can argue as follows:

Two cases are possible.

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(i) *K is scattered*. Then K contains a convergent sequence (x_n) of distinct points. The linear map $T : C(K) \to c, f \mapsto (f(x_n))$ is a continuous surjection. Hence the quotient $C(K)/T^{-1}(0)$ is isomorphic to c.

(ii) *K is not scattered*. Then K is continuously mapped onto $[0,1]$. The space ℓ_2 is isomorphic to a closed subspace of $L[0,1]$ and $l_1[0,1]$ is isomorphic to a closed subspace $L_1(K, \mathfrak{B}_K, \mu)$, where μ is some nonnegative finite regular Borel measure on K. The latter space is isomorphic to a closed subspace of the norm dual Y of $C(K)$. Therefore the reflexive space ℓ_2 is a subspace of Y that is weakly*-closed, and then a quotient of $C(K)$ is isomorphic to ℓ_2, see [30, Corollary 1.6, Proposition 1.2].

If E is a Banach space with SQ, then the spaces $C_p(K,E)$ and $C(K,E)$ of E-valued continuous functions over a non-empty compact K have SQ. In fact, $C_p(K,E)$ has a complemented copy of E and $C(K,E)$ has a complemented copy of c_0, see [3].

In [22, Theorem 18] we proved

Theorem 1. *For any completely regular Huasdorff space X the following are equivalent:*

(i) *$C_p(X)$ has an infinite-dimensional separable quotient algebra.*
(ii) *$C_k(K)$ has an infinite-dimensional separable quotient algebra.*
(iii) *X contains an infinite countable closed subset.*

Consequently $C_p(\beta\mathbb{N})$ does not admit an infinite-dimensional separable quotient algebra. This motivates the following natural problem, formally posed in [22].

Problem 2. *Does $C_p(K)$ have SQ for every infinite compact space K? Particularly, does $C_p(\beta\mathbb{N})$ admit SQ?*

If K contains a non-trivial convergent sequence, say $x_n \to x_0$, then for $A := \{x_n : n \in \mathbb{N}\} \cup \{x_0\}$, the space $C_p(K)$ has a quotient isomorphic to the infinite-dimensional separable (and metrizable) space $C_p(A)$. Many compact spaces contain non-trivial convergent sequences; particularly Valdivia compact spaces, by Kalenda's result [23]. They are plentiful, indeed: metrizable compact $\Rightarrow$ Eberlein compact $\Rightarrow$ Talagrand compact $\Rightarrow$ Gulko compact $\Rightarrow$ Corson compact $\Rightarrow$ Valdivia compact.

Let K be an infinite compact space. If K is scattered, then it contains a non-trivial convergent sequence. If K is not scattered, then there exists a continuous map from K onto $[0,1]$ but this property seems to be not so helpful for $C_p(K)$. Nevertheless, we show that a stronger condition (+): *K is continuously mapped onto* $[0,1]^{\mathfrak{c}}$, implies that $C_p(K)$ has SQ. Recall that the condition (+) is equivalent to the fact that K contains a copy of $\beta\mathbb{N}$, see [32].

On the other hand, we have the following easy fact (compare with (**)).

Proposition 3. *For any infinite compact K the space $C_p(K)$ can be mapped onto an infinite-dimensional separable metrizable locally convex space by a continuous linear map.*

Proof. If K is separable, $C_p(K)$ has countable pseudocharacter [1, Theorem 1.1.4]. Hence $C_p(K)$ admits a weaker metrizable and separable locally convex topology, see [15, Lemma

3.2]. If K is arbitrary, choose a compact separable infinite subset L and apply the previous case using the restriction surjective map $C_p(K) \to C_p(L)$. □

The main result of the paper is the following

Theorem 4. *Let X be a completely regular space with a sequence (K_n) of non-empty compact subsets such that for any $n \geq 1$ the set K_n contains two disjoint subsets homeomorphic to K_{n+1}. Then $C_p(X)$ has SQ. Consequently, if K is a compact space which contains a copy of $\beta\mathbb{N}$, then $C_p(K)$ has SQ.*

This implies the following

Corollary 5. *Let X be a normal topological space with a sequence (S_n) of non-empty closed subsets such that for any $n \geq 1$ the set S_n contains two disjoint closed subsets S'_n and S''_n that are homeomorphic to S_{n+1}. Then $C_p(\beta X)$ has SQ.*

