



INSTITUTE OF MATHEMATICS

THE CZECH ACADEMY OF SCIENCES

**Reduction cohomology
of Riemann surfaces**

Alexander Zuevsky

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REDUCTION COHOMOLOGY OF RIEMANN SURFACES

A. ZUEVSKY

ABSTRACT. We study the algebraic conditions leading to the chain property of complexes for vertex algebra n -point functions with differential being defined through reduction formulas. The notion of the reduction cohomology of Riemann surfaces is introduced. Algebraic, geometrical, and cohomological meanings of reduction formulas is clarified. A counterpart of the Bott-Segal theorem for Riemann surfaces in terms of the reductions cohomology is proven. It is shown that the reduction cohomology is given by the cohomology of n -point connections over the vertex algebra bundle defined on a genus g Riemann surface $\Sigma^{(g)}$. The reduction cohomology for a vertex algebra with formal parameters identified with local coordinates around marked points on $\Sigma^{(g)}$ is found in terms of the space of analytical continuations of solutions to Knizhnik-Zamolodchikov equations. For the reduction cohomology, the Euler-Poincare formula is derived. Examples for various genera and vertex cluster algebras are provided.

1. INTRODUCTION

The natural problem of computation continuous cohomologies for non-commutative structures on manifolds has proven to be a subject of great geometrical interest [BS, Kaw, PT, Fei, Fuks, Wag]. As it was demonstrated in [Fei, Wag], the ordinary Gelfand-Fuks cohomology of the Lie algebra of holomorphic vector fields on complex manifolds turns to be not the most effective and general one. For Riemann surfaces, and even for higher dimension complex manifolds, the classical cohomology of vector fields becomes trivial [Kaw]. The Lie algebra of holomorphic vector fields is not always an interesting Lie algebra. For example, it is zero for a compact Riemann surface of genus greater than one, and one looks for other algebraic objects having locally the same cohomology. In [Fei] Feigin obtained various results concerning (co)-homology of the Lie algebra cosimplicial objects of holomorphic vector fields $Lie(M)$. In spite results in previous approaches, it is desirable to find a way to enrich cohomological structures which motivates constructions of more refined cohomology description for non-commutative algebraic structures. In the seminal paper [BS], the authors have been proven that the Gelfand-Fuks cohomology $H^*(Vect(M))$ of vector fields on a smooth compact manifold M is isomorphic to the singular cohomology of the space of continuous cross sections of a certain fibre bundle over M . An important problem of revealing relations between non-commutative structures and geometrical objects on complex manifolds still remains underinvestigeted in the literature [PT].

Key words and phrases. Cohomology; Vertex algebras; Riemann surfaces; Cluster algebras.

The main idea of this paper is to introduce and compute the reduction cohomology of vertex algebras [B, FLM, FHL, K] with formal parameters considered as local coordinates on a genus g compact Riemann surface [FK, Bo, Gu, A] to study cohomology of spaces of converging functions with respect to adding new sets of pairs of vertex algebra elements and corresponding formal parameters [BZF, DGM, DL, EO, FMS]. Due to structure of correlations functions [FMS] and reduction relations [Y, Zhu, MTZ, GT, TW] among them, one can form chain complexes of n -point functions that are fine enough to describe local geometry of Riemann surfaces. Another meaning of the reduction cohomology of Riemann surfaces is how sections of certain bundles with values in higher genus generalizations of elliptic functions change along growing of number of marked points on a surface. In addition to that we prove a version of the Bott-Segal [BS, Wag, PT] theorem for compact Riemann surfaces of arbitrary genus. It relates the reduction cohomology with cohomology of a space of sections of certain vertex algebra bundle [BZF]. Our new vertex algebra approach to cohomology of compact Riemann surfaces involves Lie-algebraic formal series with applications of techniques used in surgery of spheres [Huang]. In contrast to more geometrical methods, e.g., in ordinary cosimplicial cohomology for Lie algebras [Fei, Wag], the reduction cohomology pays more attention to the analytical structure of elements of chain complex spaces. Computational methods involving reduction formulas proved their effectiveness in conformal field theory, geometrical descriptions of intertwined modules for Lie algebras, and differential geometry of integrable models. In section 2 we give the definition of the reduction cohomology and prove a proposition relating it to cohomology of a vertex algebra bundle in terms of n -point connections. In section 3 the main proposition expressing the reduction cohomology in terms of spaces of auxiliary functions on Riemann surfaces is proven. Results of this paper are useful for cosimplicial cohomology theory of smooth manifolds, generalizations of the Bott-Segal theorem have their consequences in conformal field theory [Fei, Wag], deformation theory [O], non-commutative geometry, modular forms, and the theory of foliations.

2. CHAIN COMPLEX FOR VERTEX ALGEBRA n -POINT FUNCTIONS

2.1. Spaces of n -point correlation functions. In this section we recall definitions and some properties of correlation functions for vertex algebras on Riemann surfaces [FHL, Zhu, FMS, DGM]. Let us fix a vertex algebra V . Depending on its commutation relations and configuration of a genus g Riemann surface $\Sigma^{(g)}$, the space of all V n -point functions can contain various forms of complex functions defined on $\Sigma^{(g)}$.

We denote by $\mathbf{v}_n = (v_1, \dots, v_n) \in V^{\otimes n}$ a tuple of vertex algebra elements (see Appendix 5 for definition of a vertex algebra). Mark n points $\mathbf{p}_n = (p_1, \dots, p_n)$ on a Riemann surface of genus $\Sigma^{(g)}$. Denote by $\mathbf{z}_n = (z_1, \dots, z_n)$ local coordinates around $\mathbf{p}_n \in \Sigma^{(g)}$. Let us introduce the notation: $\mathbf{x}_n = (\mathbf{v}_n, \mathbf{z}_n)$. On a genus g Riemann surfaces an $n \geq 0$ -point correlation function $\mathcal{F}_n^{(g)}(\mathbf{x}_n, B^{(g)})$ has certain specific form depending on g , construction of a Riemann surface $\Sigma^{(g)}$, type of conformal field theory model used for definitions of n -point functions, and the type of commutation

relations for V -elements. In addition to that, it depends on a set of moduli parameters $B^{(g)} \in \mathcal{B}^{(g)}$ where we denote by $\mathcal{B}^{(g)}$ a subset of the moduli space of genus g Riemann surfaces $\Sigma^{(g)}$ obtained by specific ways of sewing of lower genus Riemann surfaces. In particular, for n -point functions considered on Riemann surfaces, $B^{(g)}$ is an element of the space $[\mathbf{Y}] \mathcal{B}^{(g)} = (\Omega^{(i)}; (\epsilon_i); (\rho_i); (mu_i))$, $0 \leq i \leq g$, where vectors (ϵ_i) , (ρ_i) , are Riemann surface sewing parameters $[\mathbf{Y}]$, (μ_i) are any further modular space parameters, and $\Omega^{(i)}$ is the period matrix of corresponding Riemann surface $\Sigma^{(g_i)}$ used in the procedure of $\Sigma^{(g)}$ construction.

Definition 1. For a Riemann surface of genus g , and a V -module W , we consider the spaces of n -point correlation functions

$$C_{(g)}^n(W) = \left\{ \mathcal{F}_{W,n}^{(g)}(\mathbf{x}_n, B^{(g)}), n \geq 0, \right\}.$$

In what follows, we will omit where possible W and $B^{(g)}$. The co-boundary operator $\delta_{(g)}^n(v_{n+1})$ on $C_{(g)}^n(W)$ -space is defined according to the reduction formulas for V -module W on a genus g Riemann surface (cf. particular examples in subsections 4.1–4.5, [Zhu, MTZ, GT, TW]).

Definition 2. For $g \geq 0$, $n \geq 0$, and any $x_{n+1} \in V \times \mathbb{C}$, define

$$\begin{aligned} \delta_{(g)}^n : C_{(g)}^n(W) &\rightarrow C_{(g)}^{n+1}(W), \\ \delta_n^{(g)} = H^{(g)} &= H_1^{(g)} + H_2^{(g)}, \end{aligned} \quad (2.1)$$

with non-commutative operators $H_1^{(g)}(x_{n+1})$, $H_2^{(g)}(x_{n+1})$ given by

$$\begin{aligned} H_1^{(g)}(x_{n+1}) \mathcal{F}_n^{(g)}(\mathbf{x}_n) &= \sum_{l=1}^{l(g)} f_1^{(g)}(\mathbf{x}_{n+1}, l) T_l^{(g)} \mathcal{F}_n^{(g)}(\mathbf{x}_n), \\ H_2^{(g)}(x_{n+1}) \mathcal{F}_n^{(g)}(\mathbf{x}_n) &= \sum_{k=1}^n \sum_{m \geq 0} f_2^{(g)}(\mathbf{x}_{n+1}, k, m) \\ &\quad \cdot T_k^{(g)}(v_{n+1}(m)) \mathcal{F}_{W,n}^{(g)}(\mathbf{x}_n), \end{aligned} \quad (2.2)$$

where $l(g) \geq 0$ is a constant depending on g , and the meaning of indexes $1 \leq k \leq n$, $1 \leq l \leq l(g)$, $m \geq 0$ explained below.

Operator-valued functions $f_1^{(g)}(\mathbf{x}_{n+1}, l)$, $T_l^{(g)}$, $f_2^{(g)}(\mathbf{x}_{n+1}, k, m)$, $T_k^{(g)}(v_{n+1}(m))$ depend on genus of a Riemann surface $\Sigma^{(g)}$. $T_l^{(g)}$ and $T_k^{(g)}(v(m))$ are operators of insertion of certain function of vertex algebra modes into $\mathcal{F}_{W,n}^{(g)}(\mathbf{x}_n)$ at the k -th entry:

$$T_{l,k}^{(g)}(v(m)) \mathcal{F}_{W,n}^{(g)}(\mathbf{x}_n) = \mathcal{F}_{W,n}^{(g)}\left(T_l^{(g)}(v(m)).x_n\right),$$

where we use the notation

$$(\gamma \cdot)_k x_n = (x_1, \dots, \gamma \cdot x_k, \dots, x_n),$$

for an operator γ acting on k -th entry. Note that commutation properties of $H_1^{(g)}(x_{n+1})$ and $H_2^{(g)}(x_{n+1})$ depend on genus g . The reduction formulas have the form:

$$\mathcal{F}_{W,n+1}^{(g)}(\mathbf{x}_{n+1}) = H^{(g)}(x_{n+1}) \mathcal{F}_{W,n}^{(g)}(\mathbf{x}_n), \quad (2.3)$$

for $n \geq 0$.

