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DISERTAČNÍ PRÁCE

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Forcing, deskriptivní teorie množin, analýza

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Chtěl bych poděkovat svému školiteli Jindřichu Zapletalovi za jeho vedení mého doktorského studia a všeobecnou podporu. Všechny výsledky obsažené v této práci byly motivovány jeho otázkami a náměty a dále formovány při společných diskusích.

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Název práce: **Forcing, deskriptivní teorie množin, analýza**
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Abstrakt: Disertační práce je tematicky rozdělena na dva celky. První část, tj. kapitoly 2,3 a 4, obsahuje výsledky tématicky související s novou knihou školitele a spoluautorů V. Kanoveie a M. Saboka ,,Canonical Ramsey Theory on Polish Spaces". V kapitole 2 je dokázana kanonizace všech ekvivalenčních relací Borelovsky redukovatelných na ekvivalence definované z analytických P-ideálů pro Silverův ideál. Dále jsou zde vyšetřovány a klasifikovány podekvivalence ekvivalenční relace E_0. V kapitole 3 je dokázána kanonizace všech ekvivalenčních relací Borelovsky redukovatelných na ekvivalence definované z F_σ P-ideálů pro Laverův ideál a v kapitole 4 je dokázána kanonizace všech analytických ekvivalenčních relací pro ideál odvozený z Carlsonovy-Simpsonovy (duální Ramseyho) věty.

Druhý tematický celek, tvořený kapitolou 5, se zabývá existencí univerzálních a ultrahomogenních Polských metrických struktur. Konstruuje se například univerzální a ultrahomogenní Polský metrický prostor vybavený navíc spočetně mnoha uzavřenými relacemi nebo vybavený Lipschitzovskou funkcí do libovolně stanoveného Polského metrického prostoru. Práci zde obsaženou je možné chápat jako rozšíření klasického výsledku P. Urysohna, který zkonstruoval univerzální a ultrahomogenní Polský metrický prostor.

Klíčová slova: deskriptivní teorie množin, Borelovské (analytické) ekvivalenční relace, forcing, Urysohnův univerzální prostor

Title: **Forcing, descriptive set theory, analysis**
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Abstract: The dissertation thesis consists of two thematic parts. The first part, i.e. chapters 2, 3 and 4, contains results concerning the topic of a new book of the supervisor and coauthors V. Kanovei and M. Sabok "Canonical Ramsey Theory on Polish Spaces". In Chapter 2, there is proved a canonization of all equivalence relations Borel reducible to equivalences definable by analytic P-ideals for the Silver ideal. Moreover, it investigates and classifies subequivalences of the equivalence relation E_0. In Chapter 3, there is proved a canonization of all equivalence relations Borel reducible to equivalences definable by F_σ P-ideals for the Laver ideal and in Chapter 4, we prove the canonization for all analytic equivalence relations for the ideal derived from the Carlson-Simpson (Dual Ramsey) theorem.

The second part, consisting of Chapter 5, deals with the existence of universal and ultrahomogeneous Polish metric structures. For instance, we construct a universal Polish metric space which is moreover equipped with countably many closed relations or with a Lipschitz function to an arbitrarily chosen Polish metric space. This work can be considered as an extension of the result of P. Urysohn who constructed a universal and ultrahomogeneous Polish metric space.

Keywords: descriptive set theory, Borel (analytic) equivalence relations, forcing, the Urysohn universal space

Contents

Introduction

This dissertation thesis contains several results of the author that lie on the borderline of descriptive set theory, forcing and analysis. Thematically, one can divide the thesis into two parts. The first one, comprised of chapters 2-4, is related to the topic of the recent book [18] of author's supervisor. The second one, contained in the last chapter, investigates universality and homogeneity in Polish metric structures and it is related to descriptive set theory, (continuous) model theory, and metric geometry.

We give a brief summary of all chapters here.

The first chapter is just a preliminary chapter that reviews the material that the reader should know in order to understand the rest of the thesis. It contains no results due to the author and it is written in a rather dense "definition-fact" style. With a single exception for which we could not find a reference (we do not claim that the result is not known among the experts though) we do not provide proofs there and always refer to the literature for them. The main area that we review there is mainly the basics of descriptive set theory that are essential for the whole work. A large part of that chapter is also devoted to definable equivalence relations, a fundamental notion from chapters 2, 3 and 4. There are sections devoted to "Idealized forcing" and "Canonical Ramsey theory on Polish spaces", a specific topic to which the work from those three chapters (2, 3 and 4) directly relates. Included is a small section on the method forcing as we explicitly use it in the fourth chapter. The last section of the preliminary chapter deals with Fraïssé theory, a subarea of model theory that is used for constructions in the last chapter. The reader needs to understand the notions and facts (resp. the statements, knowledge of their proofs is not essential) from this section in order to read the last chapter.

The main part of the thesis begins with Chapter 2. That chapter gives a canonization result (in the sense of [18]) for the classical Silver forcing/ideal. It extends the results from the author's article [3]. We note that in [3] we prove the canonization for equivalences defined by F_σ P-ideals, while in Chapter 2 we extend it to equivalences defined by arbitrary analytic P-ideals. Moreover, Chapter 2 gives some classification, resp. anti-classification, results concerning subequivalences of E_0 on the Silver forcing.

Chapter 3 deals with another classical forcing notion (resp. σ-ideal on a Polish space) - Laver forcing/ideal. We prove the canonization for equivalences defined by F_σ P-ideals. We note that the content of this chapter is written up in an article [4] that was under peer review when this thesis was finished.

Chapter 4 deals with a σ-ideal that is derived from the Carlson-Simpson (Dual Ramsey) theorem. We prove a total canonization for all analytic equivalence relations in case when we deal with a two element alphabet. For larger alphabets we identify a finite set of canonical equivalence relations and sketch a proof that every analytic equivalence relation can be canonized to one of them. One can view it as a generalization of some weak form of the Dual Ramsey theorem. The article [5] containing results from this chapter was in preparation when the thesis was submitted. It will contain a full proof for the case of alphabets of more than two elements which is only sketched here.

The second part of the thesis, thematically different than chapters 2-4, is contained in Chapter 5. That chapter extends the universality and homogeneity properties of the Urysohn universal metric space. We enrich the Urysohn space by some additional structure and prove that this enriched Urysohn space is still universal and ultrahomogeneous for (Polish) metric spaces equipped with the same type of structure. Namely, we enrich the Urysohn space by adding finitely or countably many closed relations of an arbitrary arity; by adding a closed subset of the product of the Urysohn space and an arbitrary other Polish metric space; finally, by adding a Lipschitz function (with an arbitrary Lispchitz constant) from the Urysohn space to an arbitrary compact metric (thus again Polish) space.

The motivation is to provide a way of (Borel) coding of such Polish metric structures; i.e. a method how to use descriptive set theory in classification of such structures. Simiarly as in case with Chapter 4, the article [6] containing results from this chapter was in preparation when the thesis was submitted.

Chapter 1

Preliminaries

1.1 Descriptive set theory

Here we review the concepts of descriptive set theory that will be used through the rest of the thesis. We divide this section into two subsections. The first one summarizes the basics of descriptive set theory that have something to do with the next chapters. It contains mainly definitions but also a list of basic facts. The second one gives a basic overview of the theory of definable equivalence relations.

1.1.1 Basics

Definition 1.1.1 (Polish spaces and Polish metric spaces)**.** A topological space (X, τ) is called a Polish space if it is separable and completely metrizable (i.e. there exists a metric on X that induces the topology τ and X is complete with respect to this metric). In the rest of the text, we will always omit τ from the notation.

We shall also use the term Polish metric space. In such a case we assume that some fixed metric is given together with the space.

Examples.

- All finite or countable spaces with discrete topology.
- $\mathbb{R}$ with the standard Euclidean topology.
- Finite or countable products of Polish spaces with the product topology;

e.g. ω^ω or 2^ω (where the topology on ω and 2 is discrete). We call the former space the Baire space, the latter the Cantor space.

- All compact metrizable spaces.

- All separable Banach spaces.

- Let X be compact metrizable, Y a separable metrizable. Then $C(X,Y)$, the space of all continuous functions from X to Y endowed with the compact-open topology, is Polish. If d_Y is a compatible metric on Y then we can define a compatible metric on $C(X,Y)$ as follows: if $f,g \in C(X,Y)$ then $d(f,g) = \sup\{d_Y(f(x),g(x)) : x \in X\}$.

- Let (X,d_X) and (Y,d_Y) be two Polish metric spaces. Then $L(X,Y)$, the space of all Lipschitz functions from X to Y endowed with the pointwise-convergence topology, is Polish. There is no canonical metric on $L(X,Y)$ though; i.e. that can be defined directly from d_X and d_Y.

- Let X be Polish. Then $K(X)$, the space of all compact subsets of X endowed with the Vietoris topology, is Polish.

We have the following characterization of all Polish spaces.

Fact 1.1.2 (see [20]; Theorem 4.17)**.** *Every Polish space is homeomorphic to some closed subspace of* $\mathbb{R}^\omega$.

Definition 1.1.3 (Borel sets)**.** Let X be a Polish space, $\mathcal{B} \subseteq \mathcal{P}(X)$ a countable base of topology on X. A subset $B \subseteq X$ is Borel if it belongs to the σ-algebra $\mathrm{Borel}(X)$ on X generated by $\mathcal{B}$.

Let $\Sigma^0_1(X)$ denote the set of all open subsets of X. Let α be an arbitrary countable ordinal. Then we set $\Sigma^0_{\alpha+1}(X) = \{\bigcup_{n\in\omega}(X \setminus A_n) : (A_n)_{n\in\omega} \subseteq \Sigma^0_\alpha(X)\}$. If α is limit then we set $\Sigma^0_\alpha(X) = \bigcup_{\beta<\alpha}\Sigma^0_\beta(X)$. Moreover, we set $\Pi^0_\alpha(X) = \{X \setminus A : A \in \Sigma^0_\alpha(X)\}$.

$\Pi^0_1(X)$ are closed subsets of X, of course. Moreover, $\Sigma^0_2(X)$ sets, resp. $\Pi^0_2(X)$ sets, will be called F_σ sets, resp. G_δ sets. Occasionally, we will refer to $\Pi^0_3(X)$ sets as $F_{\sigma\delta}$ sets.

Fact 1.1.4 (Lebesgue; see [27]; Theorem 2.5)**.** *If X is an uncountable Polish space then for every $\alpha < \omega_1$ $\Sigma^0_\alpha(X) \neq \Sigma^0_{\alpha+1}(X)$.*

Fact 1.1.5 (see [20]; Theorem 3.11)**.** *Let X be a Polish space, $Y \subseteq X$ some subspace. Then Y is Polish iff Y is G_δ in X.*

Fact 1.1.6 (see [20]; Theorem 7.9)**.** *Let X be a Polish space. Then there exists a continuous surjection $\pi : \omega^\omega \to X$. In addition, there exists a closed subset $F \subseteq \omega^\omega$ and a continuous bijection $\pi_F : F \to X$.*

For compact metrizable spaces there is a similar fact involving the Cantor space.

Fact 1.1.7 (see [20]; Theorem 4.18)**.** *Let X be a compact metrizable space. Then there exists a continuous surjection $\pi : 2^\omega \to X$.*

Definition 1.1.8 (Borel function)**.** Let X, Y be two Polish spaces and $A \subseteq X, B \subseteq Y$ two Borel subsets. A function $f : A \to B$ is Borel if preimages of all (relatively) open subsets of B are Borel.

Definition 1.1.9 (Analytic and coanalytic sets)**.** Let B be a Borel set. A subset $A \subseteq B$ is analytic if it is a Borel image of a Borel set; i.e. there exist a Borel set C and a Borel function $f : C \to B$ such that $f[C] = A$.

$A \subseteq B$ is coanalytic if $B \setminus A$ is analytic.

We shall also denote $\Sigma^1_1(B)$, resp. $\Pi^1_1(A)$, the classes of analytic, resp. coanalytic, subsets of B.

Usually, some other definition of analytic set is given and the definition above is shown to be equivalent with it. For some other definitions of analytic sets and equivalences between them we refer to [20]. Compare the previous definition with the following fact.

Fact 1.1.10 (Luzin-Souslin; see [20]; Theorem 15.1)**.** *Let B and C be Borel sets. Let $f : B \to C$ be an injective Borel function. Then $f[B]$ is Borel.*

Fact 1.1.11 (Perfect set theorem; see [20]; Theorem 29.1)**.** *Let X be a Polish space and $A \subseteq X$ an uncountable analytic subset. Then there exists a non-empty perfect (i.e. closed without isolated points) subset $P \subseteq A$.*

Fact 1.1.12 (see [20]; Theorem 14.12). *Let A, B be two Borel sets. A function $f : A \to B$ is Borel iff the graph of f is an analytic subset of $A \times B$.*

Fact 1.1.13 (the Borel isomorphism theorem; see [20]; Theorem 15.6). *Let A, B be two Borel sets of the same cardinality. Then they are Borel isomorphic; i.e. there exists a bijection between A and B that is a Borel function (note that the previous fact implies that the inverse is also a Borel function).*

In particular, the previous fact says that any two uncountable Polish spaces are Borel isomorphic. Thus if we are interested in Borel (not topological) properties of some uncountable Polish space then it does not matter which one we choose to work with.

It also justifies the following definition.

Definition 1.1.14 (Standard Borel space). Let (X, Σ_X) be a measurable space. We call X (we omit the symbol for the σ-algebra on it) a standard Borel space if it is Borel isomorphic to some (equivalently any) uncountable Borel set. The elements of Σ_X are then called Borel subsets of X.

An important example of a standard Borel space is provided in the following definition. We shall use it in the last chapter.

Definition 1.1.15 (the Effros-Borel structure). Let X be a Polish space. Let us denote $F(X)$ the set of all closed subsets of X. Consider a σ-algebra Σ on $F(X)$ generated by the following sets $G_U = \{F \in F(X) : F \cap U \neq \emptyset\}$ where U varies over all basic open subsets U of X.

We call $(F(X), \Sigma)$ the Effros-Borel structure of $F(X)$.

Fact 1.1.16 (see [20]; Theorem 12.6). *For any infinite Polish space X the Effros-Borel structure $(F(X), \Sigma)$ is a standard Borel space.*

In the rest of this subsection we shall deal with Polish groups.

Definition 1.1.17 (Polish group). A topological group G is Polish if its group topology is Polish.

Examples.

- $(\mathbb{R}, +)$ or $(\mathbb{R}^+, \cdot)$.
- The additive group of any separable Banach space.
- The group of permutations of ω, usually denoted as S_∞. Note that S_∞ is a G_δ subset of ω^ω.
- The group of homeomorphisms of any compact metrizable space X. Note that such a group is a G_δ subset of $C(X, X)$.
- The group of surjective isometries of some Polish metric space X. Note that such a group is a G_δ subset of $L(X, X)$.
- Any G_δ (which is in fact always closed) subgroup of a Polish group. Also, any finite or countable product of Polish groups.
- Any quotient G/H, where G is a Polish group and H a closed normal subgroup, is a Polish group.

We also provide a definition of a more general type of groups that somehow connect to the results in the next two chapters.

Definition 1.1.18 (Polishable Borel groups)**.** A group G is called a standard Borel group if it is both a group and a standard Borel space and moreover, the group operations are Borel.

A standard Borel group G is Polishable if there exists a Polish topology on G that produces the same Borel structure on G and such that the group operations become continuous.

We conclude this subsection by defining Polish and Borel actions of Polish groups on Polish, resp. standard Borel spaces.

Definition 1.1.19 (Polish and Borel G-space)**.** Let G be a Polish group, X a Polish space and Z a standard Borel space. A group action $a_X : G \times X \to X$ is called a Polish action if the action is continuous. X together with this action is then called a Polish G-space.

Similarly, a group action $a_Z : G \times Z \to Z$ is called a Borel action if the action is Borel. Z together with this action is then called a Borel G-space.

We shall usually write just $g \cdot x$ instead of $a_X(g, x)$.

1.1.2 Definable equivalence relations

Definition 1.1.20 (Definable equivalence relations)**.** Let Γ be some "definable" pointclass, i.e. class of definable subsets (of Polish, or sometimes even standard Borel, spaces) of some sort, usually closed under continuous preimages; e.g. Σ^0_α sets for some $\alpha < \omega_1$ (such a pointclass is defined only on Polish spaces, not on standard Borel spaces), Borel sets, analytic sets, etc.

Let X be a Polish (or standard Borel) space. We say that $E \subseteq X \times X$ is a Γ-equivalence relation if it is an equivalence relation and $E \in \Gamma(X \times X)$.

Examples.

- Let X be a standard Borel space. By $\mathrm{id}(X)$ we denote the identity relation on X. It is a Borel equivalence relation. By $\mathrm{ev}(X)$ we denote the "full" relation on X, i.e. a Borel equivalence relation with a single equivalence class.

- Let X be a standard Borel space. Let Γ be a definable pointclass closed under finite products (i.e. if $A, B \in \Gamma(X)$ then also $A \times B \in \Gamma(X \times X)$) and under countable unions. Let $(P_n)_{n\in\omega}$ be a partition of X into subsets from $\Gamma(X)$. Let E be a relation on X where xEy if $\exists n(x, y \in P_n)$. It is a Γ-equivalence relation.

- Consider the Cantor space 2^ω. Let us define a relation E_0 on it. For any $x, y \in 2^\omega$ we have xE_0y if $\{n : x(n) \neq y(n)\}$ is finite. It is an F_σ-equivalence relation.

- More generally, let $\mathcal{I}$ be an ideal on ω such that $\mathcal{I} \in \Gamma(2^\omega)$ for some pointclass Γ (note that we identify the ideal $\mathcal{I}$ with a subset of 2^ω via the function sending a subset of ω to its characteristic function). Then we define a relation $E_\mathcal{I}$ on 2^ω as follows: for any $x, y \in 2^\omega$ we have $xE_\mathcal{I}y$ if $\{n : x(n) \neq y(n)\} \in \mathcal{I}$. It is a Γ-equivalence relation. Since this is an important class of equivalence relations we provide some particular examples here.

  - If $\mathcal{I} = \mathrm{Fin}$, i.e. the ideal of all finite subsets of ω, then $E_\mathcal{I} = E_0$.

- Let $\mathcal{I} = \emptyset \otimes \mathrm{Fin} = \{A \subseteq \omega \times \omega : \{n : A_n \notin \mathrm{Fin}\} = \emptyset\}$, where $A_n = \{m : (n,m) \in A\}$. Then $E_{\emptyset\otimes\mathrm{Fin}}$ is an $F_{\sigma\delta}$-equivalence relation on $2^{\omega\times\omega}$ which is usually denoted as E_3.
- Let $\mathcal{I} = \mathrm{Fin} \otimes \emptyset = \{A \subseteq \omega \times \omega : \{n : A_n \neq \emptyset\} \in \mathrm{Fin}\}$. Then $E_{\mathrm{Fin}\otimes\emptyset}$ is an F_σ-equivalence relation which is usually denoted as E_1.
- Let $\mathcal{I}_S$ be the summable ideal; i.e. $\mathcal{I}_S = \{A \subseteq \omega : \sum\{1/(n+1) : n \in A\} < \infty\}$. Then $E_{\mathcal{I}_S}$ is an F_σ-equivalence relation which is usually denoted as E_2.
- Let $\mathcal{Z}_0$ be the density zero ideal; i.e. $\mathcal{Z}_0 = \{A \subseteq \omega : \lim_{n\to\infty} \frac{|A\cap[0,n+1]|}{n+1} = 0\}$. Then $E_{\mathcal{Z}_0}$ is an $F_{\sigma\delta}$-equivalence relation.

- Similarly, let G be a subgroup of $(\mathbb{R}^\omega, +)$ such that $G \in \Gamma(\mathbb{R}^\omega)$ for some pointclass Γ. It determines a relation E_G on $\mathbb{R}^\omega$ as follows: for any $x, y \in \mathbb{R}^\omega$ we have xE_Gy if $x - y \in G$. It is a Γ-equivalence relation.
  - Let $G = \ell_p$, where $p \in [1,\infty]$ and $\ell_p = \{x \in \mathbb{R}^\omega : \sum_n x(n)^p < \infty\}$ if p is finite, and $\ell_\infty = \{x \in \mathbb{R}^\omega : \exists B \in \mathbb{R}\forall n \in \omega(x(n) \leq B)\}$. Then E_{ℓ_p} is an F_σ-equivalence relation. For $p, q \in [1,\infty]$ such that $p < q$ the following holds (see [7]): $E_{\ell_p} <_B E_{\ell_q}$.
  - Let $G = c_0$, where $c_0 = \{x \in \mathbb{R}^\omega : \lim_{n\to\infty} x(n) = 0\}$. Then E_{c_0} is an $F_{\sigma\delta}$-equivalence relation.

- Let P be the Polish space of all probability Borel measures on $[0,1]$ (see [20] 17.E for details about this space). For any two measures $\mu, \nu \in P$ we define $\mu \equiv_m \nu$ if they produce the same null ideals; i.e. $I_\mu = \{A \subseteq [0,1] : \mu(A) = 0\} = \{A \subseteq [0,1] : \nu(A) = 0\} = I_\nu$. It is an equivalence relation which can be shown to be $F_{\sigma\delta}$ ([11] p. 200).

- Let G be a Polish group and X a Polish or Borel G-space. We define a relation E_G on X where for any $x, y \in X$ we have xE_Gy if $\exists g \in G(y = g\cdot x)$. We call it an orbit equivalence relation of G on X.

  In general, this is an analytic equivalence relation: note that E_G is a projection on the first two coordinates of the Borel (closed if we work with a

Polish G-space) set $\{(x, y, g) \in X^2 \times G : g \cdot x = y\}$. However, it was shown by Miller (see [11] Theorem 3.3.2) that all equivalence classes are Borel.

Definable equivalence relations are compared in their complexity. Such a comparison will be an important concept in the next three chapters. From now on, we restrict only on analytic equivalence relations.

Definition 1.1.21 (Definable reducibility between analytic equivalences)**.** Let X, Y be two Polish (or standard Borel) spaces and $E \subseteq X^2$, $F \subseteq Y^2$ two analytic equivalence relations. We say that a function $f : X \to Y$ is a reduction of E to F if $\forall x, y \in X(xEy \Leftrightarrow f(x)Ff(y))$.

We say that E is Borel reducible to F, $E \leq_B F$, if there exists a Borel reduction of E to F. We say that E and F are bireducible, $E \approx_B F$, if $E \leq_B F$ and $F \leq_B E$. Moreover, we write $E <_B F$ if $E \leq_B F$ but not $F \leq_B E$.

We note that there are other types of reductions in literature, e.g. Baire measurable reduction, continuous reduction, etc, with obvious definitions. We restrict only on Borel reductions.

We state here a theorem which we will refer to in subsequent chapters.

Theorem 1.1.22 (Rosendal)**.** *E_{ℓ_∞} is a universal K_σ equivalence relation.*

See for example [11], Theorem 8.4.2, for the proof.

That means that for any equivalence relation E that is a countable union of compact sets there exists a Borel reduction of E to E_{ℓ_∞}. Note that E_{ℓ_∞} itself is K_σ.

Dichotomies for Borel equivalence relations.