Proof. Let $n \geq 1$. Denote by K_n, K'_n, K''_n and K_{n+1} the closures in βX of the sets S_n, S'_n, S''_n and S_{n+1}, respectively. Then K'_n and K''_n are compact and disjoint subsets of K_n that are homeomorphic to K_{n+1} by [12, Corollaries 3.6.4 and 3.6.8]. Using the last theorem, we infer that $C_p(\beta X)$ has SQ. □

Corollary 6. *If K is an infinite compact space and every infinite closed set in K contains two infinite disjoint homeomorphic closed sets, then $C_p(K)$ has SQ.*

Problem 2 combined with our main result is also connected with the following question of Efimov (posed in [10]):

Does every infinite compact space contain a non-trivial convergent sequence or a copy of $\beta\mathbb{N}$?

So far, there have been known several counterexamples to the above problem - called *Efimov spaces*, see e.g. Fedorchuk [13] and [14], Dow [8], [7], or Dow and Shelah [9], Geschke [16]; however no ZFC counterexample is known. We refer also to [17] for classes of compact spaces containing a copy of $\beta\mathbb{N}$. Since (as will be shown) $C_p(\beta\mathbb{N})$ admits SQ, Problem 2 reduces to the case when K is an Efimov space. Nevertheless, under $\diamondsuit$ there exists an Efimov space K such that $C_p(K)$ has SQ, see Example 15 below. Our approach to Problem 2 suggests a question whether every Efimov space contains two disjoint homeomorphic infinite closed subsets. It will be answered in the negative in Example 17.

2. The proof of Theorem 4

We need the following

Lemma 7. *Let X be a Tychonoff space. The following assertions are equivalent:*

(i) *$C_p(X)$ has SQ.*
(ii) *$C_p(X)$ admits a strictly increasing sequence of closed vector subspaces whose union is dense.*

(iii) *There exists a sequence (G_n) of non-zero continuous linear functionals on $C_p(X)$ such that the subspace $E := \bigcup_{n=1}^{\infty}\bigcap_{m=n}^{\infty}\ker G_n \subset C_p(X)$ is dense in $C_p(X)$.*

(iv) *There exists a sequence (F_n) of finite subsets of X and a sequence (f_n) of non-zero functions $f_n : F_n \to \mathbb{R}$ such that for every finite subset G of X and any function $g : G \to \mathbb{R}$ and any $\epsilon > 0$, there exists f in $C(X)$ having the following properties:*

(1) $\sum_{x\in F_n} f_n(x)f(x) = 0$ for almost all $n \in \mathbb{N}$.

(2) $|f(x) - g(x)| < \epsilon$ for all $x \in G$.

Conditions (i), (ii) and (iii) are equivalent for any locally convex space, see [20, Proposition 1], or [34]. The equivalence between (iii) and (iv) is easy to prove by using the description of the topological dual of $C_p(X)$. In fact, (iv) means that there exists a sequence (G_n) of non-zero continuous linear functionals over $C_p(X)$ such that the subspace $H := \{f \in C_p(X) : G_n(f) = 0 \text{ for almost all } n \in \mathbb{N}\}$ is dense in $C_p(X)$.

We are at position to prove Theorem 4.

Proof of Theorem 4. By Lemma 7 it is enough to show that for $C_p(X)$ the item (iii) holds

Let $F_0^1 = X$. By assumptions there exists a family $\{F_n^i : n \geq 1, 1 \leq i \leq 2\}$ of non-empty compact subsets of X such that for any $n \geq 1$ we have

(1) $F_n^1 \cup F_n^2 \subset F_{n-1}^1$;
(2) $F_n^1 \cap F_n^2 = \emptyset$;
(3) F_n^1 is homeomorphic to F_n^2.

Let $h_n^1 : F_n^1 \to F_n^1$ be the identity map and $h_n^2 : F_n^1 \to F_n^2$ be a homeomorphism for any $n \geq 1$.

Inductively with respect to $n \in \mathbb{N}$ we can define homeomorphisms h_n^i for $n \geq 2$ and $3 \leq i \leq 2^n$ such that

$$h_n^{2i-t} = (h_{n-1}^i|F_n^{2-t}) \circ h_n^{2-t}$$

for $n \geq 2$ and $1 \leq i \leq 2^{n-1}$ and $0 \leq t \leq 1$.