Remark 1. The author's conjecture (based on [Y, MT, MTZ, TZ, GT, TW, BKT]) is that for all possible configurations of a genus g Riemann surface, the form of the reduction relations (2.2) is coinvariant and it is given by a sum of operators acting on the an n -point correlation function

$$\mathcal{F}_{W,n}^{(g)}(\mathbf{x}_{n+1}) = \sum_{\substack{1 \leq l \leq l(g) \\ m \geq 0, k \geq 0}} f^{(g)}(\mathbf{x}_{n+1}, k, l, m) T^{(g)}(k, l, m) \mathcal{F}_{W,n}^{(g)}(\mathbf{x}_n).$$

Remark 2. The reductions formulas have an interpretation in terms of torsors [BZF] (Chapter 6). In such formulation \mathbf{x}_n is a torsor with respect to the group of transformation of the space $V^{\otimes n} \times \mathbb{C}^n$. In particular, from (2.3) we see that $T_k^{(g)}(u(m))$ -operators act on $V^{\otimes n}$ -entries of \mathbf{x}_n in $\mathcal{F}_{W,n}^{(g)}(\mathbf{x}_n, B^{(g)})$, while $T_{k,l}^{(g)}$ -operators act on \mathbf{z}_n of $\mathcal{F}_{W,n}^{(g)}(\mathbf{x}_n, B^{(g)})$ as a complex function.

For $n \geq 0$, let us denote by \mathfrak{A}_n the subsets of all $\mathbf{x}_{n+1} \in V^{\otimes n} \times \mathbb{C}^n$, such that the chain condition

$$H^{(g)}(x_{n+1}) H^{(g)}(x_n) \mathcal{F}_{W,n}^{(g)}(\mathbf{x}_n) = 0, \quad (2.4)$$

for the coboundary operators (2.2) for complexes $C_{(g)}^n(W)$ is satisfied.

Explicitly, the chain condition (2.4) leads to an infinite $n \geq 0$ set of equations involving functions $f_1^{(g)}(\mathbf{x}_{n+1}, l)$, $f_2^{(g)}(\mathbf{x}_{n+1}, k, m)$, and $\mathcal{F}_{W,n}^{(g)}(\mathbf{x}_n)$:

$$\begin{aligned} & \left(\sum_{l'=1, l=1}^{l(g)} f_1^{(g)}(\mathbf{x}_{n+2}, l') f_1^{(g)}(\mathbf{x}_{n+1}, l; B^{(g)}) T_{l'}^{(g)} T_l^{(g)} \right. \\ & + \sum_{l'=1, k=1}^{l(g)} \sum_{m \geq 0}^n f_1^{(g)}(\mathbf{x}_{n+2}, l') f_2^{(g)}(m, k) T_{l'}^{(g)} T_k^{(g)}(v_{n+1}(m)) \\ & + \sum_{k'=1}^{n+1} \sum_{m' \geq 0}^{l(g)} f_2^{(g)}(k', m') f_1^{(g)}(\mathbf{x}_{n+2}, l) T_{k'}^{(g)}(v_{n+1}(m')) T_l^{(g)} \\ & \left. + \sum_{k'=1}^{n+1} \sum_{k=1}^n \sum_{\substack{m' \\ m \geq 0}} f_2^{(g)}(k', m') f_2^{(g)}(k, m) T_{k'}^{(g)}(v_{n+2}(m')) T_k^{(g)}(v_{n+1}(m)) \right) \\ & \mathcal{F}_{W,n}^{(g)}(\mathbf{x}_n) = 0. \end{aligned} \quad (2.5)$$

Remark 3. (2.5) contain finite series and narrows the space of compatible n -point functions. The subspaces of $C_{(g)}^n(W)$, $g \geq 0$, $n \geq 0$, of genus g n -point functions such that the condition (2.5) is fulfilled for reduction cohomology complexes are non-empty. Indeed, for all g , the condition (2.5) represents an infinite $n \geq 0$ set of functional-differential equations (with finite number of summands) on converging complex functions $\mathcal{F}_{W,n}^{(g)}(\mathbf{x}_n)$ defined for n local complex variables on a Riemann surface of genus g with functional coefficients $f_1^{(g)}(v_{n+1}, l)$ and $f_2^{(g)}(k, m)$ (in our examples in subsection 4.1–4.5, these are generalizations of genus g elliptic functions) on $\Sigma^{(g)}$. Note that (see

examples in Section 4), all vertex algebra elements of $\mathbf{v}_n \in V^{\otimes n}$, as non-commutative parameters are not present in final form of functional-differential equations since they incorporated into either matrix elements, traces, etc. According to the theory of such equations [FK, Gu], each equation in the infinite set of (2.5) always have a solution in domains they are defined. Thus, there always exist solutions of (2.5) defining $\mathcal{F}_{W,n}^{(g)} \in C_{(g)}^n(W)$, and they are not empty.

Definition 3. The spaces with conditions (2.5) constitute a semi-infinite chain complex

$$0 \longrightarrow C_{(g)}^0 \xrightarrow{\delta_{(g)}^0} C_{(g)}^1 \xrightarrow{\delta_{(g)}^1} \dots \longrightarrow C_{(g)}^{n-1} \xrightarrow{\delta_{(g)}^{n-1}} C_{(g)}^n \longrightarrow \dots \quad (2.6)$$

For $n \geq 1$, we call corresponding cohomology

$$H_{(g)}^n = \text{Ker } \delta_{(g)}^n / \text{Im } \delta_{(g)}^{n-1}, \quad (2.7)$$

the n -th reduction cohomology of a vertex algebra V -module W on a compact Riemann surface $\Sigma^{(g)}$ of genus g .

In particular, the operators $T_l^{(g)}$, $0 \leq l \leq l(g)$, $T_k^{(g)}(u(m))$, $m \geq 0$, $1 \leq k \leq n$, form a set of generators of an infinite-dimensional continual Lie algebra $\mathfrak{g}(V)$ endowed with a natural grading indexed by l and m .

Indeed, we set the space of functions $\mathcal{F}_{W,n}^{(g)}$ as the base algebra [Sav] for the continual Lie algebra $\mathfrak{g}(V)$, and the generators as

$$\begin{aligned} X_{0,l} \left(\mathcal{F}_{W,n}^{(g)}(\mathbf{x}_n) \right) &= T_l^{(g)} \left(\mathcal{F}_{W,n}^{(g)}(\mathbf{x}_n) \right), \\ X_{k,m} \left(\mathcal{F}_{W,n}^{(g)}(\mathbf{x}_n) \right) &= T_k^{(g)}(u(m)) \left(\mathcal{F}_{W,n}^{(g)}(\mathbf{x}_n) \right). \end{aligned} \quad (2.8)$$

for $0 \leq l \leq l(g)$, $m \geq 0$, $1 \leq k \leq n$. Then the commutation relations for vertex algebra modes $v_{n+1}(m)$. in the action of operators $T_l^{(g)}$ and $T_{k,l}^{(g)}$ on v_k , $1 \leq k \leq n$ inside $\mathcal{F}_{W,n}^{(g)}$ represent the commutation relations of the continual Lie algebra $\mathfrak{g}(V)$. Jacobi identities for $\mathfrak{g}(V)$ follow from Jacobi identities (5.4) for a vertex algebra V .

Remark 4. Recall that we consider genus g Riemann surfaces resulting from combinations of sewing procedures of [Y]. Accordingly, due to [MT, TZ, TZ1, TZ2, GT, TW], corresponding genus g n -point functions are obtained coherently by combining lower genus functions. Then, relations among n -point functions of various genera appear. One is able to consider a cohomology theory taking into account such relations. For instance, for the ϵ -formalism of [Y] one has

$$\begin{aligned} C_n^{(g)}(W) &\rightarrow C_{n_1+1}^{(g_1)}(W_1) \times C_{n_2+1}^{(g_2)}(W_2), \\ \mathcal{F}_{W,n}^{(g)}(\mathbf{x}_n) &= \sum_{l \geq 0} \epsilon^l \mathcal{F}_{W,n+1}^{(g_1)}(\bar{u}, z, \mathbf{x}_n; B^{(g_1)}) \mathcal{F}_{w,n+1}^{(g_2)}(u, z, \mathbf{x}'_n; B^{(g_2)}), \end{aligned} \quad (2.9)$$

for $g = g_1 + g_2$, $n = n_1 + n_2$, and $W = W_1 \otimes W_2$. In the ρ -formalism [Y] one has

$$\begin{aligned} C_n^{(g)}(W) &\rightarrow C_{n_1+1}^{(g-1)}(W), \\ \mathcal{F}_{W,n}^{(g)}(\mathbf{x}_n) &= \sum_{l \geq 0} \rho^l \mathcal{F}_{W,n+2}^{(g-1)}(u, z, \mathbf{x}_n; \bar{u}, z, \mathbf{x}'_n; B^{(g-1)}). \end{aligned} \quad (2.10)$$

2.2. Geometrical meaning of reduction formulas and conditions (2.5). In this section we show that the reduction formulas have the form of multipoint connections generalizing ordinary holomorphic connections on complex curves [BZF].

2.2.1. Holomorphic n -point connections. Let us define the notion of a multipoint connection which will be usefull for identifying reduction cohomology in section 3. Motivated by the definition of a holomorphic connection for a vertex algebra bundle (cf. Section 6, [BZF] and [Gu]) over a smooth complex curve, we introduce the definition of the multiple point connection over $\Sigma^{(g)}$.

Definition 4. Let \mathcal{V} be a holomorphic vector bundle over $\Sigma^{(g)}$, and $\mathcal{X}_0 \subset \Sigma^{(g)}$ be its subdomain. Denote by \mathcal{SV} the space of sections of \mathcal{V} . A multi-point connection \mathcal{G} on \mathcal{V} is a \mathbb{C} -multi-linear map

$$\mathcal{G} : \left(\Sigma^{(g)}\right)^{\times n} \times V^{\otimes n} \rightarrow \mathbb{C},$$

such that for any holomorphic function f , and two sections $\phi(p)$ and $\psi(p')$ of \mathcal{V} at points p and p' on $\mathcal{X}_0 \subset \Sigma^{(g)}$ correspondingly, we have

$$\sum_{q, q' \in \mathcal{X}_0 \subset \Sigma^{(g)}} \mathcal{G}(f(\psi(q)).\phi(q')) = f(\psi(p')) \mathcal{G}(\phi(p)) + f(\phi(p)) \mathcal{G}(\psi(p')), \quad (2.11)$$

where the summation on left hand side is performed over locuses of points q, q' on \mathcal{X}_0 . We denote by Con_n the space of n -point connections defined over $\Sigma^{(g)}$.

Geometrically, for a vector bundle \mathcal{V} defined over $\Sigma^{(g)}$, a multi-point connection (2.11) relates two sections ϕ and ψ at points p and p' with a number of sections on $\mathcal{X}_0 \subset \Sigma^{(g)}$.

Definition 5. We call

$$G(\phi, \psi) = f(\phi(p)) \mathcal{G}(\psi(p')) + f(\psi(p')) \mathcal{G}(\phi(p)) - \sum_{q, q' \in \mathcal{X}_0 \subset \mathcal{X}} \mathcal{G}(f(\psi(q')).\phi(q)), \quad (2.12)$$

the form of a n -point connection \mathcal{G} . The space of n -point connection forms will be denoted by G^n .

Here we prove the following

Proposition 1. n -point correlation functions of the space $\left\{ \mathcal{F}_{W,n}^{(g)}(\mathbf{x}_n), n \geq 0 \right\}$ form n -point connections on the space of sections of the vertex algebra bundle \mathcal{V} associated to V . For $n \geq 0$, the reduction cohomology of a compact Riemann surface of genus g is

$$H_{(g)}^n(W) = H_{(g)}^n(\mathcal{SV}) = Con^n / G^{n-1}, \quad (2.13)$$

is isomorphic to the cohomology of the space of \mathcal{V} -sections.