It is clear that any Borel equivalence relation E with α-many equivalence classes, where $\alpha \leq \omega$, is Borel reducible to any equivalence relation F (which does need to be Borel or in any other sense definable) with at least α equivalence classes: just choose α many pairwise F-inequivalent elemenets $(x_\beta)_{\beta<\alpha}$, enumerate equivalence classes of E as $(C_\beta)_{\beta<\alpha}$ and define f which maps C_β onto $\{x_\beta\}$. It is clearly a Borel reduction.

It is no longer clear for Borel equivalence relations with more than countably many classes; however, there is the following dichotomy due to Silver.

Theorem 1.1.23 (Silver's dichotomy)**.** *Let X be a Polish (or just standard Borel) space and E a coanalytic equivalence relation on X. Then either E has at most countably many classes of equivalence or* $\mathrm{id}(2^\omega) \leq_B E$.

See [11] or [17] for proofs. The former uses forcing in the proof, the latter does not. However, both rely on methods from efective descriptive set theory. Recently, B. Miller discovered that the graph-theoretic methods, based on work [22], can be used to prove the Silver's theorem only by "classical" methods, see [28].

The Silver's theorem says that there is no coanalytic equivalence relation strictly between $\mathrm{id}(\omega)$ and $\mathrm{id}(2^\omega)$. We say that an equivalence relation E is smooth if $E \leq_B \mathrm{id}(2^\omega)$ (note that this automatically implies that E is Borel). So every coanalytic (in fact Borel) smooth equivalence relation has either at most countably many classes or it is bireducible with $\mathrm{id}(2^\omega)$.

Fact 1.1.24 (see [11]; Proposition 6.1.7)**.** *The F_σ equivalence relation E_0 is not smooth.*

Thus $\mathrm{id}(2^\omega) <_B E_0$. The following dichotomy says that E_0 is the minimal non-smooth Borel equivalence relation. We shall use it in the next chapter.

Theorem 1.1.25 (Glimm-Effros dichotomy)**.** *Let X be a Polish (or just standard Borel) space and let E be a Borel equivalence relation on X. Then either $E \leq_B$* $\mathrm{id}(2^\omega)$ *or $E_0 \leq_B E$.*

See again [11] or [17] for proofs. The next corollary immediately follows, it will also be important in the next chapter.

Corollary 1.1.26. *Let E be a Borel equivalence relation on 2^ω such that $E \subseteq E_0$. Then either $E \leq_B$* $\mathrm{id}(2^\omega)$ *or $E \approx_B E_0$.*

Particular equivalence relations.

Here we investigate some particular Borel equivalence relations in detail. In fact, all equivalence relations discussed here are $F_{\sigma\delta}$.

In particular, we will be interested in equivalences of the form $E_{\mathcal{I}}$, where $\mathcal{I}$ is an ideal on ω. We need one set-theoretic definition.

Definition 1.1.27 (P-ideal)**.** An ideal $\mathcal{I}$ on an infinite set X is called P-ideal if for any countable sequence $(X_n)_{n\in\omega} \subseteq \mathcal{I}$ of elements of the ideal there is an element of the ideal almost containing every element of the sequence, i.e. $\exists X \in \mathcal{I} \forall n (X_n \subseteq^* X)$, where $\subseteq^*$ denotes "almost inclusion", i.e. inclusion modulo a finite set.

It is immediate that Fin is a P-ideal. Also, $\emptyset \otimes \mathrm{Fin}$ is a P-ideal, while $\mathrm{Fin} \otimes \emptyset$ is not. Recall the summable ideal $\mathcal{I}_S$ and the density zero ideal $\mathcal{Z}_0$. It is an easy exercise that they are also P-ideals. The following fact connect the equivalence relations determined by them with equivalence relations from another group, those defined by subgroups of $(\mathbb{R}^\omega, +)$.

Fact 1.1.28. *We have $E_2 = E_{\mathcal{I}_S} \approx_B E_{\ell_1}$ and $E_{\mathcal{Z}_0} \approx_B E_{c_0}$.*

See [17] for a proof.

The previous fact for E_{ℓ_1} generalizes for $p \in (1, \infty)$ (not for $p = \infty$ though).

Fact 1.1.29. *For any $p \in (1, \infty)$ there exists an F_σ P-ideal $\mathcal{I}_p$ such that $E_{\mathcal{I}_p} \approx_B E_{\ell_p}$.*

Since we were unable to find a reference for this we provide a proof here; although that fact is known among experts.

Proof. Fix a bijection $\pi : \omega \times \omega \to \omega$. Let $p \in [1, \infty)$ be arbitrary. First, we show that $E_{\ell_p} \upharpoonright [0,1]^\omega \approx_B E_{\ell_p}$. Clearly $E_{\ell_p} \upharpoonright [0,1]^\omega \leq_B E_{\ell_p}$; the embedding of $[0,1]^\omega$ into $\mathbb{R}^\omega$ is the desired reduction. For the other direction, we define a reduction $f : \mathbb{R}^\omega \to [0,1]^\omega$. We set $f(x)(\pi(n, 2i-1))$ to be 1 if $x(n) \geq i+1$, 0 if $x(n) \leq i$ and $x(n) - i$ otherwise; similarly, we set $f(x)(\pi(n, 2i))$ to be 1 if $x(n) \geq i + 3/2$, 0 if $x(n) \leq i + 12$ and $x(n) - i - 1/2$ otherwise. It is straightforward to verify that f is a Borel reduction.

We define $\mathcal{I}_p$ as $\{A \subseteq \omega : \sum_{n\in\omega} (\sum_{i\in\omega} \chi_A(\pi(n,i))/2^{i+1})^p < \infty\}$. The function μ sending $A \subseteq \omega$ to $\sum_{n\in\omega} (\sum_{i\in\omega} \chi_A(\pi(n,i))/2^{i+1})^p$ is a lower semicontinuous submeasure witnessing that $\mathcal{I}_p$ is an F_σ P-ideal. To show that $E_{\mathcal{I}_p} \leq_B E_{\ell_p} \upharpoonright [0,1]^\omega$ we define a Borel reduction $f : 2^\omega \to [0,1]^\omega$ as follows: we set $f(x)(n) =$

$\sum_{i\in\omega} x(\pi(n,i))/2^{i+1}$, and to show that $E_{\ell_p} \restriction [0,1]^\omega \leq_B E_{\mathcal{I}_p}$ we define a Borel reduction $g : [0,1]^\omega \to 2^\omega$ as follows: we set $g(x)(\pi(n,i)) = 1$ iff $x(n) \geq 1/2^{i+1}$. It is easy to check that these are the desired reductions.

□

We state a theorem due to S. Solecki that characterizes ideals on ω that are analytic P-ideals. We need to define a notion of lower semicontinuous submeasure.

Definition 1.1.30 (Lower semicontinuous submeasure)**.** A submeasure on $\mathcal{P}(\omega)$ (we shall say "a submeasure on ω") is a function $\mu : \mathcal{P}(\omega) \to [0,\infty]$ with the following properties:

1. $\mu(\emptyset) = 0$
2. for $A \subseteq B$ we have $\mu(A) \leq \mu(B)$
3. for any A, B we have $\mu(A \cup A) \leq \mu(A) + \mu(B)$

Moreover, μ is lower semicontinuous if it is a lower semicontinuous function with respect to the Cantor space topology (recall we can identify elements of $\mathcal{P}(\omega)$ with elements of 2^ω); which is equivalent to the statement that for any increasing chain $A_0 \subseteq A_1 \subseteq A_2 \subseteq \dots$ we have $\mu(\bigcup_n A_n) = \sup_n \mu(A_n)$.

Definition 1.1.31 ($\mathrm{Exh}(\mu)$ and $\mathrm{Fin}(\mu)$)**.** Let μ be a lower semicontinuous submeasure on ω. We define the exhaustive and finite parts of μ:
$\mathrm{Fin}(\mu) = \{A \subseteq \omega : \mu(A) < \infty\}$.
$\mathrm{Exh}(\mu) = \{A \subseteq \omega : \lim_n \mu(A \setminus n) = 0\}$.

A simple computation gives that for such μ $\mathrm{Fin}(\mu)$ is always an F_σ set and $\mathrm{Exh}(\mu)$ an $F_{\sigma\delta}$ set. It is also immediate that $\mathrm{Exh}(\mu) \subseteq \mathrm{Fin}(\mu)$. The converse is generally not true; that will be one of the consequences of the next theorem.

Theorem 1.1.32 (Solecki; see [31])**.** *For an ideal $\mathcal{I}$ on ω the following are equivalent:*

1. *$\mathcal{I}$ is an analytic P-ideal.*
2. *There exists a lower semicontinuous submeasure μ on ω such that $\mathcal{I} = \mathrm{Exh}(\mu)$.*

3. The group $(\mathcal{I}, \triangle)$ is Polishable.

In particular, every analytic P-ideal is in fact an $F_{\sigma\delta}$ ideal. It also follows that for any analytic P-ideal $\mathcal{I}$ the equivalence relation $E_{\mathcal{I}}$ is an orbit equivalence relation.

We state the next proposition separately from Theorem 1.1.32. It will be directly used in Chapter 3.

Proposition 1.1.33 (Solecki; see again [31]). *An F_σ ideal $\mathcal{I}$ on ω is a P-ideal if and only if there exists a lower semicontinuous submeasure μ on ω such that $\mathcal{I} = \mathrm{Fin}(\mu) = \mathrm{Exh}(\omega)$.*

Though not important for our purposes we note that K. Mazur proved a statement of a similar flavor.

Fact 1.1.34 ([26]). *Any F_σ ideal $\mathcal{I}$ on ω is of the form $\mathrm{Fin}(\mu)$ for some lower semicontinuous submeasure μ on ω.*

We conclude the part of this section concerning equivalence relations and ideals on ω by the following interesting theorem which will not be used in the rest of the thesis though.

Theorem 1.1.35 (Rosendal; see [30]). *Let E be any Borel equivalence relation (on some Polish or standard Borel space). Then there exists a Borel ideal $\mathcal{I}$ on ω such that $E \leq_B E_{\mathcal{I}}$.*

Thus the equivalences of the form $E_{\mathcal{I}}$, where $\mathcal{I}$ is some Borel ideal, are cofinal in the ordering of Borel equivalences with $\leq_B$. We note that the (minimal possible) cardinality of such a cofinal family is $\aleph_1$.

The last two notions we define here and which will be important in the next chapters are those of countable Borel equivalence relations and equivalences classifiable by countable structures.

Definition 1.1.36 (Countable Borel equivalence relations). We say that a Borel equivalence relation E is countable if every E-class is countable.

Definition 1.1.37 (Equivalence relations classifiable by countable structures). Let L be a language consisting of (at most) countably many relations (enumerated as $(R_i)_{i\in\omega})$. Let Mod(L) be the set of all countable models of L with ω as the underlying universe. We can view Mod(L) as $\Pi_i 2^{\omega^{n_i}}$, where n_i is the arity of R_i, thus as a Polish space (homeomorphic to the Cantor space). We consider a relation of isomorphism on Mod(L) which we denote E_{L}. Note that it is induced by an action of S_∞ where for any $M \in \mathrm{Mod(L)}$, $g \in S_\infty$, $i \in \omega$ and $\bar{x} \in \omega^{n_i}$ we have $R_i^{g\cdot M}(x_1, \ldots, x_{n_i}) \Leftrightarrow R_i^M(g^{-1}(x_1), \ldots, g^{-1}(x_{n_i}))$.

Let E be an arbitrary equivalence relation (on some Polish or standard Borel space). We say that E is classifiable by countable structures if there exists a language L consisting of (at most) countably many relations such that $E \leq_B E_{\mathrm{L}}$.

Fact 1.1.38. *Every countable Borel equivalence relation is classifiable by countable structures.*

For the proof see [17] Lemma 6.1.3.

1.2 Set theory

This section reviews some notions from set theory that will be used in the thesis. Mainly some basic forcing facts and then the main concepts of Idealized forcing ([33]) and Canonical Ramsey theory on Polish spaces ([18]).

1.2.1 Forcing

In all but the fourth chapter (in the second and third; the last chapter does not deal with notions from forcing at all) we use the term "forcing" as a synonym for an ordering which is based on the fact that the orderings considered there were originally investigated in forcing theory. Similarly, the term "condition" is a synonym for an element of that particular ordering.

However, the fourth chapter uses forcing explicitly. It is not possible to introduce here all notions from forcing used there so we need to assume that a reader of that chapter has a basic knowledge of forcing; we refer the reader to [25] or [16] for a general exposition on forcing. Especially, the knowledge of the Forcing theorem ([16], Theorem 14.6) is necessary.

Let us highlight one particular concept that is used in Chapter 4 and which is in fact the only reason why we use forcing there. It is stated in the following theorem.

Theorem 1.2.1 (Analytic absoluteness)**.** *Let M be a transitive model of set theory, $A \subseteq \omega^\omega$ an analytic set that lies (resp. its code) in M. Then for any $x \in M$ we have $x \in A$ iff $M \models x \in A$.*

See for example [16] (Theorem 25.4). This theorem illustrates the idea we use in Chapter 4: We can use forcing method to show that some statement holds in some forcing extension, however that statement is simple enough that it also necessarily holds in the universe.

1.2.2 Idealized forcing

We state some basic concepts from [33] that will be useful in the next chapters.

Definition 1.2.2. Let X be a Polish space, $I \subseteq \mathcal{P}(X)$ a σ-ideal on X (i.e. closed under taking countable unions). By P_I we denote the ordering $(\mathrm{Borel}(X) \setminus I, \subseteq)$ of I-positive Borel sets ordered by inclusions. Similarly, $\mathrm{Borel}(X)/I$ denotes the quotient σ-algebra of Borel subsets of X modulo the σ-ideal I.

The orderings P_I and $\mathrm{Borel}(X)/I$ are forcing equivalent.

Proposition 1.2.3 ([33]; Proposition 2.1.2)**.** *The ordering P_I adds (as a forcing notion) an element $\dot{x}_{\mathrm{gen}} \in X$. If $G \subseteq P_I$ is the generic filter then $\dot{x}_{\mathrm{gen}} = \bigcap G$ and $G = \{A \in \mathrm{Borel}(X) : \dot{x}_{\mathrm{gen}} \in A\}$.*

In particular, $\dot{x}_{\mathrm{gen}}$ does not lie in any ground model Borel set from I.

Recall the definition of proper forcing (see for example [16] Definition 31.1). The book [33] focuses almost entirely on forcing notions P_I that are proper. We state the characterization of this type of forcing notions that are proper.

Proposition 1.2.4 ([33]; Proposition 2.2.2)**.** *Let X be a Polish space and I a σ-ideal on it. Then P_I is proper iff for every countable elementary submodel M of some H_λ, where λ is "large enough", and every $B \in P_I \cap M$ we have $\{x \in B : x \text{ is } M\text{-generic}\} \notin I$.*

We state one more proposition from [33] and then define a related important notion that will be used in the next two chapters.

Proposition 1.2.5 ([33]; Proposition 2.3.1). *Let X, I be as above and suppose that P_I is proper. Let Y be some other Polish space and suppose that $B \in P_I$ is a condition forcing that $\dot{y} \in Y$. Then there exist a subcondition $C \subseteq B$ and a Borel function $f : C \to Y$ such that $f(\dot{x}_{\text{gen}}) = \dot{y}$.*

Definition 1.2.6 (Continuous reading of names; see [33] Definition 3.1.1). Let X, I be again as above and suppose that P_I is proper. P_I has continuous reading of names if for every Borel function $f : B \to Y$, where $B \in P_I$ and Y is some Polish space, there exists a subcondition $C \subseteq B$ such that $f \restriction C$ is continuous.

1.2.3 Canonical Ramsey theory on Polish spaces

The theme of the next three chapters is directly related to the topic of the new book of V. Kanovei, M. Sabok and J. Zapletal "Canonical Ramsey Theory on Polish Spaces" ([18]). It is thus essential to introduce the main ideas of the book here.

Let us begin with the following classical theorem that should serve as a motivation for what follows.

Theorem 1.2.7 (Canonical Ramsey Theorem; Erdös-Rado [8]). *For any n and any partition (finite or infinite) of $[\omega]^n$, equivalently any equivalence relation on $[\omega]^n$, there exist a subset $I \subseteq \{0, \ldots, n-1\}$ and an infinite subset $H \subseteq \omega$ such that $\forall a, b \in [H]^n$, a and b lie in the same part of partition (in the same equivalence class) if and only if $\forall i \in I(a(i) = b(i))$, where $a(i)$ is the i-th element of a in the standard enumeration of ω.*

It follows from the Ramsey theorem that the collection of subsets A of $[\omega]^n$ with the property that there is no infinite set $B \in [\omega]^\omega$ such that $[B]^n \subseteq A$ forms an ideal; let us denote it I. Thus the Erdös-Rado canonical Ramsey theorem can be restated as follows: *For any n there are finitely many (2^n) canonical equivalence relations $(E_i)_{i<2^n}$ such that for any equivalence relation E on $[\omega]^n$ there are $i < 2^n$ and a set H positive with respect to I such that $E \restriction H = E_i \restriction H$.*

In [18], they consider "Polish versions" of the previous theorem. We start with the crucial definition.

Definition 1.2.8 (Spectrum of a σ-ideal)**.** Let X be a Polish space and $I \subseteq \mathcal{P}(X)$ a σ-ideal on it. The spectrum of I is the set of all analytic equivalence relations $E \subseteq X^2$ for which there exists an I-positive Borel set $B \in P_I$ such that for every I-positive Borel subset $C \subseteq B$ we have $E \restriction C \approx_B E$; i.e. the complexity of E remains the same on every I-positive subset of B.

If some analytic equivalence relation $E \subseteq X^2$ is not in the spectrum of I then its complexity can be decreased on every I-positive set. Typically, we would like to have some simple class of "canonical" equivalence relations (see the Erdös-Rado theorem) such that every other equivalence relation from some bigger class (i.e. class of all smooth equivalences, class of all Borel equivalences, etc.) can be canonized to one of these on every I-positive Borel set. This is the content of the next definition.

Definition 1.2.9 (Canonization of equivalence relations)**.** Let X and I be as before. Let $\mathcal{C}$ be some set of (canonical) equivalence relations and let $\mathcal{D}$ be some class of analytic equivalence relations on X (resp. I-positive subsets of X). We say that I has a canonization for all equivalences from $\mathcal{D}$ to $\mathcal{C}$ if for every $E \in \mathcal{D}$ and for every I-positive Borel set B on which E is defined there is some equivalence $F \in \mathcal{C}$ and an I-positive subset $C \subseteq B$ such that $E \restriction B \approx_B F$.

In fact, we shall usually consider a stronger form of canonization. We will have some set of (canonical) equivalence relations $\mathcal{C}$ on X^2 and some class of analytic equivalence relations on X (resp. I-positive subsets of X) $\mathcal{D}$. For every $E \in \mathcal{D}$ and for every I-positive Borel set B on which E is defined there is some equivalence $F \in \mathcal{C}$ and an I-positive subset $C \subseteq B$ such that $E \restriction C = F \restriction C$.

The following is the strongest form of canonization.

Definition 1.2.10 (Total canonization)**.** Let X and I be as above. We say that I has a total canonization for some class $\mathcal{D}$ of analytic equivalence relations on X (resp. I-positive subsets of X) if for every $E \in \mathcal{D}$ and for every I-positive Borel set B on which E is defined there is an I-positive subset $C \subseteq B$ such that either $E \restriction C = \mathrm{id}(C)$ or $E \restriction C = C \times C$.

It follows the spirit of canonization results like that of (for example) H.J. Prömel and B. Voigt [29] and O. Klein and O. Spinas [23].

Example. Let $X = 2^\omega$ and I be the σ-ideal of all countable subsets. It follows from Silver's theorem 1.1.23 that I has a total canonization for all Borel equivalence relations. To see this, let E be a Borel equivalence relation defined on some uncountable Borel set B. We define $\bar{E}$ on 2^ω as follows: for $x, y \in 2^\omega$, $x\bar{E}y$ if either both x and y lie in $2^\omega \setminus B$ or xEy. $\bar{E}$ is a Borel equivalence relation on 2^ω, thus we may use Theorem 1.1.23. If $\bar{E}$ has at most countably many classes then so does E and at least one of the classes has an uncountable intersection with B. This uncountable class C is then an I-positive subset such that $E \restriction C = C \times C$. If $\mathrm{id}(2^\omega) \leq_B \bar{E}$ and it is witnessed by $f : 2^\omega \to 2^\omega$ then $P = f[2^\omega]$ is an uncountable Borel (we use Fact 1.1.10) subset of pairwise $\bar{E}$-inequivalent elements. However, $\bar{E}$ contains only one more equivalence class than E, namely $2^\omega \setminus B$, thus $P \cap B$ is still uncountable and $E \restriction (P \cap B) = \mathrm{id}(P \cap B)$.

We conclude this section by stating the following theorem from [18] that will be directly used later.

Theorem 1.2.11 (see [18]; Corollary 4.3.3.)**.** *Let I be a σ-ideal on a Polish space X such that the forcing notion P_I is proper, nowhere ccc and adds a minimal forcing extension. Then I has a total canonization for equivalence relations classifiable by countable structures.*

1.3 Fraïssé theory

In 1954, R. Fraïssé published a seminal paper ([10]) where he describes how the rational numbers with their order relation can be viewed as a certain limit of all finite linear orderings. Realize that not only does the structure $(\mathbb{Q}, \leq)$ contain all finite linear orderings as substructures but it also contains all countable linear orderings; moreover, any finite order isomorphism between two finite subsets of $\mathbb{Q}$ can be extended to an isomorphism of the whole structure $(\mathbb{Q}, \leq)$, and the structure with these properties is unique up to isomorphism.

This Fraïssé's construction is applicable in other cases too and we shall use it in the last chapter, thus we give an introduction to this subarea of model theory here.

Definition 1.3.1 (Age of a structure)**.** Let L be a relational language and let A be some L-structure. By age of A, $\mathcal{A}(A)$, we denote the class of all isomorphism types of finite substructures of A.

As usual in mathematics, we shall work with elements of $\mathcal{A}(A)$ as if they were actual finite substructures of A rather than just their isomorphism types. This simplifies the notation. We fix some countable relational language L from now on.

Next, instead of starting with some (L-)structure A and considering its age we start with some class $\mathcal{K}$ of isomorphism types of finite L-structures and investigate whether this class is an age of some L-structure A. We need some definitions.

Definition 1.3.2 (HP and JEP)**.** Let $\mathcal{K}$ be a class of isomorphism types of some finite L-structures. We say $\mathcal{K}$ has the hereditary property (HP) if whenever $B \in \mathcal{K}$ and A is a substructure of B then $A \in \mathcal{K}$.

We say $\mathcal{K}$ has the joint-embedding property (JEP) if whenever $A, B \in \mathcal{K}$ then there exists some $C \in \mathcal{K}$ such that both A and B embedd into C.

The following fact is easy to prove.

Fact 1.3.3 (see [15]; Theorem 7.1.1)**.** *Suppose that $\mathcal{K}$ is a countable class of isomorphism types of finite* L*-structures that has the HP and the JEP. Then there exists an* L*-structure A such that $\mathcal{A}(A) = \mathcal{K}$.*

Realize that $\mathcal{A}(\mathbb{Q}) = \mathcal{A}(\mathbb{N})$. So passing from A to $\mathcal{A}(A)$ and then back to an L-structure via the previous fact need not give the original structure. Observe that $\mathbb{N}$ does not have the homogeneity property of $\mathbb{Q}$ we stated at the beginning, i.e. any finite order isomorphism between two finite subsets of $\mathbb{Q}$ can be extended to an isomorphism of the whole structure $(\mathbb{Q}, \leq)$. This property is in Fraïssé theory called ultrahomogeneity. The definition follows.