Put $F_n^i = h_n^i(F_n^1)$ for $n \geq 2$ and $3 \leq i \leq 2^n$; clearly $F_n^i = h_n^i(F_n^1)$ for $n \geq 1$ and $1 \leq i \leq 2$. For $n \geq 1$ and $1 \leq i \leq 2^n$ we have

$$F_{n+1}^{2i-1} = h_{n+1}^{2i-1}(F_{n+1}^1) = h_n^i(F_{n+1}^1) \subset h_n^i(F_n^1) = F_n^i$$

and

$$F_{n+1}^{2i} = h_{n+1}^{2i}(F_{n+1}^1) = h_n^i(F_{n+1}^2) \subset h_n^i(F_n^1) = F_n^i.$$

Hence $F_{n+1} \subset F_n$ for $n \geq 1$, where $F_n = \bigcup_{i=1}^{2^n} F_n^i$, and $F_n^i \cap F_n^j = \emptyset$ for $n \geq 1$ and $1 \leq i, j \leq 2^n$ with $i \neq j$.

For $n \geq 1$ and $1 \leq i, j \leq 2^n$ the map

$$h_n^{i,j} = h_n^j \circ (h_n^i)^{-1} : F_n^i \to F_n^j$$

is a homeomorphism.

By induction with respect to k we prove the following

$$(*)\quad h_{n+k}^{2^k i-t,2^k j-t} = h_n^{i,j}|F_{n+k}^{2^k i-t} \text{ for } k \geq 1, n \geq 1, 1 \leq i, j \leq 2^n \text{ and } 0 \leq t \leq 2^k - 1.$$

Let $k = 1, n \geq 1, 1 \leq i, j \leq 2^n$ and $0 \leq t \leq 1$. Then

$$h_{n+1}^{2i-t,2j-t} = h_{n+1}^{2j-t} \circ (h_{n+1}^{2i-t})^{-1} = [(h_n^j|F_{n+1}^{2-t}) \circ h_{n+1}^{2-t}] \circ [(h_n^i|F_{n+1}^{2-t}) \circ h_{n+1}^{2-t}]^{-1} =$$
$$(h_n^j|F_{n+1}^{2-t}) \circ (h_n^i|F_{n+1}^{2-t})^{-1} = [h_n^j \circ (h_n^i)^{-1}]|h_n^i(F_{n+1}^{2-t}) = h_n^{i,j}|F_{n+1}^{2i-t},$$

since

$$h_n^i(F_{n+1}^{2-t}) = h_{n+1}^{2i-t} \circ (h_{n+1}^{2-t})^{-1}(F_{n+1}^{2-t}) = h_{n+1}^{2i-t} \circ (h_{n+1}^{2-t})^{-1}(h_{n+1}^{2-t}(F_{n+1}^1)) = h_{n+1}^{2i-t}(F_{n+1}^1) = F_{n+1}^{2i-t}.$$

Assume now that (*) holds for some $k \geq 1$.

We prove that (*) holds for $k+1$.

Let $n \geq 1, 1 \leq i, j \leq 2^n$ and $0 \leq t \leq 2^{k+1} - 1$. Let $0 \leq t_1 \leq 2^k - 1$ and $0 \leq t_2 \leq 1$ with $2t_1 + t_2 = t$. Then we have

$$h_{n+k+1}^{2^{k+1}i-t,2^{k+1}j-t} = h_{(n+k)+1}^{2(2^k i-t_1)-t_2,2(2^k j-t_1)-t_2} =$$
$$h_{n+k}^{2^k i-t_1,2^k j-t_1}|F_{n+k+1}^{2(2^k i-t_1)-t_2} = (h_n^{i,j}|F_{n+k}^{2^k i-t_1})|F_{n+k+1}^{2^{k+1}i-t} = h_n^{i,j}|F_{n+k+1}^{2^{k+1}i-t}.$$

Let $(x_n) \subset X$ be a sequence with $x_n \in F_n^1$ for $n \geq 1$. Put $x_n^i = h_n^i(x_n)$ for $n \geq 1$ and $1 \leq i \leq 2^n$. Then

(1) $x_n^i \in F_n^i \subset F_n$ for $n \geq 1$ and $1 \leq i \leq 2^n$ and
(2) $h_n^{i,j}(x_n^i) = x_n^j$ for $n \geq 1$ and $1 \leq i, j \leq 2^n$.