Remark 5. Proposition 1 is a vertex algebra version of the main proposition of [BS, Wag], i.e., the Bott-Segal theorem for Riemann surfaces.

Proof. In [BZF] (Chapter 6, subsection 6.5.3) the vertex operator bundle \mathcal{V} was explicitly constructed. It is easy to see that n -point connections are holomorphic connection on the bundle \mathcal{V} with the following identifications. For non-vanishing $f(\phi(p))$ let us write (2.3) as

$$\mathcal{G}(\psi(p')) = -\frac{f(\psi(p'))}{f(\phi(p))} \mathcal{G}(\phi(p)) + \frac{1}{f(\phi(p))} \sum_{q, q' \in \mathcal{X}_0 \subset \mathcal{X}} \mathcal{G}(f(\psi(q)) \cdot \phi(q')). \quad (2.14)$$

Let us set

$$\begin{aligned} \mathcal{G} &= \mathcal{F}_{W,n}^{(g)}(\mathbf{x}_n), \\ \psi(p') &= (\mathbf{x}_{n+1}), \\ \phi(p) &= (\mathbf{x}_n), \\ \mathcal{G}(f(\psi(q)) \cdot \phi(q')) &= T_k^{(g)}(v(m)) \mathcal{F}_{W,n}^{(g)}(\mathbf{x}_n), \\ -\frac{f(\psi(p'))}{f(\phi(p))} \mathcal{G}(\phi(p)) &= \sum_{l=1}^{l(g)} f_1^{(g)(g)}(v_{n+1}, l) T_l^{(g)} \mathcal{F}_{W,n}^{(g)}(\mathbf{x}_n), \\ \frac{1}{f(\phi(p))} \sum_{\substack{q_n, q'_n \in \\ \mathcal{X}_0 \subset \Sigma^{(g)}}} \mathcal{G}(f(\psi(q)) \cdot \phi(q')) &= \sum_{k=1}^n \sum_{m \geq 0} f_2^{(g)(g)}(k, m) \\ &\quad \cdot T_k^{(g)}(v(m)) \mathcal{F}_{W,n}^{(g)}(\mathbf{x}_n). \end{aligned} \quad (2.15)$$

Thus, the formula (2.15) gives (2.3). Recall [BZF] the construction of the vertex algebra bundle \mathcal{V} . According to Proposition 6.5.4 of [BZF], one canonically (i.e., coordinate independently) associates $\text{End}\mathcal{V}$ -valued sections \mathcal{Y}_p of \mathcal{V}^* (the bundle dual to \mathcal{V}) to matrix elements of a number of vertex operators on appropriate punctured disks around points with local coordinates \mathbf{z}_n on $\Sigma^{(g)}$. The spaces of such \mathcal{V} -sections for each n of is described by identifications (2.15). Taking into account the construction of Section 6 (subsection 6.6.1, in particular, construction 6.6.4, and Proposition 6.6.7) of [BZF], we see that n -point functions are connections on the space of sections of \mathcal{V} , and the reduction cohomology (2.7) is represented by (2.13). \square

For the chain condition (2.5) we have

$$\begin{aligned} 0 = \mathcal{G}(\chi(p'')) &= -\frac{f(\chi(p''))}{f(\psi(p'))} \mathcal{G}(\psi(p')) + \frac{1}{f(\psi(p'))} \sum_{\tilde{q}_n, \tilde{q}'_n \in \mathcal{X}_0 \subset \Sigma^{(g)}} \mathcal{G}(f(\chi(\tilde{q})) \cdot \psi(\tilde{q}')), \\ 0 &= \mathcal{G}(\chi(p'')) \\ &= \frac{f(\chi(p''))}{f(\phi(p))} \mathcal{G}(\phi(p)) - \frac{f(\chi(p''))}{f(\psi(p')) f(\phi(p))} \sum_{q_n, q'_n \in \mathcal{X}_0 \subset \Sigma^{(g)}} \mathcal{G}(f(\psi(q)) \cdot \phi(q')) \\ &\quad + \frac{1}{f(\psi(p'))} \sum_{\tilde{q}_n, \tilde{q}'_n \in \mathcal{X}_0 \subset \Sigma^{(g)}} \mathcal{G}(f(\chi(\tilde{q})) \cdot \psi(\tilde{q}')), \end{aligned} \quad (2.16)$$

The geometrical meaning of (2.5) consists in the following. Since in (2.2) operators act on vertex algebra elements only, we can interpret it as a relation on modes of V with functional coefficients. In particular, all operators T change vertex algebra elements by action either of $o(v) = v_{wtv-1}$. or positive modes of $v(m)$., $m \geq 0$. Recall that for all g the n -point functions possess certain modular properties with respect to groups depending on genus g and modular space $\mathcal{B}^{(g)}$ parameters. Moreover, the reduction formulas (2.3) are used to prove modular invariance for n -point functions. Due to automorphic properties of n -point functions, (2.5) can be also interpreted as relations among modular forms. The condition (2.3) defines a complex variety in $\mathbf{z}_n \in \mathbb{C}^n$ with non-commutative parameters $\mathbf{v}_n \in V^{\otimes n}$. As most identities (e.g., trisecant identity [Fa, TZ1] and triple product identity [K, MTZ, TZ2]) for n -point functions (2.5) has its algebraic-geometrical meaning. The condition (2.5) relates finite series of vertex algebra correlations functions on a genus g Riemann surface with rational function coefficients (at genus $g = 0$) [FHL], or elliptic functions (at genus $g = 1$) [Zhu, MT, MTZ], or generalizations of classical elliptic functions (at genus $g \geq 2$) [GT, TW]. Since in all cases n -point functions possess certain modular properties, we treat (2.5) as a source of new identities on modular forms at corresponding genus of Riemann surfaces.

3. COHOMOLOGY

In this section we compute the reduction cohomology defined by (2.6)–(2.7). The main result of this paper is the following.

Proposition 2. *The n -th reduction cohomology of a vertex algebra V -module W is the space of analytical continuations of solutions $\mathcal{F}_{W,n}^{(g)}(\mathbf{x}_n)$ to a Knizhnik-Zamolodchikov equation, and provided by series of auxiliary functions recursively generated by reduction formulas (2.3) with $x_i \notin \mathfrak{V}_i$, for $1 \leq i \leq n$.*

Remark 6. The first cohomology is given by the space of transversal (i.e., with vanishing sum over q, q') one-point connections $\mathcal{F}_{W,1}^{(g)}(x_1)$ provided by coefficients in terms of series of special functions. The second cohomology is given by a space of generalized higher genus complex kernels corresponding to V and $\Sigma^{(g)}$.

Proof. By definition (2.7), the n -th reduction cohomology is defined by the subspace of $C_n^{(g)}(W)$ of functions $\mathcal{F}_{W,n}^{(g)}(\mathbf{x}_n)$ satisfying

$$\begin{aligned} & \left(\sum_{l=1}^{l(g)} f_1^{(g)}(\mathbf{x}_{n+1}, l) T_l^{(g)}(v_{n+1}) \right. \\ & \left. + \sum_{k=1}^n \sum_{m \geq 0} f_2^{(g)}(k, m) T_k^{(g)}(v_{n+1}(m)) \right) \mathcal{F}_{W,n}^{(g)}(\mathbf{x}_n) = 0, \end{aligned} \tag{3.1}$$

modulo the subspace of $C_n^{(g)}(W)$ n -point functions $\mathcal{F}_{W,n}^{(g)}(\mathbf{x}'_n)$ resulting from:

$$\begin{aligned} \mathcal{F}_{W,n}^{(g)}(\mathbf{x}'_n) &= \left(\sum_{l=1}^{l(g)} f_1^{(g)}(\mathbf{x}'_n, l) T_l^{(g)} \right. \\ &\quad \left. + \sum_{k=1}^{n-1} \sum_{m \geq 0} f_2^{(g)}(k, m) T_k^{(g)}(v'_n(m)) \right) \mathcal{F}_{W,n-1}^{(g)}(\mathbf{x}'_{n-1}). \end{aligned} \quad (3.2)$$

We assume that, subject to other fixed parameters, n -point functions are completely determined by all choices $\mathbf{x}_n \in V^{\otimes n} \times \mathbb{C}^n$. Thus, the reduction cohomology can be treated as depending on set of \mathbf{x}_n only with appropriate action of endomorphisms generated by x_{n+1} . Consider a non-vanishing solution $\mathcal{F}_{W,n}^{(g)}(\mathbf{x}_n)$ to (3.1) for some \mathbf{x}_n . Let us use the reduction formulas (2.3) recursively for each x_i , $1 \leq i \leq n$ of \mathbf{x}_n in order to express $\mathcal{F}_{W,n}^{(g)}(\mathbf{x}_n)$ in terms of the partition function $\mathcal{F}_{W,0}^{(g)}(B^{(g)})$, i.e., we obtain

$$\mathcal{F}_{W,n}^{(g)}(\mathbf{x}_n) = \mathcal{D}^{(g)}(\mathbf{x}_n) \mathcal{F}_{W,0}^{(g)}(B^{(g)}), \quad (3.3)$$

as in [MT, MTZ, TZ, TZ1, TZ2]. It is clear that $x_i \notin \mathfrak{V}_i$ for $1 \leq i \leq n$, i.e., at each stage of the recursion procedure towards (3.3), otherwise $\mathcal{F}_{W,n}^{(g)}(\mathbf{x}_n)$ would be zero. Thus, $\mathcal{F}_{W,n}^{(g)}(\mathbf{x}_n)$ is explicitly known and is represented as a series of auxiliary functions $\mathcal{D}^{(g)}$ depending on V , genus g , and moduli space parameters $B^{(g)}$. Consider now $\mathcal{F}_{W,n}^{(g)}(\mathbf{x}'_n)$ given by (3.2). It is either vanishes when $v_{n-i} \in \mathfrak{V}_{n-i}$, $2 \leq i \leq n$, or given by (3.3) with \mathbf{x}'_n arguments.

The way the reduction relations (2.3) were derived in [MT, MTZ, GT, TW, BKT] is exactly the same as for the vertex algebra derivation [KZ, TK] for the Knizhnik-Zamolodchikov equations. The general idea is consider the double integration of $\mathcal{F}_{W,n}^{(g)}(\mathbf{x}_n)$ along small circles around two auxiliary variables with the action of reproduction kernels inserted. Then, these procedure leads to recursion formulas relating $\mathcal{F}_{W,n+1}^{(g)}$ and $\mathcal{F}_{W,n}^{(g)}$ with functional coefficients depending on the nature of the vertex algebra V , and the way a Riemann surface $\Sigma^{(g)}$ was constructed. Thus, in our context, (3.1) is seen as a version of the Knizhnik-Zamolodchikov equation. In [Y, MT, MTZ] formulas to n -point functions in various specific examples of V and configuration of Riemann surfaces were explicitly and recursively obtained.