Definition 1.3.4 (Ultrahomogeneity)**.** Let A be an L-structure. We say that A is ultrahomogeneous if any finite L-isomorphism between two finite substructures of A can be extended to an isomorphism of the whole structure A.

We need another requirement on $\mathcal{K}$.

Definition 1.3.5 (AP)**.** We say $\mathcal{K}$ has the amalgamation property (AP) if whenever $A, B, C \in \mathcal{K}$ such that A is embeddable into B via some embedding ϕ_B and embeddable into C via some embedding ϕ_C, then there exists $D \in \mathcal{K}$ and embeddings of B via ψ_B into D and of C via ψ_C into D such that $\psi_C \circ \phi_C = \psi_B \circ \phi_B$.

If a countable class of isomorphism types of some finite L-structures $\mathcal{K}$ has the HP, the JEP and the HP, then we call it a Fraïssé class.

We can now state the main theorem of Fraïssé theory.

Theorem 1.3.6 (Fraïssé's theorem; see [15]; Theorem 7.1.2)**.** *Let $\mathcal{K}$ be a countable class of isomorphism types of finite* L*-structures that has the HP, the JEP and the AP. Then there exists a unique up to isomorphism countable structure A such that $\mathcal{A}(A) = \mathcal{K}$ and A is ultrahomogeneous.*

We call such A a Fraïssé limit of $\mathcal{K}$.

The converse is now true too.

Fact 1.3.7 (see [15]; Theorem 7.1.7)**.** *If A is a non-empty at most countable* L*-structure that is ultrahomogenous, then $\mathcal{A}(A)$ is countable, has the HP, the JEP and the AP.*

Let us review the properties of a Fraïssé limit A of some Fraïssé class $\mathcal{K}$.

- Not only does A contain as substructures all finite structures from $\mathcal{K}$, it also contains as substructures all L-structures B such that $\mathcal{A}(B) \subseteq \mathcal{K}$. This does not immediately follow from Theorem 1.3.6 but it is easy to prove (see [15]; Lemma 7.1.3). This is a generalization of the fact that the rationals contains as substructures all countable linear orderings.

- A is ultrahomogeneous.

- A is unique up to isomorphism with the property that $\mathcal{A}(A)$ is countable, has the HP, the JEP and the AP.

We finish this section and this chapter by stating one single property of a Fraïssé limit A that implies all of those three properties above.

Definition 1.3.8 (One-point extension property)**.** Let A be an L-structure. We say that A has the one-point extension property if for any finite substructure $B_0 \subseteq A$ and any one-point extension $B_1 \in \mathcal{A}(A)$ of B_0, i.e. $|B_1| = |B_0| + 1$ and there exists an embedding $\phi : B_0 \hookrightarrow B_1$, there exists an embedding $\psi : B_1 \hookrightarrow A$ such that $\mathrm{id} = \psi \circ \phi$.

Fact 1.3.9 (see [15]; Lemma 7.1.4 (b))**.** *Let A be a (at most) countable* L-*structure. $\mathcal{A}(A)$ is a Fraïssé class iff A has the one-point extension property.*

Chapter 2

Silver forcing

Introduction

Recall that the Silver forcing is the set $\{f : A \subseteq \omega \to 2 : |\omega \setminus A| = \omega\}$ ordered by the reverse inclusion. Though not important for our purposes, we note that Silver forcing can be presented in the $\mathrm{Borel}(2^\omega)/I$ way as follows. We define I as the σ-ideal generated by Borel G-independent sets, where G is the graph on 2^ω such that there is an edge between x and y iff there is exactly one n such that $x(n) \neq y(n)$ (see [33, p. 212]). We will never use this information. From now on, I is fixed as the Silver ideal.

S. Grigorieff proved that the Silver forcing adds a minimal real degree and it follows that it canonizes all smooth equivalences (see [14, Cor. 5.5] and [18] for the latter). However, one can see that E_0 is in the spectrum of the Silver ideal, so when canonizing an equivalence relation E which is above E_0 in the Borel reducibility order, the best we can hope is that it can be reduced to the identity, a full equivalence relation or an equivalence relation bireducible with E_0, on some positive subset.

Here we focus on and canonize to the full eqivalence relation or a subset of E_0 $E_{\mathcal{I}}$ relations where $\mathcal{I}$ is an analytic P-ideal on ω. Recall that this class includes the E_{ℓ_p} relations for $p \in [1, \infty)$, $E_2 \approx_B \ell_1$ or E_{c_0}, where only the last one is $F_{\sigma\delta}$, the preceding ones are F_σ.

We will work with sets of type B_f, where $f : \omega \to 2$ is a partial function with a coinfinite domain, defined as $B_f = \{x \in 2^\omega : x \supseteq f\}$. These sets form a dense subset of $\mathrm{Borel}(2^\omega)/I$ isomorphic with the original Silver forcing, so they are called conditions throughout this chapter. By H_f we will denote the set of "holes" of the condition B_f, thus H_f is the complement of the domain of f. $2^{<H_f}$ in analogy with $2^{<\omega}$ denotes the set of finite functions to $\{0,1\}$ with domain contained in H_f. For $x \in 2^\omega$ and $A \subseteq \omega$, $x \odot A$ denotes the element $y \in 2^\omega$ such that $y(n) = x(n)$ for $n \in \omega \setminus A$ and $y(n) = 1 - x(n)$ for $n \in A$ (we write $x \odot n$ instead of $x \odot \{n\}$). When s, t, u (s, t finite, u may be infinite) are sequences, stu is their concatenation. We will occasionally use the term "Silver tree" for the condition B_f when we are interested in properties of initial subsequences of elements of B_f, i.e. $x \restriction n$'s, for $x \in B_f, n \in \omega$.

We state a proposition from [18] that we will use later in the proof of the main theorem. It is an other information information about the spectrum of the Silver ideal beyond that of Grigorieff mentioned above.

Proposition 2.0.10 ([18]; Theorem 8.2.3.)**.** *Let B be a condition in the Silver forcing and let E be an equivalence relation on B that is classifiable by countable structures. Then there exists a Silver subcondition $C \subseteq B$ such that $E \restriction C$ is C^2 or a subset of E_0.*

The last assertion follows from Fact 1.1.38. Note that we cannot use Theorem 1.2.11 as the Silver ideal does not add a minimal forcing extension.

2.1 Canonization results

Theorem 2.1.1. *Let B be a condition in the Silver forcing, $\mathcal{I}$ an analytic P-ideal and $E \subseteq B^2$ an equivalence relation Borel reducible to $E_{\mathcal{I}}$. Then there exists a Silver subcondition $C \subseteq B$ such that $E \restriction C$ is C^2 or a subset of E_0.*

Proof of the theorem. We start with a basic observation.

Claim 2.1.2. *There is a subcondition $B_g \subseteq B$ such that $f \restriction B_g$ is determined by a function $p : 2^{<H_g} \to 2^{<\omega}$ from finite subsets of H_g to finite subsets of ω, which is monotonous, i.e. $p(t) \supseteq p(s)$ for $t \supseteq s$, $|p(t)| = |t|$ for all $t \in 2^{<H_g}$ and $f(h) = \bigcup_n p(h \restriction n)$.*

We will also denote $f(x) = \bigcup_n p(x \upharpoonright n)$ for $x \in B_g$ as $p(x)$, it should not make any confusion.

Proof. Silver forcing has the continuous reading of names (see [33, Theorem 3.3.2 and the fact the Silver forcing is bounding]) thus we can find a subcondition B_{r_0} of B on which f is continuous. Pick a hole $h_0 \in H_{r_0}$. It follows from the continuity of f that we can find a subcondition B_{r_1} with $h_0 \in H_{r_1}$ such that the value of $f(x)(0)$ depends only on the value $x(h_0)$ for $x \in B_{r_1}$. We pick next hole h_1 and find again a subcondition such that the value on h_1 decides the value of $f(x)(1)$ for both possible values on h_0. Generally, when we have the holes $h_0, \ldots, h_{n-1}$ deciding the corresponding finite part of $f(x)$ we pick the next least hole h_n, 2^n times apply the continuous reading of names and find a subcondition so that for every configuration on holes $h_0, \ldots, h_{n-1}$ the value on h_n decides the value of $f(x)(n)$. We end up with a condition B_g, which is an intersection of conditions obtained along the construction, with $H_g = \{h_0, h_1, \ldots\}$ from the statement of the claim. $\square$

We will WLOG assume that $H_g = \omega$.

By Theorem 1.1.32 we have a lower semicontinuous submeasure $\mu : \mathcal{P}(\omega) \to [0, \infty]$ such that $\mathcal{I} = \mathrm{Exh}(\mu)$.

From now on, we also reserve the letters x, y, z to denote infinite binary sequences and other letters, if it is not said otherwise, to denote finite binary sequences.

We define $\Delta_n(x, y)$, for $x, y \in 2^{\leq\omega}$, as $\mu((p(x) \setminus n) \triangle (p(y) \setminus n))$ $(= \mu((p(x) \triangle p(y)) \setminus n))$, where we identify $p(x)$ with the corresponding (finite or infinite) subset of ω. $\Delta_0(x, y)$ may be denoted as $\Delta(x, y)$.

Note that $\Delta(x, y)$ (resp. $\Delta_k(x, y)$) is a pseudometric (which may attain an infinite value though). Symmetricity is obvious; triangle inequality $\Delta(x, z) \leq \Delta(x, y) + \Delta(y, z)$ follows from the inclusion $p(x) \triangle p(z) \subseteq (p(x) \triangle p(y)) \cup (p(y) \triangle p(z))$ and monotonicity from subadditivity of μ (similarly for $\Delta_k(x, y)$). We will frequently use this triangle inequality.

We extend the predicate E to finite sequences as follows: sEt iff $|s| = |t|$ and $\forall x(sxEtx)$.

Moreover, for $\delta \in \mathbb{R}^+$ we define $xE_\delta y$ (resp. $sE_\delta t$) when $\Delta(x,y) < \delta$ (resp. $\forall x(\Delta(sx,tx) < \delta)$) and for finite sequences s, t (of the same length) we will write $sE^\delta t$ when $\forall x(\Delta_{|s|}(sx,tx) < \delta)$. For the rest of the proof the relation E with a subscript will always denote one of those just defined and it should not be confused with E_0, E_2, etc.

The proof splits into three cases.

Case 1 There exists $\varepsilon > 0$ such that the set

$$S = \{s \in 2^{<\omega} : \exists t(\Delta(s0t, s1t) \geq \varepsilon \wedge s0tEs1t)\}$$

is somewhere dense (in $2^{<\omega}$ ordered by reverse inclusion).

Assume that S is dense above some $d \in 2^{<\omega}$ and start with some $s_0 \supseteq d$ in S. There is some t_0 such that $\Delta(s_0 0 t_0, s_0 1 t_0) \geq \varepsilon$ and for every $x \in 2^\omega$ $\Delta(s_0 0 t_0 x, s_0 1 t_0 x)$ is finite, so we may in fact assume that t_0 is extended enough so that $s_0 0 t_0 E^{\frac{\varepsilon}{8}} s_0 1 t_0$. Otherwise, there would be $t_1, t_2, \ldots$ such that

$$\forall n(s_0 0 t_0 t_1 \ldots t_n \not{E}^{\frac{\varepsilon}{8}} s_0 1 t_0 t_1 \ldots t_n)$$

which would (from the exhaustivity of μ) imply that $s_0 0 x_t \not{E} s_0 1 x_t$, where $x_t = t_0 t_1 \ldots$, a contradiction.

Then find $s_1 \in 2^{<\omega}$ such that $s_0 0 t_0 s_1 \in S$. There is t_1 such that $\Delta(s 0 t_0 s_1 0 t_1, s_0 0 t_0 s_1 1 t_1) \geq \varepsilon$. That automatically implies that also $\Delta(s_0 1 t_0 s_1 0 t_1, s_0 1 t_0 s_1 1 t_1) \geq \frac{3\varepsilon}{4}$. It follows from the fact that $\Delta_{|s_0 0 t_0|}(s 0 t_0 s_1 i t_1, s_0 1 t_0 s_1 i t_1) < \frac{\varepsilon}{8}$, for $i \in \{0,1\}$, and the triangle inequality.

Again, we may assume that t_1 is extended enough that $s_0 i t_0 s_1 0 t_1 E^{\frac{\varepsilon}{16}} s_0 i t_0 s_1 1 t_1$, for $i \in \{0,1\}$.

Then we find s_2 such that $s_0 0 t_0 s_1 0 t_1 s_2 \in S$, obtain t_2 so that $s_0 \ldots 0 s_2 t_2 E^{\frac{\varepsilon}{32}} s_0 \ldots 1 s_2 t_2$ and continue in the same manner.

The way we have chosen t_n's guarantees that

$$\Delta_{|s_0\dots s_n|}(s_0 i_0 \dots i_{n-1} t_{n-1} s_n 0 t_n, s_0 j_0 \dots j_{n-1} t_{n-1} s_n 1 t_n) \geq \frac{\varepsilon}{2}$$

where $i_m, j_m \in \{0,1\}$ for $m < n$. To see this, notice that

$$\Delta_{|s_0\dots s_n|}(s_0 i_0 \dots i_{n-1} t_{n-1} s_n 0 t_n, s_0 0 \dots 0 t_{n-1} s_n 0 t_n) < \frac{\varepsilon}{8} + \dots + \frac{\varepsilon}{2^{2+n}} < \frac{\varepsilon}{4}$$

$$\Delta_{|s_0\dots s_n|}(s_0 j_0 \dots j_{n-1} t_{n-1} s_n 1 t_n, s_0 0 \dots 0 t_{n-1} s_n 1 t_n) < \frac{\varepsilon}{8} + \dots + \frac{\varepsilon}{2^{2+n}} < \frac{\varepsilon}{4}$$

and finally

$$\Delta_{|s_0\dots s_n|}(s_0 0 \dots 0 t_{n-1} s_n 0 t_n, s_0 0 \dots 0 t_{n-1} s_n 1 t_n) > \varepsilon$$

and use the triangle inequality.

Now let

$$x = s_0 i_0 t_0 s_1 i_1 t_1 \dots i_n t_n \dots$$

$$y = s_0 j_0 t_0 s_1 j_1 t_1 \dots j_n t_n \dots$$

where $i_m, j_m \in \{0,1\}$ for $m \in \omega$. If $i_m \neq j_m$ for infinitely many m's then it follows from the construction that there are infinitely many disjoint intervals $[k_m, l_m]$ such that $\Delta_{k_m}^{l_m}(x,y) \geq \frac{\varepsilon}{2}$, thus $p(x) \triangle p(y) \notin \mathrm{Exh}(\mu)$.

On the contrary, if the set $\{m : i_m \neq j_m\}$ is finite, then by transitivity of E, xEy. It follows that we just found a condition $B_h = C$ on which E is equal to E_0, where

$$B_h = \{x \in 2^\omega : x = s_0 i_0 t_0 s_1 i_1 t_1 \dots i_n t_n \dots, i_m \in \{0,1\}\}$$

If Case 1 does not hold then for every $\varepsilon > 0$

$$S_\varepsilon = \{s \in 2^{<\omega} : \exists t(\Delta(s0t, s1t) \geq \varepsilon \wedge s0tEs1t)\}$$

is nowhere dense. For a particular ε and $s \notin S_\varepsilon$ that implies that either for every v there is an infinite extension $x \supseteq v$ such that $s0x\not{E}s1x$ or there is t such that

$s0tE_\varepsilon s1t$. Let S^1_ε denote the set of all s's from the latter case, i.e.

$$S^1_\varepsilon = \{s : \exists t(s0tE_\varepsilon s1t)\}$$

and S^2_ε the set of all s's from the "either" case, i.e.

$$S^2_\varepsilon = \{s : \forall v \exists x \supseteq v(s0x \not{E} s1x)\}$$

Here we split into the remaining two cases.

Case 2 Assume that $S^1_{\frac{1}{n}}$ is dense for infinitely many $n \in \omega$. Then let us start with some $s_0 \in S^1_{\frac{1}{m_0}}$, where $m_0 \geq 2$, and we obtain appropriate t_0, i.e. $s_0 0 t_0 E_{\frac{1}{m_0}} s_0 1 t_0$, which may be again sufficiently extended so that $s_0 0 t_0 E^{\frac{1}{4}} s_0 1 t_0$. Then we find s_1 such that $s_0 0 t_0 s_1 \in S^1_{\frac{1}{m_1}}$ for some $m_1 \geq 4$. We again obtain appropriate t_1 and extend it if necessary so that $s_0 0 t_0 s_1 0 t_1 E^{\frac{1}{8}} s_0 i t_0 s_1 j t_1$ for $i, j \in \{0, 1\}$. Generally, we look for s_n such that $s_0 0 \ldots 0 t_{n-1} s_n \in S^1_{\frac{1}{m_n}}$ for some $m_n \geq 2^{n+1}$ and t_n is again sufficiently extended, so we always have $s_0 i_0 t_0 \ldots i_n t_n E^{(\frac{1}{m_n} + \frac{1}{2^n})} s_0 j_0 \ldots j_n t_n$, for $i_0, \ldots, i_n, j_0, \ldots, j_n \in \{0, 1\}$.

Now let

$$x = s_0 i_0 t_0 s_1 i_1 t_1 \ldots i_n t_n \ldots$$

$$y = s_0 j_0 t_0 s_1 j_1 t_1 \ldots j_n t_n \ldots$$

where $i_m, j_m \in \{0, 1\}$ for $m \in \omega$. Then by the construction $\Delta(x, y) < \sum_{\{n : x(n) \neq y(n)\}} (\frac{1}{2^{n+1}} + \frac{1}{2^n})$. Hence we found a condition $B_h = C$, where

$$B_h = \{x \in 2^\omega : x = s_0 i_0 t_0 s_1 i_1 t_1 \ldots i_n t_n \ldots, i_m \in \{0, 1\}\}$$

on which E is the full relation, i.e. $E \restriction C = C \times C$.

Case 3 The remaining case is when there is $s_0 \in 2^{<\omega}$ such that $\forall s \supseteq s_0 \forall v \exists x \supseteq v(s0x \not{E} s1x)$. We assume that E does not have I-positive classes and we find a condition on which E is a subset of E_0.

For $s \supseteq s_0$ and $\varepsilon > 0$, let us denote

$$T_s^\varepsilon = \{t : \exists e(\Delta_{|s0t|-1}(s0te, s1te) \geq \varepsilon)\}$$

Note that T_s^ε is closed under initial segments.

Claim 2.1.3. *For every $s \supseteq s_0$ there exists $\varepsilon > 0$ such that the set T_s^ε is somewhere dense above s.*

Proof. Otherwise, let us assume that the set

$$N = \{s \supseteq s_0 : \forall\varepsilon(T_s^\varepsilon \text{ is nowhere dense above } s)\}$$

is dense (above s_0). If it were not dense, we could extend s_0 so that there would be no element of N above s_0.

Pick some $n_0 \in N$. Since $T_{n_0}^1$ is nowhere dense above s_0 we can find v_0 so that there is no element of $T_{n_0}^1$ above v_0, thus $\forall t \supseteq v_0 \forall e$

$$\Delta_{|n_0 0t|-1}(n_0 0te, n_0 1te) < 1$$

Then find n_1 such that $n_0 0 v_0 n_1 \in N$. We prove that from triangle inequality we also have $n_0 1 v_0 n_1 \in N$.

To see this, denote for simplicity $n_0 0 v_0 n_1$ as m_0 and $n_0 1 v_0 n_1$ as m_1. Suppose that there is ε such that $T_{m_1}^\varepsilon$ is dense above some $k \supseteq m_1$. Since $T_{n_0}^{\varepsilon/4}$ is nowhere dense above n_0, there exists $\bar{k} \supseteq k$ such that $\forall l \supseteq \bar{k} \forall e$

$$\Delta_{|m_0 0l|-1}(m_0 0le, m_1 0le) \leq \varepsilon/4$$

and

$$\Delta_{|m_0 1l|-1}(m_0 1le, m_1 1le) \leq \varepsilon/4$$

Then for every $t \in T_{m_1}^\varepsilon$, $t \supseteq \bar{k}$, there is e such that

$$\Delta_{|m_1 0t|-1}(m_1 0te, m_1 1te) \geq \varepsilon$$

so we have from the triangle inequality

$$\Delta_{|m_0 0t|-1}(m_0 0te, m_0 1te) \geq \Delta_{|m_1 0t|-1}(m_1 0te, m_1 1te)$$

$$-\Delta_{|m_0 0t|-1}(m_0 0te, m_1 0te) - \Delta_{|m_0 1t|-1}(m_0 1te, m_1 1te) \geq \varepsilon/2$$

Thus $T^{\varepsilon/2}_{m_0}$ is somewhere dense above m_0 which contradicts that $m_0 \in N$.

We can find some v_1 so that both $v_0 n_1 i v_1$ are outside of $T^{2^{-2}}_{n_0}$ (no element of $T^{2^{-2}}_{n_0}$ is above $v_0 n_1 i v_1$), for $i \in \{0, 1\}$, which guarantess that $\forall t \supseteq v_1 \forall e$

$$\Delta_{|m_0 0t|-1}(m_0 0te, m_1 0te) < 2^{-2}$$

and

$$\Delta_{|m_0 1t|-1}(m_0 1te, m_1 1te) < 2^{-2}$$

and if necessary we can extend v_1 so that it is outside of both $T^{2^{-2}}_{n_0 i v_0 n_1}$, for $i \in \{0, 1\}$, which guarantess that $\forall t \supseteq v_1 \forall e$

$$\Delta_{|n_0 0 v_0 n_1 0t|-1}(n_0 0 \ldots 0te, n_0 0 \ldots 1te) < 2^{-2}$$

and

$$\Delta_{|n_0 1 v_0 n_1 0t|-1}(n_0 1 \ldots 0te, n_0 1 \ldots 1te) < 2^{-2}$$

Finally, from triangle inequality $\forall t \supseteq v_1 \forall e$

$$\Delta_{|n_0 0 v_0 n_1 0t|-1}(n_0 0 \ldots 0te, n_0 1 \ldots 1te) < 2^{-1}$$

and

$$\Delta_{|n_0 0 v_0 n_1 1t|-1}(n_0 0 \ldots 1te, n_0 1 \ldots 0te) < 2^{-1}$$

In summary, we have the following inequalities: $\forall t \supseteq v_1 \forall e$

$$\Delta_{|n_0 j_0 \ldots j_1 t|}(n_0 i_0 v_0 n_1 i_1 te, n_0 j_0 v_0 n_1 j_1 te) < 2^{-1}$$

In general, once we have n_{m-1}, v_{m-1} we choose n_m so that $n_0 0 \ldots n_{m-1} 0 v_{m-1} n_m \in N$ and it again follows from triangle inequality that also

$n_0 i_0 \dots n_{m-1} i_{m-1} v_{m-1} n_m \in N$ for $i_k \in \{0,1\}$. As above, we can find some v_m so that for every $t \supseteq v_m$ and every e

$$\Delta_{|n_0 j_0 \dots j_m t|}(n_0 i_0 v_0 n_1 i_1 v_1 \dots i_m te, n_0 j_0 \dots j_m te) < 2^{-m}$$

for $i_k, j_l \in \{0,1\}$. We obtain a Silver subcondition

$$B = \{x \in 2^\omega : x = n_0 i_0 \dots n_m i_m v_m \dots \text{ where } i_m \in 2 \forall m \in \omega\}$$

where the relation reduces to the full relation. To see this, let $x, y \in B$ and $n, l \in \omega$. Let $\bar{m} = |n_0 0 \dots 0 v_{n+1}|$ and $I = \{k_0, k_1, \dots, k_d\}$ be the finite set of indices of holes where the finite segments $x \restriction l$ and $y \restriction l$ differ and that come after $n_0 0 \dots 0 v_{n+1}$. We want to prove the following inequality

$$\forall m \geq \bar{m}(\Delta_m(x \restriction l, y \restriction l) \leq 2^{-n}$$

Since n and l were arbitrary, it would give us precisely the condition for $p(x) E_{\mathcal{I}} p(y)$, thus xEy.