Hence, by $(*)$, we deduce that

$$(**) \;\; h_n^{i,j}(x_{n+k}^{2^k i-t}) = x_{n+k}^{2^k j-t} \;\text{ for }\; k \geq 1, n \geq 1 \;\text{ and }\; 1 \leq i, j \leq 2^n.$$

Let $P_x : C_p(X) \to K$ be a map defined by the formula $f \to f(x)$, for $x \in X$. Next define

$$G_n : C_p(X) \to K, G_n = \sum_{i=1}^{2^n} P_{x_n^i}$$

for all $n \geq 1$. Clearly (G_n) is a sequence of non-zero continuous linear functionals on the space $C_p(X)$.

We prove that the subspace

$$E = \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} \ker G_n$$

is dense in $C_p(X)$. To prove this it it is enough to show that for any $n \geq 1$ and for all n different points $z_1, \ldots, z_n$ of X there exists $g \in E$ with $g(z_1) = 1$ and $g(z_i) = 0$ for $1 < i \leq n$.

Consider two cases.

Case 1: $z_1 \notin F_k$ for some $k \geq 1$. Then there exists $g \in C(X)$ with $g(z_1) = 1, g(z_i) = 0$ for $1 < i \leq n$ and $g|F_k = 0$. For $m \geq k$ and $1 \leq i \leq 2^m$ we have

$$x_m^i \in F_m^i \subset F_m \subset F_k,$$

so $g(x_m^i) = 0$. Hence $G_m(g) = 0$ for $m \geq k$, so $g \in E$.

Case 2: $z_1 \in \bigcap_{k=1}^{\infty} F_k$. Then there exist $1 \leq i_0, j_0 \leq 2^n$ with $z_1 \in F_n^{i_0}$ and $z_i \notin F_n^{j_0}$ for $1 \leq i \leq n$. Thus there exists $g_1 \in C(F_n^{i_0})$ with $g_1(z_1) = 1$ and $g_1(z_i) = 0$ for $1 < i \leq n$ with $z_i \in F_n^{i_0}$.

Let $g_2 \in C(F_n)$ with

$$g_2|F_n^{i_0} = g_1, \;\; g_2|F_n^{j_0} = -g_1 \circ (h_n^{i_0,j_0})^{-1}$$

and $g_2|F_n^i = 0$ for $1 \le i \le 2^n$ with $i_0 \ne i \ne j_0$. Then there exists $g \in C(X)$ with $g|F_n = g_2$ such that $g(z_i) = 0$ for $1 < i \le n$ with $z_i \notin F_n$. Clearly $g(z_1) = 1$ and $g(z_i) = 0$ for $1 < i \le n$. For $k \ge 1$ we have

$$G_{n+k}(g) = \sum_{i=1}^{2^{n+k}} g(x_{n+k}^i) = \sum_{t=0}^{2^k-1} g_1(x_{n+k}^{2^k i_0 - t}) - \sum_{t=0}^{2^k-1} g_1 \circ (h_n^{i_0,j_0})^{-1}(x_{n+k}^{2^k j_0 - t}) = 0,$$

since $h_n^{i_0,j_0}(x_{n+k}^{2^k i_0 - t}) = x_{n+k}^{2^k j_0 - t}$ for $0 \le t \le 2^k - 1$. Thus one gets $g \in E$. □

Since every extremally disconnected compact space K is an F-space, K contains a copy of $\beta\mathbb{N}$, so we have

Corollary 8. *If K is an extremally disconnected compact space, then $C_p(K)$ has SQ.*

Let X be a completely regular space that is not pseudocompact. It is well-known that $\beta X \setminus X$ contains a copy of $\beta\mathbb{N} \setminus \mathbb{N}$. Moreover (as easily seen), $C_p(X)$ contains a complemented copy of $\mathbb{R}^{\mathbb{N}}$.

Corollary 9. *If X is a completely regular space that is not pseudocompact, then $C_p(X)$ and $C_p(\beta X \setminus X)$ have SQ.*

Theorem 4 yields also the following

Corollary 10. *Let X be a completely regular space containing an infinite compact subset K such that:*

1. *For any infinite closed subset S of K there exists an infinite closed subset T of S and a non-trivial continuous injection $\phi : T \to S$; or*
2. *For any non-empty open subset U of K there exists an open subset V of U and a non-trivial continuous injection $\phi : V \to U$.*

Then $C_p(X)$ has SQ.