In terms of x_{n+1} , by using (5.1)–(5.6), we are able to transfer in (3.1) the action of v_{n+1} -modes into an analytical continuation of $\mathcal{F}_{W,n}^{(g)}(\mathbf{x}_n)$ multi-valued holomorphic functions to domains $D_n \subset \Sigma^{(g)}$ with $z_i \neq z_j$ for $i \neq j$. Namely, in (3.1), the operators $T_l^{(g)}$ and $T_k^{(g)}(v_n(m))$ act by certain modes $v_{n+1}(m)$ of a vertex algebra element v_{n+1} on $\mathbf{v}_n \in V^{\otimes n}$. Using vertex algebra associativity property (5.6) we express the action of operators $T_l^{(g)}$ and $T_k^{(g)}(v_n(m))$ in terms of modes $v_{n+1}(m)$ inside vertex operators in actions of V -modes on the whole vertex operator at expense of a shift of their formal parameters \mathbf{z}_n by z_{n+1} , i.e., $\mathbf{z}'_n = \mathbf{z}_n + z_{n+1}$. Note that under such associativity transformations v -part of \mathbf{x}_n , i.e., \mathbf{v}_n remains the untouched. Thus,

the n -th reduction cohomology of a V -module W is given by the space of analytical continuations of n -point functions $\mathcal{F}_{W,n}^{(g)}(\mathbf{x}_n)$ with $\mathbf{x}_{n-1} \notin \mathfrak{A}_{n-1}$ that are solutions to the Knizhnik-Zamolodchikov equations (3.1). The above analytic extensions for the Knizhnik-Zamolodchikov equations generated by x_{n+1} and with coefficients provided by functions $f_1^{(g)}(\mathbf{x}_{n+1}, n)$ and $f_2^{(g)}(\mathbf{x}_{n+1}, k, m)$ on the Riemann surface $\Sigma^{(g)}$ \square

This result is coherent with considerations of [TUY]. We illustrate the proposition above in cases of zero, one, two, and higher genera in section 4. One can make connection with the first cohomology of grading-restricted vertex algebras in terms of derivations, and to the second cohomology in terms of square-zero extensions of V by W [Huang2].

Euler-Poincare formula. In [Fuks], for a Lie algebra, we find a celebrated formula relating sums over dimensions of chain complex spaces and dimensions of homologies for a graded infinite-dimensional Lie algebra. Suppose a V -module W is endowed with a complex grading $W = \bigcup_{m \in \mathbb{C}} W_{(m)}$, i.e., one could consider $m = m' + \beta(m)$, where $m' \in \mathbb{Z}$, and $\beta(m) \in \mathbb{C}$ [MTZ]. In our case all spaces $C_{(g)}^n(W)$ are infinite-dimensional, but, according to definition of a vertex algebra, $\dim W_{(m)} \leq \infty$, $m \in \mathbb{C}$. Thus, for a fixed g , each $C_{(g)}^n(W)$ can be endowed with separation with respect to m . Then the complex (2.6) decomposes into sum of complexes

$$0 \longrightarrow C_{(g)}^0(m) \xrightarrow{\delta_{(g)}^0} C_{(g)}^1(m) \xrightarrow{\delta_{(g)}^1} \dots \longrightarrow C_{(g)}^{n-1}(m) \xrightarrow{\delta_{(g)}^{n-1}} C_{(g)}^n \longrightarrow \dots \quad (3.4)$$

For $g \geq 0$, $l \geq 0$, let

$$q_{n,m} = \dim C_{(g)}^n(V_{(m)}) = \dim \left\{ \mathcal{F}_{W,n}^{(g)}(\mathbf{x}_n) : \mathbf{x}_n \in V_{(m)}^{\otimes n}, m \in \mathbb{C} \right\}.$$

Let

$$p_{m,n} = \dim H^n(V_{(m)}) = \dim \left\{ \left(\text{Ker } \delta_{(g)}^n / \text{Im } \delta_{(g)}^{n-1} \right) |_{V_{(m)}}, v_{n+1} \in V_{(m)} \right\},$$

be dimensions of corresponding cohomology spaces. In our context we find that for fixed $m \in \mathbb{C}$, and $n \geq 0$,

$$\sum_{n \geq 0}^N (-1)^n (q_{n,m} - p_{n,m}) = 0, \quad (3.5)$$

Indeed, let us consider relations of chain complex spaces and cohomology spaces for vertex algebra elements that belong to fixed $V_{(m)}$ grading subspace of V . Recall that according to the definition of a vertex algebra (see Appendix 5) subspaces $W_{(m)}$ are finite dimensional. Thus, as a functional space, $\left\{ \mathcal{F}_{W,n}^{(g)}(\mathbf{x}_n), \mathbf{v}_n \in V_{(m)}^{\otimes n} \right\}$ is finite-dimensional. Consider now the cohomology spaces $\left\{ H_{(g)}^n, \mathbf{v}_{n+1} \in V_{(m)}^{\otimes(n+1)} \right\}$. Let us fix $v_{n+1} \in V_{(m)}$. For the same reason as above, $\text{rank} \left(\text{Ker } \delta_{(g)}^n \right) < \infty$, $\text{rank} \left(\text{Im } \delta_{(g)}^{n-1} \right) < \infty$, as subspaces of $n+1$ - and n -point functions. Thus, $\text{rank} H_{(g)}^n |_{V_{(m)}} < \infty$. Using then the standard Euler-Poincare formula [Fuks] considerations for rank we obtain (3.5). The formula (3.5) has deep number-theoretical meanings as equality of generating functions for series expansions for automorphic forms.

4. EXAMPLES

The reduction cohomology depends on actual coefficients of the Knizhnik-Zamolodchikov equations (3.1). Note that for $n = 0$, $\mathcal{F}_{W,0}^{(g)}(B^{(g)})$, $g \geq 0$, is the partition function (or graded dimension for $g = 1$), for a vertex algebra V module W . In this section we provide examples of vertex algebras considered on Riemann surfaces of various genus g .

4.1. Vertex algebra n -point functions on the sphere. For $u, \mathbf{v}_n \in V$, and a homogeneous $u \in V$, we find the formula [FHL, FLM] for the n -point functions

$$\mathcal{F}_{W,n}^{(0)}(\mathbf{x}_n) = \langle u', \mathbf{Y}(\mathbf{x}_n)u \rangle,$$

where

$$\mathbf{Y}(\mathbf{x}_n) = Y(x_1) \dots Y(x_n),$$

The partition function is given by

$$\mathcal{F}_{W,0}^{(0)} = \langle u'_{(a)}, u_{(b)} \rangle = \delta_{a,b}.$$

The reduction operators of (2.2) are [FHL]

$$H_1^{(0)}(\mathbf{x}_{n+1}) \mathcal{F}_{W,n}^{(0)}(\mathbf{x}_n) = T_1(o(v)) \mathcal{F}_{W,n}^{(0)}(\mathbf{x}_n), \quad (4.1)$$

$$H_2^{(0)}(\mathbf{x}_{n+1}) \mathcal{F}_{W,n}^{(0)}(\mathbf{x}_n) = \sum_{k=1}^n \sum_{m \geq 0} f_{wt(v_{n+1}),m}(z_{n+1}, z_r) T_k(v(m)) \mathcal{F}_{W,n}^{(0)}(\mathbf{x}_n),$$

where we define $f_{wt(v,m)}^{(0)}(z, w)$ is a rational function defined by

$$f_{n,m}^{(0)}(z, w) = \frac{z^{-n}}{m!} \left(\frac{d}{dw} \right)^m \frac{w^n}{z-w},$$

$$\iota_{z,w} f_{n,m}^{(0)}(z, w) = \sum_{j \in \mathbb{N}} \binom{n+j}{m} z^{-n-j-1} w^{n+j-1}.$$

Let us take x_{n+1} as the variable of expansion. The n -th reduction cohomology $H_{(0)}^n(W)$ is given the space of solutions to the Knizhnik-Zamolodchikov equation (4.4) of the n -point function $\mathcal{F}_{W,0}^{(0)}(\mathbf{x}_n)$ (not given by $\delta_0^{n-1} \mathcal{F}_{W,n-1}^{(0)}(\mathbf{x}_{n-1})$)

$$\begin{aligned} \mathbf{Y}(\mathbf{x}_n) &\mapsto o(v_{n+1}) \mathbf{Y}(\mathbf{x}_n), \\ \mathbf{x}_k &\mapsto v_{(m)} \cdot \mathbf{x}_k, \end{aligned} \quad (4.2)$$

(i.e., generated by x_{n+1} -endomorphisms) of solutions $\mathcal{F}_{V,n}^{(0)}$ of the Knizhnik-Zamolodchikov equation

$$0 = \mathcal{F}_{W,n}^{(0)}(o(v_{n+1}) \mathbf{x}_n) + \sum_{k=1}^n \sum_{m \geq 0} f_{wt(v_{n+1}),m}^{(0)}(z_{n+1}, z_k) \mathcal{F}_{W,n}^{(0)}((v_{n+1}(m))_k \cdot \mathbf{x}_n), \quad (4.3)$$

with rational function coefficients $f_{wt(v_{n+1}),m,k}^{(0)}(z_{n+1}, z_k)$, modulo the space of n -point functions obtained by the recursion procedure. Using (5.1)–(5.6) we obtain from (4.3),

and the standard Virasoro algebra representation $L_V(0) = z_{n+1}\partial_{z_{n+1}}$, we obtain

$$\left(\partial_{z_{n+1}} + \sum_{k=1}^n \tilde{f}_{wt(v_{n+1}),m}^{(0)}(z_{n+1}, z_k) \right) \mathcal{F}_{W,n}^{(0)}(\mathbf{x}_n + (z_{n+1})_k) = 0, \quad (4.4)$$

which is the Knizhnik-Zamolodchikov equation on analytical continuation of $\mathcal{F}_{W,n}^{(0)}(\mathbf{x}_n + (z_{n+1})_k)$ with a different $\tilde{f}^{(0)}$. Using the reduction formulas (2.3) we obtain

$$\mathcal{F}_{W,n}^{(0)}(\mathbf{x}_n + (z_{n+1})_k) = \mathcal{D}^{(0)}(\mathbf{x}_{n+1}),$$

where $\mathcal{D}^{(0)}(\mathbf{x}_{n+1})$ is given by the series of rational-valued functions in $\mathbf{x}_{n+1} \notin \mathfrak{A}_n$ resulting from the recursive procedure starting from n -point function to the partition function. Thus, in this example, the n -th cohomology is the space of analytic extensions [FHL, FLM] of rational function solutions to the equation (4.3) with rational function coefficients.