Note that from inequalities that we have it holds that

$$\forall m \geq \bar{m}(\Delta_m(x \restriction l, y \restriction l) \leq \Delta_{\bar{m}}(x \restriction l, y \restriction l) \leq \Delta_{\bar{m}}(x \restriction l, (y \odot I) \restriction l) +$$

$$\Delta_{|\dots v_{k_0} i_{k_0}|}(x \restriction l, (y \odot (I \setminus \{i_{k_0}\})) \restriction l)) + \Delta_{|\dots v_{k_1} i_{k_1}|}(x \restriction l, (y \odot (I \setminus \{i_{k_0}, i_{k_1}\}) \restriction l))$$

$$+ \dots \leq 2^{-n-1} + 2^{-n-2} + 2^{-n-3} + \dots + 2^{-n-d-1} \leq 2^{-n})$$

That is a contradiction with the assumption that E has no I-positive class. $\square$

So far we have proved that for any $s \supseteq s_0$ there is ε such that T_s^ε is somewhere dense above s; i.e. there is some $v \supseteq s$ such that the set $\{t \supseteq v : \exists e(\Delta_{|s0t|-1}(s0te, s1te) \geq \varepsilon)\}$ is dense above v. However, the ε from the statement need not to be optimal. So for example we could have both sets $\{t \supseteq v : \exists e(\Delta_{|s0t|-1}(s0te, s1te) \geq \varepsilon)\}$ and $\{t \supseteq v : \exists e(\Delta_{|s0t|-1}(s0te, s1te) \geq 2\varepsilon)\}$ being dense above v.

Let U (unbounded) denote the set of those $s \supseteq s_0$ for which T_s^ε is dense above

s for an arbitrarily large ε; i.e.

$$U = \{s \supseteq s_0 : \forall\varepsilon \in \mathbb{R}^+ (T_s^\varepsilon \text{ is dense above } s)\}$$

The complement is the set of those $s \supseteq s_0$ such that there is some v and some ε such that

1. for every e we have $\Delta_{|s|}(s0ve, s1ve) \leq \varepsilon$

2. for an arbitrarily small $\delta > 0$ the set $T_s^{\varepsilon-\delta}$ is dense above v

Let B (bounded) denote this complement; i.e.

$$B = \{s \supseteq s_0 : \exists v \exists \varepsilon \text{ such that } \forall\delta > 0 T_s^{\varepsilon-\delta} \text{ is dense above } v \text{ and}$$

$$\forall e(\Delta_{|s|}(s0ve, s1ve) \leq \varepsilon)\}$$

If the set B is dense above s_0 then by the same argument using triangle inequality that we used in the proof of Claim 2.1.3 one can show that B is even symmetric. This means that for any $b \in B$ and $b0u \in B$ we also have $b1u \in B$, and moreover if v (used in the definition of B) witnesses that $b0u \in B$ then the same v witnesses that $b1u \in B$. To see this, just note that we are again using the fact that when $\{\Delta(b0e, b1e) : e\}$ is bounded, $\{\Delta(b0v0t, b0v1t) : t\}$ is bounded, then by the triangle inequality $\{\Delta(b1v0w, b1v1w) : w\}$ cannot be unbounded.

Thus if B is dense then we can form a Silver subtree such that every splitting node lies in B. If B is not dense, then we can extend the initial segment s_0 sufficiently enough so that we will work only with nodes from U.

So we have two cases. One that we have a Silver subtree with splitting nodes from U, the second that we have a Silver subtree with splitting nodes from B. For the further use, let us denote $sp(S)$ the set of splitting nodes of a Silver tree S. We shall build a tree T.

- We have a Silver subtree with splitting nodes from U: Start with the first splitting node from U, for simplicity again denoted s_0. Choose arbitrarily some $\varepsilon_{s_0} > 1$. Since $T_{s_0}^{\varepsilon_{s_0}}$ is dense above s_0 we cand find some e_0 and such that $\Delta_{|s_0|-1}(s_0 0 e_0, s_0 1 e_0) \leq \varepsilon_{s_0})$. Denote s_1^0, resp. s_1^1, the nodes

$s_0 0 e_0$, resp. $s_0 1 e_0$. Again choose some $\varepsilon_{s_1^0} > 1$ and $\varepsilon_{s_1^1} > 1$ such that $T_{s_1^0}^{\varepsilon_{s_1^0}}$, resp. $T_{s_1^1}^{\varepsilon_{s_1^1}}$, is dense above s_1^0, resp. s_1^1. We can find some t_1 such that $t_1 \in T_{s_1^0}^{\varepsilon_{s_1^0}} \cap T_{s_1^1}^{\varepsilon_{s_1^1}}$ and moreover $0 e_0 t_1, 1 e_0 t_1 \in T_{s_0}^{\varepsilon_{s_0}}$ (this is possible since these T_s^ε-sets are closed under initial segments). We extend t_1 into some e_1 so that $\Delta_{|s_1^i|-1}(s_1^i 0 e_1, s_1^i 1 e_1) \leq \varepsilon_{s_0^i})$, for $i \in \{0,1\}$, and moreover $\Delta_{|s_1^0|-1}(s_1^0 0 e_1, s_1^1 0 e_1) > \varepsilon_{s_0}$ and $\Delta_{|s_1^0|-1}(s_1^0 1 e_1, s_1^1 1 e_1) > \varepsilon_{s_0}$.

Denote obtained nodes $s_2^0, \ldots, s_2^3$ ($s_2^i = s_1^0 i e_1$ for $i \in \{0,1\}$ and $s_2^i = s_1^1(3 - i) e_1$ for $i \in \{2,3\}$).

Similarly, for every $n \in \omega$, we get s_n^i and $\varepsilon_{s_n^i}$, $0 \leq i < 2^n$, where each s_n^i is $s_0 i_0 e_0 \ldots i_k \ldots i_{n-1} e_{n-1}$ for some values $i_0, \ldots, i_{n-1}$, and we have the following inequalities

$$\Delta_{|s_0 i_0 e_0 \ldots i_{n-1} t_{n-1}|-1}(s_0 i_0 \ldots (i_k) \ldots i_{n-1} e_{n-1},$$

$$s_0 i_0 \ldots (1 - i_k) \ldots i_{n-1} e_{n-1}) \geq \varepsilon_{s_0 i_0 \ldots e_{k-1}}$$

where $i_j \in \{0,1\}$ for $j < n$ and $k < n$. This finishes the construction of T.

- We have a Silver tree with splitting nodes from B: Recall that for every $s \in B$ we have some v and ε_s such that $\forall e(\Delta_{|s|}(s0ve, s1ve) \leq \varepsilon_s)$, however for an arbitrarily small δ we have that $T_s^{\varepsilon_s - \delta}$ is dense above v. From now on for every $s \in B$ and corresponding ε_s we shall always consider $\delta_s = \varepsilon_s/4$, abuse the notation and write $T_s^{\varepsilon_s}$ for $T_s^{\varepsilon_s - \delta_s}$. Such δ's are small enough with respect to the proof that we will write $\forall t \in T_s^{\varepsilon_s} \exists e(\Delta_{|s0t|-1}(s0te, s1te) \doteq \varepsilon_s$ which is means $\forall t \in T_s^{\varepsilon_s} \exists e(\Delta_{|s0t|-1}(s0te, s1te) \in [\varepsilon_s - \delta_s, \varepsilon_s]$.

  We again start with some element from B, for simplicity again denoted s_0, determine the corresponding ε_{s_0} and v_0 and find some e_0 so that $\Delta_{|s0v|-1}(s0ve, s1ve) \doteq \varepsilon_{s_0}$. Denote s_1^0, s_1^1 the nodes $s_0 0 v_0 e_0, s_0 1 v_0 e_0$. WLOG we may assume they are splitting nodes, i.e. $s_1^0, s_1^1 \in B$. Determine $\varepsilon_{s_1^0}$ and $\varepsilon_{s_1^1}$ and recall that there is a common v_1 (from the definition of B) for both s_1^0 and s_1^1. Similarly as in the previous item we can, if necessary, extend v_1

and find e_1 so that the following equalities hold

$$\Delta_{|s_1^i 0 v_1|-1}(s_1^i 0 v_1 e_1, s_1^i 1 v_1 e_1) \doteq \varepsilon_{s_1^i}$$

for $i \in \{0, 1\}$ and

$$\Delta_{|s_0 0 v_0 e_0 i v_1|-1}(s_0 0 \dots i v_1 e_1, s_0 1 \dots i v_1 e_1) \doteq \varepsilon_{s_0}$$

for $i \in \{0, 1\}$. Denote obtained nodes $s_2^0, \dots, s_2^3$ ($s_2^i = s_1^0 i v_1 e_1$ for $i \in \{0, 1\}$ and $s_2^i = s_1^1(3 - i) v_1 e_1$ for $i \in \{2, 3\}$).

Similarly, for every $n \in \omega$, we get s_n^i and $\varepsilon_{s_n^i}$, $0 \le i < 2^n$, where each s_n^i is $s_0 i_0 v_0 \dots i_k \dots i_{n-1} v_{n-1} e_{n-1}$ for some values $i_0, \dots, i_{n-1}$, and we have the following inequalities

$$\Delta_{|s_0 i_0 t_0 \dots i_{n-1} v_{n-1}|-1}(s_0 i_0 \dots (i_k) \dots i_{n-1} v_{n-1} e_{n-1},$$

$$s_0 i_0 \dots (1 - i_k) \dots i_{n-1} v_{n-1} e_{n-1}) \doteq \varepsilon_{s_0 i_0 \dots t_{k-1} e_{k-1}}$$

where $i_j \in \{0, 1\}$ for $j < n$ and $k < n$. This finishes the construction of T.

Let us now consider two possible cases that may happen. Each of them directly leads to some form of canonization. We will then prove that we can obtain a Silver subtree satisfying one of them.

Subcase 3a Assume there is a Silver subtree S of T such that the set $\{\varepsilon_s : s \in sp(S)\} \subseteq \mathbb{R}$ is bounded from below. Then we prove that the equivalence relation restricted to $[S]$ (a Silver subcondition given by branches of the Silver tree S) is countable (and we are done by Fact 1.1.38 Proposition 2.0.10). For this, let ε be the lower bound for this set and suppose for some $x \in [S]$ the set $V = \{y \in [S] : xEy\}$ is uncountable. Then there is an uncountable subset $V_n^m \subseteq V$ such that for every $y \in V_n^m$ n is the least number where x and y differ and $\Delta_m(x, y) < \frac{\varepsilon}{2}$. Let $y, z \in V_n^m$ be any two branches that split above the m-th level. Then it follows from the construction that $\Delta_m(y, z) > \varepsilon$ and thus from the triangle inequality either $\Delta_m(x, y) > \varepsilon/2$ or $\Delta_m(x, z) > \varepsilon/2$, a contradiction.

Thus V_n^m must be finite, so V was not uncountable.

Remark 2.1.4. Notice that if we built T from a Silver subtree with splitting nodes from U we always end up in Subcase 3a and the theorem is proved. Just observe that in the construction we have guaranteed that for every $s \in sp(T)$ we have $\varepsilon_s > 1$. Thus 1 is the lower bound.

Subcase 3b Assume there is a Silver subtree S such that the set $\{\varepsilon_s : s \in sp(S)\} \subseteq \mathbb{R}$ is a sequence converging to zero. We may refine the tree so that for every $s \in sp(S)$ $\varepsilon_s \geq 4\varepsilon_{s_0}, 4\varepsilon_{s_1}$ where s_0, s_1 are immediate splitting successors of s in S. Then we claim that the equivalence relation restricted to $[S]$ is the identity. Let x, y be two branches of S and let $s \in S$ be their last common node. Then since ε_t's, for $t \in S$, are decreasing quickly, it follows from the triangle inequality that there are infinitely many n's such that $\Delta_n(x,y) \geq \frac{5}{12}\varepsilon_s$. To see this, let $n_0 = |s|$, then

$$\Delta_{n_0}(x,y) \doteq \varepsilon_s \geq \frac{5}{12}\varepsilon_s$$

Let $n_1 > n_0$ be the length of some next splitting node $s\prime$ and let $i_0, \ldots, i_k$ be the holes between s and $s\prime$ where x and y differ, then

$$\Delta_{n_1}(x,y) \geq \Delta_{n_1}(x, y \odot \{i_0, \ldots, i_k\}) - \Delta_{n_1}(y \odot \{i_0, \ldots, i_k\}, y \odot \{i_1, \ldots, i_k\}) -$$

$$\ldots - \Delta_{n_1}(y \odot i_k, y) \geq (\varepsilon_s - \delta_s) - \varepsilon_s/4 - \ldots - \varepsilon/4^k \geq \frac{5}{12}\varepsilon_s$$

And so on. So we found a Silver subcondition (given by a subtree) $[S]$ such that E here is ev, i.e. $E \restriction [S] = [S] \times [S]$.

Thus to finish the proof of Theorem 2.1.1 it remains to prove the following lemma. By Remark 2.1.4 we assume that T was built from a Silver subtree with splitting nodes from B.

Lemma 2.1.5. *There exists a Silver subtree of T satisfying either the condition from 3a, or the condition from 3b.*

Proof. We will try to build a Silver subtree satisfying the condition from Subacase 3b. If we find an obstacle preventing us from doing that, then we will be able to build a tree satisfying the condition from Subcase 3a.

We start with picking v_0 such that $\varepsilon_{v_0} < 1$. Next we want to find v_1 such that $\varepsilon_{v_0 0 v_1}, \varepsilon_{v_0 1 v_1} < 2^{-1}$. If such v_1 does not exist, then whenever $\varepsilon_{v_0 0 v} < 2^{-1}$ for some v, then $\varepsilon_{v_0 1 v} \geq 2^{-1}$. But we show that we can build a Silver subtree P with stem $v_0 0$ such that $\forall s \in P(\varepsilon_s < 2^{-1})$. But then the symmetric tree above $v_0 1$ (i.e. for every s $v_0 1 s$ is in this tree iff $v_0 0 s \in T$) will satisfy the condition from Subcase 3a with the bound 2^{-1}. The following simple claim will be the main tool.

Claim 2.1.6. *$\varepsilon_{siv} \leq 2\varepsilon_s + \varepsilon_{s(1-i)v}$ for any $s \in T$, $i \in \{0,1\}$, $siv \in T$.*

Proof. We use the triangle inequality:

$$\forall t \in T_{siv}^{\varepsilon_{siv}} \forall e \Delta_{|siv0t|-1}(siv0te, siv1te) \leq \Delta_{|siv0t|-1}(siv0te, s(1-i)0te)+$$

$$+\Delta_{|siv0t|-1}(siv1te, s(1-i)1te) + \Delta_{|siv0t|-1}(s(1-i)v0te, s(1-i)v1te) \leq$$

$$\leq 2\varepsilon_s + \varepsilon_{s(1-i)v}$$

which is what we wanted to prove.

Building the tree We will be looking for nodes and building the left-most branch, other nodes of the Silver tree (denoted P) will be determined automatically. We will ensure that $\varepsilon_t < 2^{-1}$ for any t in the tree.

Pick any node t_0 above $v_0 0$ such that ε_{t_0} is very small compared to 2^{-1}, less than 2^{-4} suffices. Than find any t_1 such that $\varepsilon_{t_0 0 t_1} < 2^{-8}$, then t_2 such that $\varepsilon_{t_0 0 t_1 0 t_2} < 2^{-16}$ and so on.

Observe that

$$\varepsilon_{t_0 0 t_1 1 t_2} < 2 \cdot 2^{-8} + 2^{-16} < 2^{-1}$$

$$\varepsilon_{t_0 1 t_1 0 t_2} < 2 \cdot 2^{-4} + 2^{-16} < 2^{-1}$$

and

$$\varepsilon_{t_0 1 t_1 1 t_2} < 2 \cdot 2^{-4} + 2 \cdot 2^{-8} + 2^{-16} < 2^{-1}$$

Generally, let $s \in P$ be arbitrary and let t_s be the node of the same length lying on the leftmost branch. Let n be the number of bits where s and t_s differ and $m(\geq n)$ the number of splitting nodes in P up to the length of t_s. Then

$$\varepsilon_s < 2 \cdot 2^{-4} + 2 \cdot 2^{-8} + \ldots + 2 \cdot 2^{-2^{n+1}} + 2^{-2^{m+1}} < 2^{-1}$$

Hence we really are able to find v_1 so that $\varepsilon_{v_0 0 v_1}, \varepsilon_{v_0 1 v_1} < 2^{-1}$. Next we want to find v_2 such that $\varepsilon_{v_0 i v_1 j v_2} < 2^{-2}$ for $i, j \in \{0, 1\}$.

As in the previous paragraph we can build a Silver subtree P_0 above the node $v_0 0 v_1 0$ such that for any node $t \in P_0$ we have $\varepsilon_t < 2^{-2}$. Now put the same tree above $v_0 0 v_1 1$. Either the set of its ε's is bounded, we are then in Subcase 3a, or we can refine it and obtain a Silver subtree P_1 such that for any node $t \in P_1$ we have $\varepsilon_t < 2^{-2}$. Do the same for remaining two nodes and obtain P_3 which is a Silver subtree of P_i, $i < 3$. We can then pick any node from $v_0 0 v_1 0 P_3$ above $v_0 0 v_1 0$ as v_2.

Then build the next level with ε-value 2^{-3}. It is now clear that we either end up with a Silver tree satisfying the condition from Subcase 3b, or we fail on some level and then build a Silver subtree satisfying the condition from Subcase 3a.

This finishes the proof of Lemma 2.1.5 and of Theorem 2.1.1. $\square$

Although it may seem that the previous proof can be possibly generalized to all F_σ ideals using the Mazur's theorem (1.1.34), it is not the case as we essentially used the exhaustivity of μ associated to analytic P-ideals. In fact, there is a counter-example among F_σ non-P-ideals. Zapletal found a K_σ equivalence relation on the Cantor space which is in the spectrum of the Silver ideal [18]. The relation (denoted here as E_{K_σ}) is defined as

$$x E_{K_\sigma} y \equiv \exists n \forall m (|\sharp\{k \leq m : x(k) = 1\} - \sharp\{k \leq m : y(k) = 1\}| \leq n)$$

We remark that for a finite set A, $\sharp A$ denote the number of elements of A.

This relation is Borel bireducible to $E_{\mathcal{I}_W}$ where
$\mathcal{I}_W = \{A \subseteq \omega : A$ does not contain arbitrarily large arithmetic progressions$\}$ is the van der Waerden ideal which is F_σ non-P. This relation is moreover Borel

bireducible to E_{ℓ_∞}, i.e. a universal K_σ equivalence relation (see 1.1.22). We present both proofs.

Fact 2.1.7. *The relation E_{K_σ} is Borel bireducible with E_{ℓ_∞}.*

For the simplicity we shall consider $E_{\ell_\infty} \restriction (\mathbb{R}^+)^\omega$ which is clearly bireducible with E_{ℓ_∞}: one direction is simple, for the other consider $f : \mathbb{R}^\omega \to (\mathbb{R}^+)^\omega$ such that if $x(n) \geq 0$ then $f(x)(2n) = x(n)$ and $f(x)(2n+1) = 0$ and if $x(n) < 0$ then $f(x)(2n) = 0$ and $f(x)(2n+1) = |x(n)|$.

Proof. To define a Borel function $f : (\mathbb{R}^+)^\omega \to 2^\omega$ witnessing $E_{\ell_\infty} \leq_{\mathrm{B}} E_{K_\sigma}$ we split ω into intervals $(I_k)_{k\geq 1}$ such that $|I_k| = 2k$,
$I_k = \{i_k^0, i_k^1, \ldots, i_k^{2k-1}\}$, for every $k \in \omega$.

Let $\pi : \omega \to (\omega \setminus \{\emptyset\})$ be some surjection such that the preimage $\pi^{(-1)}(k)$ is infinite for every $k \in \omega \setminus \{\emptyset\}$.

Let $x \in \mathbb{R}^{+\omega}$ and $n \in \omega$ be given. For $k < n$, $f(x)(i_n^k) = 1$ iff $x(\pi(n)) \geq k$. And for $n \leq k < 2n$ $f(x)(i_n^k) = 1$ iff $x(\pi(n)) < k$. So for every $x \in (\mathbb{R}^+)^\omega$, $\sharp\{i \in I_k : f(x)(i) = 1\}$ is always equal to k for every k.

It is clear that f is Borel and it is easy to check that $xE_{\ell_\infty}y \equiv f(x)E_{K_\sigma}f(x)$.

For the other direction, one can use the general Rosendal's result (1.1.22) that E_{ℓ_∞} is the universal K_σ equivalence relation. It can be directly shown as follows: for $x \in 2^\omega$ let $f(x)(n) = \sharp\{k \leq n : x(k) = 1\}$. This f is Borel and witnesses the reduction $E_{K_\sigma} \leq_{\mathrm{B}} E_{\ell_\infty}$. $\square$

Fact 2.1.8. *The relation E_{K_σ} is Borel bireducible with $E_{\mathcal{I}_W}$.*

Proof. Since we have $E_{\mathcal{I}_W} \leq E_{\ell_\infty}$ again by 1.1.22 and from the previous fact we have $E_{\ell_\infty} \leq_B E_{K_\sigma}$, it suffices to show that $E_{K_\sigma} \leq_B E_{\mathcal{I}_W}$. We define the Borel reduction $f : 2^\omega \to 2^\omega$ as follows: for any $x \in 2^\omega$ $f(x)(n) = 1 \equiv \exists k \in \omega(n \in [2^{k+1} - 2, 2^{k+1} - 2 + k] \wedge n - 2^{k+1} + 1 \leq \sharp\{m \leq k : x(m) = 1\})$. We leave the verification to the reader. We just note that $\exists n \geq 3 \forall m(|\sharp\{k \leq m : x(k) = 1\} - \sharp\{k \leq m : y(k) = 1\}| \leq n)$ iff $f(x) \triangle f(y)$ does not contain an arithmetic progression of length $\max\{n-1, 1\}$. $\square$

Zapletal conjectured that all analytic equivalence relations reducible to equivalence relations induced by an action of a Polish group should be canonized for the

Silver forcing (to the full relation or to a subset of E_0). It fits with the fact that equivalences given by analytic P-ideals are of this kind: by Solecki's results, every analytic P-ideal $\mathcal{I}$ is Polishable, i.e. there is a topology on $\mathcal{I}$ (which produces the same Borel sets as the Cantor topology restricted on $\mathcal{I}$) such that $(\mathcal{I}, \triangle)$ is a Polish group; and with the fact that E_{K_σ} is not an orbit equivalence (see [21] that E_1 is not Borel reducible to any orbit equivalence relation, however by Theorem 1.1.22 and Fact 2.1.7 we have that $E_1 \leq_B E_{K_\sigma}$) and is in the spectrum. On the other hand, Zapletal showed that the non-orbit equivalence relation E_1 is not in the spectrum (note that E_1 is defined by $\mathrm{Fin} \otimes \emptyset$ which is an F_σ non P-ideal). That follows either from results from [18] on hyper-smooth equivalences or there is an argument that uses a technique from the previous proof. Indeed, for $x, y \in (2^\omega)^\omega$ set $\Delta_n(x, y) = 1$ iff $\exists m > n(x(m) \neq y(m))$, apply the previous proof with this Δ and check that it works.

