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**Cohomology of Jacobi forms**

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Preprint No. 18-2021

PRAHA 2021



# COHOMOLOGY OF JACOBI FORMS

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ABSTRACT. We define and compute a cohomology of the space of Jacobi forms based on precise analogues of Zhu reduction formulas derived in [Zhu, BKT, MTZ]. A counterpart of the Bott-Segal theorem for the reduction cohomology of Jacobi forms on the torus is proven. It is shown that the reduction cohomology for Jacobi forms is given by the cohomology of  $n$ -point connections over a deformed vertex algebra bundle defined on the torus. The reduction cohomology for Jacobi forms for a vertex algebra is determined in terms of the space of analytical continuations of solutions to Knizhnik-Zamolodchikov equations.

## 1. INTRODUCTION

The natural problem of computation of continuous cohomologies for non-commutative structures on manifolds has proven to be a subject of great geometrical interest [BS, Fei, Fuks, Wag]. For Riemann surfaces, and even for higher dimension complex manifolds, the classical cohomology of holomorphic vector fields is often trivial [Kaw, Wag]. In [Fei] Feigin has obtained various results concerning (co)-homology of cosimplicial objects associated to holomorphic vector fields  $Lie(M)$ . Vertex algebra [BZF, FHL, K] theory of automorphic forms [Fo] goes back to celebrated Moonshine problem [MT]. Most of  $n$ -point characteristic functions [FS, FHL, KZ, MT, Zhu] for vertex algebras deliver examples of modular forms with respect to appropriate groups attached to geometry of corresponding underlying manifolds.  $n$ -point functions are subject to action of differential operators with specific analytical behavior [BKT, GK, GN, Ob]. In this paper we develop ideas and previous results on cohomology of Jacobi forms originating from algebraic and geometrical procedures in conformal field theory [FS, TUY]. This paper aims at developing algebraic, differential geometry, and topological methods for the investigation of cohomology theories of Jacobi forms generated by vertex algebras, with applications in algebraic topology, number theory and mathematical physics.

In most cases of lower genera Riemann surfaces, there exist algebraic formulas relating  $n$ -point functions with  $n - 1$ -point functions in a linear way for fixed genus  $g$  [Zhu, MT, MTZ]. The reduction cohomology is defined via reduction formulas [Zhu, BKT] relating  $n$ -point characteristic functions with  $(n - 1)$ -functions. Our new algebraic and geometrical approach for computation of reduction (co)homology involves vertex algebras and applications of techniques [Huang, Y] used in conformal field theory. Computation of moduli forms reduction cohomology is useful in further

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*Key words and phrases.* Cohomology; Jacobi forms; Vertex algebras; Knizhnik-Zamolodchikov equations.

studies of constructions in algebraic topology, analytical and geometrical structure of spaces of modular forms originating from the description of vertex algebras by means of characteristic functions on manifolds. The main aim of the reduction cohomology is to describe non-commutative structures in terms of commutative ones. In contrast to more geometrical methods in classical cohomology for Lie algebras [Fuks], the reduction cohomology pays more attention to the differential, analytical, and automorphic structure of chain complex elements constructed by means of characteristic functions for non-commutative elements of vertex algebras with complex parameters. Computational methods involving reduction formulas proved their effectiveness in conformal field theory [KMI, KMII, MT, MT1, MTZ, TZ, DLM, Miy]. Though the Zhu reduction formulas were obtained for ordinary  $n$ -point functions of vertex operators, it also works for multi-parametric automorphisms inserted into traces written for the torus case. Then coefficients in the reduction formulas are expressed in terms of quasi-modular forms. Since quasi-modular forms are holomorphic on the complex upper half-plane  $\mathbb{H}$ , then it follows that  $n$ -point Jacobi functions are also holomorphic. The plan of this paper is the following. We define the reduction cohomology, chain condition, and co-boundary operator for complexes of Jacobi forms. Specific examples of coboundary operators are provided subject to various conditions on vertex algebra elements. A statement relating  $n$ -th reduction cohomology with analytic extensions of solutions to a counterpart of Knizhnik–Zamolodchikov equation is proven, and its geometrical meaning is found. In appendixes we recall the notions of quasi-modular forms, reduction formulas for Jacobi functions, and vertex algebras. Quasi-Jacobi forms have found applications in vertex algebra theory in [HE], for characteristic functions of topological  $N = 2$  vertex algebras, Gromov-Witten potentials [Kaw], computation of elliptic genera [Lib] related to Jacobi zero-point functions, Landau-Ginzburg orbifolds [KYY].

## 2. CHAIN COMPLEX FOR VERTEX ALGEBRA $n$ -POINT FUNCTIONS

In this section we will give definition of a chain complex associated to the space of Jacobi forms [BKT] defined by vertex algebras.

**2.1. Spaces of  $n$ -point Jacobi functions via vertex operators.** Let us fix a vertex algebra  $V$ . We denote by  $\mathbf{v}_n = (v_1, \dots, v_n) \in V^{\otimes n}$  a tuple of vertex algebra elements (see Appendix 3.4 for definition of a vertex algebra). Mark  $n$  points  $\mathbf{p}_n = (p_1, \dots, p_n)$  on the torus  $\mathcal{T}$ . Denote by  $\mathbf{z}_n = (z_1, \dots, z_n)$  local coordinates around  $\mathbf{p}_n \in \mathcal{T}$ . Let us introduce the notation:  $\mathbf{x}_n = (\mathbf{v}_n, \mathbf{z}_n)$ .

In [MTZ] we considered the orbifold Jacobi  $n$ -point functions associated with a vertex operator superalgebra [K] (see Appendix 3.4), with an automorphism inserted in traces. Let  $\sigma \in \text{Aut}(V)$  denote the parity automorphism

$$\sigma a = (-1)^{p(a)} a. \quad (2.1)$$

Let  $g \in \text{Aut}(V)$  denote any other automorphism which commutes with  $\sigma$ . Let  $W$  be a  $V$ -module. Assume that  $W$  is stable under both  $\sigma$  and  $g$ , i.e.,  $\sigma$  and  $g$  act on  $W$ .

**Definition 1.** The  $n$ -point Jacobi function on  $W$  for  $\mathbf{x}_n \in V^{\otimes n} \times \mathbb{C}^n$ , and  $g \in \text{Aut}(V)$  is defined by

$$\mathcal{Z}_W^J(\mathbf{x}; g, \tau) = \text{STr}_W \left( \mathbf{Y}_W \left( \mathbf{q}^{L_V(0)} \mathbf{v}, \mathbf{q} \right)_n g q^{L_V(0)-c/24} \right), \quad (2.2)$$

$q = \exp(2\pi i\tau)$ ,  $q_i = \exp(z_i)$ ,  $1 \leq i \leq n$ .

Here  $\text{STr}_W$  denotes the supertrace defined by

$$\text{STr}_W(X) = \text{Tr}_W(\sigma X) = \text{Tr}_{W_0}(X) - \text{Tr}_{W_1}(X). \quad (2.3)$$

The orbifold Jacobi zero-point function for general  $g$  is then

$$\mathcal{Z}_W(g, \tau) = \text{STr}_W \left( g q^{L_V(0)-c/24} \right). \quad (2.4)$$

In particular, when  $V$  is a vertex operator algebra with Virasoro vector  $\omega$  of central charge  $c_V$ . Consider an element  $J \in V_1$  such that  $J(0)$  acts semisimply on  $V$ . For  $\mathbf{v}_n \in V^{\otimes n}$ , on  $\mathcal{T}$ , and a weak  $V$ -module  $W$  [MT], the Jacobi  $n \geq 0$ -point function is

$$\mathcal{Z}_W^J(\mathbf{x}_n; B) = \text{Tr}_W \left( \mathbf{Y} \left( e^{\mathbf{z} L_V(0)} \mathbf{v}, e^{\mathbf{z}} \right)_n \zeta^{J(0)} q^{L(0)} \right), \quad (2.5)$$

where  $B$  denotes parameters of  $\mathcal{Z}_W^J$  including  $z$  and  $\tau$ , and

$$\zeta = q_z = e^{2\pi iz}.$$

The Jacobi one-point function, for  $v \in V$ , is given by

$$\mathcal{Z}_W^J(x_1; B) = \text{Tr}_W \left( o_0(v_1) \zeta^{J(0)} q^{L(0)} \right),$$

which does not depend on  $z_1$ . Here  $o_0(v_1) = v_1(\text{wt } v_1 - 1)$  (see Appendix 3.4), with  $z \in \mathbb{C}$ , and  $\tau$  being the modular parameter of  $\mathcal{T}$ .