Proof. By (1) any infinite compact subset S of K contains two infinite disjoint compact and homeomorphic subsets. By (2) any non-empty open subset U of K contains two disjoint compact and homeomorphic subsets such that at least one of them has non-empty interior with respect to K. Hence there exists a sequence (K_n) of infinite compact subsets of X such that K_n contains two disjoint subsets homeomorphic to K_{n+1} for any $n \in N$. Now Theorem 4 applies. □

Corollary 11. *Let X be a locally compact space. Assume that X contains a non-empty open subset U such that every non-empty open subset V of U has two different points x, y that have some homeomorphic disjoint neighbourhoods V_x and V_y. Then $C_p(X)$ has SQ.*

Let X be a topological space. For any $x \in X$ we assign the set $J(x)$ of all maps ϕ defined as follows: $\phi \in J(x)$ iff there exits an open neighbourhood V of x such that

$\phi : V \to X$ is injective and continuous. The set $O(x) = \{\phi(x) : \phi \in J(x)\}$ will be called the *orbit* of x.

Another consequence of Theorem 4 is the following

Theorem 12. *Let X be an infinite compact space. Assume that the orbit $O(x)$ of every element x of X is infinite. Then $C_p(X)$ has SQ.*

We need the following

Lemma 13. *Let X be a completely regular space. Assume that X contains an element x such that: (1) x is an accumulation point of its orbit $O(x)$; (2) x has a neighbourhood V_0 such that its closure $K_0 = \overline{V_0}$ is compact. Then $C_p(X)$ has SQ.*

Proof. Let $\phi_1 \in J(x)$ with $\phi_1(x) \in (V_0 \setminus \{x\})$. Let V_1 and W_1 be neighbourhoods of x such that

$$\overline{V_1} \subset W_1 \subset V_0 \cap \phi_1^{-1}(V_0) \text{ and } W_1 \cap \phi_1(W_1) = \emptyset.$$

Next, let $\phi_2 \in J(x)$ with $\phi_2(x) \in (V_1 \setminus \{x\})$. Let V_2 and W_2 be neighbourhoods of x such that

$$\overline{V_2} \subset W_2 \subset V_1 \cap \phi_2^{-1}(V_1) \text{ and } W_2 \cap \phi_2(W_2) = \emptyset.$$

Continuing on this manner we obtain a sequence $(\phi_n) \subset J(x)$ and two sequences (V_n) and (W_n) of neighbourhoods of x such that

$$\overline{V_n} \subset W_n \subset V_{n-1} \cap \phi_n^{-1}(V_{n-1}) \text{ and } W_n \cap \phi_n(W_n) = \emptyset.$$

Set $K_n = \overline{V_n}$ and $K'_n = \phi_n(K_n)$ for all $n \in N$. The sets K_n and K'_n are disjoint, compact and homeomorphic subsets of K_{n-1} for $n \in N$. By Theorem 4 we conclude that $C_p(X)$ has SQ. □

Proof of Theorem 12. By A^d we denote the set of all accumulation points of a subset A of X.

If $x \in X$ and $y \in O(x)^d$, then $O(y)^d \subset O(x)^d$. Indeed, let $z \in O(y)^d$. Let W_z be a neighbourhood of z. Then there exists a neighbourhood V_y of y and a continuous injection $\phi : V_y \to X$ with $\phi(y) \in (W_z \setminus \{0\})$. Then $W_y = \phi^{-1}(W_z)$ is a neighbourhood of y. Since $y \in O(x)^d$, there exists a neighbourhood V_x of x and a continuous injection $\psi : V_x \to X$ with

$$\psi(x) \in (W_y \setminus \{y, \phi^{-1}(z)\}).$$

The set $W_x = \psi^{-1}(\phi^{-1}(W_z))$ is a neighbourhood of x and $\phi \circ \psi | W_x$ is a continuous injection from W_x to X. Clearly, $\phi \circ \psi(x) \in (W_z \setminus \{z\})$. Consequently $z \in O(x)^d$. We proved that $O(y)^d \subset O(x)^d$.