2.2 Subequivalences of E_0 on Silver forcing

Consider this basic subequivalence $E_0^{\text{even}}$of E_0 where x and y are equivalent if $\{n : x(n) \neq y(n)\}$ is of even finite cardinality. Obviously it is not equal to E_0 on any Silver subcondition. More generally, for any $n \in \omega$ let us denote $E_0^n \subseteq E_0$ the equivalence relation, where

$$xE_0^n y \equiv \exists m(\forall j > m(x(j) = y(j)) \wedge \exists k \in \mathbb{Z}(|\{i \leq m : x(i) = 1\}|$$

$$-|\{i \leq m : y(i) = 1\}| = k \cdot n))$$

Clearly, E_0, resp. E_0^{even} is equal to E_0^1, resp. E_0^2 in this notation and any E_0^n remains the same on any Silver subcondition (in the sense that it is defined there by the same formula, just quantifying over the set of holes instead of the whole ω).

These subequivalences have the property that they are homogeneous, where by homogeneity we mean the following.

Definition 2.2.1. The subequivalence $E \subseteq E_0$ is homogeneous if whenever $xEx \odot \{n_0, \dots, n_m\}$ then also $yEy \odot \{n_0, \dots, n_m\}$ provided that $y(n) = x(n)$

for $n \in \{n_0, \dots, n_m\}$.

It turns out that every homogeneous subequivalence of E_0 is in fact one of them.

Theorem 2.2.2. *Let B_g be a condition in Silver forcing and E a Borel equivalence relation on B_g that is either a homogeneous subequivalence of $E_0 \restriction B_g$ or $E \restriction B_g \supseteq E_0^0 \restriction B_g$. Then there is a subcondition B_h such that $E \restriction B_h$ is equal to either* $\mathrm{id} \restriction B_h$ *or $E_0^n \restriction B_h$ for some n.*

Proof. By $\bar{0}$ we denote the element $x \in 2^\omega$ such that for every $n \in \omega$ $x(n) = 0$.

We begin the proof just assuming that E is homogeneous. The case when $E \restriction B_g \supseteq E_0^0 \restriction B_g$ is treated in Step 3.

Step 1

We start by proving the following claim by which we decide whether E is on some subcondition equal to E_0 or to some proper subequivalence.

Claim 2.2.3. *There is a subcondition $B_f \subseteq B_g$ such that either $E \restriction B_f = E_0 \restriction B_f$ or $\forall x \in B_f \forall n \in H_f (x \not{E} x \odot n)$.*

Proof.

Case 1: There is an infinite subset $I \subseteq H_g$ such that for every $n \in I$ $\bar{0} E \bar{0} \odot n$. Then we fill the holes from $H_g \setminus I$ arbitrarily and obtain a subcondition $B_f \subseteq B_g$ such that $I = H_f$ and since E is homogeneous we have $\forall x \in B_f \forall n \in H_f (x E x \odot n)$. It follows from transitivity of E that $\forall x, y \in B_f (x E_0 y \Rightarrow x E y)$.

Case 2: The set I from Case 1 is finite. We fill these finitely many holes arbitrarily and obtain a subcondition $B_f \subseteq B_g$ such that $\forall x \in B_f \forall n \in H_f (x \not{E} x \odot n)$. Suppose it is not true. Then for some $x \in B_f$ and $n \in H_f$ we have $x E x \odot n$. However, since E is homogeneous we would have that also $\bar{0} E \bar{0} \odot n$ which is a contradiction. $\square$

Step 2

We now work only with the case that we have a condition B_f such that $\forall x \in B_f \forall n \in H_f (x \not{E} x \odot n)$.

Claim 2.2.4. *Either there is a subcondition of B_f on which E is a superset of E_0^0 or there is a subconidition on which E is the identity.*

Proof. For any $n \in \omega \setminus \{0\}$ let f_i^n, $i < 3^n - 1$, be an enumeration of all function from n to 3 such that for at least one $m < n$ $f_i^n(m) \neq 2$. Moreover, for any $n \in \omega \setminus \{0\}$ and for any $\bar{d} \in [H_f]^n$ we shall write $\bar{d} = \{d_0, \ldots, d_{n-1}\}$ where the elements of $\bar{d}$ are enumerated according to enumeration of ω. For any such $\bar{d} \in [H_f]^n$ let us denote $F_i(\bar{d})$, $i < 3^n - 1$, the set $\{j \in H_f \setminus \bar{d} : \bar{0} \odot \{d_m : f_i^n(m) = 1\} E \bar{0} \odot \{j\} \cup \{d_m : f_i^n(m) = 0\}\}$.

Case 1: Suppose that there exist $n \in \omega \setminus \{0\}$, $i < 3^n - 1$ and $\bar{d} \in [H_f]^n$ such that $F_i(\bar{d}) \cap H_f$ is infinite. Then WLOG we may assume that $H_f \subseteq F_i(\bar{d})$. Let us fill the hole d_j from $\bar{d}$ by 0 if $f_i^n(j) = 0$ or $f_i^n(j) = 2$ and by 1 if $f_i^n(j) = 1$, and denote B_h the obtained condition; i.e. $H_h = H_f \setminus \bar{d}$ and $\forall x \in B_h \forall j < n((f_i^n(j) \leq 1 \Rightarrow x(d_j) = f_i(j)) \wedge f_i^n(j) = 2 \Rightarrow x(d_j) = 0)$.

We claim that $E \restriction B_h \supseteq E_0^0 \restriction B_h$. Because of transitivity of E it suffices to check that for any $x \in B_h$ and any $n_1, n_2 \in H_h$ such that $x(n_1) = 0$ and $x(n_2) = 1$ we have $xEx \odot \{n_1, n_2\}$. However, since $n_1, n_2 \in F_i(\bar{d})$ and E is homogeneous we have $x \odot Ex \odot \{n_1\} \cup \{j : f_i^n(j) \leq 1\}$ and similarly $x \odot \{n_1, n_2\} Ex \odot \{n_1\} \cup \{j : f_i^n(j) \leq 1\}$. The claim then follows from transitivity.

Case 2: Suppose that for every $n \in \omega \setminus \{0\}$, $i < 3^n - 1$ and $\bar{d} \in [H_f]^n$ $F_i(\bar{d}) \cap H_f$ is finite.

By induction we construct a condition B_h on which E is the identity. Let us describe the step 1. Pick some $d_0 \in H_f$. Since neither $F_0(d_0) \cap H_f$ nor $F_1(d_0) \cap H_f$ is infinite, we can find a subcondition $B_{h_1} \subseteq B_f$ such that $H_{h_1} \cap F_0(d_0) = \emptyset$ and $H_{h_1} \cap F_1(d_0) = \emptyset$. It follows from homogeneity of E that $\forall x \in B_{h_1} \forall n \in H_{h_1}(x \not\!\!E x \odot \{d_0, n\})$. Since we are going to do a fusion the hole $d_0 \in H_{h_1}$ will be fixed, i.e. it will remain as a hole in all subsequent conditions.

We describe one more step. Pick next hole, different than d_0, $d_1 \in H_{h_1}$. Since for every $i < 3^2 - 1$ $F_i(\{d_0, d_1\}) \cap H_{h_1}$ is finite we can find a subcondition $B_{h_2} \subseteq B_{h_1}$ such that $H_{h_2} \cap F_i(\{d_0, d_1\}) = \emptyset$ for every $i < 3^2 - 1$ (and $\{d_0, d_1\} \subseteq H_{h_2}$, of course). It again follows from homogeneity of E that for every subset

$D \subseteq \{d_0, d_1\}$, every $n \in H_{h_2}$ and every $x \in B_{h_2}$ we have $x \not\!E x \odot D \cup \{n\}$.

Let us describe the general n-th step of the induction. We have condition $B_{h_{n-1}}$. The first n holes $d_0, \dots, d_{n-2}$ were already fixed. Pick next different hole $d_{n-1} \in H_{h_{n-1}}$. Since for every $i < 3^n - 1$ $F_i(\{d_0, \dots, d_{n-1}\}) \cap H_{h_{n-1}}$ is finite we can find a subcondition $B_{h_n} \subseteq B_{h_{n-1}}$ such that $H_{h_n} \cap F_i(\{d_0, \dots, d_{n-1}\}) = \emptyset$ for every $i < 3^n - 1$ (and again $\{d_0, \dots, d_{n-1}\} \subseteq H_{h_n}$, of course). It again follows from homogeneity of E that for every subset $D \subseteq \{d_0, \dots, d_{n-1}\}$, every $m \in H_{h_n}$ and every $x \in B_{h_n}$ we have $x \not\!E x \odot D \cup \{m\}$.

Once the fusion is finished we have a condition B_h with $H_h = \{d_0, d_1, \dots\}$ and we claim that $E \restriction B_h = \mathrm{id}(B_h)$. Let $x, y \in B_h$ be such that $x E_0 y$. Let $d_{n-1} \in H_h$ be the last hole where they differ. However, x and y then belong to B_{h_n} and in the n-th step of the induction we have guaranteed that $x \not\!E y$. □

Step 3

In the last step we treat the case when we have a condition B_h such that $E \restriction B_h \supseteq E_0^0 \restriction B_h$. This is one possible output of Step 2 or one possibility from the statement of Theorem 2.2.2.

For any pair $i, j \in H_p$, let $Z_{i,j}$ be the set $\{x \in B_p : x(i) = x(j) = 0, xE(x \odot \{i, j\})\}$. If for some pair this set is positive with respect to the Silver ideal, then we find a condition B_r on which E is equal to E_0^2 ($\forall x \in B_r(xEx \odot x\{i, j\})$ and since $E \restriction B_p \supseteq E_0^0 \restriction B_p$ it follows that for any pair). Otherwise, we subtract all $Z_{i,j}$ from B_p and find a condition which for notational simplicity again denote B_p. Next, for any triple $i, j, k \in H_p$, let $Z_{i,j,k}$ be the set $\{x \in B_p : x(i) = x(j) = x(k) = 0, xE(x \odot \{i, j, k\})\}$. Again, if one of these sets is positive, we find a condition B_r on which E is equal to E_0^3; otherwise, we subtract all these sets from the ideal. We continue similarly and find a subcondition such that E is equal to E_0^n on it for some $n \in \omega, n > 0$ or subtract all these countable sets from ideal and get a condition on which E is equal to E_0^0. Note that we used the fact that if n is the least number such that $E \supseteq E_0^n$, then if we found $m > n$ such that $E \supseteq E_0^m$ then E would contain E_0^k, where k is the greatest common divisor of m and n. That would be a contradiction. This finishes the proof of the proposition. □

Definition 2.2.5. Let $E_0^{0:n} \subseteq E_0^0$, $n \in \mathbb{N}$ and $n > 1$, be the equivalence relation where $xE_0^{0:n}y$ if xE_0^0y and $|\{k < m : x(k) = 1\}| = |\{k < m : y(k) = 1\}|$ is divisible by n, where m is the least number such that $\forall l \geq m(x(l) = y(l))$.

To check that it is an equivalence relation, let $xE_0^{0:n}y$ and $yE_0^{0:n}z$. Let m be the least number such that $\forall l \geq m(x(l) = y(l))$ and k the least number such that $\forall l \geq k(y(l) = z(l))$. Assume that $m \leq k$. Then this k works for the pair x, z and $|\{i < k : x(i) = 1\}| = |\{i < k : y(i) = 1\}| = |\{i < k : z(i) = 1\}|$ is divisible by n. We checked the transitivity, the symmetricity and reflexivity are obvious.

The definition is obviously made so that the equivalence is non-homogeneous as this basic example witnesses: let $x \neq y$ and assume that for some $n > 1$ $xE_0^{0:n}y$, then $(x \odot 0) \not E_0^{0:n} (y \odot 0)$ as $|\{k < m : (x \odot 0)(k) = 1\}| = |\{k < m : (y \odot 0)(k) = 1\}|$, where m is the least number such that $\forall l \geq m((x \odot 0)(l) = (y \odot 0)(l))$, now cannot be divisible by n.

Note that also the union $E_0^{0:n_0,\ldots,n_m} = E_0^{0:n_0} \cup \ldots \cup E_0^{0:n_m}$ is a non-homogeneous subequivalence of E_0^0. To check transitivity, just observe that if $xE_0^{0:n_i}y$ and $yE_0^{0:n_j}z$, $i, j \leq m$, m_0 is the least number such that $\forall l \geq m_0(x(l) = y(l))$ and similarly m_1 for the pair y, z, then if $m_1 \geq m_0$ we have that $xE_0^{0:n_j}z$ since m_1 is the least number such that $\forall l \geq m_1(x(l) = z(l))$ and $|\{k < m_1 : x(k) = 1\}| = |\{k < m_1 : z(k) = 1\}|$ is divisible by n_j.

The definition can be generalized so that there are non-homogeneous relations $E_0^{p:q}$, where p is divisible by q, and $xE_0^{p:q}y$ if xE_0^py and $|\{k < m : x(m) = 1\}|$ is divisible by q, where m is the least number such that $\forall l \geq m(x(l) = y(l))$.

It turns out that the class of non-homegeneous subequivalences of E_0 seems not to be easily classifiable. We call two E and F, essentially different if they remain different as subsets on every Silver condition, i.e. $E \restriction B_f \neq F \restriction B_f$ for every Silver condition B_f. We can show the following.

Theorem 2.2.6. *There are perfectly many essentially different non-homogeneous subequivalences of E_0.*

Proof. We will use the non-homogeneous equivalence relations defined above as a base for our construction. Moreover we define the relation $E_0^{-0:n}$ where $xE_0^{-0:n}y$ if xE_0^0y and $|\{k < m : x(k) = 0\}| = |\{k < m : y(k) = 0\}|$ is divisible by n, where m is the least number such that $\forall l \geq m(x(l) = y(l))$.

Let $\{p_1, p_2, \ldots\}$ be the set of all primes and let $z \in 2^{\mathbb{N}}$ be given. In the following, we assume that $0 \notin \mathbb{N}$. We define an equivalence relation F_z as follows:

$$xF_zy \equiv x = y \text{ or}$$

$$\exists n \in \mathbb{N}(z(n) = 1 \wedge xE_0^{0:p_{2n}}y \wedge \forall m \in [1, n)(z(m) = 0 \Rightarrow x\not{E}_0^{0:p_{2m-1}}y))$$

To check that it is an equivalence relation, first one can easily observe that it is reflexive and symmetric. For transitivity let $x_1F_zx_2$ and $x_2F_zx_3$, m_1 is the least number such that $\forall l \geq m_1(x_1(l) = x_2(l))$ and m_2 the least number such that $\forall l \geq m_2(x_2(l) = x_3(l))$. Moreover, let $v \in \mathbb{N}$ be the number such that $z(v) = 1$, $x_1E_0^{0:p_{2v}}x_2 \wedge \forall m \in [1, v)(z(m) = 0 \Rightarrow x_1\not{E}_0^{0:p_{2m-1}}x_2))$. Similarly $w \in \mathbb{N}$ the number such that $z(w) = 1$, $x_2E_0^{0:p_{2w}}x_3 \wedge \forall m \in [1, w)(z(m) = 0 \Rightarrow x_2\not{E}_0^{0:p_{2m-1}}x_3)$. We describe the case $m_1 \leq m_2$, the other case is symmetric. Obviously, m_2 is the least number such that $\forall l \geq m_2(x_1(l) = x_3(l))$. Since $x_1E_0^0x_3$, $|\{k < m_2 : x_1(k) = 0\}| = |\{k < m_2 : x_3(k) = 0\}|$ and $|\{k < m_2 : x_1(k) = 1\}| = |\{k < m_2 : x_3(k) = 1\}|$. Thus $x_1E_0^{0:p_{2w}}x_3$ and $\forall m \in [1, w)(z(m) = 0 \Rightarrow x_1\not{E}_0^{0:p_{2m-1}}x_3))$, so $x_1F_zx_3$.

Now let $z, z' \in 2^{\mathbb{N}}$ be different and B_g be a Silver condition. Let n be the least number such that $z(n) \neq z'(n)$, let us say $z(n) = 1$,$z'(n) = 0$. It suffices to find $x \in B_g$ and $h_0, h_1 \in H_g$ such that $xF_zx \odot \{h_0, h_1\}$ but $x\not{F}_{z'}x \odot \{h_0, h_1\}$.

It follows from the definition of the relations F_z and $F_{z'}$, that this will be done if we find $x \in B_g$ and $h_0, h_1 \in H_g$ with $x(h_0) = 0$, $x(h_1) = 1$ such that $|\{m \leq \max\{h_0, h_1\} : x(m) = 1\}|$ is divisible by p_{2n} but not divisible by p_{2k} $\forall k \in [1, n)$ for which $z(k) = 1$, and $|\{m \leq \max\{h_0, h_1\} : x(m) = 0\}|$ is divisible by p_{2n-1} but not divisible by p_{2k-1} $\forall k \in [1, n)$ for which $z(k) = 0$. To see this, notice that in that case it is fulfilled that $xE_0^{0:p_{2n}}x \odot \{h_0, h_1\}$ and $\forall k \in [1, n)(z(k) = 0 \Rightarrow x\not{E}_0^{0:p_{2k-1}}x \odot \{h_0, h_1\})$, thus $xF_zx \odot \{h_0, h_1\}$. On the contrary, suppose that also $xF_{z'}x \odot \{h_0, h_1\}$. Then there is m such that $z'(m) = 1$ and $xE_0^{0:p_{2m}}x \odot \{h_0, h_1\}$. It follows that $m > n$. However, if we put $k = n$, then we get $k < m$ such that $z'(k) = 0$ and $xE_0^{-0:p_{2k-1}}x \odot \{h_0, h_1\}$, thus $x\not{F}_{z'}x \odot \{h_0, h_1\}$.

Finding such x and holes h_0, h_1 is just elementary number theory. Let $p = (\prod_{i=1}^{2n} p_i) + 2$ and $d_1, \ldots, d_p$ first p holes in B_g, i.e. elements of H_g. We denote

$h_0 = d_1$ and $h_1 = d_p$ and let $x' \in B_g$ be an element such that $x'(h_1) = 1$ and $x'(h_0) = x(d_2) = \ldots = x'(d_{p-1}) = 0$. Let $a = |\{k \leq h_1 : x'(k) = 1\}|$. Chinese remainder theorem says that the following system of congruences has a solution $b \leq p$.

$$a + b \equiv 0 \pmod{p_{2n}}$$

$$h_1 - a - b \equiv 0 \pmod{p_{2n-1}}$$

for $k \in [1, n)$ such that $z(k) = 1$

$$a + b \equiv 1 \pmod{p_{2k}}$$

and for $k \in [1, n)$ such that $z(k) = 0$

$$h_1 - a - b \equiv 1 \pmod{p_{2k-1}}$$

We set $x = x' \odot \{d_2, d_3, \ldots, d_{b+1}\}$ and it follows that this x satisfies the required conditions. $\square$

Chapter 3

Laver forcing

Introduction

Let us recall that a Laver tree $T \subseteq \omega^{<\omega}$ is a tree with stem s, the maximal node such that every other node is compatible with it, such that every node above s (and including s) splits into infinitely many immediate successors. The set of all branches of T is denoted as $[T]$.

We can now state the main result of this chapter.

Theorem 3.0.7. *Let T be a Laver tree, $\mathcal{I}$ an F_σ P-ideal on ω and $E \subseteq [T] \times [T]$ be an equivalence relation Borel reducible to $E_{\mathcal{I}}$. Then there is a Laver subtree $S \leq T$ such that $E \restriction [S]$ is either* $\mathrm{id}([S])$ *or* $[S] \times [S]$.

We note that the subtree S in general cannot be found as a direct extension of T.

Recall (Fact 1.1.29) that the list of equivalence relations Borel bireducible with $E_{\mathcal{I}}$ for $\mathcal{I}$ an F_σ P-ideal includes for instance E_{ℓ_p} equivalences for $p \in [1, \infty)$ on $\mathbb{R}^\omega$; or $E_2 (= E_{\mathcal{I}_S})$.

Before proving the main theorem we state existing knowledge about the spectrum of Laver ideal and some results about Laver ideal that we will need in the proof of the theorem.

We add some notation concerning Laver trees and Laver ideal. We say that a Laver tree S is a direct extension of a Laver tree T, $S \leq_0 T$ in symbols, if the stem of S is the same as the stem of T. If $s \in T$ is a node above the stem then by T_s we denote the induced subtree with s as the stem, i.e. $T_s = \{t \in T : t \text{ is compatible with } s\}$.

We use the definition of Laver ideal I from [33, p.200]; $I \subseteq \mathcal{P}(\omega^\omega)$ is the σ-ideal generated by sets $A_g = \{f \in \omega^\omega : \exists^\infty n(f(n) \in g(f \restriction n))\}$, where g is a function from $\omega^{<\omega}$ to ω.

The following proposition, resp. its corollary, will be used extensively.

Proposition 3.0.8 ([33]; Proposition 4.5.14)**.** *Let $A \subseteq \omega^\omega$ be analytic. Then either A contains all branches of some Laver tree or $A \in I$.*

We will provide a proof of the following corollary. Recall that a barrier B in a Laver tree T is a subset of nodes such that $\forall x \in [T] \exists n(x \restriction n \in B)$.

Corollary 3.0.9. *Let T be a Laver tree and let $A \subseteq [T]$ be analytic. Then there exists a direct extension $S \leq_0 T$ such that either $[S] \subseteq A$ or $[S] \cap A = \emptyset$.*

Proof. It follows from the proposition above that there is always $S \leq T$ with that property which is in general not a direct extension though. The use of "direct extension property" will give us the desired tree. Let t be the stem of T. If there exist infinitely many immediate successors s of t such that there exists a direct extension $S \leq_0 T_s$ with the property above, then for infinitely many of them it holds that $[S] \subseteq A$, or for infinitely many of them it holds that $[S] \cap A = \emptyset$, and we use them. So suppose that not, we erase these finitely many exceptions and proceed to the next level and do the same. At the end we obtain a Laver tree $T' \leq_0 T$. We apply the proposition above and get a node $t \in T'$ and a direct extension $S \leq_0 T'_t$ such that either $[S] \subseteq A$ or $[S] \cap A = \emptyset$. That is a contradicition since such a node was erased during the construction of T'. $\square$

Recall Theorem 1.2.11 from the first chapter. As Laver ideal fulfils these conditions we immediately get the following corollaries.

Corollary 3.0.10 ([18])**.** *Let T be a Laver tree, E an equivalence classifiable by countable structures. Then there is a Laver subtree on which E is either the identity relation or the full relation.*

Since every countable equivalence relation is classifiable by countable structures (see Fact 1.1.38) we have the corollary that we shall use in this chapter.