4.2. Vertex algebra n -point functions on the torus. In order to consider modular-invariance of n -point functions at genus one, Zhu introduced [Zhu] a second square-bracket VOA $(V, Y[., .], \mathbf{1}_V, \tilde{\omega})$ associated to a given VOA $(V, Y(., .), \mathbf{1}_V, \omega)$. The new square bracket vertex operators are

$$Y[v, z] = \sum_{n \in \mathbb{Z}} v[n] z^{-n-1} = Y(q_z^{L(0)} v, q_z - 1),$$

with $q_z = e^z$, while the new conformal vector is

$$\tilde{\omega} = \omega - \frac{c_V}{24} \mathbf{1}_V.$$

For v of $L(0)$ weight $wt(v) \in \mathbb{R}$ and $m \geq 0$,

$$v[m] = m! \sum_{i \geq m} c(wt(v), i, m) v(i),$$

where

$$\sum_{m=0}^i c(wt(v), i, m) x^m = \binom{wt(v) - 1 + x}{i}.$$

For $\mathbf{v}_n \in V^{\otimes n}$ the genus one n -point function [Zhu] has the form

$$F_n^{(1)}(\mathbf{x}_n; \tau) = Tr_V \left(\mathbf{Y} \left(\mathbf{q}^{L(0)} \mathbf{v}_n, \mathbf{q}_n \right) q^{L(0) - c_V/24} \right),$$

for $q = e^{2\pi i \tau}$ and $q_i = e^{z_i}$, where τ is the torus modular parameter, and c_V is the central charge of V -Virasoro algebra. Then the genus one Zhu recursion formula is given by the following [Zhu]. For any v_{n+1} , $\mathbf{v}_n \in V^{\otimes n}$ we find for an $n+1$ -point function

$$\begin{aligned} H_1^{(1)}(\mathbf{x}_{n+1}) &= F_n^{(1)}(o(v_{n+1}) \mathbf{x}_n; \tau) \\ H_2^{(1)}(\mathbf{x}_{n+1}) &= \sum_{k=1}^n \sum_{m \geq 0} P_{m+1}(z_{n+1} - z_k, \tau) F_V^{(1)}((v[m])_k, \mathbf{x}_n; \tau). \end{aligned} \quad (4.5)$$

In this theorem $P_m(z, \tau)$ denote higher Weierstrass functions defined by

$$P_m(z, \tau) = \frac{(-1)^m}{(m-1)!} \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{n^{m-1} q_z^n}{1 - q^n}.$$

The partition function

$$\mathcal{F}_{W,0}^{(1)}(\tau) = \text{Tr}_V \left(q^{L_V(0) - c/24} \right), \quad (4.6)$$

is called the graded dimension for V . Let us fix $\mathbf{x}_{n+1} \in V^{\otimes n} \times \mathbb{C}^n$. The n -th reduction cohomology H^n is the space of extensions of the Knizhnik-Zamolodchikov equation solutions

$$0 = F_V^{(1)}(o(v_{n+1}) \mathbf{x}_n; \tau) + \sum_{k=1}^{n+1} \sum_{m \geq 0} P_{m+1}(z_{n+1} - z_k, \tau) F_n^{(1)}((v_{n+1}[m])_k \cdot \mathbf{x}_n; \tau). \quad (4.7)$$

This space can be described by complex kernels (i.e., prime forms [Mu]) given by sums of elliptic functions.

4.3. Genus two n -point functions. In this subsection we recall [GT] the construction and reduction formulas for vertex operator algebra n -point functions on a genus two Riemann surface formed in the sewing procedure of two tori due to [Y]. Let us assume that a vertex operator algebra V is of strong-type. Thus, it possess a non-degenerate bilinear form. For a V -basis $\{u^{(a)}\}$ we define the dual basis $\{\bar{u}^{(a)}\}$ with respect to the bilinear form where

$$\langle \bar{u}^{(a)}, u^{(b)} \rangle_{\text{sq}} = \delta_{ab}.$$

Definition 6. The genus two partition function (zero-point function) for V is defined by

$$\mathcal{F}_{V,0}^{(2)}(B^{(2)}) = \sum_{r \geq 0} \epsilon^r \sum_{\bar{u} \in V_{[r]}} \mathcal{F}_{V,1}^{(1)}(u; \tau_1) \mathcal{F}_{V,1}^{(1)}(\bar{u}; \tau_2), \quad (4.8)$$

where $B^{(g)} = (\tau_1, \tau_2, \epsilon)$ in the ϵ -sewing procedure for constructing a genus two Riemann surface [Y, MT, TZ1], and the internal sum is taken over any $V_{[n]}$ -basis, and \bar{u} is the dual of u with respect to the bilinear form on V .

We then recall [GT] a formal genus two reduction formulas for all n -point functions.

Definition 7. Let $v_{n+1} \in V$ be inserted at $x_{n+1} \in \hat{\Sigma}_1^{(1)}$, $\mathbf{v}_k \in V^{\otimes k}$ be inserted at $\mathbf{y}_k \in \hat{\Sigma}_1^{(1)}$ and $\mathbf{v}'_l \in V^{\otimes l}$ be inserted at $\mathbf{y}'_l \in \hat{\Sigma}_2^{(1)}$, a punctured torus

$$\hat{\Sigma}_a^{(1)} = \Sigma_a^{(1)} \setminus \{z_a, |z_a| \leq |\epsilon|/r_a\}, \quad (4.9)$$

where we here we use the convention

$$\bar{1} = 2, \quad \bar{2} = 1. \quad (4.10)$$

We consider the corresponding genus two n -point function

$$\begin{aligned} & \mathcal{F}_{V,n}^{(2)}(x_{n+1}; \mathbf{x}_k; \mathbf{x}'_l; B^{(2)}) \\ &= \sum_{r \geq 0} \epsilon^r \sum_{u \in V_{[r]}} \mathcal{F}_{V,k+1}^{(1)}(Y[x_{n+1}] \mathbf{Y}[\mathbf{x}_k]_k u; \tau_1) \mathcal{F}_{V,l}^{(1)}(\mathbf{Y}[\mathbf{x}'_l]_l \bar{u}; \tau_2). \end{aligned} \quad (4.11)$$

where the sum as in (4.8).

Remark 7. (4.8) is independent of the choice of V -basis. One could also write a similar expression by inserting the $Y[x]$ vertex operator at $z \in \widehat{\Sigma}_2^{(1)}$ on the right hand side of (4.11).

First, one defines the functions $\mathcal{F}_{V,n,a}^{(2)}$ for $a \in \{1, 2\}$, by

$$\begin{aligned} \mathcal{F}_{V,n,1}^{(2)}(\mathbf{x}_{n+1}; B^{(2)}) &= \sum_{r \geq 0} \epsilon^r \sum_{u \in V_{[r]}} \text{Tr}_V \left(o(v_{n+1}) \mathbf{Y}(\mathbf{q}_{\mathbf{z}_k}^{\mathbf{L}(0)} \mathbf{v}_k, \mathbf{q}_{\mathbf{z}_k}) q_1^{L_V(0) - c_V/24} \right) \\ &\quad \cdot \mathcal{F}_{V,n-k}^{(1)}(\mathbf{Y}[\mathbf{x}_{k+1,n}] \bar{u}; \tau_2), \end{aligned}$$

$$\begin{aligned} \mathcal{F}_{V,n,2}^{(2)}(\mathbf{x}_{n+1}; B^{(2)}) &= \sum_{r \geq 0} \epsilon^r \sum_{u \in V_{[r]}} \mathcal{F}_{V,k}^{(1)}(\mathbf{Y}[\mathbf{x}_k] u; \tau_1) \\ &\quad \cdot \text{Tr}_V \left(o(v_{n+1}) \mathbf{Y}(\mathbf{q}_{\mathbf{z}_{k+1,n}}^{\mathbf{L}(0)} \mathbf{v}_{k+1}, \mathbf{q}_{\mathbf{z}_{k+1}}) q_2^{L_V(0) - c_V/24} \right), \end{aligned}$$

and $\mathcal{F}_{V,n,3}^{(2)}(\mathbf{x}_{n+1}; B^{(2)}) = \mathbb{X}_1^\Pi$ of (6.7). We also define

Definition 8. Let $f_a^{(2)}(p; z_{n+1})$, for $p \geq 1$, and $a = 1, 2$ be given by

$$f_a^{(2)}(p; z_{n+1}) = 1^{\delta_{ba}} + (-1)^{p\delta_{b\bar{a}}} \epsilon^{1/2} \left(\mathbb{Q}(p; z_{n+1}) \left(\tilde{\Lambda}_{\bar{a}} \right)^{\delta_{ba}} \right) (1), \quad (4.12)$$

for $z_{n+1} \in \widehat{\Sigma}_b^{(1)}$. Let $f_3^{(2)}(p; z_{n+1})$, for $z_{n+1} \in \Sigma_a^{(1)}$, be an infinite row vector given by

$$f_3^{(2)}(p; z_{n+1}) = \left(\mathbb{R}(z_{n+1}) + \mathbb{Q}(p; z_{n+1}) \left(\tilde{\Lambda}_{\bar{a}} \Lambda_a + \Lambda_{\bar{a}} \Gamma \right) \right) \Pi. \quad (4.13)$$

In [GT] it is proven that the genus two $n = k + l$ -point function for a quasi-primary vector v_{n+1} of weight $p = \text{wt}[v_{n+1}]$ inserted at $x \in \widehat{\Sigma}_1^{(1)}$, and general vectors \mathbf{v}_k and \mathbf{v}'_l inserted at $\mathbf{x}_k \in \widehat{\Sigma}_1^{(1)}$, respectively, has the following operators $H_1^{(2)}(\mathbf{x}_{n+1})$ and $H_2^{(2)}(\mathbf{x}_{n+1})$:

$$\begin{aligned} H_1^{(2)}(x_{n+1}) \mathcal{F}_{V,n}^{(2)}(\mathbf{x}_n; B^{(2)}) &= \sum_{l=1}^3 f_l^{(2)}(p; z_{n+1}) \mathcal{F}_{V,n,l}^{(2)}(\mathbf{x}_{n+1}; B^{(2)}), \\ H_2^{(2)}(x_{n+1}) \mathcal{F}_{V,n}^{(2)}(\mathbf{x}_n; B^{(2)}) &= \sum_{i=1}^n \sum_{j \geq 0} \mathcal{P}_{j+1}(p; z_{n+1}, z_i) \mathcal{F}_{V,n}^{(2)}\left((v_{n+1}[j])_i \cdot \mathbf{x}_n; B^{(2)} \right), \end{aligned} \quad (4.14)$$

with $\mathcal{P}_{j+1}(p; x, y)$ of (6.10). A similar expression corresponds to v_{n+1} inserted on $x \in \widehat{\Sigma}_2^{(1)}$.

Remark 8. The functions $\mathcal{P}_{j+1}(p; x, y)$ are holomorphic for $x \neq y$ on the sewing domain in the cases $p = 1, 2$.

4.4. Vertex algebra n -point functions and reduction formulas in genus g Schottky parameterization. In this section we recall [TZ, TW, T2] the construction and reduction relations for vertex operator algebra n -point functions defined on a genus g Riemann surface formed in the Schottky parameterization. In particular, the formal partition and n -point correlation functions for a vertex operator algebra associated to a genus g Riemann surface Σ^g are introduced in the Schottky scheme. All expressions here are functions of formal variables $w_{\pm a}, \rho_a \in \mathbb{C}$ and vertex operator parameters. Then we recall the genus g reduction formula with universal coefficients that have a geometrical meaning and are meromorphic on a Riemann surface $\Sigma^{(g)}$. These coefficients are generalizations of the elliptic Weierstrass functions [L]. For a $2g$ vertex algebra V states

$$\mathbf{b} = (b_{-1}, b_1; \dots; b_{-g}, b_g),$$

and corresponding local coordinates

$$\mathbf{w} = (w_{-1}, w_1; \dots; w_{-g}, w_g),$$

of $2g$ points $(p_{-1}, p_1; \dots; p_{-g}, p_g)$ on the Riemann sphere $\Sigma^{(0)}$, consider the genus zero $2g$ -point correlation function

$$\begin{aligned} Z^{(0)}(\mathbf{b}, \mathbf{w}) &= Z^{(0)}(b_{-1}, w_{-1}; b_1, w_1; \dots; b_{-g}, w_{-g}; b_g, w_g) \\ &= \prod_{a \in \mathcal{I}_+} \rho_a^{\text{wt}(b_a)} Z^{(0)}(\bar{b}_1, w_{-1}; b_1, w_1; \dots; \bar{b}_g, w_{-g}; b_g, w_g). \end{aligned}$$

where $\mathcal{I}_+ = \{1, 2, \dots, g\}$. Let

$$\mathbf{b}_+ = (b_1, \dots, b_g),$$

denote an element of a V -tensor product $V^{\otimes g}$ -basis with dual basis

$$\mathbf{b}_- = (b_{-1}, \dots, b_{-g}),$$

with respect to the bilinear form $\langle \cdot, \cdot \rangle_{\rho_a}$ (cf. Appendix 5).