**Definition 2.** For a  $V$ -module  $W$ , we consider the spaces of  $n$ -point Jacobi forms

$$C^n(W) = \{ \mathcal{Z}_W^J(\mathbf{x}_n, ; B), n \geq 0, \}.$$

The coboundary operator  $\delta^n(\mathbf{x}_{n+1})$  on  $C^n(W)$ -space is defined according to the reduction formulas (see section 3 and Appendix 6) [BKT, MTZ] for  $V$ -module  $W$  Jacobi forms.

**Definition 3.** For  $n \geq 0$ , and any  $x_{n+1} \in V \times \mathbb{C}$ , define

$$\delta^n(\mathbf{x}_{n+1}) : C^n(W) \rightarrow C^{n+1}(W), \quad (2.6)$$

with non-commutative operators  $T_j(v[m].)$ ,  $j \geq 0$ , is given by the reduction formulas (2.7)

$$\delta^n(\mathbf{x}_{n+1}) \mathcal{Z}_W^J(\mathbf{x}_n; B) = \sum_{\substack{k=0 \\ m \geq 0}}^n f_{k,m}(\mathbf{x}_{n+1}; B) T_k(v_{n+1}[m].) \mathcal{Z}_W^J(\mathbf{x}_n; B), \quad (2.7)$$

The operators  $T_k(v[m].)$  are insertion operators of vertex algebra modes  $v[m].$ ,  $m \geq 0$ , into  $\mathcal{Z}_W^J(\mathbf{x}_n; B)$  at the  $k$ -th entry:

$$T_k(v[m].) \mathcal{Z}_W^J(\mathbf{x}_n; B) = \mathcal{Z}_W^J(T_k(v[m]) . \mathbf{x}_n; B),$$

where we use the notation

$$(\Gamma)_k \mathbf{x}_n = (x_1, \dots, \Gamma.x_k, \dots, x_n),$$

for an operator  $\Gamma$  acting on  $k$ -th entry.

*Remark 1.* The reductions formulas have an interpretation in terms of torsors [BZF] (Chapter 6). In such formulation  $\mathbf{x}_n$  is a torsor with respect to the group of transformation of the space  $V^{\otimes n} \times \mathbb{C}^n$ . In particular, from (2.7) we see that  $T_k(u[m].)$ -operators act on  $V^{\otimes n}$ -entries of  $\mathbf{x}_n$ , while  $f_{k,m}(\mathbf{x}_{n+1}; B)$ -functions act on  $\mathbf{z}_n$  of  $\mathcal{Z}_W^J(\mathbf{x}_n; B)$  as a complex function.

For  $n \geq 0$ , let us denote by  $\mathfrak{Y}_n$  the subsets of all  $\mathbf{x}_n \in V^{\otimes n} \times \mathbb{C}^n$ , such that the chain condition

$$\delta^{n+1}(\mathbf{x}_{n+1}) \delta^n(\mathbf{x}_n) \mathcal{Z}_W^J(\mathbf{x}_n; B) = 0, \quad (2.8)$$

for the coboundary operators (2.7) for complexes  $C^n(W)$  is satisfied. Explicitly, the chain condition (2.8) leads to an infinite  $n \geq 0$  set of equations involving functions  $f_{k,m}(\mathbf{x}_{n+1}; B)$  and  $\mathcal{Z}_W^J(\mathbf{x}_n; B)$ :

$$\left( \sum_{\substack{k', k=0 \\ m', m \geq 0}}^{n+1, n} f_{k', m'}(\mathbf{x}_{n+1}; B) f_{k, m}(\mathbf{x}_n; B) T_{k'}(v_{n+2}[m']). T_k(v_{n+1}[m].) \right) \mathcal{Z}_W^J(\mathbf{x}_n; B) = 0. \quad (2.9)$$

*Remark 2.* (2.9) contain finite series and narrows the space of compatible  $n$ -point functions. It follows from considerations of [BKT], the subspaces of  $C^n(W)$ ,  $n \geq 0$ , of  $n$ -point Jacobi forms such that the condition (2.9) is fulfilled for reduction cohomology complexes are non-empty. Indeed, the condition (2.9) represents an infinite  $n \geq 0$  set of functional-differential equations (with finite number of summands) on converging complex functions  $\mathcal{Z}_W^J(\mathbf{x}_n; B)$  defined for  $n$  local complex variables on  $\mathcal{T}$  with functional coefficients  $f_{k,m}(\mathbf{x}_{n+1}; B)$  (in our examples in subsection 3.1–3.4, these are generalizations of elliptic functions) on  $\mathcal{T}$ . Note that all vertex algebra elements of  $\mathbf{v}_n \in V^{\otimes n}$ , as non-commutative parameters are not present in final form of functional-differential equations since they incorporated into either matrix elements, traces, etc. According to the theory of such equations [FK, Gu], each equation in the infinite set of (2.9) always have a solution in domains they are defined. Thus, there always exist solutions of (2.9) defining  $\mathcal{Z}_W^J \in C^n(W)$ , and they are not empty.

**Definition 4.** The spaces with conditions (2.9) constitute a semi-infinite chain complex

$$0 \longrightarrow C^0 \xrightarrow{\delta^0(x_1)} C^1 \xrightarrow{\delta^1(\mathbf{x}_2)} \dots \xrightarrow{\delta^{n-2}(\mathbf{x}_{n-1})} C^{n-1} \xrightarrow{\delta^{n-1}(\mathbf{x}_n)} C^n \xrightarrow{\delta^n(\mathbf{x}_{n+1})} \dots \quad (2.10)$$

For  $n \geq 1$ , we call corresponding cohomology

$$H_J^n(W) = \text{Ker } \delta^n(\mathbf{x}_{n+1}) / \text{Im } \delta^{n-1}(\mathbf{x}_n), \quad (2.11)$$

the  $n$ -th reduction cohomology of a vertex algebra  $V$ -module  $W$  on  $\mathcal{T}$ .

### 3. REDUCTION FORMULAS AND EXAMPLES OF COBOUNDARY OPERATORS FOR JACOBI $n$ -POINT FUNCTIONS

**3.1. The main formula for coboundary operator.** In this subsection, using Propositions 5 and 6 of [BKT] (see Appendix 6), we introduce the definition of a coboundary operator associated to the most general (up to certain assumptions) reduction formulas available for Jacobi forms. Recall the definition of square bracket vertex operators from Appendix 6.4. Summing (6.10) over  $l$  multiplied by  $z_{n+1}^{l-1}$ ,  $Z_W^J(v_{n+1}[-l].x_1, \mathbf{x}_{2,n}; B)$ . and using associativity of vertex operators we formulate the following definition of the coboundary operator