Corollary 3.0.11 ([18]). *Let T be a Laver tree, E a countable equivalence relation (i.e. with countable classes). Then there is a Laver subtree on which E is either the identity relation or the full relation.*

J. Zapletal found the following F_σ equivalence relation (with K_σ classes) that is in the spectrum of Laver.

Definition 3.0.12. For $x, y \in \omega^\omega$, we set xKy if $\exists b \forall n \exists m_x, m_y \leq b(y(n+m_y) \geq x(n) \wedge x(n+m_x) \geq y(n))$.

The following lemma gives us basic properties of K. The proof may be found in [18], we provide here the proof of the last item as it is stated slightly differently in [18]. Notice the difference between E_{ℓ_p} for $p \in [1, \infty)$ and E_{ℓ_∞} as the former can be canonized according to the main theorem.

Lemma 3.0.13.

(a) For any two Laver trees T, S there are branches $x_1, x_2 \in [T]$ and $y_1, y_2 \in [S]$ such that $x_1 K y_1$ and $x_2 \not{K} y_2$.

(b) K is in the spectrum of Laver.

(c) K is Borel bireducible with $E_{\ell_\infty} \subseteq \mathbb{R}^\omega \times \mathbb{R}^\omega$, where $xE_{\ell_\infty}y \equiv x - y \in \ell_\infty$.

Proof.

(a)

(b) We refer to [18] for the proof of the first two items.

(c) • $E_{\ell_\infty} \leq_B K$: We will prove $E_{\ell_\infty} \leq_B E_{\ell_\infty} \upharpoonright (\mathbb{R}^+)^\omega \leq_B K$. To prove the first inequality, consider $f : \mathbb{R}^\omega \to (\mathbb{R}^+)^\omega$ such that if $x(n) \geq 0$ then $f(x)(2n) = x(n)$ and $f(x)(2n+1) = 0$ and if $x(n) < 0$ then $f(x)(2n) = 0$ and $f(x)(2n+1) = |x(n)|$.

For the second, let $\pi : \omega^2 \to \omega$ be a bijection and $(I_{\pi(i,j)})_{i,j}$ a partition of ω into intervals such that $|I_{\pi(i,j)}| = j+1$ and $I_{\pi(i,j)} = \{p_0^{i,j}, p_1^{i,j}, \ldots, p_j^{i,j}\}$.

We define $g : (\mathbb{R}^+)^\omega \to \omega^\omega$ as follows: Let $x \in (\mathbb{R}^+)^\omega$ be given, $g(x)(p_k^{i,j}) = \min\{j, \lfloor x(i) \rfloor + k\}$ for $k < j$ and $g(x)(p_j^{i,j}) = j$.

If $xE_{\ell_\infty}y$ and $\forall n(|x(n)-y(n)| \leq m)$, then $\forall k, ij \exists m_1, m_2 \leq m(g(x)(p_k^{i,j}) \leq g(y)(p_{k+m_1}^{i,j}) \wedge g(y)(p_k^{i,j}) \leq g(x)(p_{k+m_2}^{i,j}))$, thus $g(x)Kg(y)$. Just observe that $g(x)(p_k^{i,j}) \leq j$ and either $j-k \leq m$ and we have $g(y)(p_j^{i,j}) = j$, or since $x(i) - y(i) = m_1 \leq m$ we have $g(y)(p_{k+m_1}^{i,j}) = y(i) + k + m_1 = x(i) + k = g(x)(p_k^{i,j})$.

Suppose $x \not{E}_{\ell_\infty} y$, let m be arbitrary and let n be such that $|x(n)-y(n)| > m$, let us assume that $y(n) - x(n) > m$. Then $\forall b \leq m(g(x)(p_{k+b}^{n,m}) < g(y)(p_k^{n,m}))$. Since m was arbitrary we have $g(x) \not{K} g(y)$.

- $K \leq_B E_{\ell_\infty}$: Let $(s_n)_n$ be an enumeration of $\omega^{<\omega}$. We define $f : \omega^\omega \to \mathbb{R}^\omega$ as follows: $f(x)(n) =$
  $\min\{b : \exists y \supseteq s_n(xKy \wedge b \text{ is the bound from the definition that works})\}$. One can easily check that f is Borel. Let xKy such that a bound b works for this pair and let n be arbitrary. Let $z \supseteq s_n$ be arbitrary such that xKz and b_1 works for the pair and yKz and b_2 works for the pair. Then one can check that $|b_1 - b_2| \leq b$ so $f(x)E_{\ell_\infty}f(y)$.

  Suppose that $x \not{K} y$ and let m be arbitrary. Then there exists n such that $x(n) > y(n+k)$ for $k < m$ (or vice versa). Let $s_i = x \upharpoonright (n+1)$, then $f(x)(n) = 0$, however $f(y)(n) \geq m$, thus $f(x) \not{E}_{\ell_\infty} f(y)$.

$\square$

3.1 Proof of the theorem

We can now start proving the main theorem, we provide its statement here again for the convenience.

Theorem 3.1.1. *Let T be a Laver tree, $\mathcal{I}$ an F_σ P-ideal on ω and $E \subseteq [T] \times [T]$ be an equivalence relation Borel reducible to $E_\mathcal{I}$. Then there is a Laver subtree $S \leq T$ such that $E \upharpoonright [S]$ is either* $\mathrm{id}([S])$ *or* $[S] \times [S]$.

Proof. Let $f : [T] \to 2^\omega$ be the Borel reduction and let μ be the lower semicontinuous submeasure for $\mathcal{I}$ guaranteed by Theorem 1.1.33. The submeasure

μ induces a pseudometrics (which may attain infinite value though) which we denote d, i.e. $d(x,y) = \mu(x \triangle y)$ for $x, y \in 2^\omega$. Moreover, we define $d_n^k(x,y)$ as $\mu(x \restriction (n,k) \triangle y \restriction (n,k))$. When n or k is omitted it means that $n = 0$, resp. $k = \infty$.

We need to refine T to obtain a Laver tree with some special properties. This will be done in a series of claims. To simplify the notation, after applying each one of these claims we will still denote the tree as T.

Claim 3.1.2. *There exist a direct extension $T' \leq_0 T$ and a function $p : T' \to 2^{<\omega}$ which is monotone and preserves length of sequences, i.e. if $s \subseteq t$, then $p(s) \subseteq p(t)$, and $|s| = |p(s)|$, such that $\forall x \in [T'](f(x) = \bigcup_n p(x \restriction n))$.*

In other words, f on $[T']$ is Lipschitz.

Proof of the Claim. We will find a direct extension of T and p defined on it from the statement of the claim. For simplicity we assume the stem of T is the empty sequence.

Consider the following sets

$$A_i = \{x \in [T] : f(x)(0) = i\}$$

for $i \in \{0,1\}$. They are Borel and according to Corollary 3.0.9 one of them contains a direct extension S of T. We replace T by S, set $p(\emptyset) = i$ and fix the first level above the stem. Then for any immediate successor s of the stem we again consider sets $A_i^1 = \{x \in [S_s] : f(x)(1) = i\}$. One of them contains direct extension and we continue similarly. The final tree is obtained by fusion. □

Observation 3.1.3. Let $s \in T$ be a node above (or equal to) the stem of T. Then for every n there is a direct extension $T_s^n \leq_0 T_s$ such that $\forall x, y \in [T_s^n] \forall m \leq n(p(x \restriction m) = p(y \restriction m))$. We will call such a tree homogeneous up to level n.

We may also suppose that we have $T_s^n \subseteq T_s^m$ for $n \geq m$. Define then $x_s \in 2^\omega$ such that $x_s(n) = p(x \restriction n+1)(n)$ for $x \in [T_s^m]$, where $m \geq n+1$. This definition does not depend on $m \geq n+1$ and $x \in [T_s^m]$.

The following can be done by a basic fusion argument.

Fact 3.1.4. *There exists a direct extension $T' \leq_0 T$ such that $\forall s \in T'$ above the stem if $S \leq_0 T'_s$ is homogeneous up to some level n then $\forall x \in [S](p(x \restriction n) = x_s \restriction n)$.*

Moreover, if $s \in S$ and $\{s_0, s_1, \ldots\}$ is a set of its immediate successors then $\forall n \exists m_0 \forall m \geq m_0 (x_s \restriction n = x_{s_m} \restriction n)$. In other words, $\lim_{n\to\infty} x_{s_n} \to x_s$.

Let $s \in T$ be any node above (or equal to) the stem of T. Let $\{s_0, s_1, \ldots\}$ be the set of its immediate successors. We reduce this set so that precisely one of the following two possibilities happens: $\forall n (x_{s_n} E_{\mathcal{I}} x_s)$ or $\forall n (x_{s_n} \not{E}_{\mathcal{I}} x_s)$.

Definition 3.1.5. If the former case holds then we mark s as "convergent", if the latter then we mark it as "divergent".

Moreover, for every $s \in S$ strictly above the stem we define ε_s as follows: if the immediate predecessor t of s is marked as convergent, then we set $\varepsilon_s = d(x_t, x_s)$; otherwise, we set $\varepsilon_s = \infty$.

Splitting into cases

We split into two complementary cases (i.e. one holds if and only if the other does not).

- **Case 1** There exists $S \leq T$ such that every $s \in S$ above the stem is marked as convergent.

- **Case 2** For every $s \in T$ above the stem there is a barrier $B \subseteq T_s$ of elements above s that were marked as divergent.

Proof of canonization assuming Case 1. We will do a fusion. Let us denote the stem of S as s. We will inductively build U_n, S_n, m_n for every n such that $S_n \leq_0 S_{n-1}$, $U_n \subseteq S_m$, for every $n \leq m$, is an $n+1$-element subtree $\{u_0, \ldots, u_n\}$ of S and $m_n \in \omega$. At the end we will get a direct extension $U = \bigcup_n U_n = \bigcap_n S_n$ together with pairwise disjoint sets $C_{u_1}, C_{u_2}, \ldots$, where $C_{u_i} \subseteq (m_{i-1}, m_i)$, such that $\forall x \in [U]$ $(f(x) \triangle x_s) \cap (m_{i-1}, m_i) = C_{u_i}$ if $u_i \subseteq x$ and $\mu(\bigcup_{\{i>0: u_i \not\subseteq x\}} f(x) \cap (m_{i_1}, m_i)) < 1$. The following conditions will be satisfied during the n-th step of the fusion.

- For every $0 < i \leq n$ and any branch $x \in [S_n]$ going through u_i $|d_{m_{i-1}}^{m_i}(f(x), x_s) - \varepsilon_{u_i}| < 1/2^i$; more precisely there will be some finite set $C_{u_i} \subseteq (m_{i-1}, m_i)$ always defined as $(x_{u_i} \triangle x_s) \cap (m_{i-1}, m_i)$ such that for any branch $x \in [S_n]$ going through u_i we will have $(f(x) \triangle x_s) \cap (m_{i-1}, m_i) = C_{u_i}$ and $|\mu(C_{u_i}) - \varepsilon_{u_i}| < 1/2^i$. And for every branch $y \in [U_n]$ not going through u_i but going through some other u_j, $d_{m_{i-1}}^{m_i}(f(y), x_s) < 1/2^i$; thus it will follow from the triangle inequality that $|d_{m_{i-1}}^{m_i}(f(x), f(y)) - \varepsilon_{u_i}| < 1/2^{i-1}$; resp. $\mu((f(x) \triangle f(y) \triangle C_{u_i}) \cap (m_{i-1}, m_i)) < 1/2^i$.

- For every $i \leq n$ $d_{m_n}(x_{u_i}, x_s) < 1/2^{n+2}$.

Suppose at first that such U has been already constructed. Let us consider the set

$$A = \{x \in [U] : \mu(\bigcup_{i=|s|+1}^{\infty} C_{x \restriction i}) < \infty\}$$

It is Borel and by Corollary 3.0.9 either there is a Laver subtree $V \leq_0 U$ such that $[V] \subseteq A$ or there is a Laver subtree $V \leq_0 U$ such that $[V] \cap A = \emptyset$. In the former case, V is a Laver subtree such that $\forall x, y \in [V](xEy)$; while in the latter case, V is a Laver subtree such that $\forall x, y \in [V](x \not{E} y)$. This follows immediately from the condition above. Let $x, y \in [V]$ be two different branches splitting on the n-th level. Then $\max\{\mu(\bigcup_{i=n}^{\infty} C_{x\restriction i}), \mu(\bigcup_{i=n}^{\infty} C_{y\restriction i})\} - \sum_{j=n-|s|+1}^{\infty} 1/2^j \leq d(f(x), f(y)) \leq \mu(\bigcup_{i=n}^{\infty} C_{x\restriction i}) + \mu(\bigcup_{i=n}^{\infty} C_{y\restriction i}) + \sum_{j=n-|s|+1}^{\infty} 1/2^{j-1}$.

Let s be the stem of S. Set $S_0 = S$, $U_0 = \{s\}$, $m_0 = |s|$. Before treating the general step let us describe the case $n = 1$. We pick some immediate successor of the stem s, denote it as u_1 and we set $U_1 = \{s = u_0, u_1\}$. Since $d(x_{u_1}, x_s) = \varepsilon_{u_1}$ there is some $m > m_0$ such that $d^m(x_{u_1}, x_s) > \varepsilon_{u_1} - 1/2$. There is some $m_1 \geq m$ such that $d_{m_1}(x_{u_1}, x_s) < 1/2^3$. Then there exist direct extensions $E_1 \leq_0 S_{0u_1}$ and $E_0 \leq_0 S_0$ such that for all branches $x \in [E_1]$ we have $f(x)(m) = x_{u_1}(m)$ for $m \leq m_1$, and for all branches $y \in [E_0]$ we have $f(y)(m) = x_s(m)$ for $m \leq m_1$. We set $S_1 = E_0 \cup E_1$, i.e. we replace S_{0u_1} in S_0 by its direct extension E_1 and we replace $S_0 \setminus S_{0u_1}$ by its direct extension E_0. The required conditions are satisfied and we proceed to a general step.

Now let us suppose that we have already found S_{n-1},$U_{n-1} = \{s = u_0, u_1, \ldots, u_{n-1}\}$

and $m_0 = |s|, m_1, \ldots, m_{n-1}$. Choose some next node $u_n \in S_{n-1}$ for the fusion such that it is an immediate successor of some u_i $i < n$ and $x_{u_i} \restriction m_{n-1} = x_{u_n} \restriction m_{n-1}$ (recall Observation 3.1.3). Set $U_n = U_{n-1} \cup \{u_n\}$. There is some m such that $d^m_{m_{n-1}}(x_{u_n}, x_{u_i}) > \varepsilon_{u_m} - 1/2^{n+1}$. Since we have from the inductive assumption that $d^m_{m_{n-1}}(x_{u_i}, x_s) < 1/2^{n+1}$, we get from the triangle inequality $d^m_{m_{n-1}}(x_{u_n}, x_s) > \varepsilon_{u_m} - 1/2^n$. Let m_n be the $\max\{m, \max\{k_i : i \leq n\}\}$, where k_i is any number such that $d_{k_i}(x_{u_i}, x_s) < 1/2^{n+2}$. Note that such k_i exist because $x_{u_i} E_{\mathcal{I}} x_s$. We then find direct extensions $E_i \leq_0 S_{n-1,u_i}$ such that for every branch $x \in [E_i]$ we have $f(x)(m) = x_{u_i}(m)$ for $m \leq m_n$. We refine them so that they are mutually disjoint, i.e. $E_i \cap E_j = \emptyset$ for $i \neq j$ and we set $S_n = \bigcup_{i \leq n} E_i$. The induction step is done, all required conditions are satisfied. That finishes the proof of this case.

Proof of canonization assuming Case 2. We will assume that we have a Laver tree $S \leq T$ such that for every $s \in S$ above the stem if s is marked as convergent then there is a barrier $B \subseteq S_s$ of elements above s that are marked as divergent (if we assume that Case 1 does not hold then we may take $S = T$).

The following lemma will be the main tool.

Lemma 3.1.6. *For any Laver subtree $P \leq S$ there is its direct extension $Q \leq_0 P$ such that for any two branches $x, y \in [Q]$ splitting from the stem of Q we have $d(f(x), f(y)) > 1$.*

Once the lemma is proved the rest will be rather easy. We will do a fusion in which we will be fixing levels. We will construct direct extensions of S $S = V_0 \geq_0 V_1 \geq_0 V_2 \geq_0 \ldots$ such that for $i < j$ the i-th level of V_i is equal to the i-th level of V_j in such a way that the resulting tree $S \geq_0 V = \bigcap_i V_i$ will have the property that for any two different branches $x, y \in [V]$ we will have $d(f(x), f(y)) > 1$.

This is not hard to do. We start with stem s of $S = V_0$. We find a direct extension $V_1 \leq_0 V_0$ guaranteed by the lemma. We fix the first level $\{s_0, s_1, \ldots\}$ (the set of all immediate successors of s) above the stem. Then for every immediate successor $s_i \in V_1$ of s we apply the lemma with V_{1s_i} as P and obtain a direct extension Q_i. We set $V_2 = \bigcup_i Q_i \leq_0 V_1$, fix the second level above the stem and continue similarly.

Then we are done by the following claim and Corollary 3.0.11.

Claim 3.1.7. *E on $[V]$ is countable.*

Proof. Suppose for contradiction that there is some $x \in [V]$ that has uncountably many equivalent branches $(y_\alpha)_{\alpha<\omega_1} \subseteq [V]$. For every α there is n such that $d_n(f(x), f(y_\alpha)) < 1/2$. Since the set of all y_α's is uncountable we may assume that one single n works for them all. But let $y_{\alpha_0}, y_{\alpha_1}$ be two of such branches that split above the n-th coordinate. It follows that from our construction that $d_n(f(y_{\alpha_0}), f(y_{\alpha_1})) > 1$, so for one of them, let us say y_{α_0}, must hold that $d(f(x), f(y_{\alpha_0})) > 1/2$, a contradiction. □

So what remains is to prove the lemma.

Proof of the lemma. Let $P \leq S$ be given. Denote s its stem. There are two cases.

- s is marked as divergent: Pick its immediate successor s_0. Since s is marked as divergent, there is n_0 such that $d^{n_0}(x_{s_0}, x_s) > 1$ and there are direct extensions $Q_0 \leq_0 P_{s_0}$, $P_0 \leq_0 P \setminus P_{s_0}$ such that $\forall x \in [Q_0] \forall y \in [P_0] \forall m \leq n_0 (f(x)(m) = x_{s_0}(m) \wedge f(y)(m) = x_s(m))$.

  We then pick next immediate successor $s_1 \in P_0$ of s. There is again some n_1 such that $d^{n_1}(x_{s_1}, x_s) > 1$ and we find direct extensions $Q_1 \leq_0 P_{0s_1}$, $P_1 \leq_0 P_0 \setminus P_{0s_1}$ such that $\forall x \in [Q_1] \forall y \in [P_1] \forall m \leq n_1 (f(x)(m) = x_{s_0}(m) \wedge f(y)(m) = x_s(m))$.

  We continue similarly until we pick infinitely many immediate successors of s and find corresponding direct extensions Q_i. Then we set $Q = \bigcup_i Q_i$. It is easy to check it has the required properties.

- s is marked as convergent: There is a barrier $B \subseteq P$ of elements that were marked as divergent. We may assume that for every $b \in B$ and every $s \leq t < b$, t is marked as convergent. We will do a similar fusion to that in the proof of canonization assuming Case 1. We will inductively build Q_n, P_n, m_n such that $P_n \leq_0 P_{n-1}$, $Q_n = (\{q_0 = s, \ldots, q_n\} \cup R) \subseteq P_m$ for $n \leq m$, $m_n \in \omega$. Let $\{q_i : i \in C\} \subseteq \{q_0, \ldots, q_n\}$ be the (possibly empty) set of those elements that are immediate successors of some element from B. Then $R = \bigcup_{i \in C} P_{i,q_i}$. The final tree is again obtained as $Q = \bigcup_i Q_i = \bigcap_i P_i$. Conditions that must be satisfied during the n-th step of the fusion are the following.

- For every $i < n$ if $i \notin C$, i.e. q_i is not an immediate successor of an element from B, then for any branch $x \in [P_n]$ going through q_i we have $d^{m_n}_{m_{n-1}}(f(x), x_s) < 1/2^n$. And if $n \in C$, i.e. q_n is an immediate successor of an element from B, then for any branch $y \in [P_n]$ going through q_n we have $d^{m_n}_{m_{n-1}}(f(y), x_s) > 2$; thus it will follow from the triangle inequality that $d^{m_n}_{m_{n-1}}(f(x), f(y)) > 1$.
- For every $i \leq n$ if $i \notin C$, i.e. q_i is not an immediate successor of an element from B, then $d_{m_n}(x_{q_i}, x_s) < 1/2^{n+2}$.

Suppose at first that such Q has been constructed. We need to prove that for any two branches $x, y \in [Q]$ with s as the last common node we have $d(f(x), f(y)) > 1$. It follows from the assumption that x goes through some u_i which is an immediate successor of some element from B, similarly y goes through some different u_j with the same property. Assume $i < j$. Then we get from the inductive assumption that $d^{m_i}_{m_{i-1}}(f(x), f(y)) > 1$ and we are done.

In the first step of the induction we set $Q_0 = \{s\}$, $P_0 = P$ and $m_0 = |s|$; the set R is empty.

Suppose we have already found $Q_{n-1}, P_{n-1}, m_{n-1}$. We choose some q_n that is an immediate successor of some q_i. We have two cases.

- $q_i \notin B$, i.e. q_n is not an immediate successor of an element from B. Then we set $m_n = \max\{k_i : i \leq n, i \notin C\}$, where k_i, for $i \notin C$, is any number such that $d_{k_i}(x_{q_i}, x_s) < 1/2^{n+2}$. Note that such k_i exist because $x_{u_i} E_{\mathcal{I}} x_s$. We then find direct extensions $E_i \leq_0 P_{n-1,q_i}$ for $i \leq n, i \notin C$ such that for every branch $x \in [E_i]$ we have $f(x)(m) = x_{q_i}(m)$ for $m \leq m_n$. We refine them so that they are mutually disjoint, i.e. $E_i \cap E_j = \emptyset$ for $i \neq j$ and we set $P_n = (\bigcup_{i \notin C} E_i) \cup R$.

– $q_i \in B$, i.e. q_n is an immediate successor of an element from B. We add n to C. There is some m such that $d^m_{m_{n-1}}(x_{q_n}, x_{q_i}) > 2 + 1/2^{n+1}$ since $q_i \in B$ is marked as divergent. Since from the inductive assumption we have $d^m_{m_{n-1}}(x_{q_i}, x_s) < 1/2^{n+1}$ we get from the triangle inequality that $d^m_{m_{n-1}}(x_{q_n}, x_s) > 2$. We then set $m_n = \max\{m, \max\{k_i : i \leq n, i \notin C\}\}$, where k_i's are defined exactly the same as in the first case. We then again find direct extensions $E_i \leq_0 P_{n-1,q_i}$ for $i = n$ and $i < n, i \notin C$ such that for every branch $x \in [E_i]$ we have $f(x)(m) = x_{q_i}(m)$ for $m \leq m_n$. We refine them so that they are mutually disjoint, i.e. $E_i \cap E_j = \emptyset$ for $i \neq j$. We add E_n to R and we set $P_n = (\bigcup_{i \notin C} E_i) \cup R$.