Let w_a for $a \in \mathcal{I}$ be $2g$ formal variables. One identify them with the canonical Schottky parameters (for details of the Schottky construction, see [TW, T2]). One can define the genus g partition function as

$$Z_V^{(g)} = Z_V^{(g)}(\mathbf{w}, \boldsymbol{\rho}) = \sum_{\mathbf{b}_+} Z^{(0)}(\mathbf{b}, \mathbf{w}), \quad (4.15)$$

for

$$(\mathbf{w}, \boldsymbol{\rho}) = (w_{\pm 1}, \rho_1; \dots; w_{\pm g}, \rho_g).$$

Now we recall the formal reduction formulas for all genus g Schottky n -point functions. One defines the genus g formal n -point function for n vectors $\mathbf{v}_n \in V^{\otimes n}$ inserted at \mathbf{y}_n by

$$Z_V^{(g)}(\mathbf{v}, \mathbf{y}) = Z_V^{(g)}(\mathbf{v}, \mathbf{y}; \mathbf{w}, \boldsymbol{\rho}) = \sum_{\mathbf{b}_+} Z^{(0)}(\mathbf{v}, \mathbf{y}; \mathbf{b}, \mathbf{w}), \quad (4.16)$$

where

$$Z^{(0)}(\mathbf{v}, \mathbf{y}; \mathbf{b}, \mathbf{w}) = Z^{(0)}(v_1, y_1; \dots; v_n, y_n; b_{-1}, w_{-1}; \dots; b_g, w_g).$$

Let U be a vertex operator subalgebra of V where V has a U -module decomposition

$$V = \bigoplus_{\alpha \in A} W_\alpha,$$

for U -modules W_α and some indexing set A . Let

$$W_\alpha = \bigotimes_{a=1}^g W_{\alpha_a},$$

denote a tensor product of g modules

$$Z_{W_\alpha}^{(g)}(\mathbf{v}, \mathbf{y}) = \sum_{\mathbf{b}_+ \in W_\alpha} Z^{(0)}(\mathbf{v}, \mathbf{y}; \mathbf{b}, \mathbf{w}), \quad (4.17)$$

where here the sum is over a basis $\{\mathbf{b}_+\}$ for W_α . It follows that

$$Z_V^{(g)}(\mathbf{v}, \mathbf{y}) = \sum_{\alpha \in A} Z_{W_\alpha}^{(g)}(\mathbf{v}, \mathbf{y}), \quad (4.18)$$

where the sum ranges over $\alpha = (\alpha_1, \dots, \alpha_g) \in \mathbf{A}$, for $\mathbf{A} = A^{\otimes g}$. Finally, one defines corresponding formal n -point correlation differential forms

$$\begin{aligned} \mathcal{F}_V^{(g)}(\mathbf{v}, \mathbf{y}) &= Z^{(g)}(\mathbf{v}, \mathbf{y}) \, d\mathbf{y}^{\text{wt}(\mathbf{v})}, \\ \mathcal{F}_{W_\alpha}^{(g)}(\mathbf{v}, \mathbf{y}) &= Z_{W_\alpha}^{(g)}(\mathbf{v}, \mathbf{y}) \, d\mathbf{y}^{\text{wt}(\mathbf{v})}, \end{aligned} \quad (4.19)$$

where

$$d\mathbf{y}^{\text{wt}(\mathbf{v})} = \prod_{k=1}^n dy_k^{\text{wt}(v_k)}.$$

Recall notations and identifications given in Appendix 6.3. In [TW] they prove that the genus g $(n+1)$ -point formal differential $\mathcal{F}_{W_\alpha}^{(g)}(x; \mathbf{v}, \mathbf{y})$, for $x_{n+1} = (v_{n+1}, y_{n+1})$, for a quasiprimary vector $v_{n+1} \in U$ of weight $\text{wt}(v_{n+1}) = p$ inserted at a point p_0 , with the coordinate y_{n+1} , and general vectors \mathbf{v}_n inserted at points \mathbf{p}_n with coordinates \mathbf{y}_n satisfies the recursive identity for $\mathbf{x}_n = (\mathbf{v}, \mathbf{y})$

$$\begin{aligned} \mathcal{F}_{W, n+1, \alpha}^{(g)}(x_{n+1}; \mathbf{x}_n) &= \left(H_1^{(g)} + H_2^{(g)} \right) \mathcal{F}_{W, n}^{(g)}(\mathbf{x}_n), \quad (4.20) \\ H_1^{(g)}(\mathbf{x}_{n+1}) \mathcal{F}_{W, n}^{(g)}(\mathbf{x}_n) &= \sum_{a=1}^g \Theta_a(y_{n+1}) O_a^{W_\alpha}(v_{n+1}; \mathbf{x}_n), \\ H_2^{(g)}(\mathbf{x}_{n+1}) \mathcal{F}_{W, n+1}^{(g)}(x_{n+1}) &= \sum_{k=1}^n \sum_{j \geq 0} \partial^{(0, j)} \Psi_p(y_{n+1}, y_k) \\ &\quad \cdot \mathcal{F}_{W, n, \alpha}^{(g)}((u(j))_k, \mathbf{x}_n) \, dy_k^j \end{aligned}$$

Here $\partial^{(0, j)}$ is given by

$$\partial^{(i, j)} f(x, y) = \partial_x^{(i)} \partial_y^{(j)} f(x, y),$$

for a function $f(x, y)$, and $\partial^{(0,j)}$ denotes partial derivatives with respect to x and y_j . The forms $\Psi_p(y_{n+1}, y_k) dy_k^j$ given by (6.24), $\Theta_a(x)$ is of (6.26), and $O_a^{W\alpha}(v_{n+1}; \mathbf{x}_n)$ of (6.27).

4.5. Vertex cluster algebras. Finally, we would like to describe the example of vertex cluster algebras introduced in [Zu] and reveal their cohomological nature. The condition (2.5) for mutation (4.26) to be involutive gives us a cohomological condition (vanishing square of the n -th coboundary operator) on modular forms. Cluster algebras introduced in [FZ1] have numerous applications in various areas of mathematics [FG1, FG2, FG3, FG4, GSV1, GSV2, GSV3, FST, KS, N1, N2, DFK, GLS, HL, Ke1, Ke2, Na, Sch, N1, N2, Sch]. In this subsection we recall [Zu] definition of a vertex cluster algebra and show its cohomological nature. Let us fix a vertex operator algebra V . Chose n -marked points p_i , $i = 1, \dots, n$ on a genus g compact Riemann surface formed by one of procedures [Y, TZ] mentioned in Introduction. In the vicinity of each marked point p_i define a local coordinate z_i with zero at p_i . For n -tuples of elements \mathbf{x}_n let us denote

$$\mathcal{Y}(\mathbf{x}_n) = (Y(x_1), \dots, Y(x_n)).$$

Definition 9. We define a vertex operator cluster algebra seed

$$\left(\mathbf{v}_n, \mathcal{Y}(\mathbf{x}_n), \mathcal{F}_{W,n}^{(g)}(\mathbf{x}_n) \right). \quad (4.21)$$

The mutation is defined as follows:

Definition 10. For \mathbf{v}_n , we define the mutation \mathbf{v}'_n of \mathbf{v}_n in the direction $k \in 1, \dots, n$ as

$$\mathbf{v}'_n = \mu_k(x_{n+1}, m) \mathbf{v}_n = ((F_k(v_{n+1}(m)))_k \mathbf{v}_n), \quad (4.22)$$

for some $m \geq 0$, and V -valued functions $F_k^{(g)}(v_{n+1}(m))$, depending on genus g of the Riemann surface. Note that due to the property (5.3) we get a finite number of terms as a result of the action of $v_{n+1}(m)$ on v_k , $1 \leq k \leq n$. For the n -tuple of vertex operators we define

$$\mathcal{Y}(\mathbf{x}'_n) = \mu_k(x_{n+1}, m) \mathcal{Y}(\mathbf{x}_n) = \left(Y \left(G_k^{(g)}(v_{n+1}(m)) \cdot \mathbf{x}_n \right) \right), \quad (4.23)$$

where $G_k^{(g)}(v_{n+1}(m))$ are other V -valued functions. For $u \in V$, $w \in \mathbb{C}$, the mutation $\mu(x_{n+1})$ of $\mathcal{F}_n^{(g)}(\mathbf{x}_n)$,

$$\mathcal{F}_{W,n}^{(g)'}(\mathbf{x}'_n) = \mu(x_{n+1}) \mathcal{F}_{W,n}^{(g)}(\mathbf{x}_n), \quad (4.24)$$

where $\mu(x_{n+1})$ is given by the coboundary operator

$$\mu(x_{n+1}) = \delta_{(g)}^n,$$

defined by summation over mutations in all possible directions k , $1 \leq k \leq n$, with auxiliary functions $f_1^{(g)}(x_{n+1}, l)$ and $f_2^{(g)}(m, k)$, $1 \leq l \leq l(g)$, $1 \leq k \leq n$, $m \geq 1$,

$$\mathcal{F}_{W,n}^{(g)'}(\mathbf{x}') = \delta_{(g)}^n \mathcal{F}_{W,n}^{(g)}(\mathbf{x}_n). \quad (4.25)$$

We also require the involutivity condition

$$\mu(x_{n+1}) \mu(x_{n+1}) = \text{Id}, \quad (4.26)$$

define the mutation of the seed (4.21) in the direction $k \in 1, \dots, n$ of $x_{n+1} \in V$. We denote by $\mathcal{C}_{(G)}^n$ the space of seeds (4.21) for particular $n \geq 1$, satisfying condition (4.26).

The involutivity condition can be expressed in terms of coboundary operator $\delta_{(g)}^n$ as (2.5) with $v_{n+2} = v_{n+1}$ with extra term Id on the right hand side.

Definition 11. For a fixed g , definitions 9–10 and involutivity condition (4.26) for mutation determine the structure of a vertex operator cluster algebra \mathcal{CG}_n of dimension n . We call the full vertex operator cluster algebra the union $\bigcup_{n \geq 0} \mathcal{CG}_n$.