For any $x \in X$ the set $O(x)^d$ is non-empty and compact. The set $\Phi = \{O(x)^d : x \in X\}$ is ordered by inclusion. By Kuratowski-Zorn Lemma the family Ψ of all linearly ordered subsets of Φ has a maximal element. Let $\Omega = \{O(x_\gamma)^d : \gamma \in \Gamma\}$ be a maximal linearly ordered subset of Φ. The set $K = \bigcap_{\gamma \in \Gamma} O(x_\gamma)^d$ is non-empty. Let $x \in K$ and $y \in O(x)^d$. Then $O(y)^d \subset O(x)^d \subset O(x_\gamma)^d$ for any $\gamma \in \Gamma$. By maximality of Ω we have $O(y)^d = O(x_\gamma)^d$ for some $\gamma \in \Gamma$. Hence $O(y)^d = O(x)^d$, so $y \in O(y)^d$. By Lemma 13, $C_p(X)$ has SQ. □

3. Remarks, questions and examples

(A) A Banach space E is called a *Grothendieck space* [6] if every null sequence in the weak*-dual of E converges to zero in the weak topology of the dual of E. It is well know that $C(K)$ is not a Grothendieck space if a compact space K contains a non-trivial convergent sequence. If K is extremally disconnected (in that case K contains a copy of $\beta\mathbb{N}$), the space $C(K)$ is a Grothendieck space, see again [6]. On the other hand, Talagrand [35] constructed under (CH) a compact space K such that $C(K)$ is a Grothendieck space and yet $C(K)$ does not admit any quotient isomorphic to ℓ_∞, particularly K does not contain a copy of $\beta\mathbb{N}$. Hence this K is an Efimov space. This example combined with our main Theorem 4 motivates the following

Problem 14. *Does $C_p(K)$ admit SQ if $C(K)$ is a Grothendieck Banach space?*

(B) From Lemma 7 (iv) and Theorem 4 we deduce for any compact space K:

$$\textit{non (iv)} \Leftrightarrow C_p(K) \textit{ fails } SQ \Rightarrow K \textit{ is Efimov.}$$

Example 15. *There exists (under $\diamondsuit$) an Efimov space K such that $C_p(K)$ has SQ.*

Proof. De la Vega [5, Theorem 3.22] (we refer also to [4]) constructed (under $\diamondsuit$) a compact zero-dimensional S-space K (hence not containing a copy of $\beta\mathbb{N}$) and such that:

(i) K does not contain non-trivial convergent sequences.
(ii) K has a base of clopen pairwise hemeomorphic sets.
(iii) K contains non homeomorphic clopen subsets.

It is easy to see that K admits a sequence (K_n) of infinite compact subsets such that each K_n contains two disjoint subsets homeomorphic to K_{n+1}; therefore by our main Theorem 4 the space $C_p(K)$ has SQ. □

There exist however compact zero-dimensional spaces K without non-trivial convergent sequences for which no disjoint open sets are homeomorphic, see [28, Theorem], see also [2]. Last Example 15 motivates the following

Problem 16. *Does there exist an Efimov space K such that $C_p(K)$ does not admit SQ?*

Moreover, Corollary 6 may suggest also the following variant of Efimov's problem: *Does every infinite compact space K without a non-trivial convergent sequence contain two infinite disjoint homeomorphic closed subsets?*

The authors were kindly informed by Professor P. Koszmider about the following example answering the above problem. Recall that a compact space K is a *Koszmider space*, see [11], if all operators on $C(K)$ have the form $gI + S$, where $g \in C(K)$ and S is weakly compact. *If K is a connected Koszmider space then $C(K)$ is* indecomposable*, i.e. there are no infinite-dimensional closed subspaces Y and Z such that $C(K) = Y \oplus Z$*, see [11, Lemma 2.6].

Example 17. *Under $\diamondsuit$ there exists a separable Efimov space F such that F is a Koszmider space and does not admit two disjoint homeomorphic infinite closed subsets.*

Proof. Let K be the compact connected space as in [11, Theorem 5.2]. Let F be an infinite separable closed subset of K. Assume that F contains two closed infinite disjoint homeomorphic subsets L_1 and L_2. Put $L := L_1 \cup L_2$. This generates a homeomorphism $\phi : L \to L$ which is not the identity. Then the composition operator $C_\phi : C(L) \to C(L)$, $C_\phi(g) := g \circ \phi$, provides an operator which contradicts [11, Theorem 5.3]. Then F does not contain $\beta\mathbb{N}$. Moreover, F does not contain non-trivial convergent sequences. Indeed, otherwise $C(K)$ is not a Grothendieck space, so $C(K)$ contains a complemented copy of c_0, see [3, Corollary 2], so $C(K)$ is not indecomposable, a contradiction with the above remark. □