**Definition 5.** Let  $v_{n+1} \in V$  such that

$$v_{n+1}[l].v_k = 0,$$

for  $l \geq 1$ ,  $1 \leq k \leq n$ , and such that

$$J(0)v_{n+1} = \alpha v_{n+1}.$$

with  $\alpha \in \mathbb{C}$ . Then the coboundary operator is given by (2.7) with the summation over  $l \in \mathbb{Z}$ , i.e.,

$$\delta^n(\mathbf{x}_{n+1}) \mathcal{Z}_W^J(\mathbf{x}_n; z, \tau) = \sum_{\substack{l \in \mathbb{Z} \\ m \geq 0, \\ k=0}}^n f_{k,m}(\mathbf{x}_{n+1}; B) T_k(v_{n+1}[m].) \mathcal{Z}_W^J(\mathbf{x}_n; z, \tau), \quad (3.1)$$

$$\begin{aligned} f_0(\mathbf{x}_{n+1}; B) T_0(v[m].) &= \sum_{l \in \mathbb{Z}} (-1)^{l+1} \delta_{\alpha z, \mathbb{Z}\tau + \mathbb{Z}} \frac{\lambda^{l-1}}{(l-1)!} z_{n+1}^{l-1} T_0(o_\lambda(v_{n+1})), \\ f_{k,m}(\mathbf{x}_{n+1}; B) &= \sum_{l \in \mathbb{Z}} (-1)^{m+1} \binom{m+l-1}{m} z_{n+1}^{l-1} F_{k,m}(\mathbf{x}_{n+1}; l, \alpha z, \tau), \end{aligned} \quad (3.2)$$

where

$$\begin{aligned} F_{k,m}(\mathbf{x}_{n+1}; l, \alpha z, \tau) &= \delta_{0,m} T^{1-\delta_{\alpha z, \mathbb{Z}\tau + \mathbb{Z}}} \cdot \tilde{E}_{m+l,\lambda}((1 - \delta_{\alpha z, \mathbb{Z}\tau + \mathbb{Z}}) \alpha z, \tau) \\ &\quad + T^{1-\delta_{\alpha z, \mathbb{Z}\tau + \mathbb{Z}}} \cdot \tilde{P}_{m+l, (1-\delta_{\alpha z, \mathbb{Z}\tau + \mathbb{Z}})\lambda} \left( \frac{z_1 - z_k}{2\pi i}, (1 - \delta_{\alpha z, \mathbb{Z}\tau + \mathbb{Z}}) \alpha z, \tau \right), \end{aligned}$$

with tilde applying operator  $T$ , i.e.,

$$\begin{aligned} T.E_{m+l,\lambda} &= \tilde{E}_{m+l,\lambda}, \\ T.P_{m+l,\lambda} &= \tilde{P}_{m+l,\lambda}, \end{aligned} \quad (3.3)$$

and  $\tilde{E}_{m+k,\lambda}(\alpha z, \tau)$ ,  $\tilde{P}_{m+l,\lambda}(z', \alpha z, \tau)$  given by (5.6) and (5.5) correspondingly.

**3.2. The simplest coboundary operator.** For certain further restriction on  $v_{n+1}$ , we are able to define the simplest version of coboundary operator for the reduction cohomology. Using propositions 3 and 4 proven in [BKT] (see Appendix 6), we obtain

**Definition 6.** For  $v_{n+1}$ , with  $J(0)v_{n+1} = \alpha v_{n+1}$ ,  $\alpha \in \mathbb{C}$ , the coboundary operator is defined by (2.7) with

$$\begin{aligned} f_0(\mathbf{x}_{n+1}; \alpha z, \tau) T_0(v_{n+1}[m]) &= \delta_{\alpha z, \lambda \tau + \mu \in \mathbb{Z}\tau + \mathbb{Z}} e^{-z_{n+1}\lambda} T_0(o_\lambda(v_{n+1})), \\ f_{k,m}(\mathbf{z}_{n+1}; \lambda, k, \alpha z, \tau) &= T^{1-\delta_{\alpha z, \lambda \tau + \mu \in \mathbb{Z}\tau + \mathbb{Z}}} P_{m+1, \lambda} \left( \frac{z_{n+1} - z_k}{2\pi i}, (1 - \delta_{\alpha z, \lambda \tau + \mu \in \mathbb{Z}\tau + \mathbb{Z}}) \alpha z, \tau \right), \end{aligned} \quad (3.4)$$

with  $\tilde{P}_{m+1, \lambda}(\mathbf{z}_{n+1}, \alpha z, \tau)$  defined in (5.5).

**3.3. Coboundary operator for a shifted Virasoro vector.** Suppose that  $J(0)a = \alpha a$  for  $\alpha \notin \mathbb{Z} \setminus \{0\}$ , and define a  $V$ -automorphism  $g \in \text{Aut}(V)$  by

$$g = e^{2\pi i \frac{\mu}{\alpha} J(0)},$$

for  $\mu \in \mathbb{Z}$  for which  $ga = a$ . Then Corollary 2 follows from Proposition 6 of [MTZ] which states that

$$\sum_{k=1}^n \text{Tr}_W \left( T_k(a[0].) \mathbf{Y} \left( e^{z L_V(0)} v, e^{\mathbf{z}} \right)_n g q^{L(0)} \right) = 0. \quad (3.5)$$

For

$$J(0)v_k = \alpha_k v_k,$$

for  $k = 1, \dots, n$ , For the case of shifted Virasoro vector [MTZ] (see Appendix 3.4) we find In a similar fashion, we can relate Proposition 4 to Theorem 2 of [MTZ] for the above shifted Virasoro grading  $L_h(0)$  and with  $g = e^{2\pi i \frac{\mu}{\alpha} J(0)}$ . Using Theorem 2 of [MTZ] we give the following

**Definition 7.** The shifted coboundary operator for the shifted Jacobi form

$$\mathcal{Z}_W^J(\mathbf{x}_{n+1}; h, \mu, \alpha, z, \tau) = \text{Tr}_W \left( \mathbf{Y} \left( e^{z \mathbf{L}_h(0)} \mathbf{v}, e^{\mathbf{x}} \right)_{n+1} g q^{L_h(0)} \right), \quad (3.6)$$

is given by (2.7) with

$$f_0(\mathbf{x}_{n+1}; B) T_0(v_{n+1}[m].) = T_0(o_h(v_{n+1})), \quad (3.7)$$

$$f_{k,m}(\mathbf{x}_{n+1}; B) = P_{m+1} \left( \frac{z_{n+1} - z_k}{2\pi i}, \tau \right) \quad (3.8)$$

and  $T_k(v_{n+1}[m]_{h.})$ , where

$$o_h(v_{n+1}) = v_{n+1}(\text{wt}_h(v_{n+1}) - 1) = v_{n+1}(\text{wt}(v_{n+1}) - 1 + \mu) = o_\mu(v_{n+1}). \quad (3.9)$$

**3.4. Vertex operator superalgebra case.** For the case of orbifold Jacobi  $n$ -point functions, we have the following. Let  $v_{n+1}$  be homogeneous of weight  $\text{wt}(v_{n+1}) \in \mathbb{R}$  and define  $\phi \in U(1)$  by

$$\phi = \exp(2\pi i \text{wt}(v_{n+1})). \quad (3.10)$$

We also take  $v_{n+1}$  to be an eigenfunction under  $g$  with

$$g v_{n+1} = \theta^{-1} v_{n+1} \quad (3.11)$$

for some  $\theta \in U(1)$  so that

$$g^{-1} v_{n+1}(k) g = \theta v_{n+1}(k). \quad (3.12)$$



Then we have Let  $v, \theta$  and  $\phi$  be as as above. Then the coboundary operator is defined by

$$\begin{aligned} f_0(\mathbf{x}_{n+1}; B) T_0(v_{n+1}[m]) &= \delta_{\theta,1} \delta_{\phi,1} T_0(o_0(v_{n+1})), \\ f_{k,m}(\mathbf{x}_{n+1}; B) &= p(v_{n+1}, \mathbf{v}_{k-1}) P_{m+1} \begin{bmatrix} \theta \\ \phi \end{bmatrix} (z_{n+1} - z_k, \tau) \end{aligned} \quad (3.13)$$

where the deformed Weierstrass functions are defined in (5.9) (see Appendix 5.2).

Note that, as it was shown in [MTZ], the orbifold Jacobi function case is related to the shifted Virasoro vector case above.

#### 4. COHOMOLOGY

In this section we compute the reduction cohomology defined by (2.10)–(2.11).

**4.1. The  $n$ -th cohomology and analytic extensions of solutions to Knizhnik-Zamolodchikov equations.** The main result of this paper is the following.