It is naturally graded by n . of \mathcal{CG}_n . The cohomology of vertex cluster algebras will be considered elsewhere. In [Zu] we have proven the following

Proposition 3. For a vertex operator algebra V such that $\dim V_k = 1$, $k \in \mathbb{Z}$, and with $u = \mathbf{1}_V$, $w \in \mathbb{C}$, and

$$F_k(u(m)).v = G_k(u(m)).v = \xi_{u,v} u[-1].v,$$

$$T_k^{(g)}(u(m)) = u[m],$$

for $m \geq 0$, and $\xi_{u,v} \in \mathbb{C}$, $\xi_{u,v}^2 = 1$, depending on u and v , in (4.22), (4.23), and (4.25), the mutation

$$\mu = (\mu_k(\mathbf{1}_V, -1), \mu_k(\mathbf{1}_V, -1), \mu(\mathbf{1}_V, w)),$$

$$\left(\mathbf{v}'_n, \mathcal{Y}(\mathbf{x}'_n), \mathcal{F}_{W,n}^{(g)'}(\mathbf{x}'_n) \right) = \mu \left(\mathbf{v}, \mathcal{Y}(\mathbf{x}_n), \mathcal{F}_{W,n}^{(g)}(\mathbf{x}_n) \right), \quad (4.27)$$

defined by (4.22), (4.23), (4.25) is an involution, i.e.,

$$\mu \mu = \text{Id}.$$

□

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5. APPENDIX: VERTEX OPERATOR ALGEBRAS

In this subsection we recall the notion of a vertex operator algebra [B, DL, FHL, FLM, K, LL, MN].

Definition 12. A vertex operator algebra is determined by a quadruple $(V, Y, \mathbf{1}_V, \omega)$, where V is a linear space endowed with a \mathbb{Z} -grading with

$$V = \bigoplus_{r \in \mathbb{Z}} V_r,$$

with $\dim V_r < \infty$. The state $\mathbf{1}_V \in V_0$, $\mathbf{1}_V \neq 0$, is the vacuum vector and $\omega \in V_2$ is the conformal vector with properties described below. The vertex operator Y is a linear map

$$Y : V \rightarrow \text{End}(V)[[z, z^{-1}]],$$

for formal variable z so that for any vector $u \in V$ we have a vertex operator

$$Y(u, z) = \sum_{n \in \mathbb{Z}} u(n) z^{-n-1}. \quad (5.1)$$

The linear operators (modes) $u(n) : V \rightarrow V$ satisfy creativity

$$Y(u, z) \mathbf{1}_V = u + O(z), \quad (5.2)$$

and lower truncation

$$u(n)v = 0, \quad (5.3)$$

conditions for each $u, v \in V$ and $n \gg 0$. Finally, the vertex operators satisfy the Jacobi identity

$$\begin{aligned} z_0^{-1} \delta \left(\frac{z_1 - z_2}{z_0} \right) Y(u, z_1) Y(v, z_2) - z_0^{-1} \delta \left(\frac{z_2 - z_1}{-z_0} \right) Y(v, z_2) Y(u, z_1) \\ = z_2^{-1} \delta \left(\frac{z_1 - z_0}{z_2} \right) Y(Y(u, z_0)v, z_2). \end{aligned} \quad (5.4)$$

These axioms imply locality, skew-symmetry, associativity and commutativity conditions:

$$(z_1 - z_2)^N Y(u, z_1) Y(v, z_2) = (z_1 - z_2)^N Y(v, z_2) Y(u, z_1), \quad (5.5)$$

$$Y(u, z)v = e^{zL(-1)} Y(v, -z)u,$$

$$(z_0 + z_2)^N Y(u, z_0 + z_2) Y(v, z_2)w = (z_0 + z_2)^N Y(Y(u, z_0)v, z_2)w, \quad (5.6)$$

$$u(k)Y(v, z) - Y(v, z)u(k) = \sum_{j \geq 0} \binom{k}{j} Y(u(j)v, z)z^{k-j}, \quad (5.7)$$

for $u, v, w \in V$ and integers $N \gg 0$.

For the conformal vector ω one has

$$Y(\omega, z) = \sum_{n \in \mathbb{Z}} L(n) z^{-n-2}, \quad (5.8)$$

where $L(n)$ satisfies the Virasoro algebra for some central charge C

$$[L(m), L(n)] = (m - n)L(m + n) + \frac{C}{12}(m^3 - m)\delta_{m, -n}\text{Id}_V, \quad (5.9)$$

where Id_V is identity operator on V . Each vertex operator satisfies the translation property

$$\partial_z Y(u, z) = Y(L(-1)u, z). \quad (5.10)$$

The Virasoro operator $L(0)$ provides the \mathbb{Z} -grading with

$$L(0)u = ru,$$

for $u \in V_r, r \in \mathbb{Z}$.

For $v = \mathbf{1}_V$ one has

$$Y(\mathbf{1}_V, z) = \text{Id}_V. \quad (5.11)$$

Note also that modes of homogeneous states are graded operators on V , i.e., for $v \in V_k$,

$$v(n) : V_m \rightarrow V_{m+k-n-1}. \quad (5.12)$$

In particular, let us define the zero mode $o(v)$ of a state of weight $\text{wt}(v) = k$, i.e., $v \in V_k$, as

$$o(v) = v(\text{wt}(v) - 1), \quad (5.13)$$

extending to V additively.

Definition 13. Given a vertex operator algebra V , one defines the adjoint vertex operator with respect to $\alpha \in \mathbb{C}$, by

The adjoint operators defined by the mapping $z \mapsto 1/z$, i.e., with $\alpha = \epsilon$,

$$Y_\epsilon^\dagger[v, z] = Y \left[\exp \left(\frac{z}{\epsilon} L[1] \right) \left(-\frac{\epsilon}{z^2} \right)^{L[0]} v, \frac{\epsilon}{z} \right].$$

associated with the formal Möbius map [FHL]

$$z \mapsto \frac{\alpha}{z}.$$

Definition 14. An element $u \in V$ is called quasiprimary if

$$L(1)u = 0.$$

For quasiprimary u of weight $\text{wt}(u)$ one has

$$u^\dagger(n) = (-1)^{\text{wt}(u)} \alpha^{n+1-\text{wt}(u)} u(2\text{wt}(u) - n - 2).$$

Definition 15. A bilinear form

$$\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{C},$$

is called invariant if [FHL, Li]

$$\langle Y(u, z)a, b \rangle = \langle a, Y^\dagger(u, z)b \rangle, \quad (5.15)$$

for all $a, b, u \in V$.

Notice that the adjoint vertex operator $Y^\dagger(\cdot, \cdot)$ as well as the bilinear form $\langle \cdot, \cdot \rangle$, depend on α . In terms of modes, we have

$$\langle u(n)a, b \rangle = \langle a, u^\dagger(n)b \rangle. \quad (5.16)$$

Choosing $u = \omega$, and for $n = 1$ implies

$$\langle L(0)a, b \rangle = \langle a, L(0)b \rangle.$$

Thus,

$$\langle a, b \rangle = 0,$$

when $\text{wt}(a) \neq \text{wt}(b)$.

Definition 16. A vertex operator algebra is called of strong-type if

$$V_0 = \mathbb{C}\mathbf{1}_V,$$

and V is simple and self-dual, i.e., V is isomorphic to the dual module V' as a V -module.

It is proven in [Li] that a strong-type vertex operator algebra V has a unique invariant non-degenerate bilinear form up to normalization. This motivates

Definition 17. The form $\langle \cdot, \cdot \rangle$ on a strong-type vertex operator algebra V is the unique invariant bilinear form $\langle \cdot, \cdot \rangle$ normalized by

$$\langle \mathbf{1}_V, \mathbf{1}_V \rangle = 1.$$

Given a vertex operator algebra $(V, Y(\cdot, \cdot), \mathbf{1}_V, \omega)$, one can find an isomorphic vertex operator algebra $(V, Y[\cdot, \cdot], \mathbf{1}_V, \tilde{\omega})$ called [Zhu] the square-bracket vertex operator algebra. Both algebras have the same underlying vector space V , vacuum vector $\mathbf{1}_V$, and central charge. The vertex operator $Y[\cdot, \cdot]$ is determined by

$$Y[v, z] = \sum_{n \in \mathbb{Z}} v[n] z^{-n-1} = Y\left(q_z^{L(0)} v, q_z - 1\right).$$

The new square-bracket conformal vector is

$$\tilde{\omega} = \omega - \frac{c}{24} \mathbf{1}_V,$$

with the vertex operator

$$Y[\tilde{\omega}, z] = \sum_{n \in \mathbb{Z}} L[n] z^{-n-2}.$$

The square-bracket Virasoro operator mode $L[0]$ provides an alternative \mathbb{Z} -grading on V , i.e., $\text{wt}[v] = k$ if

$$L[0]v = kv,$$

where $\text{wt}[v] = \text{wt}(v)$ for primary v , and $L(n)v = 0$ for all $n > 0$. We can similarly define a square-bracket bilinear form $\langle \cdot, \cdot \rangle_{\text{sq}}$.

6. APPENDIX: GENUS g GENERALIZATIONS OF ELLIPTIC FUNCTIONS

In this Appendix we recall [T2] genus g generalizations of classical elliptic functions.

6.1. Classical elliptic functions. In this subsection we recall the classical elliptic functions and modular forms [Se, La].

Definition 18. For an integer $k \geq 2$, the Eisenstein series is given by

$$E_k(\tau) = E_k(q) = \delta_{n, \text{even}} \left(-\frac{B_k}{k!} + \frac{2}{(k-1)!} \sum_{n \geq 1} \sigma_{k-1}(n) q^n \right),$$

where $\tau \in \mathbb{H}$, $q = e^{2\pi i \tau}$,

$$\sigma_{k-1}(n) = \sum_{d|n} d^{k-1},$$

and k -th Bernoulli number B_k .

If $k \geq 4$ then $E_k(\tau)$ is a modular form of weight k on $\text{SL}(2, \mathbb{Z})$, while $E_2(\tau)$ is a quasi-modular form.