Remark 18. *As every separable compact space is a continuous image of $\beta\mathbb{N}$, the space F from above Example 17 enjoys this property. Therefore we conclude that the construction provided by Theorem 4 (which applies to $\beta\mathbb{N}$) is not inherited by continuous open surjections.*

Having in mind that $C_p(\beta\mathbb{N})$ has SQ we note also the following

Proposition 19. *The following assertions are equivalent:*

(i) $C_p(b\mathbb{N})$ *has SQ for any compactification $b\mathbb{N}$ of* $\mathbb{N}$.
(ii) $C_p(K)$ *has SQ for any infinite compact K.*

Proof. Assume that (i) holds. Then for any infinite compact space K the space $C_p(K)$ has SQ. Indeed, as K contains a discrete infinite subset (in the induced topology), hence homeomorphic to $\mathbb{N}$, so its closure in K provides some compactification $b\mathbb{N}$. Clearly $C_p(K)$ has a quotient isomorphic to $C_p(S)$, so $C_p(K)$ has SQ. The converse is trivial. □

(C) A topological space X is a σ-space, see [26], if X has a network composing a σ-locally finite family of subsets of X. Recall also that the Alexandrov-Urysohn compacta (AU-compacta) are separable uncountable compact spaces whose set of all accumulation points has exactly one non-isolated point, see [26]. In [26, Theorem 3.4, Section 3.4] the authors proved that $C_p(K)$, where K is the AU-compacta $K(2^{<\omega})$ associated with the Cantor tree, is a σ-space. There exist however AU-compacta $K := K(\omega^{<\omega})$ associated with a Baire tree such that $C_p(K)$ is not perfect, hence not a σ-space, [26, Theorem 3.4]. Also by [26, Theorem 5.11] the space $C_p(K)$ over a dyadic separable compacta is a σ-space and yet K has non-trivial convergent sequences.

For this cases we know that $C_p(K)$ can be mapped by a continuous and open linear map onto a separable and metrizable infinite dimensional locally convex space, and clearly every metrizable and separable space is a σ-space. One may ask *whether for every infinite separable compact space K not containing non-trivial convergent sequences and such that $C_p(K)$ has SQ the space $C_p(K)$ is a σ-space.* The answer is negative, as our Theorem 4 shows that $C_p(\beta\omega)$ has SQ while $C_p(\beta\omega)$ *is not a σ-space;* the latest follows from the results in [26, Sec. 3], cf. also [27, Prop. 5.2]. So, there exist compact spaces K for which $C_p(K)$ have SQ (even metrizable) and some of those $C_p(K)$ are perfect (even σ-spaces) while some are not. We conclude with the following.

Problem 20. *Is $C_p(K)$ a σ-space, if K is a separable Efimov space?*

References

[1] A. V. Arkhangel'skii, *Topological function spaces*, Math. and its Appl., Kluwer, 1992.

[2] G. Brenner, *A simple construction for ridig and weakly homogenous Bollean algebras answering a question of Rubin*, Proc. Amer. Math. Soc. **87** (1983), 601–606.

[3] P. Cembranos, *$C(K, E)$ contains a complemented copy of c_0*. Proc. Amer. Math. Soc. **91** (1984), 556-558.

[4] De la R. Vega, *Basic homogeneity properties in the class of zero-dimensional compact spaces*, Topology Appl. **155** (2008), 225–232.

[5] De la R. Vega, *Homogeneity properties on compact spaces*, Disserationes (2005) University of Wisconsin-Madison.

[6] H. G. Dales, F. K . Dashiell, M. Lau, D. Strauss, *Banach Spaces of Continuous Functions a Dual space*, Can. Math. Soc. 2016.

[7] A. Dow, *Efimov spaces and the splitting number*, Topology Proc. **29** (2005), 105-113.

[8] A. Dow, *Compact sets without converging sequences in the random real model*, Acta Math. Univ. Comenianae **76** (2007), 161-171

[9] A. Dow, S. Shelah, *An Efimov space from Martins Axiom*, Houston J. Math. **39** (2013), 1423-1435.