In both cases it is easy to check that all required conditions are satisifed.

□

3.2 Corollaries

Theorem 3.2.1. *Let $E \subseteq \omega^\omega \times \omega^\omega$ be an equivalence relation containing K, i.e. $E \supseteq K$, which is Borel reducible to $E_{\mathcal{I}}$ for some F_σ P-ideal. Then there exists a Laver large set contained in one equivalence class.*

Recall that K was defined in Definition 3.0.12.

Proof. Consider the set

$$X = \{x \in \omega^\omega : [x]_E \text{ contains all branches of some Laver tree}\}$$

We use Theorem 3.1.1 to prove that X is non-empty. Suppose it is empty, then by Theorem 3.1.1 there exists a Laver tree T such that $E \restriction [T] = \mathrm{id}([T])$. However, there must be two branches $x, y \in [T]$ such that xKy and since $K \subseteq E$, also xEy, a contradiction.

Thus X is non-empty. We show that it is also E-equivalent, i.e. there is no pair $x, y \in X$ such that $x \not{E} y$. Suppose the contrary. Then $[x]_E$ contains all branches of some Laver tree T_x and $[y]_E$ contains all branches of Laver tree T_y and there are branches $b_x \in T_x$ and $b_y \in T_y$ such that $b_x K b_y$ and since $K \subseteq E$, also $b_x E b_y$, a contradiction.

So X is a single equivalence class, containing all branches of some Laver tree T, and thus it is Borel. If it were not Laver large, then the complement would be a Borel Laver positive set, so by Proposition 3.0.8 it would contain all branches of some Laver tree S. But we would again have that there are a branch $x \in [T] \subseteq X$ and a branch $y \in [S]$ such that xKy, thus xEy, a contradiction. $\square$

Theorem 3.2.2 (Silver dichotomy - under "$\forall x \in \mathbb{R}(\omega_1^{L[x]} < \omega_1)$"). *Let $E \subseteq \omega^\omega \times \omega^\omega$ be an equivalence relation Borel reducible to $E_{\mathcal{I}}$ for F_σ P-ideal $\mathcal{I}$. Then either $\omega^\omega = (\bigcup_{n\in\omega} E_n) \cup J$, where E_n for every n is an equivalence class of E and J is a set in the Laver ideal, or there exists a Laver tree T such that $E \restriction [T] = \mathrm{id}([T])$.*

This is just a combination of Theorem 3.1.1 and the results from the section on Silver dichotomy from [18]. It is not known if the assumption "$\forall x \in \mathbb{R}(\omega_1^{L[x]} < \omega_1)$" is necessary.

Corollary 3.2.3 (under the same assumption). *Let $E \subseteq \omega^\omega \times \omega^\omega$ be an equivalence relation Borel reducible to $E_{\mathcal{I}}$ for F_σ P-ideal $\mathcal{I}$ and let $X \subseteq \omega^\omega$ be an arbitrary Laver-positive subset (not necessarily definable) such that $\forall x, y \in X(x \not\!E y)$. Then there exists a Laver tree T such that $E \restriction [T] = \mathrm{id}([T])$.*

Proof. Just use the Silver dichotomy from the previous theorem and notice that the first possibility cannot happen. If $\omega^\omega = (\bigcup_{n\in\omega} E_n) \cup J$ as in the statement of the previous theorem, then $X \setminus J$ is still not in the Laver ideal and is uncountable. $\square$

Chapter 4

Carlson-Simpson forcing

Introduction

In [1], Timothy J. Carlson and Stephen G. Simpson prove a strong combinatorial theorem concerning finite partitions of natural numbers that is in some sense dual to the classical Ramsey theorem. It is usually called the Dual Ramsey theorem or the Carlson-Simpson theorem. In this chapter we define a forcing notion, resp. a σ-ideal on a certain Polish space, that corresponds to the object studied in the Dual Ramsey theorem and prove a canonization result for this σ-ideal. More specifically, we identify a finite set of equivalence relations that are in the spectrum of this ideal and any other analytic equivalence relation canonizes to one of them.

Let us state one immediate interesting consequence of the result from this chapter.

Theorem 4.0.4. *Let E be any analytic equivalence relation on $\mathcal{P}(\omega)$ (we identify elements of $\mathcal{P}(\omega)$ with elements of 2^ω). Then there exists an infinite sequence $(A_n)_{n\in\omega}$ of pairwise disjoint non-empty subsets of ω (finite or infinite) such that either for any two different arbitrary unions of such sets (both containing A_0 though) they are E-equivalent, or for any two different arbitrary unions both containing A_0 they are E-inequivalent.*

Now we introduce the original notation of Carlson and Simpson from [1] and state their theorem. Then we define the forcing notion and our theorem in their language in order to motivate it. However, we will slightly change the notation in the proof.

Definition 4.0.5. Let A be a finite (at least two-element) alphabet. As in [1], by $(\omega)^\alpha_A$, where $\alpha \in (\omega \setminus |A|) \cup \{\omega\}$, we denote the set of all partitions of $A \cup \omega$ into α pieces such that two different elements $a \neq b \in A$ lie in two different pieces of such partitions. For any $X \in (\omega)^\alpha_A$, a piece containing some $a \in A$ is called an a-block, a piece not containing any element of A is called a free block.

For $Y \in (\omega)^\beta_A$ and $X \in (\omega)^\alpha_A$, where $\beta \leq \alpha$, we say Y is coarser than X, $Y \preceq X$, if every block of X is contained in some block of Y. For any $X \in (\omega)^\alpha_A$ by $(X)^\beta_A$, where $\beta \leq \alpha$, we denote the set $\{Y \in (\omega)^\beta_A : Y \preceq X\}$.

Definition 4.0.6 (Space $(\omega)^0_A$)**.** Consider the set $(\omega)^0_A$. We look at it as a set of all partitions of ω into $|A|$ pieces indexed by A. There is a natural correspondence between $(\omega)^0_A$ and A^ω. The latter carries a product topology if we consider A as a discrete space which is homeomorphic to the topology of the Cantor space. From now on we will not distinguish between these two sets and thus be able to speak about topological properties of $(\omega)^0_A$.

Definition 4.0.7 (Carlson-Simpson forcing/ideal)**.** We shall consider $((\omega)^\omega_A, \preceq)$ as a forcing notion. For $X \in (\omega)^\omega_A$ we shall write $[X]$ to denote the set $(X)^0_A$. Note that for any such X, $[X]$ is a closed subset of $(\omega)^0_A$ (or A^ω).

Let $I_{C_n} \subseteq \mathcal{P}(A^\omega)$, where n denotes the cardinality of A, be the set of all Borel subsets of A^ω that do not contain $[X]$ for some $X \in (\omega)^\omega_A$.

The following proposition gives some properties of I_{C_n}.

Proposition 4.0.8.

1. *I_{C_n} is a σ-ideal.*

2. *$P_{I_{C_n}}$ is forcing equivalent to $((\omega)^\omega_A, \preceq)$.*

3. *$P_{I_{C_n}}$ is proper.*

We postpone the proof until we have proved the main theorem 4.1.1. The reason for that is that the first item of Proposition 4.0.8 will follow easily. We do not need any part of the proposition in the proof of the main theorem. However, let us mention that all items of Proposition 4.0.8 could be proved by a direct argument without applying the main theorem.

Let us state a restricted version of the Carlson-Simpson (Dual Ramsey) theorem for partitions without free blocks.

Theorem 4.0.9 (Carlson-Simpson [1])**.** *For any $X \in (\omega)^\omega_A$ and any finite partition $[X] = C_0 \cup \ldots \cup C_n$ into pieces having the Baire property there exists $Y \in (X)^\omega_A$ and $i \leq n$ such that $[Y] \subseteq C_i$.*

4.1 Canonization

Let A be a finite alphabet such that $|A| = n \geq 3$. Let $B \subseteq A$ be a proper subset of A such that $|B| \geq 2$. Then we can consider the following equivalence relation E_B on A^ω: for $x, y \in A^\omega$ we set xE_By iff $\forall n \in \omega((x(n) \in B \Leftrightarrow y(n) \in B) \wedge (x(n) \notin B \Rightarrow y(n) = x(n)))$.

It is easy to check that E_B is a closed equivalence relation that is in the spectrum of I_{C_n}. For a finite alphabet A let $\mathcal{B}_A = \{B_i : i < 2^n - n - 2\}$ denote the set of all proper subsets of A of cardinality at least 2.

Theorem 4.1.1. *Fix some finite alphabet A with at least two elements. Let $X \in (\omega)^\omega_A$ be a condition in the Carlson-Simpson forcing and E an analytic equivalence relation on $[X]$ (i.e. an analytic subset of $[X]^2$). Then there exists a subcondition $Y \in (X)^\omega_A$ such that $E \upharpoonright [Y]$ is equal to $[Y] \times [Y]$ or to $\mathrm{id}([Y])$ or there exists $B \in \mathcal{B}_A$ such that $E \upharpoonright [Y] = E_B \upharpoonright [Y]$.*

In particular, we have a total canonization for I_{C_2}.

Remark 4.1.2. This is an "almost generalization" of Theorem 4.0.9 as this theorem can be viewed as a canonization result for equivalence relations having finitely many classes. We used the term "almost generalization" as the Theorem 4.0.9 holds for partitions into pieces having the Baire property whereas Theorem 4.1.1 generalizes only the case with analytic partitions.

As mentioned in Introduction, we prove the theorem for a two element alphabet in full detail. Then we sketch how to obtain it for general A. The more detailed proof of the general case is in the article [5] which is in preparation.

As announced, we will slightly change the notation unfortunately. From now on we consider a two element alphabet A. We consider a forcing notion which is equivalent to $((\omega)^\omega_A, \preceq)$ for two element A but will be notationally easier to deal with in our proof. Few definitions follow.

Definition 4.1.3 (Forcing notion $\mathbb{CS}$)**.** A condition p in the Carlson-Simpson forcing $\mathbb{CS}$ is a pair (a_p, b_p) where $a_p \subseteq \omega$ is a coinfinite subset of ω and $b_p : \omega \to \mathcal{P}(\omega)$ is an infinite partition of an infinite set B_p which is disjoint with a_p; i.e. $\bigcup_{i\in\omega} b_p(i) = B_p$ and $a_p \cap B_p = \emptyset$. We will moreover assume that if $i < j$ then $\min\{b_p(i)\} < \min\{b_p(j)\}$.

We define $p \leq q$ if $a_p \supseteq a_q \wedge a_p \setminus a_q = \bigcup_{i\in C} b_q(i)$, where $C \subseteq \omega$ is an arbitrary coinfinite set, $\forall i \exists D \in [\omega]^{\leq\omega}(b_p(i) = \bigcup_{j\in D} b_q(j))$.

By $[p]$ we will denote the set of all subsets of ω that can be obtained from the condition p; i.e. $[p] = \{x \subseteq \omega : x \supseteq a_p \wedge \exists C \subseteq \omega(x \setminus a_p = \bigcup_{i\in C} b_p(i))\}$.

Remark 4.1.4. Let us describe the correspondence between $\mathbb{CS}$ and $((\omega)^\omega_A, \preceq)$. Any $p \in \mathbb{CS}$ can be viewed as an infinite partition where the free blocks are $b_p(i)$, for $i \in \omega$, and the two non-free blocks are a_p and $\omega \setminus (a_p \cup \bigcup_{i\in\omega} b_p(i))$. On the other hand, any $X \in (\omega)^\omega_A$ (let us say that $A = \{a, b\}$) can be viewed as $p_X \in \mathbb{CS}$ such that a_{p_X} is the non-free block containing b and $b_{p_X}(i)$, for $i \in \omega$, are free blocks of X ordered by their minimal elements.

Definition 4.1.5. Let $q \in \mathbb{CS}$ be a condition and $s \in 2^{<\omega}$ a finite binary sequence. By q^s we denote the condition for which $a_{q^s} = a_q \cup \{b_q(i) : s(i) = 1\}$ and $B_{q^s} = B_q \setminus \bigcup_{i<|s|} b_q(i) \wedge \forall i(b_{q^s}(i) = b_q(i + |s|))$.

Note that whenever for some q and s there is $r \leq q^s$, then there is in fact a condition t such that $t \leq q$ and $t^s = r$; i.e. $a_t = a_r \setminus \bigcup_{i<|s|} b_q(i)$, $b_t(i) = b_q(i)$ for $i < |s|$ and $b_t(i) = b_r(i - |s|)$ for $i \geq |s|$. From that reason for a condition q and a finite binary sequence s when we write $t^s \leq q^s$ then by t we mean the condition $(\leq q)$ described above.

We would like to use fusion of conditions, so in the next definition we define what fusion sequence is.

Definition 4.1.6 (Fusion sequence)**.** We define the suborder $\leq_n \subseteq \leq$ for every

n. For $p, q \in \mathbb{CS}$ and $n \in \omega$ $p \leq_n q$ if $p \leq q$ and $\forall m < n(b_p(m) \supseteq b_q(m))$. In particular, $\leq_0 = \leq$.

A sequence $(p_n)_{n\in\omega} \subseteq \mathbb{CS}$ is a fusion sequence if $\forall n > 0(p_n \leq_n p_{n-1})$. Then we define the fusion of such a sequence to be the condition p where $a_p = \bigcup_n a_{p_n}$ and for every i $b_p(i) = \bigcup_{n\geq i} b_{p_n}(i)$.

It is easy to check that $p \leq_{n+1} p_n$ for every n.

Definition 4.1.7 (Reduced products)**.** We define two reduced products of two copies of the Carlson-Simpson forcing.

We set $\mathbb{CS} \times_{R0} \mathbb{CS} = (\{(p, q) \in \mathbb{CS} \times \mathbb{CS} : b_p = b_q \wedge \min\{a_p \setminus a_q\} \leq \min\{a_q \setminus a_p\}\}, \leq_{R0})$. The order relation $\leq_{R0}$ is induced from the usual product. We denote it differently to emphasize that we are working with the reduced product.

Moreover, we define a second reduced product $\mathbb{CS} \times_{R1} \mathbb{CS}$ as follows $\mathbb{CS} \times_{R1} \mathbb{CS} = (\{(p, q) \in \mathbb{CS} \times \mathbb{CS} : b_p = b_q \wedge a_p \subseteq a_q\}, \leq_{R1})$. The order relation $\leq_{R1}$ is a subrelation of $\leq_{R0}$ defined as follows: $(p, q) \leq_{R1} (r, t)$ if $(p, q) \leq_{R0} (r, t)$ and $a_p \setminus a_r \subseteq a_q \setminus a_t$.

We now restate Theorem 4.1.1 in this new language for two element alphabet.

Theorem 4.1.8. *Let $p \in \mathbb{CS}$ be a condition in the Carlson-Simpson forcing and E an analytic equivalence relation on $[p]$. Then there exists a subcondition $q \leq p$ such that $E \restriction [p]$ is equal to $[q] \times [q]$ or to $\mathrm{id}([q])$.*

Proof. Consider $\mathbb{CS} \times_{R0} \mathbb{CS}$ and $\mathbb{CS} \times_{R1} \mathbb{CS}$ as forcing notions. They both add a pair (x_L, x_R) of infinite subsets of ω. More specifically, one can easily check that $\mathbb{CS} \times_{R0} \mathbb{CS} \Vdash |x_L \setminus x_R| = |x_R \setminus x_L| = \omega$ while $\mathbb{CS} \times_{R1} \mathbb{CS} \Vdash x_L \subseteq x_R$. We will use the symbol $\Vdash_1$ to specify that we are forcing with $\mathbb{CS} \times_{R1} \mathbb{CS}$. Similarly, we will use $\Vdash_0$ to specify that we are forcing with $\mathbb{CS} \times_{R0} \mathbb{CS}$.

The following lemma is the main tool that immediately implies Theorem 4.1.8. It is more general than Theorem 4.1.8, however we do not have any other application of it besides that theorem.

The statement is divided into three items. It could be stated at once but it is probably more convenient and transparent to have these items separately.

Lemma 4.1.9.

(i) *Let $p \in \mathbb{CS}$ be any condition and let M be a countable elementary submodel of some H_λ, where H_λ is sufficiently large, which contains p and E. Then there exists $q \le p$ such that $\forall x, y \in [q]$ if $x \subseteq y$ then the pair (x, y) is M-generic for $\mathbb{CS} \times_{R1} \mathbb{CS}$, and $\forall x, y \in [q]$ if $|x \setminus y| = |y \setminus x| = \omega$ and $\min\{x \setminus y\} < \min\{y \setminus x\}$ then the pair (x, y) is M-generic for $\mathbb{CS} \times_{R0} \mathbb{CS}$.*

(ii) *Let $(s, t) \le_{R1} (p, p)$ be any condition and let M be again a countable elementary submodel of a large enough structure containing (s, t) and E. Then there exists $(q, r) \le_{R1} (s, t)$ such that $\forall x \in [q] y \in [r]$ if $x \subseteq y$ then the pair (x, y) is M-generic for $\mathbb{CS} \times_{R1} \mathbb{CS}$.*

(iii) *Let $(s, t) \le_{R0} (p, p)$ be any condition and let M be again a countable elementary submodel of a large enough structure containing (s, t) and E. Then there exists $(q, r) \le_{R0} (s, t)$ such that $\forall x \in [q] y \in [r]$ the pair (x, y) is M-generic for $\mathbb{CS} \times_{R1} \mathbb{CS}$.*

We postpone the proof for later. First, we show how the theorem follows.

Let us consider the two following cases.

- **Case 1** $(p, p) \Vdash_0 x_L \not{E} x_R$ and $(p, p) \Vdash_1 x_L \not{E} x_R$.
- **Case 2** Either $\exists (s, t) \le_{R1} (p, p)((s, t) \Vdash_1 x_L E x_R)$ or $\exists (s, t) \le_{R0} (p, p)((s, t) \not\le_{R1} (p, p) \wedge (s, t) \Vdash_0 x_L E x_R)$.

If **Case 1** holds then we fix a countable elementary submodel M of some H_λ, where H_λ is sufficiently large, which contains p and E and we apply Lemma 4.1.9 (i) to obtain corresponding $q \le p$. It follows that for any $x, y \in [q]$ $M[x, y] \vDash x \not{E} y$ and since E is analytic it follows from the analytic absoluteness (see Theorem 1.2.1) that $x \not{E} y$.

If **Case 2** holds then either there exists $(s, t) \le_{R0} (q, q)$ such that $(s, t) \Vdash_0 x_L E x_R$, or there exists $(s, t) \le_{R1} (q, q)$ such that $(s, t) \Vdash_1 x_L E x_R$.

In the former case, we again fix a suitable countable elementary submodel M and apply Lemma 4.1.9 (iii) to obtain the corresponding (q, r). We have that for any $x \in [q], y \in [r]$ $M[x, y] \vDash x E y$ and it again follows from the analytic absoluteness that $x E y$. It immediately follows from the transitivity of E that $E \restriction [q] = [q] \times [q]$ (or similarly $E \restriction [r] = [r] \times [r]$).

In the latter case, we again use some suitable countable elementary submodel M and Lemma 4.1.9 (ii) to obtain the corresponding (q,r). We now have that for any $x \in [q], y \in [r]$ if $x \subseteq y$ then $M[x,y] \models xEy$ and by the analytic absoluteness xEy. We use transitivity of E to show that for any $x \in [q], y \in [r]$ xEy. Let a pair $x \in [q], y \in [r]$ such that $x \setminus y \neq \emptyset$ be given. Denote $d = x \setminus y$. It is easy to check that $y \cup d \in [r]$; similarly $x \setminus d \in [q]$ (just note that $(q,r) \leq_{R1} (q,q)$, i.e. $a_q \subseteq a_r$ and $b_q = b_r$). Thus we have $xE(y \cup d)E(x \setminus d)Ey$ and from the transitivity xEy. It again follows using transivity that $E \restriction [q] = [q] \times [q]$ (or similarly $E \restriction [r] = [r] \times [r]$).

Proof of the lemma. We prove only (i), proofs of the other items are just a routine modification.

Let us enumerate all open dense subsets of $\mathbb{CS} \times_{R0} \mathbb{CS}$ lying in M as $(D_n)_{n\in\omega}$, and all open dense subsets of $\mathbb{CS} \times_{R1} \mathbb{CS}$ lying in M as $(E_n)_{n\in\omega}$.

Step 1 We find a subcondition p_∞ of p such that for all different $x, y \in [p_\infty]$ such that $x \subseteq y$ the pair (x,y) is M-generic for $\mathbb{CS} \times_{R1} \mathbb{CS}$.

Claim 4.1.10. *For any $r \leq p$ and any finite binary sequence u there is $s \leq_{|u|+1} r$ such that $\forall x \in [s^{u0}] y \in [s^{u1}]$ such that $x \subseteq y$ we have that the pair (x,y) is M-generic for $\mathbb{CS} \times_{R1} \mathbb{CS}$.*

Suppose the claim is proved. The fusion producing the condition p_∞ goes as follows. According to the claim there exists $p_0 \leq_1 p$ such that $\forall x \in [p_0^0] y \in [p_0^1]$ if $x \subseteq y$ then the pair (x,y) is M-generic for $\mathbb{CS} \times_{R1} \mathbb{CS}$. Then using the claim two times there exists $p_1 \leq_2 p_0$ such that $\forall x \in [p_1^{00}] y \in [p_1^{01}]$ if $x \subseteq y$ then the pair (x,y) is M-generic for $\mathbb{CS} \times_{R1} \mathbb{CS}$, and similarly $\forall x \in [p_1^{10}] y \in [p_1^{11}]$ if $x \subseteq y$ then the pair (x,y) is M-generic for $\mathbb{CS} \times_{R1} \mathbb{CS}$.

In general, when we have already found p_{n-1} then using the claim 2^n-times we find a condition $p_n \leq_{n+1} p_{n-1}$ such that for any binary sequence u of length n and $\forall x \in [p_n^{u0}] y \in [p_n^{u1}]$ if $x \subseteq y$ then the pair (x,y) is M-generic for $\mathbb{CS} \times_{R1} \mathbb{CS}$.

Let p_∞ be the fusion of that sequence. Then for any $x \in [p_\infty] y \in [p_\infty]$ such that $x \subseteq y$ there exists some i such that the block $b_{p_\infty}(i)$ lies in y but not in x. Let i be the minimal such index. Then it follows that $x \in [p_i^{u0}] y \in [p_i^{u1}]$, for some binary sequence u of length i and so we have guaranteed during the fusion that (x, y) is an M-generic pair for $\mathbb{CS} \times_{R1} \mathbb{CS}$.

To prove the claim, again a fusion is needed. Let $r \leq p$ and a finite binary sequence u be given. Since E_0 is dense there exists $s_0 \leq_1 r^u$ such that $(s_0^0, s_0^1) \in E_0$. Note that we are looking for a pair $(s, t) \leq_{R1} (r^{u0}, r^{u1})$ lying in E_0 and this pair (s, t) is of the form (s_0^0, s_0^1) for some $s_0 \leq_1 r^u$. Also note that $s_0 = s_0'^u$ for some $s_0' \leq_{1+|u|} r$.