**Proposition 1.** *Under assumptions of subsections 3.1–3.4, the  $n$ -th reduction cohomology of the space of Jacobi forms for  $V$ -module  $W$  is given by space of analytical continuations of solutions  $\mathcal{Z}_W^J(\mathbf{x}_n; B)$  to the Knizhnik-Zamolodchikov equation*

$$\sum_{k=0}^n \sum_{m \geq 0} f_{k,m}(\mathbf{x}_n; B) T_k(v_{n+1}[m]_{\beta}) \mathcal{Z}_M^J(\mathbf{x}_n; B) = 0, \quad (4.1)$$

with  $x_i \notin \mathfrak{V}_i$ ,  $\beta = h$  for a shifted Virasoro element and zero otherwise, for  $1 \leq i \leq n$ . These are given by the spaces of quasi-modular forms in terms of series of deformed Weierstrass functions, defined in Appendix 5.2, recursively generated by reduction formulas (2.7).

**Proof:** The  $n$ -th reduction cohomology is defined by the subspace of  $C^n(W)$  of functions  $\mathcal{Z}^J(\mathbf{x}_n; B)$  satisfying (4.1), modulo the subspace of  $C^n(W)$   $n$ -point functions  $\mathcal{Z}_W^J(\mathbf{x}'_n; B)$  resulting from:

$$\mathcal{Z}_W^J(\mathbf{x}'_n; B) = \left( \sum_{k=1}^{n-1} \sum_{m \geq 0} f_{k,m}(\mathbf{x}_n; B) T_k^{(g)}(v'_n[m]_{\beta}) \right) \mathcal{Z}_W^J(\mathbf{x}'_{n-1}; B). \quad (4.2)$$

Subject to other fixed parameters,  $n$ -point functions are completely determined by all choices  $\mathbf{x}_n \in V^{\otimes n} \times \mathbb{C}^n$  which does not belong to  $\mathfrak{V}$ . Thus, the reduction cohomology can be treated as depending on set of  $\mathbf{x}_n$  only with appropriate action of endomorphisms generated by  $x_{n+1}$ . Consider a non-vanishing solution  $\mathcal{Z}_W^J(\mathbf{x}_n; B)$  to (4.1) for some  $\mathbf{x}_n$ . Let us use the reduction formulas (2.7) recursively for each  $x_i$ ,  $1 \leq i \leq n$  of  $\mathbf{x}_n$  in order to express  $\mathcal{Z}_W^J(\mathbf{x}_n; B)$  in terms of the partition function  $\mathcal{Z}_W^J(B)$ , i.e., we obtain

$$\mathcal{Z}_W^J(\mathbf{x}_n; B) = \mathcal{D}(\mathbf{x}_n; B) \mathcal{Z}_W^J(B), \quad (4.3)$$

as in [MT, MTZ, TZ]. Thus,  $x_i \notin \mathfrak{V}_i$  for  $1 \leq i \leq n$ , i.e., at each stage of the recursion procedure reproducing (4.3), otherwise  $\mathcal{Z}_W^J(\mathbf{x}_n; B)$  is zero. Therefore,  $\mathcal{Z}_W^J(\mathbf{x}_n; B)$ ,

is explicitly known and is represented as a series of auxiliary functions  $\mathcal{D}(\mathbf{x}_n; B\tau)$  depending on  $V$ . Consider now  $\mathcal{Z}_W^J(\mathbf{x}'_n; B)$  given by (4.2). It is either vanishes when  $v_{n-i} \in \mathfrak{V}_{n-i}$ ,  $2 \leq i \leq n$ , or given by (4.3) with  $\mathbf{x}'_n$  arguments.

The way the reduction relations (2.7) were derived in [Y] is exactly the same as for the vertex algebra derivation [KZ, TK] for the Knizhnik-Zamolodchikov equations. Namely, one considers a double integration of  $\mathcal{Z}_W^J(\mathbf{x}_n; B)$  along small circles around two auxiliary variables with the action of appropriate reproduction kernels inserted. Then, these procedure leads to recursion formulas relating  $\mathcal{Z}_W^J(\mathbf{x}_{n+1}; B)$  and  $\mathcal{Z}_W^J(\mathbf{x}_n; B)$  with functional coefficients depending on the nature of the vertex algebra  $V$ . Thus, (4.1) can be seen as a version of the Knizhnik-Zamolodchikov equation. In [Y, MT, MTZ, BKT] formulas for  $n$ -point functions in various specific examples of  $V$  and configuration of Riemann surfaces were explicitly obtained.

In terms of  $x_{n+1}$ , by using (6.11), one transfers in (4.1) the action of  $v_{n+1}$ -modes into an analytical continuation of  $\mathcal{Z}_W^J(\mathbf{x}_n; B)$  multi-valued holomorphic functions to domains  $\mathcal{T}_n \subset \mathcal{T}$  with  $z_i \neq z_j$  for  $i \neq j$ . Namely, in (4.1), the operators  $T_k(v_{n+1}[m]_\beta)$  act by certain modes  $v_{n+1}[m]$  of a vertex algebra element  $v_{n+1}$  on  $\mathbf{v}_n \in V^{\otimes n}$ . Using vertex algebra associativity we express the action of operators  $T_k(v_{n+1}[m]_\beta)$  in terms of modes  $v_{n+1}[m]$  inside vertex operators in actions of  $V$ -modes on the whole vertex operator at expense of a shift of their formal parameters  $\mathbf{z}_n$  by  $z_{n+1}$ , i.e.,  $\mathbf{z}'_n = \mathbf{z}_n + z_{n+1}$ . Note that under such associativity transformations  $v$ -part of  $\mathbf{x}_n$ , i.e.,  $\mathbf{v}_n$  do not change. Thus, the  $n$ -th reduction cohomology of a  $V$ -module  $W$  is given by the space of analytical continuations of  $n$ -point functions  $\mathcal{Z}_W^J(\mathbf{x}_n; B)$  with  $\mathbf{x}_{n-1} \notin \mathfrak{V}_{n-1}$  that are solutions to the Knizhnik-Zamolodchikov equations (4.1). The above analytic extensions for the Knizhnik-Zamolodchikov equations generated by  $x_{n+1}$  and with coefficients provided by functions  $f_{k,m}(\mathbf{x}_{n+1}; B)$  on  $\mathcal{T}$ .  $\square$

One can make connection with the first cohomology of grading-restricted vertex algebras in terms of derivations, and to the second cohomology in terms of square-zero extensions of  $V$  by  $W$  [Huang]. In certain cases of coboundary operators, we are able to compute the  $n$ -th cohomology even more explicitly by using reduction formulas in terms of generalized elliptic functions. In particular, for orbifold  $n$ -point Jacobi functions associated to a vertex operator superalgebra described in Appendix 3.4, we obtain from [MTZ]

**Corollary 1.** *For  $\mathbf{v}_n \notin \mathfrak{V}_n$ , the  $n$ -th cohomology is given by the space of determinants of  $n \times n$ -matrices containing deformed elliptic functions depending on  $z_i - z_j$ ,  $1 \leq i, j \leq n$ , for all possible combinations of  $\mathbf{v}_n$ -modes.*

**4.2. Geometrical meaning of reduction formulas and conditions (2.9).** In this section we show that the Jacobi forms reduction formulas (2.7) appear as of multipoint connections on a vector bundle over  $\mathcal{T}$  generalizing ordinary holomorphic connections on complex curves [BZF]. Let us recall the notion of a multipoint connection which will be useful for further identifying reduction cohomology in this subsection. Motivated by the definition of a holomorphic connection for a vertex algebra bundle (cf. Section 6, [BZF] and [Gu]) over a smooth complex curve, we introduce the definition of the multiple point connection over  $\mathcal{T}$ .