Definition 19. For integer $k \geq 1$, define elliptic functions $z \in \mathbb{C}$:

$$P_k(z, \tau) = \frac{(-1)^{k-1}}{(k-1)!} \partial_z^{k-1} P_1(z, \tau),$$

$$P_1(z, \tau) = \frac{1}{z} - \sum_{k \geq 2} E_k(\tau) z^{k-1}.$$

In particular

$$P_2(z, \tau) = \wp(z, \tau) + E_2(\tau),$$

for Weierstrass function $\wp(z, \tau)$ with periods $2\pi i$ and $2\pi i\tau$. $P_1(z, \tau)$ is related to the quasi-periodic Weierstrass σ -function with

$$P_1(z + 2\pi i\tau, \tau) = P_1(z, \tau) - 1.$$

6.2. Genus two counterparts of Weierstrass functions. In this section we recall the definition of genus two Weierstrass functions [GT]. For $m, n \geq 1$, we first define a number of infinite matrices and row and column vectors:

$$\begin{aligned} \Gamma(m, n) &= \delta_{m, -n+2p-2}, \\ \Delta(m, n) &= \delta_{m, n+2p-2}. \end{aligned} \tag{6.1}$$

We also define the projection matrix

$$\Pi = \Gamma^2 = \begin{bmatrix} \mathbb{1}_{2p-1} & 0 \\ 0 & \ddots \end{bmatrix}, \tag{6.2}$$

where Id_{2p-3} denotes the $2p-3$ dimensional identity matrix and $\text{Id}_{-1} = 0$. Let Λ_a for $a \in \{1, 2\}$ be the matrix with components

$$\begin{aligned} \Lambda_a(m, n) &= \Lambda_a(m, n; \tau_a, \epsilon) \\ &= \epsilon^{(m+n)/2} (-1)^{n+1} \binom{m+n-1}{n} E_{m+n}(\tau_a). \end{aligned} \tag{6.3}$$

Note that

$$\Lambda_a = S A_a S^{-1}, \tag{6.4}$$

for A_a given by

$$A_a = A_a(k, l, \tau_a, \epsilon) = \frac{(-1)^{k+l} \epsilon^{(k+l)/2}}{\sqrt{kl}} \frac{(k+l-1)!}{(k-1)!(l-1)!} E_{k+l}(\tau_a).$$

introduce the infinite dimensional matrices for S a diagonal matrix with components

$$S(m, n) = \sqrt{m} \delta_{mn}. \tag{6.5}$$

Let $\mathbb{R}(x)$ for $x \in \widehat{\Sigma}_a^{(1)}$ be the row vector with components

$$\mathbb{R}(x; m) = \epsilon^{\frac{m}{2}} P_{m+1}(x, \tau_a). \tag{6.6}$$

Let \mathbb{X}_a for $a \in \{1, 2\}$ be the column vector with components

$$\begin{aligned}\mathbb{X}_1(m) &= \mathbb{X}_1 \left(m; v_{n+1}, \mathbf{x}_n; B^{(2)} \right) \\ &= \epsilon^{-m/2} \sum_{u \in V} \mathcal{F}_{V,k}^{(1)}(\mathbf{Y}[\mathbf{x}_k]v[m]u; \tau_1) \mathcal{F}_{V,n-k}^{(1)}(\mathbf{Y}[\mathbf{x}_{k+1,n}]\bar{u}; \tau_2), \\ \mathbb{X}_2(m) &= \mathbb{X}_2 \left(m; v_{n+1}, \mathbf{x}_n; B^{(2)} \right) \\ &= \epsilon^{-m/2} \sum_{u \in V} \mathcal{F}_{V,k}^{(1)}(\mathbf{Y}[\mathbf{x}_k]u; \tau_1) \mathcal{F}_{V,m-k}^{(1)}(\mathbf{Y}[\mathbf{x}_{n-k}]v[m]\bar{u}; \tau_2).\end{aligned}\quad (6.7)$$

Introduce also $\mathbb{Q}(p; x)$ an infinite row vector defined by

$$\mathbb{Q}(p; x) = \mathbb{R}(x)\Delta \left(\mathbb{1} - \tilde{\Lambda}_{\bar{a}}\tilde{\Lambda}_a \right)^{-1}, \quad (6.8)$$

for $x \in \widehat{\Sigma}_a^{(1)}$. Notice that

$$\tilde{\Lambda}_a = \Lambda_a \Delta.$$

On introduces

$$\mathbb{P}_{j+1}(x) = \frac{(-1)^j}{j!} \mathbb{P}_1(x),$$

for $x \in \widehat{\Sigma}_a^{(1)}$, and $j \geq 0$, is the column with components

$$\mathbb{P}_{j+1}(x; m) = \epsilon^{\frac{m}{2}} \binom{m+j-1}{j} (P_{j+m}(x, \tau_a) - \delta_{j0} E_m(\tau_a)). \quad (6.9)$$

Definition 20. One defines

$$\mathcal{P}_1(p; x, y) = \mathcal{P}_1(p; x, y; \tau_1, \tau_2, \epsilon),$$

for $p \geq 1$ by

$$\begin{aligned}\mathcal{P}_1(p; x, y) &= P_1(x-y, \tau_a) - P_1(x, \tau_a) \\ &\quad - \mathbb{Q}(p; x) \tilde{\Lambda}_{\bar{a}} \mathbb{P}_1(y) - (1 - \delta_{p1}) (\mathbb{Q}(p; x) \Lambda_{\bar{a}}) (2p-2),\end{aligned}$$

for $x, y \in \widehat{\Sigma}_a^{(1)}$, and

$$\begin{aligned}\mathcal{P}_1(p; x, y) &= (-1)^{p+1} \left[\mathbb{Q}(p; x) \mathbb{P}_1(y) + (1 - \delta_{p1}) \epsilon^{p-1} P_{2p-1}(x) \right. \\ &\quad \left. + (1 - \delta_{p1}) (\mathbb{Q}(p; x) \tilde{\Lambda}_{\bar{a}} \Lambda_a) (2p-2) \right],\end{aligned}$$

for $x \in \widehat{\Sigma}_a^{(1)}$, $y \in \widehat{\Sigma}_{\bar{a}}^{(1)}$. For $j > 0$, define

$$\mathcal{P}_{j+1}(p; x, y) = \frac{1}{j!} \partial_y^j (\mathcal{P}_1(p; x, y)),$$

i.e.,

$$\mathcal{P}_{j+1}(p; x, y) = \delta_{a,\bar{a}} P_{j+1}(x-y) + (-1)^{j+1} \mathbb{Q}(p; x) \left(\tilde{\Lambda}_{\bar{a}} \right)^{\delta_{a,\bar{a}}} \mathbb{P}_{j+1}(y), \quad (6.10)$$

for $x \in \widehat{\Sigma}_a^{(1)}$, $y \in \widehat{\Sigma}_{\bar{a}}^{(1)}$. One calls $\mathcal{P}_{j+1}(p; x, y)$ the genus two generalized Weierstrass functions.

6.3. Genus g generalizations of elliptic functions. For purposes of the formula (4.20) we recall here certain definitions [TW]. Define a column vector

$$X = (X_a(m)),$$

indexed by $m \geq 0$ and $a \in \mathcal{I}$ with components

$$X_a(m) = \rho_a^{-\frac{m}{2}} \sum_{\mathbf{b}_+} Z^{(0)}(\dots; u(m)b_a, w_a; \dots), \quad (6.11)$$

and a row vector

$$p(x) = (p_a(x, m)),$$

for $m \geq 0, a \in \mathcal{I}$ with components

$$p_a(x, m) = \rho_a^{\frac{m}{2}} \partial^{(0,m)} \psi_p^{(0)}(x, w_a). \quad (6.12)$$

Introduce the column vector

$$G = (G_a(m)),$$

for $m \geq 0, a \in \mathcal{I}$, given by

$$G = \sum_{k=1}^n \sum_{j \geq 0} \partial_k^{(j)} q(y_k) Z_V^{(g)}((u(j))_k \mathbf{x}_n),$$

where $q(y) = (q_a(y; m))$, for $m \geq 0, a \in \mathcal{I}$, is a column vector with components

$$q_a(y; m) = (-1)^p \rho_a^{\frac{m+1}{2}} \partial^{(m,0)} \psi_p^{(0)}(w_{-a}, y), \quad (6.13)$$

and

$$R = (R_{ab}(m, n)),$$

for $m, n \geq 0$ and $a, b \in \mathcal{I}$ is a doubly indexed matrix with components

$$R_{ab}(m, n) = \begin{cases} (-1)^p \rho_a^{\frac{m+1}{2}} \rho_b^{\frac{n}{2}} \partial^{(m,n)} \psi_p^{(0)}(w_{-a}, w_b), & a \neq -b, \\ (-1)^p \rho_a^{\frac{m+n+1}{2}} \mathcal{E}_m^n(w_{-a}), & a = -b, \end{cases} \quad (6.14)$$

where

$$\mathcal{E}_m^n(y) = \sum_{\ell=0}^{2p-2} \partial^{(m)} f_\ell(y) \partial^{(n)} y^\ell, \quad (6.15)$$

$$\psi_p^{(0)}(x, y) = \frac{1}{x-y} + \sum_{\ell=0}^{2p-2} f_\ell(x) y^\ell, \quad (6.16)$$

for any Laurent series $f_\ell(x)$ for $\ell = 0, \dots, 2p-2$. Define the doubly indexed matrix $\Delta = (\Delta_{ab}(m, n))$ by

$$\Delta_{ab}(m, n) = \delta_{m, n+2p-1} \delta_{ab}. \quad (6.17)$$

Denote by

$$\tilde{R} = R\Delta,$$

and the formal inverse $(I - \tilde{R})^{-1}$ is given by

$$(I - \tilde{R})^{-1} = \sum_{k \geq 0} \tilde{R}^k. \quad (6.18)$$

Define $\chi(x) = (\chi_a(x; \ell))$ and

$$o(u; \mathbf{v}, \mathbf{y}) = (o_a(u; \mathbf{v}, \mathbf{y}; \ell)),$$

are finite row and column vectors indexed by $a \in \mathcal{I}$, $0 \leq \ell \leq 2p - 2$ with

$$\chi_a(x; \ell) = \rho_a^{-\frac{\ell}{2}} (p(x) + \tilde{p}(x)(I - \tilde{R})^{-1}R)_a(\ell), \quad (6.19)$$

$$o_a(\ell) = o_a(u; \mathbf{v}, \mathbf{y}; \ell) = \rho_a^{\frac{\ell}{2}} X_a(\ell), \quad (6.20)$$

and where

$$\tilde{p}(x) = p(x)\Delta.$$

$\psi_p(x, y)$ is defined by

$$\psi_p(x, y) = \psi_p^{(0)}(x, y) + \tilde{p}(x)(I - \tilde{R})^{-1}q(y). \quad (6.21)$$

For each $a \in \mathcal{I}_+$ we define a vector

$$\theta_a(x) = (\theta_a(x; \ell)),$$

indexed by $0 \leq \ell \leq 2p - 2$ with components

$$\theta_a(x; \ell) = \chi_a(x; \ell) + (-1)^p \rho_a^{p-1-\ell} \chi_{-a}(x; 2p - 2 - \ell). \quad (6.22)$$

Now define the following vectors of formal differential forms

$$\begin{aligned} P(x) &= p(x) dx^p, \\ Q(y) &= q(y) dy^{1-p}, \end{aligned} \quad (6.23)$$

with

$$\tilde{P}(x) = P(x)\Delta.$$

Then with

$$\Psi_p(x, y) = \psi_p(x, y) dx^p dy^{1-p}, \quad (6.24)$$

we have

$$\Psi_p(x, y) = \Psi_p^{(0)}(x, y) + \tilde{P}(x)(I - \tilde{R})^{-1}Q(y). \quad (6.25)$$

Defining

$$\Theta_a(x; \ell) = \theta_a(x; \ell) dx^p, \quad (6.26)$$

and

$$O_a(u; \mathbf{v}, \mathbf{y}; \ell) = o_a(u; \mathbf{v}, \mathbf{y}; \ell) d\mathbf{y}^{\text{wt}(\mathbf{v})}, \quad (6.27)$$

Remark 9. The $\Theta_a(x)$, and $\Psi_p(x, y)$ coefficients depend on $p = \text{wt}(u)$ but are otherwise independent of the vertex operator algebra V .

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INSTITUTE OF MATHEMATICS, CZECH ACADEMY OF SCIENCES, ŽITNA 25, PRAGUE, CZECH REPUBLIC

E-mail address: zuevsky@yahoo.com