[10] B. Efimov, *Subspaces of dyadic bicompacta*, Doklady Akademiia Nauk USSR **185** (1969), 987-990 (Russian); English transl., Soviet Mathematics. Doklady **10** (1969), 453-456.

[11] R. Fajardo *Quotients of indecomposable Banach spaces of continuous functions*, Studia Math. **212** (2012), 259–283.

[12] R. Engelking, *General topology*, Heldermann Verlag, Berlin, 1989.

[13] V. V. Fedorcuk, *A bicompactum whose infinite closed subsets are all n-dimensional*, Mathematics of the USSR. Sbornik **25** (1976), 37-57.

[14] V. V. Fedorcuk, *Completely closed mappings, and the consistency of certain general topology theorems with the axioms of set theory*, Mathematics of the USSR. Sbornik **28** (1976), 3-33.

[15] S. Gabriyelyan, J. Kąkol, W. Kubiś, M. Marciszewski, *Networks for the weak topology of Banach and Fréchet spaces*, J. Math. Anal. Appl.**142** (2015), 1183–1199.

[16] S. Geschke, *The coicoinitialities of Efimov spaces* Set Theory and its Applications, Babinkostova et al., Editors, Contemporary Mathematics **533** (2011), 259–265.

[17] M. Gillman, M. Jerison, *Rings of Continuous Functions*, Grad. Texts in Math.,**43**, Springer, Berlin, 1976.

[18] P. Hart, *Efimov's problem*, Problems in Topology II Edited by E. Pearl, pages 171–177, 2007.

[19] W. B. Johnson, H. P. Rosenthal, *On weak*-basic sequences and their applications to the study of Banach spaces*, Studia Math. **43** (1975), 166–168.

[20] J. Kąkol, and W. Śliwa, *Remarks concerning the separable quotient problem*, Note Mat. **13** (1993), 277–282.

[21] J. Kąkol, S. A. Saxon and A. Tood, *Barrelled spaces with(out) separable quotients*, Bull. Aust. Math. Soc. **90** (2014), 295—303.

[22] J. Kąkol, S. A. Saxon, *Separable quotients in $C_c(X)$, $C_p(X)$ and their duals*, Proc. Amer. Math. Soc. **145** (2017), 3829–3841.

[23] O. Kalenda, *A characterization of Valdivia compact spaces*, Collect. Math. **51** (2000), 59–81.

[24] E. Lacey, *Separable quotients of Banach spaces*, An. Acad. Brasil. Ciènc. **44** (1972), 185–189.

[25] E. Lacey, P. Morris, *On spaces of the type A(K) and their duals*, Proc. Amer. Math. Soc. **23** (1969), 151-157

[26] W. Marciszewski, R. Pol, *On Banach spaces whose norm-open sets are F_σ-sets in the weak topology*, J. Math. Anal. Appl. **350** (2009), 708–722.

[27] W. Marciszewski, G. Plebanek, *On Borel structures in the Banach space $C(\beta\omega)$*, J. Funct. Anal. **266** (2014), 4026–4036.

[28] K. Mcaloon, *Consistency results about ordinal definability*, Anals of Math. Logic, **2** (1971), 449–467.

[29] J. Mujica, *Separable quotients of Banach spaces*, Rev. Mat. Complut. **10** (1997), 299–330.

[30] H. P. Rosenthal, *On quasi-complemented subspaces of Banach spaces, with an appendix on compactness of operators from $L_p(\mu)$ to $L_r(\nu)$*, J. Funct. Anal. **4** (1969), 176–214.

[31] S. A. Saxon, A. Wilansky, *The equivalence of some Banach space problems*, Colloq. Math. **37** (1977), 217–226.

[32] V. Shapirovsky, *On mappings onto Tychonoff cubes* (in Russian), Uspekhi Mat. Nauk. **35** (1980), 122-130.

[33] W. Śliwa, *The separable quotient problem and the strongly normal sequences*, J. Math. Soc. Japan **64** (2012), 387–397.

[34] W. Śliwa, M. Wójtowicz, *Separable Quotients of Locally Convex Spaces*, Bulletin of the Polish Academy of Sciences Math., **43** (1995), 175–185.

[35] M. Talagrand, *Un nouvean $C(K)$ qui possede la propriete de Grotjhendieck*, Israel J. Math. **37** (1980), 181–191.

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