**Definition 8.** Let  $\mathcal{V}$  be a holomorphic vector bundle over  $\mathcal{T}$ , and  $\mathcal{T}_0 \subset \mathcal{T}$  be its subdomain. Denote by  $\mathcal{SV}$  the space of sections of  $\mathcal{V}$ . A multi-point connection  $\mathcal{G}$  on  $\mathcal{V}$  is a  $\mathbb{C}$ -multi-linear map

$$\mathcal{G} : \mathcal{T}^{\times n} \times V^{\otimes n} \rightarrow \mathbb{C},$$

such that for any holomorphic function  $f$ , and two sections  $\phi(p)$  and  $\psi(p')$  of  $\mathcal{V}$  at points  $p$  and  $p'$  on  $\mathcal{T}_0 \subset \mathcal{T}$  correspondingly, we have

$$\sum_{q, q' \in \mathcal{T}_0 \subset \mathcal{T}} \mathcal{G}(f(\psi(q)) \cdot \phi(q')) = f(\psi(p')) \mathcal{G}(\phi(p)) + f(\phi(p)) \mathcal{G}(\psi(p')), \quad (4.4)$$

where the summation on left hand side is performed over locuses of points  $q, q'$  on  $\mathcal{T}_0$ . We denote by  $Con^n$  the space of  $n$ -point connections defined over  $\mathcal{T}$ .

Geometrically, for a vector bundle  $\mathcal{V}$  defined over  $\mathcal{T}$ , a multi-point connection (4.4) relates two sections  $\phi$  and  $\psi$  at points  $p$  and  $p'$  with a number of sections on  $\mathcal{T}_0 \subset \mathcal{T}$ .

**Definition 9.** We call

$$G(\phi, \psi) = f(\phi(p)) \mathcal{G}(\psi(p')) + f(\psi(p')) \mathcal{G}(\phi(p)) - \sum_{q, q' \in \mathcal{T}_0 \subset \mathcal{T}} \mathcal{G}(f(\psi(q')) \cdot \phi(q)), \quad (4.5)$$

the form of a  $n$ -point connection  $\mathcal{G}$ . The space of  $n$ -point connection forms will be denoted by  $G^n$ .

Here we prove the following

**Lemma 1.** *Jacobi  $n$ -point forms (2.2) generated by reduction formulas (2.7) are  $n$ -point connections on the space of automorphisms  $g$  deformed sections of the vertex algebra bundle  $\mathcal{V}$  associated to  $V$ . For  $n \geq 0$ , the  $n$ -th reduction cohomology of Jacobi forms is given by*

$$H_J^n(W) = H_J^n(\mathcal{SV}_g) = Con^n / G^{n-1}, \quad (4.6)$$

*is isomorphic to the cohomology of the space of deformed  $\mathcal{V}$ -sections.*

*Remark 3.* Proposition 1 is a deformed section vertex algebra version of the main proposition of [BS, Wag], i.e., the Bott–Segal theorem for Riemann surfaces.

**Proof:** In [BZF] (Chapter 6, subsection 6.5.3) the vertex operator bundle  $\mathcal{V}$  was explicitly constructed. It is easy to see that  $n$ -point connections are holomorphic connection on the bundle  $\mathcal{V}$  with the following identifications. For non-vanishing  $f(\phi(p))$  let us set

$$\begin{aligned} \mathcal{G} &= \mathcal{Z}_W^J(\mathbf{x}_n; B), \\ \psi(p') &= (\mathbf{x}_{n+1}), \\ \phi(p) &= (\mathbf{x}_n), \\ \mathcal{G}(f(\psi(q)) \cdot \phi(q')) &= T_k(v[m]_{\beta \cdot}) \mathcal{Z}_W^J(\mathbf{x}_n; B), \\ -\frac{f(\psi(p'))}{f(\phi(p))} \mathcal{G}(\phi(p)) &= f_0(\mathbf{x}_{n+1}; B) T_0(o_\lambda(v_{n+1})) \mathcal{Z}_W^J(\mathbf{x}_n; B), \\ f^{-1}(\phi(p)) \sum_{\substack{q_n, q'_n \\ \in \mathcal{T}_0 \subset \mathcal{T}}} \mathcal{G}(f(\psi(q)) \cdot \phi(q')) &= \sum_{\substack{k=1 \\ m \geq 0}}^n f_{k,m}(\mathbf{x}_{n+1}; B) T_k(v[m]_{\beta \cdot}) \mathcal{Z}_W^J(\mathbf{x}_n; B). \end{aligned} \quad (4.7)$$

Thus, the formula (4.7) gives (2.7). Recall [BZF] the construction of the vertex algebra bundle  $\mathcal{V}$ . Here we use a Virasoro vector shifted version of it. According to Proposition 6.5.4 of [BZF], one canonically (i.e., coordinate independently) associates End  $\mathcal{V}$ -valued sections  $\mathcal{Y}_p$  of the  $g$ -twisted bundle  $\mathcal{V}^*$  (the bundle dual to  $\mathcal{V}$ ). The intrinsic, i.e., coordinate independent, vertex algebra operators are defined by [BZF]

$$\langle u, (\mathcal{Y}_p^*(i(\mathbf{v}_n)))_n g v \rangle = \langle u, \mathbf{Y}(\mathbf{x}_n)v \rangle.$$

to matrix elements of a number of vertex operators on appropriate punctured disks around points with local coordinates  $\mathbf{z}_n$  on  $\Sigma^{(g)}$ . The spaces of such  $\mathcal{V}$ -sections for each  $n$  of is described by identifications (4.7). Taking into account the construction of Section 6 (subsection 6.6.1, in particular, construction 6.6.4, and Proposition 6.6.7) of [BZF], we see that  $n$ -point functions are connections on the space of sections of  $\mathcal{V}$ , and the reduction cohomology (2.11) is represented by (4.6).  $\square$

The geometrical meaning of (2.9) consists in the following. Since in (2.7) operators act on vertex algebra elements only, we can interpret it as a relation on modes of  $V$  with functional coefficients. In particular, all operators  $T$  change vertex algebra elements by action either of  $o(v) = v_{wtv-1}$ , or positive modes of  $v[m]$ ,  $m \geq 0$ . Recall that for  $n$ -point Jacobi forms are quasi-modular forms. Moreover, the reduction formulas (2.7) can be used to prove modular invariance for higher  $n$  Jacobipoint functions. Due to automorphic properties of  $n$ -point functions, (2.9) can be also interpreted as relations among modular forms. It also defines a complex variety in  $\mathbf{z}_n \in \mathbb{C}^n$  with non-commutative parameters  $\mathbf{v}_n \in V^{\otimes n}$ . As most identities (e.g., trisecant identity [Fay, Mu] and triple product identity [K, MTZ]) for  $n$ -point functions (2.9) has its algebraic-geometrical meaning. The condition (2.9) relates finite series of vertex algebra correlations functions on  $\mathcal{T}$  with elliptic functions [Zhu, MT, MTZ]. Since  $n$ -point Jacobi forms are quasi-modular forms, we treat (2.9) as a source of new identities on such forms.

#### ACKNOWLEDGMENTS

The author would like to thank H. V. Lê, A. Lytchak, and M. Krauel for related discussions.

#### 5. APPENDIX: QUASI-JACOBI FORMS

In this appendix we recall definitions and properties of Jacobi and quasi-Jacobi forms [BKT]. First, we provide the definition of ordinary Jacobi forms [EZ]. Let  $\mathbb{H}$  be the upper-half plane.

**Definition 10.** Let  $k, m \in \mathbb{N}_0$ , and  $\chi$  be a rational character for a one dimensional representation of the Jacobi group  $\mathrm{SL}(2, \mathbb{Z}) \ltimes \mathbb{Z}^2$ . A holomorphic Jacobi form of weight  $k$  and index  $m$  on  $\mathrm{SL}_2(\mathbb{Z})$  with rational multiplier  $\chi$  is a holomorphic function

$$\phi : \mathbb{C} \times \mathbb{H} \rightarrow \mathbb{C},$$

and

$$\gamma \cdot \tau = \frac{a\tau + b}{c\tau + d},$$

which satisfies the following conditions. Let  $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ ,

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then, for  $(\lambda, \mu) \in \mathbb{Z} \times \mathbb{Z}$ ,

$$\phi \Big|_{k,m} (\gamma, (\lambda, \mu)) = \chi(\gamma, (\lambda, \mu)) \phi, \quad (5.1)$$

where for a function

$$\phi : \mathbb{C} \times \mathbb{H} \rightarrow \mathbb{C},$$

$$\begin{aligned} & \phi \Big|_{k,m} (\gamma, (\lambda, \mu))(z, \tau) \\ &= (c\tau + d)^{-k} e \left( -\frac{cm(z + \lambda\tau + \mu)^2}{c\tau + d} + m(\lambda^2\tau + 2\lambda z) \right) \phi \left( \frac{z + \lambda\tau + \mu}{c\tau + d}, \gamma.\tau \right). \end{aligned}$$

with  $e(w) = e^{2\pi iw}$ . For a multiplier  $\chi$ ,

$$\chi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \chi \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix}, (0, 0) \right), \quad \chi(\lambda, \mu) = \chi \left( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, (\lambda, \mu) \right),$$

and  $N_1, N_2 \in \mathbb{N}$  uniquely defined by

$$\chi \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = e^{2\pi i \frac{a_1}{N_1}}, \quad \chi(0, 1) = e^{2\pi i \frac{a_2}{N_2}},$$

where for  $a_j \in \mathbb{N}$ ,  $\gcd(a_j, N_j) = 1$ . The function  $\phi$  has a Fourier expansion of the form with  $q = e(\tau)$ ,  $\zeta = e(z)$ ,

$$\phi(z, \tau) = \sum_{n \in \mathbb{N}_0 + \rho_1} \sum_{\substack{r \in \mathbb{Z} + \rho_2 \\ r, 2 \leq 4nm}} c(n, r) q^n \zeta^r, \quad (5.2)$$

where  $\rho_j = \frac{a_j}{N_j} \pmod{\mathbb{Z}}$  with  $0 \leq \rho_j < 1$ .

We next consider quasi-Jacobi forms as introduced in [Lib].

**Definition 11.** An almost meromorphic Jacobi form of weight  $k$ , index 0, and depth  $(s, t)$  is a meromorphic function in  $\mathbb{C}\{q, \zeta\}[z^{-1}, \frac{z_2}{\tau_2}, \frac{1}{\tau_2}]$  with  $z = z_1 + iz_2$ ,  $\tau = \tau_1 + i\tau_2$  satisfying (5.1), and which has degree at most  $s, t$  in  $\frac{z_2}{\tau_2}, \frac{1}{\tau_2}$ , respectively.

**Definition 12.** A quasi-Jacobi form of weight  $k$ , index 0, and depth  $(s, t)$  is defined by the constant term of an almost meromorphic Jacobi form of index 0 considered as a polynomial in  $\frac{z_2}{\tau_2}, \frac{1}{\tau_2}$ .

**5.1. Modular and elliptic functions.** For a variable  $x$ , set

$$D_x = \frac{1}{2\pi i} \frac{\partial}{\partial x},$$

and  $q_x = e^{2\pi ix}$ . Define for

$$m \in \mathbb{N} = \{\ell \in \mathbb{Z} : \ell > 0\},$$

the elliptic Weierstrass functions

$$\begin{aligned} P_1(w, \tau) &= - \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{q_w^n}{1 - q^n} - \frac{1}{2}, \\ P_{m+1}(w, \tau) &= \frac{(-1)^m}{m!} D_w^m (P_1(w, \tau)) = \frac{(-1)^{m+1}}{m!} \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{n^m q_w^n}{1 - q^n}. \end{aligned} \quad (5.3)$$

Next, we have

**Definition 13.** The modular Eisenstein series  $E_k(\tau)$ , defined by  $E_k = 0$  for  $k$  for odd and  $k \geq 2$  even

$$E_k(\tau) = -\frac{B_k}{k!} + \frac{2}{(k-1)!} \sum_{n \geq 1} \frac{n^{k-1} q^n}{1 - q^n},$$

where  $B_k$  is the  $k$ -th Bernoulli number defined by

$$(e^z - 1)^{-1} = \sum_{k \geq 0} \frac{B_k}{k!} z^{k-1}.$$

It is convenient to define  $E_0 = -1$ .  $E_k$  is a modular form for  $k > 2$  and a quasi-modular form for  $k = 2$ . Therefore,

$$E_k(\gamma\tau) = (c\tau + d)^k E_k(\tau) - \delta_{k,2} \frac{c(c\tau + d)}{2\pi i}.$$

**Definition 14.** For  $w, z \in \mathbb{C}$ , and  $\tau \in \mathbb{H}$  let us define

$$\tilde{P}_1(w, z, \tau) = - \sum_{n \in \mathbb{Z}} \frac{q_w^n}{1 - q_z q^n}.$$

We also have

**Definition 15.**

$$\tilde{P}_{m+1}(w, z, \tau) = \frac{(-1)^m}{m!} D_w^m \left( \tilde{P}_1(w, z, \tau) \right) = \frac{(-1)^{m+1}}{m!} \sum_{n \in \mathbb{Z}} \frac{n^m q_w^n}{1 - q_z q^n}. \quad (5.4)$$

It is thus useful to give

**Definition 16.** For  $m \in \mathbb{N}_0$ , let

$$P_{m+1, \lambda}(w, \tau) = \frac{(-1)^{m+1}}{m!} \sum_{n \in \mathbb{Z} \setminus \{-\lambda\}} \frac{n^m q_w^n}{1 - q^{n+\lambda}}. \quad (5.5)$$

On notes that

$$P_{1, \lambda}(w, \tau) = q_w^{-\lambda} (P_1(w, \tau) + 1/2),$$

with

$$P_{m+1, \lambda}(w, \tau) = \frac{(-1)^m}{m!} D_w^m (P_{1, \lambda}(w, \tau)).$$

We also consider the expansion

$$P_{1, \lambda}(w, \tau) = \frac{1}{2\pi i w} - \sum_{k \geq 1} E_{k, \lambda}(\tau) (2\pi i w)^{k-1},$$

where we find [Zag]

$$E_{k,\lambda}(\tau) = \sum_{j=0}^k \frac{\lambda^j}{j!} E_{k-j}(\tau). \quad (5.6)$$

**Definition 17.** We define another generating set  $\tilde{E}_k(z, \tau)$  for  $k \geq 1$  together with  $E_2(\tau)$  given by [Ob]

$$\tilde{P}_1(w, z, \tau) = \frac{1}{2\pi iw} - \sum_{k \geq 1} \tilde{E}_k(z, \tau) (2\pi iw)^{k-1}, \quad (5.7)$$

where we find that for  $k \geq 1$ ,

$$\tilde{E}_k(z, \tau) = -\delta_{k,1} \frac{q_z}{q_z - 1} - \frac{B_k}{k!} + \frac{1}{(k-1)!} \sum_{m, n \geq 1} (n^{k-1} q_z^m + (-1)^k n^{k-1} q_z^{-m}) q^{mn}, \quad (5.8)$$

and  $\tilde{E}_0(z, \tau) = -1$ .

**5.2. Deformed elliptic functions.** In this subsection we recall the definition of deformed elliptic functions [DLM, MTZ]. Let  $(\theta, \phi) \in U(1) \times U(1)$  denote a pair of modulus one complex parameters with  $\phi = \exp(2\pi i \lambda)$  for  $0 \leq \lambda < 1$ .

**Definition 18.** For  $z \in \mathbb{C}$ ,  $\tau \in \mathbb{H}$  we define deformed Weierstrass functions for  $k \geq 1$  as

$$P_k \left[ \begin{array}{c} \theta \\ \phi \end{array} \right] (z, \tau) = \frac{(-1)^k}{(k-1)!} \sum'_{n \in \mathbb{Z} + \lambda} \frac{n^{k-1} q_z^n}{1 - \theta^{-1} q^n}, \quad (5.9)$$

for  $q = q_{2\pi i \tau}$  where  $\sum'$  means we omit  $n = 0$  if  $(\theta, \phi) = (1, 1)$ .

The functions (5.9) converge absolutely and uniformly on compact subsets of the domain  $|q| < |q_z| < 1$  [DLM]. For  $k \geq 1$ ,

$$P_k \left[ \begin{array}{c} \theta \\ \phi \end{array} \right] (z, \tau) = \frac{(-1)^{k-1}}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} P_1 \left[ \begin{array}{c} \theta \\ \phi \end{array} \right] (z, \tau). \quad (5.10)$$

## 6. APPENDIX: REDUCTION FORMULAS FOR JACOBI $n$ -POINT FUNCTIONS

In this subsection we recall the reduction formulas derived in [MTZ, BKT].

**6.1. Vertex operator superalgebra case.** Recall [MTZ], the following

**Proposition 2.** *Suppose that  $v_{n+1} \in V$  is homogeneous of integer weight  $\text{wt}(v_{n+1}) \in \mathbb{Z}$ . Then we have*

$$\sum_{k=1}^n p(v_{n+1}, \mathbf{v}_{k-1}) \mathcal{Z}_w^J((v[0] \cdot)_k \mathbf{v}_n, B) = 0, \quad (6.1)$$

with  $p(v_{n+1}, \mathbf{v}_{k-1})$  given by

$$p(A, B_1 \dots B_{r-1}) = \begin{cases} 1 & \text{for } r = 1 \\ (-1)^{p(A)[p(B_1) + \dots + p(B_{r-1})]} & \text{for } r > 1 \end{cases}. \quad (6.2)$$

Let  $v_{n+1}$  be homogeneous of weight  $\text{wt}(v_{n+1}) \in \mathbb{R}$  and define  $\phi \in U(1)$  by

$$\phi = \exp(2\pi i \text{wt}(v_{n+1})). \quad (6.3)$$

We also take  $v_{n+1}$  to be an eigenfunction under  $g$  with

$$gv_{n+1} = \theta^{-1}v_{n+1} \quad (6.4)$$

for some  $\theta \in U(1)$  so that

$$g^{-1}v_{n+1}(k)g = \theta v_{n+1}(k). \quad (6.5)$$

Then we obtain the following generalization of Zhu's Proposition 4.3.2 [Zhu] for the  $n$ -point function:

**Theorem 1.** *Let  $v_{n+1}, \theta$  and  $\phi$  be as above. Then for any  $\mathbf{v}_n \in V^{\otimes n}$  we have*

$$\begin{aligned} \mathcal{Z}_W^J(\mathbf{x}_{n+1}; B) &= \delta_{\theta,1} \delta_{\phi,1} \text{STr}_W \left( o(v_{n+1}) \mathbf{Y}_W \left( \mathbf{q}^{L_V(0)} \mathbf{v}, \mathbf{q} \right)_n g q^{L(0)-c/24} \right) \\ &+ \sum_{\substack{k=1 \\ m \geq 0}}^n p(v_{n+1}, \mathbf{v}_{k-1}) P_{m+1} \left[ \begin{array}{c} \theta \\ \phi \end{array} \right] (z_{n+1} - z_k, \tau) \mathcal{Z}_W^J((v_{n+1}[m])_k \cdot \mathbf{v}_n; B). \end{aligned} \quad (6.6)$$

The deformed Weierstrass function is defined in (5.9).

**6.2. The first reduction formula.** Suppose that  $v_{n+1} \in V$  with

$$L(0)v_{n+1} = \text{wt}(v_{n+1})v_{n+1},$$

$$J(0)v_{n+1} = \alpha v_{n+1},$$

for  $\alpha \in \mathbb{C}$ . The simplest case of reduction formulas for modes

$$v_{n+1}(\text{wt}(v_{n+1}) - 1 + \beta) = o_\beta(v_{n+1}),$$

by  $\beta \in \mathbb{Z}$ , is given in [BKT]:

**Lemma 2.** *For all  $k \in \mathbb{C}$ , we have*

$$\begin{aligned} &(1 - \zeta^{-\alpha} q^\beta) T_0(o_\beta(v_{n+1})) \mathcal{Z}_W^J(\mathbf{x}_n, B) \\ &= \sum_{k=1}^n \sum_{m \geq 0} \mathcal{Z}_M^J \left( \left( e^{z_k \beta} \frac{\beta^m}{m!} v_{n+1}[m] \right)_k \cdot \mathbf{x}_n; B \right). \end{aligned}$$

Lemma 2 implies the following corollary.

**Corollary 2.** *Let  $J(0)v_{n+1} = \alpha v_{n+1}$ . If  $\alpha z = \lambda \tau + \mu \in \mathbb{Z}\tau + \mathbb{Z}$ , then*

$$\sum_{k=1}^n \sum_{m \geq 0} \mathcal{Z}_M^J \left( \left( e^{z_k \lambda} \frac{\lambda^m}{m!} v_{n+1}[m] \right)_k \cdot \mathbf{v}_n; B \right) = 0. \quad (6.7)$$

We now provide the following reduction formula for formal Jacobi  $n$ -point functions [BKT]. For eigenstates  $v_{n+1}$  with respect to  $J(0)$  we obtain:

**Proposition 3.** *Let  $\mathbf{x}_{n+1} \in V^{\otimes(n+1)} \times \mathbb{C}^{n+1}$ , with  $J(0)v_{n+1} = \alpha v_{n+1}$ ,  $\alpha \in \mathbb{C}$ . For  $\alpha z \notin \mathbb{Z}\tau + \mathbb{Z}$ , we have*

$$\mathcal{Z}_M^J(\mathbf{x}_{n+1}; B) = \sum_{k=1}^n \sum_{m \geq 0} \tilde{P}_{m+1} \left( \frac{z_{n+1} - z_k}{2\pi i}, \alpha z, \tau \right) \mathcal{Z}_W^J((v_{n+1}[m])_k \cdot \mathbf{x}_n; B). \quad (6.8)$$



**Proposition 4.** Let  $\mathbf{x}_{n+1} \in V^{\otimes(n+1)} \times \mathbb{C}^{n+1}$ , with  $J(0)v_{n+1} = \alpha v_{n+1}$ . For  $\alpha z = \lambda\tau + \mu \in \mathbb{Z}\tau + \mathbb{Z}$ , we have

$$\begin{aligned} & \mathcal{Z}_W^J(\mathbf{x}_{n+1}; B) \\ &= e^{-z_{n+1}\lambda} \text{Tr}_W \left( v_{n+1}(\text{wt}(v_{n+1}) - 1 + \lambda) \mathbf{Y} \left( e^{\mathbf{z}L(0)} \mathbf{v}, e^{\mathbf{x}} \right)_n \zeta^{J(0)} q^{L(0)} \right) \\ &+ \sum_{k=1}^n \sum_{m \geq 0} P_{m+1, \lambda} \left( \frac{z_{n+1} - z_k}{2\pi i}, \tau \right) \mathcal{Z}_M^J((v_{n+1}[m])_k \mathbf{x}_n; B), \end{aligned} \quad (6.9)$$

with  $P_{m+1, \lambda}(w, \tau)$  defined in (5.5).

Next we provide the reduction formula for Jacobi  $n$ -point functions.

**Proposition 5.** Let  $\mathbf{x}_{n+1} \in V^{\otimes(n+1)} \times \mathbb{C}^{n+1}$ , with  $J(0)v_{n+1} = \alpha v_{n+1}$ . For  $l \geq 1$  and  $\alpha z \notin \mathbb{Z}\tau + \mathbb{Z}$ , we have

$$\begin{aligned} & \mathcal{Z}_W^J(v_{n+1}[-l].x_1, \mathbf{x}_{2,n}; B) \\ &= \sum_{m \geq 0} (-1)^{m+1} \binom{m+l-1}{m} \tilde{G}_{m+l}(\alpha z, \tau) \mathcal{Z}_W^J(v_{n+1}[m].x_1, \mathbf{x}_{2,n}; B) \\ &+ \sum_{k=2}^n \sum_{m \geq 0} (-1)^{l+1} \binom{m+l-1}{m} \tilde{P}_{m+l} \left( \frac{z_1 - z_k}{2\pi i}, \alpha z, \tau \right) \mathcal{Z}_W^J(v_{n+1}[m].\mathbf{x}_n; B). \end{aligned} \quad (6.10)$$

Propositions 4 and 5 imply the next result [BKT]:

**Proposition 6.** Let  $\mathbf{x}_{n+1} \in V^{\otimes(n+1)} \times \mathbb{C}^{n+1}$ , with  $J(0)v_{n+1} = \alpha v_{n+1}$ . For  $l \geq 1$  and  $\alpha z = \lambda\tau + \mu \in \mathbb{Z}\tau + \mathbb{Z}$ , we have

$$\begin{aligned} & \mathcal{Z}_W^J(v_{n+1}[-l].x_1, \mathbf{x}_{2,n}; B) \\ &= (-1)^{l+1} \frac{\lambda^{l-1}}{(l-1)!} \text{Tr}_W \left( v_{n+1}(\lambda + \text{wt}(v_{n+1}) - 1) \mathbf{Y} \left( e^{\mathbf{x}Lv(0)} \mathbf{v}, e^{\mathbf{x}} \right)_n \zeta^{J(0)} q^{L(0)} \right) \\ &+ \sum_{m \geq 0} (-1)^{m+1} \binom{m+l-1}{m} E_{m+l, \lambda}(\tau) \mathcal{Z}_W^J(v_{n+1}[m].x_1, \mathbf{x}_{2,n}; B) \\ &+ \sum_{k=2}^n \sum_{m \geq 0} (-1)^{l+1} \binom{m+l-1}{m} P_{m+l, \lambda} \left( \frac{x_1 - x_k}{2\pi i}, \tau \right) \mathcal{Z}_W^J(v_{n+1}[m].\mathbf{x}_n; B), \end{aligned}$$

for  $E_{k, \lambda}$  given in (5.6).

*Remark 4.* In the case  $\alpha = 0$  we have that  $\lambda = \mu = 0$  and Propositions 4 and 6 imply the standard results of [Zhu] or [MTZ] with  $a(\lambda + \text{wt}(a) - 1) = o(a)$ .

**6.3. Vertex operator (super)algebras.** In this subsection we recall the notion of vertex operator (super)algebras [B, FHL, FLM, K, MN]. Let  $V$  be a superspace, i.e., a complex vector space

$$V = V_0 \oplus V_1 = \bigoplus_{\alpha} V_{\alpha},$$

with index label  $\alpha$  in  $\mathbb{Z}/2\mathbb{Z}$  so that each  $a \in V$  has a parity  $p(a) \in \mathbb{Z}/2\mathbb{Z}$ .

An  $\mathbb{C}$ -graded vertex operator superalgebra is defined by  $(V, Y, \mathbf{1}_V, \omega)$  where  $V$  is a superspace with a  $\mathbb{C}$ -grading where

$$V = \bigoplus_{r \geq r_0} V_r,$$

for some  $r_0$  and with parity decomposition

$$V_r = V_{\bar{0},r} \oplus V_{\bar{1},r}.$$

$\mathbf{1}_V \in V_{\bar{0},0}$  is the vacuum vector and  $\omega \in V_{\bar{0},2}$  the conformal vector with properties described below. The vertex operator  $Y$  is a linear map

$$Y : V \rightarrow (\text{End}V)[[z, z^{-1}]],$$

for formal variable  $z$ , so that for any vector  $x = (a, v) \in V \times \mathbb{C}$ ,

$$Y(x) = \sum_{n \in \mathbb{Z}} a(n) z^{-n-1}. \quad (6.11)$$

The component operators (modes)  $a(n) \in \text{End}V$  are such that

$$a(n)\mathbf{1}_V = \delta_{n,-1}a,$$

for  $n \geq -1$  and

$$a(n)V_\alpha \subset V_{\alpha+p(a)}, \quad (6.12)$$

for  $a$  of parity  $p(a)$ .

The vertex operators satisfy the locality property for all  $x_i = (v_i, z_i) \in V \times \mathbb{C}$ ,  $i = 1, 2$ ,

$$(z_1 - z_2)^N [Y(x_1), Y(x_2)] = 0, \quad (6.13)$$

for  $N \gg 0$ , where the commutator is defined in the graded sense, i.e.,

$$[Y(x_1), Y(x_2)] = Y(x_1)Y(x_2) - (1)^{p(v_1)p(v_2)}Y(x_2)Y(x_1).$$

The vertex operator for the vacuum is

$$Y(\mathbf{1}_V, z) = \text{Id}_V,$$

whereas that for  $\omega$  is

$$Y(\omega, z) = \sum_{n \in \mathbb{Z}} L(n) z^{-n-2}, \quad (6.14)$$

where  $L(n)$  forms a Virasoro algebra for central charge  $c$

$$[L_V(m), L_V(n)] = (m - n)L_V(m + n) + \frac{c}{12}(m^3 - m)\delta_{m,-n}\text{Id}_V. \quad (6.15)$$

$L_V(-1)$  satisfies the translation property

$$Y(L_V(-1)x) = \frac{d}{dz}Y(x). \quad (6.16)$$

$L_V(0)$  describes the  $\mathbb{C}$ -grading with

$$L_V(0)a = \text{wt}(a)a,$$

for weight  $\text{wt}(a) \in \mathbb{C}$  and

$$V_r = \{a \in V \mid \text{wt}(a) = r\}. \quad (6.17)$$

We quote the standard commutator property of vertex operator superalgebra, e.g., [K, FHL, MN], for  $x_1 = (a, z_1)$ ,  $x = (b, z_2)$

$$[a(m), Y(x)] = \sum_{j \geq 0} \binom{m}{j} Y(a(j).x) z_1^{m-j}. \quad (6.18)$$

Taking  $a = \omega$  this implies for  $b$  of weight  $\text{wt}(b)$  that

$$[L_V(0), b(n)] = (\text{wt}(b) - n - 1)b(n), \quad (6.19)$$

so that

$$b(n)V_r \subset V_{r+\text{wt}(b)-n-1}. \quad (6.20)$$

In particular, we define for  $a$  of weight  $\text{wt}(a)$  the zero mode

$$o_\lambda(a) = \begin{cases} a(\text{wt}(a) - 1 + \lambda), & \text{for } \text{wt}(a) \in \mathbb{Z} \\ 0, & \text{otherwise,} \end{cases} \quad (6.21)$$

which is then extended by linearity to all  $a \in V$ .

**6.4. Square bracket formalism.** Define the square bracket operators for  $V$  by

$$Y[x] = Y\left(e^{z L(0)}v, e^z - 1\right) = \sum_{n \in \mathbb{Z}} v[n]z^{-n-1}. \quad (6.22)$$

For  $v$  of weight  $\text{wt}(v)$  and  $k \in \mathbb{Z}$  (see [Zhu, Lemma 4.3.1]), we have

$$\sum_{j \geq 0} \binom{k + \text{wt}(v) - 1}{j} v(j) = \sum_{m \geq 0} \frac{k^m}{m!} v[m]. \quad (6.23)$$

The square bracket operators form an isomorphic vertex operator algebra with Virasoro vector

$$\tilde{\omega} = \omega - \frac{c}{24} \mathbb{1}_V.$$

Let us now introduce [DMs] the shifted Virasoro vector

$$\omega_h = \omega + h(-2) \mathbb{1}_V,$$

where

$$h = -\frac{\lambda}{\alpha} J,$$

for  $\lambda \in \mathbb{Z}$ . Then the shifted grading operator is

$$L_h(0) = L(0) - h(0) = L(0) + \frac{\lambda}{\alpha} J(0).$$

Denote the square bracket vertex operator for the shifted Virasoro vector by

$$Y[x]_h = Y\left(e^{z L_h(0)}v, e^z - 1\right) = \sum_{n \in \mathbb{Z}} v[n]_h z^{-n-1},$$

Therefore,

$$Y[a, z]_h = e^{z\lambda} Y[a, z],$$

or equivalently,

$$a[n]_h = \sum_{m \geq 0} \frac{\lambda^m}{m!} a[n+m]. \quad (6.24)$$

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