Suppose we have already found s_{n-1} (again we may write $s_{n-1} = s_{n-1}'^u$ for some $s_{n-1}' \leq_{n+|u|} s_{n-2}'$). Using that E_n is open dense we extend s_{n-1} to obtain $s_n \leq_{n+1} s_{n-1}$ such that for every pair $(v, w) \in (2^n)^2$ of binary sequences of length n such that $\{i : v(i) = 1\} \subseteq \{i : w(i) = 1\}$ we have $(s_n^{0v}, s_n^{1w}) \in E_n$. To do this, enumerate all such pairs of binary sequences of length n as $\{(v_i, w_i) : i < \sum_{i=0}^{n} 2^i \binom{n}{i}\}$ (denote $k = \sum_{i=0}^{n} 2^i \binom{n}{i}$) and set $t_0 = s_{n-1}$. When we already have t_{i-1} for $i < k-1$ we find $t_i \leq_{n+1} t_{i-1}$ such that $(t_i^{0v_{i-1}}, t_i^{1w_{i-1}}) \leq_{R1} (t_{i-1}^{0v_{i-1}}, t_{i-1}^{1w_{i-1}})$ and such that $(t_i^{0v_{i-1}}, t_i^{1w_{i-1}}) \in E_n$. Finally, set $s_n = t_{k-1}$ and the induction step is done.

Once the fusion sequence is constructed, let s' be the fusion limit. It is easy to check that s' is in fact s^u where s is the fusion limit of the sequence $s_0' \leq_{2+|u|} s_1' \leq_{3+|u|} s_2' \leq_{4+|u|} \ldots$ and this s is the desired condition.

This finishes the proof of the claim and also the proof of **Step 1**.

Step 2 Now we find a subcondition $q \leq p_\infty$ such that $\forall x, y \in [q]$ such that $|x \setminus y| = |y \setminus x| = \omega$ and $\min\{x \setminus y\} < \min\{y \setminus x\}$ the pair (x, y) is M-generic for $\mathbb{CS} \times_{R0} \mathbb{CS}$ which will finish the proof.

Claim 4.1.11. *For any $r \leq p_\infty$ and any two finite binary sequences u, v there exists $s \leq_{|u|+|v|+2} r$ such that $\forall x \in [s^{u1v0}] y \in [s^{u0v1}]$ the pair (x, y) is M-generic for $\mathbb{CS} \times_{R0} \mathbb{CS}$.*

Suppose the claim is proved. Then the final fusion producing the condition q goes as follows.

According to the claim there exists $q_0 \leq_2 p_\infty$ such that $\forall x \in [q_0^{10}] y \in [q_0^{01}]$ the pair (x, y) is M-generic for $\mathbb{CS} \times_{R0} \mathbb{CS}$. Suppose that we have already found q_{n-1}.

Then using the claim several times (precisely 2^{n+2} times) we find a condition $q_n \leq_{n+2} q_{n-1}$ such that for any two finite (including empty) binary sequences u, v such that $|u| + |v| = n$ we have that $\forall x \in [q_n^{u1v0}] y \in [q_n^{u0v1}]$ the pair (x, y) is M-generic.

Let q be the fusion limit. Then for any $x, y \in [q]$ such that $\min x \backslash y < \min y \backslash x$ there exists the least block $b_q(i)$ such that it lies in x but not in y and also there exists the least block $b_q(j)$ such that it lies in y but not in x (note that $i < j$). It follows that $x \in [q_{j-1}^{u1v0}] y \in [q_{j-1}^{u0v1}]$, where $|u| = i$ and $|v| = j - i - 1$ for some binary sequences u, v, and so we have guaranteed during the fusion that the pair (x, y) is M-generic for $\mathbb{CS} \times_{R0} \mathbb{CS}$.

It remains to prove this last claim to finish the proof of Lemma 4.1.9 and Theorem 4.1.8. The proof is similar to the proof of Claim 4.1.10.

Let $r \leq p_\infty$ and a finite binary sequences u, v be given. Since D_0 is dense there exists $s_0 \leq_{|u|+|v|+2} r$ such that $(s_0^{u1v0}, s_0^{u0v1}) \in D_0$. Note that we are looking for a pair $(s, t) \leq_{R1} (r^{u1v0}, r^{u0v1})$ lying in D_0 and this pair (s, t) is of the form (s_0^{u1v0}, s_0^{u0v1}) for some $s_0 \leq_{|u|+|v|+2} r$.

Suppose we have already found s_{n-1}. Using that D_n is open dense we extend s_{n-1} to obtain $s_n \leq_{|u|+|v|+n+2} s_{n-1}$ such that for every pair $(w, w') \in (2^n)^2$ of binary sequences of length n we have $(s_n^{u1v0w}, s_n^{u0v1w'}) \in D_n$. To do this, enumerate all such pairs of binary sequences of length n as $\{(w_i, w_i') : i < 2^{2n}\}$ and set $t_0 = s_{n-1}$. When we already have t_{i-1} for $i < 2^{2n} - 1$ we find $t_i \leq_{|u|+|v|+n+2} t_{i-1}$ such that $(t_i^{u1v0w_{i-1}}, t_i^{u0v1w'_{i-1}}) \leq_{R1} (t_{i-1}^{u1v0w_{i-1}}, t_{i-1}^{u0v1w'_{i-1}})$ and such that $(t_i^{u1v0w_{i-1}}, t_i^{u0v1w'_{i-1}}) \in D_n$. Finally, set $s_n = t_{2^{2n}-1}$ and the induction step is done.

Once the fusion sequence is constructed we set s to be the fusion limit and it is a routine to check that this is the desired s. $\square$

The proof of the generalization for an alphabet containing more than two elements is sketched here. It uses induction on the cardinality of A.

Theorem 4.1.12. *Let A be a finite alphabet such that $|A| \geq 3$. Let $X \in (\omega)_A^\omega$ be a condition in the Carlson-Simpson forcing and E an analytic equivalence relation* *on $[X]$. Then there exists a subcondition $Y \in (X)^\omega_A$ such that $E \restriction [Y]$ is equal to $[Y] \times [Y]$ or to* $\mathrm{id}([Y])$ *or there exists $B \in \mathcal{B}_A$ such that $E \restriction [Y] = E_B \restriction [Y]$.*

Sketch of the proof. Let $n = |A|$ and enumerate A as $\{a_0, \dots, a_{n-1}\}$. For every $i < n$ let us denote $\times_{Ri}$ the reduced product of $(\omega)^\omega_A$ such that $(Z, Y) \in (\omega)^\omega_A \times_{Ri} (\omega)^\omega_A$ iff for every $j \neq i$ the a_j-block of Z is contained in the a_j-block of Y and the free blocks of Z and Y are the same. Moreover, let $\times_{Rn}$ denote the reduced product of $(\omega)^\omega_A$ such that $(Z, Y) \in (\omega)^\omega_A \times_{Rn} (\omega)^\omega_A$ iff for every $j < n$ neither the a_j-block of Z is contained in the a_j-block of Y nor the a_j-block of Y is contained in the a_j-block of Z, and the free blocks of Z and Y are the same. Notice how this generalizes Definition 4.1.7 for an arbitrary alphabet.

The following lemma is proved by analogous means as Lemma 4.1.9.

Lemma 4.1.13.

(i) Let $X \in (\omega)^\omega_A$ be any condition and let M be a countable elementary submodel of some H_λ, where H_λ is sufficiently large, which contains X and E. Then there exists $Y \preceq X$ such that $\forall z, y \in [Y]$ if there is $i < n$ such that for every $j \neq i$ the a_j-block of z is contained in the a_j-block of y then the pair (z, y) is M-generic for $(\omega)^\omega_A \times_{Ri} (\omega)^\omega_A$, and $\forall x, y \in [q]$ if for every $j < n$ neither the a_j-block of z is contained in the a_j-block of y nor the a_j-block of y is contained in the a_j-block of z then the pair (z, y) is M-generic for $(\omega)^\omega_A \times_{Rn} (\omega)^\omega_A$.

(ii) Let $i < n$ and let $(Z', Y') \leq_{Ri} (X, X)$ be any condition and let M be again a countable elementary submodel of a large enough structure containing (Z', Y') and E. Then there exists $(Z, Y) \leq_{Ri} (Z', Y')$ such that $\forall z \in [Z] y \in [Y]$ if for every $j \neq i$ the a_j-block of z is contained in the a_j-block of y then the pair (z, y) is M-generic for $(\omega)^\omega_A \times_{Ri} (\omega)^\omega_A$.

(iii) Let $(Z', Y') \leq_{Rn} (X, X)$ be any condition and let M be again a countable elementary submodel of a large enough structure containing (Z', Y') and E. Then there exists $(Z, Y) \leq_{Rn} (Z', Y')$ such that $\forall z \in [Z] y \in [Y]$ if for every $j < n$ neither the a_j-block of z is contained in the a_j-block of y nor the a_j-

block of y is contained in the a_j-block of z then the pair (z, y) is M-generic for $(\omega)^\omega_A \times_{Rn} (\omega)^\omega_A$.

We again have two cases. Note that for each $i \leq n$ the forcing notion $(\omega)^\omega_A \times_{Ri} (\omega)^\omega_A$ again adds a pair of reals again denoted as x_L and x_R.

- **Case 1** $\forall i \leq n((X, X) \Vdash_i x_L \not E x_R)$.
- **Case 2** $\exists i \leq n \exists (Z, Y) \leq_{Ri} (X, X)((Z, Y) \Vdash_i x_L E x_R)$.

If **Case 2** holds then as in Theorem 4.1.8 using analytic absoluteness and transitivity of E we prove that there exists a condition $Y \preceq X$ such that $E \restriction [Y] = [Y] \times [Y]$.

Suppose that **Case 1** holds. Then as in Theorem 4.1.8 using analytic absoluteness we prove that there is $Y \preceq X$ such that $\forall z, y \in [Y]$ if there is no $i < n$ such that the a_i-block of z is equal to the a_i-block of y, then $z \not E y$. However, notice that the case when there is some $i < n$ such that the a_i-block of z is equal to the a_i-block of y is not treated by Lemma 4.1.13 since in such a case the pair (z, y) is not generic for $(\omega)^\omega_A \times_{Ri} (\omega)^\omega_A$ for any $i \leq n$.

However, consider $\{a_1, \ldots, a_{n-1}\} \in \mathcal{B}_A$. We can use by an inductive argument Theorem 4.1.12 for $A' = \{a_1, \ldots, a_{n-1}\}$ to obtain $Y_0 \preceq Y$ such that either $\forall z, y \in [Y_0](z E_{A'} y \Rightarrow z \not E y)$, or there is some subset $B \subseteq A'$ ($|B| \geq 2$, including the case when $B = A'$) such that $\forall z, y \in [Y_0](z E_B y \Rightarrow z E y)$. If the latter case holds and $B = A'$ then we are done. We found a condition Y_0 such that $E \restriction [Y_0] = E_B \restriction [Y_0]$. If B is a proper subset of A' then consider $B' = B \cup \{a_0\}$ and we can again use by an inductive argument Theorem 4.1.12 for B'. We obtain some $Z \preceq Y_0$ such that either $E \restriction [Z] = E_B \restriction [Z]$ or $E \restriction [Z] = E_{B'} \restriction [Z]$.

If the former case holds then we will succesively use Theorem 4.1.12 for $\{a_0, a_i\}$, for every $0 < i < n$. Either we end up with a condition $Z \preceq Y$ such that $\forall z, y \in [Z](z \not E y)$ or we end up with a condition $Z \preceq Y$ and some $0 < i < n$ such that $E \restriction [Z] = E_{\{a_0, a_i\}} \restriction [Z]$. $\square$

We finish by providing the proofs of Proposition 4.0.8 and Theorem 4.0.4.

Proof of Proposition 4.0.8. Fix an alphabet A with $|A| = n \geq 2$. Let us prove (1). Let $A_n \in I_{C_n}$ for all $n \in \omega$. Suppose that $A = \bigcup_{n \in \omega} A_n \notin I_{C_n}$. It must contain

$[X]$ for some $X \in (\omega)^\omega_A$. We can define a Borel equivalence relation E on $[X]$ with countably many classes such that $\forall x, y \in [X](xEy \Leftrightarrow \exists n \in \omega(x, y \in A_n))$. Applying Theorem 4.1.1 we get $Y \preceq X$ such that $E \restriction [Y]$ is the full relation, the identity relation or $E_B \restriction [Y]$ for some $B \in \mathcal{B}_A$. Since E has only countably many classes only the first case is possible. Thus $E \restriction [Y] = [Y] \times [Y]$, i.e. there is $n \in \omega$ such that $[Y] \subseteq A_n$ which is a contradiction.

The item (2) follows from (1). For any $X \in (\omega)^\omega_A$, $[X]$ is a Borel (closed) I_{C_n}-positive subset; conversely, it follows from (1) that any Borel I_{C_n}-positive subset of A^ω contains $[X]$ for some $X \in (\omega)^\omega_A$.

We prove (3) only for case $|A| = 2$ because of our notation introduced for this special case. Consider the suborders $\leq_n \subseteq \leq$ on $\mathbb{CS}$. It is easy to check that $\mathbb{CS}$ with these relations satisfies Axiom A and thus it is proper (see [16] Definition 31.10 and then Lemma 31.11). $\square$

Proof of Theorem 4.0.4. This is just a special case of Theorem 4.1.8 if we consider p to be the biggest condition in $\mathbb{CS}$, i.e. $a_p = \emptyset$ and $b_p(i) = \{i\}$ for every $i \in \omega$. Theorem 4.1.8 gives a subcondition $q \leq p$ on which E is simple. The condition q determines the sequence $(A_n)_{n\in\omega}$: $A_0 = a_q$ and $A_i = b_q(i-1)$ for $i \geq 1$.

Let us just note that we cannot eliminate the set A_0 from the statement, i.e. demand it to be empty. Just consider an equivalence relation E on $\mathcal{P}(\omega)$ where for $a, b \in \mathcal{P}(\omega)$ we have aEb if $\min a = \min b$. $\square$

Chapter 5

Universal and ultrahomogeneous metric structures

Introduction

In 1927, P. S. Urysohn constructed a metric space $\mathbb{U}$ which is now called The Urysohn universal metric space ([32]). It is a Polish metric space that is both universal and ultrahomogeneous for the class of all finite metric spaces. The universality means that every finite metric space can be isometrically embedded into $\mathbb{U}$ and the ultrahomogeneity means that any finite isometry $\phi : \{x_1, \ldots, x_n\} \subseteq \mathbb{U} \rightarrow \{y_1, \ldots, y_n\} \subseteq \mathbb{U}$ extends to an isometry $\bar{\phi} \supseteq \phi : \mathbb{U} \rightarrow \mathbb{U}$ on the whole space. These two properties imply that $\mathbb{U}$, in fact, contains an isometric copy of every separable metric space and that $\mathbb{U}$ is unique with these two properties up to isometry.

The aim of this chapter is to enrich the Urysohn space with some additional structure so that this enriched Urysohn space is still universal and ultrahomogeneous for that specific (Polish) metric structure. The definition of Polish metric structures considered here is given at the end of this section. A related work has been done by W. Kubiś in [24] (see also [13]).

Our initial motivation was to provide a general way of coding of such classes of Polish metric structures as standard Borel spaces. Let us say we are given some class of Polish metric structures and we would like to use methods of descriptive set theory to investigate (e.g. classify) this class. In order to use these methods we need to represent such a class as a Polish space or, it is sufficient, as a standard Borel space. Recall Definition 1.1.15 of an Effros-Borel structure. Effors-Borel structure is an example of a standard Borel space that can serve in this direction. Let us illustrate it on examples.

Examples.

- Recall (Fact 1.1.2) that every Polish space X is homeomorphic to a closed subset of $\mathbb{R}^{\mathbb{N}}$. Thus the Effros-Borel space of $F(\mathbb{R}^{\mathbb{N}})$ can be interpreted as a standard Borel space of all Polish spaces.

- Recall the classical result of Banach and Mazur that every separable Banach space can be be embedded by a linear isometry into the separable Banach space $C([0,1])$, i.e. the Banach space of all real-continuous functions on $[0,1]$. Consider the following subset of the standard Borel space $F(C([0,1]))$, which can be checked to be Borel, $\text{Subs} = \{X \in F(C([0,1])) : X \text{ is a linear subspace}\}$. It is a standard Borel space of all separable Banach spaces (that has been used, for instance, by V. Ferenczi, A. Louveau and C. Rosendal in [9] for a classification result of separable Banach spaces with the relation of linear isomorphism). There are a lot of Borel subsets of Subs that represent certain subclasses of separable Banach spaces (see [2] for example).

- Because of the properties of $\mathbb{U}$ the standard Borel space $F(\mathbb{U})$ can serve as a coding of all Polish metric spaces. We remark that this approach was used by Gao and Kechris in [12] in their classification of Polish metric spaces up to isometry.

The Effros-Borel structure of $F(\mathcal{S})$, where $\mathcal{S}$ will be one of the structures we investigate here, should serve in a similar way. Let us state the main definitions of this chapter.

Definition 5.0.14 (Polish metric structure)**.** Let $Z_1, \ldots, Z_k$ be a list of Polish metric spaces. A finite or countably infinite set $\mathcal{O}$ is called a signature if it consists of symbols for closed sets. Moreover, there is a function $a : \mathcal{O} \to ([0, \ldots, k] \times \mathbb{N})^{<\omega}$;

i.e. to each symbol from $\mathcal{O}$ it assigns a finite sequence of elements (a, b) where $0 \leq a \leq k$ and $b \in \mathbb{N}$. By $a_F(n, i)$, for $i \in \{1, 2\}$, we denote the i-th coordinate of the n-th element of $a(F)$.

A Polish metric structure of signature $\mathcal{O}$ is a Polish metric space (X, d) such that for every $F \in \mathcal{O}$ there is a closed set $F_X \subseteq Z_{a_F(1,1)}^{a_F(1,2)} \times \ldots \times Z_{a_F(|a(F)|,1)}^{a_F(|a(F)|,2)}$, where by Z_0 we denote X.

Definition 5.0.15. Let $(X, d, \mathcal{O}_X)$ be a Polish metric space of some signature $\mathcal{O}$. We say that $(X, d, \mathcal{O}_X)$ is universal if for any Polish metric space $(Y, d, \mathcal{O}_Y)$ of the same signature $\mathcal{O}$ there is an isometric embedding $\phi : Y \hookrightarrow X$ that moreover reduces F_Y into F_X (for every $F \in \mathcal{O}$): i.e. for any $(y_1, \ldots, y_n)$, where $I \subset \{1, \ldots, n\}$ are the coordinates such that $y_i \in Y$ iff $i \in I$, we have $(y_1, \ldots, y_n) \in F_Y \Leftrightarrow (x_1, \ldots, x_n) \in F_X$, where $x_i = \phi(y_i)$ if $i \in I$ and $x_i = y_i$ otherwise.

We say that $(X, d, \mathcal{O}_X)$ is ultrahomogeneous if any isomorphism between two finite (metric) substructures $(F_1, d, \mathcal{O}_{F_1})$ and $(F_2, d, \mathcal{O}_{F_2})$ of $(X, d, \mathcal{O}_X)$ extends to an automorphism of the whole $(X, d, \mathcal{O}_X)$.

Let us illustrate the universality and ultrahomogeneity on examples.

Examples.

- If the signature $\mathcal{O}$ is empty then $(X, d, \mathcal{O}_X)$ is just the Urysohn universal metric space $\mathbb{U}$, i.e. space containing an isometric copy of every Polish (or just separable) metric space and with the property that every finite partial isometry extends to an isometry of the whole space.

- Let us consider the case when the signature $\mathcal{O}_X$ contains a symbol for one closed subset C of X. Then for any Polish metric space (Y, d) equipped with some closed subset $D \subseteq Y$ there is an isometric embedding $\phi : Y \hookrightarrow X$ that maps D into C, i.e. $\forall y \in Y (y \in D \Leftrightarrow \phi(y) \in C)$; in other words, $\phi(Y) \cap C = \phi(D)$. Moreover, for any two finite subspaces $F_1, F_2 \subseteq X$ and an isometry $\phi : F_1 \to F_2$ respecting the closed subset, i.e. $\phi(F_1 \cap C) = F_2 \cap C$, there is an extension to an isometry on the whole space $\phi \subseteq \bar{\phi} : X \to X$ that still respects the closed subset, i.e. $\bar{\phi}(C) = C$.

- Let us consider the case when the signature $\mathcal{O}_X$ contains a symbol for a closed subset C of $X \times Z$ where Z is some fixed Polish metric space. Then for any Polish metric space (Y, d) and a closed subset $D \subseteq Y \times Z$ there is an isometric embedding $\phi : Y \hookrightarrow X$ such that $\forall y \in Y \forall z \in Z((y, z) \in D \Leftrightarrow (\phi(y), z) \in C)$. Moreover, for any two finite substructures F_1, F_2 and isometry ϕ between them respecting the structure, i.e. $\forall f \in F_1 \forall z \in Z((f, z) \in C \Leftrightarrow (\phi(f), z) \in C)$, there is an extension $\bar{\phi}$ to the whole space still respecting the closed set C.

- Let us consider the case when the signature $\mathcal{O}_X$ contains a symbol f for a closed subset of $X \times Z$ and moreover
$(X, d, \mathcal{O}_X) \models f$ is a graph of a continuous function, where Z is some fixed Polish metric space. Then for any Polish metric space (Y, d) equipped with a continuous function $g : Y \to Z$ (i.e. a Polish metric structure of that signature which also models that this closed set is in fact a graph of a continuous function) to that fixed space Z there is an isometric embedding $\phi : Y \hookrightarrow X$ that maps the graph of g into the graph of f; in other words, $\forall y \in Y(g(y) = f \circ \phi(y))$. Moreover, for any two finite metric substructures $F_1, F_2 \subseteq X$ and an isometry ϕ between them that respects the continuous function, i.e. $\forall x \in F_1(f(x) = f \circ \phi(x))$, there is an extension to the isometry on the whole space that still respects the continuous function.

In what follows we shall denote the Polish metric structures somewhat loosely. For instance the Polish metric structure with two closed sets would be denoted often as (X, F_1, F_2) instead of (X, d, F^1_X, F^2_X) where F^1, F^2 are two symbols for closed sets.

Definition 5.0.16 (Almost universal and ultrahomogeneous structures)**.** Suppose now that the signature $\mathcal{O}$ on (X, d) consists of countably many symbols for closed sets of the same type, e.g. countably many closed subsets of X or countably many continuous functions (resp. graphs of them) from X to some fixed metric spaces. In such a case we will usually not be able to maintain universality and ultrahomogeneity in the full strength. Let us have $\mathcal{O}$ enumerated as $\{O_n : n \in \mathbb{N}\}$. We say that $(X, d, (O_n)_{n\in\mathbb{N}})$ is almost universal and ultrahomogeneous if for any Polish metric space $(Y, d, (F_n)_{n\in\mathbb{N}})$, where (F_n) are of the same type as (O_n) there is an isometric embedding $\phi : Y \hookrightarrow X$ and an injection $\pi : \mathbb{N} \to \mathbb{N}$ such that ϕ maps F_n into $O_{\pi(n)}$. Moreover, let F_1, F_2 be two finite subspaces of X such that there is a finite isometry ϕ between F_1 and F_2 and two sets of indices $\{k_1, \ldots, k_n\} \subseteq \mathbb{N}$ and $\{l_1, \ldots, l_n\} \subseteq \mathbb{N}$ such that ϕ maps the restriction $O_{k_i} \upharpoonright F_1$ into the restriction $O_{l_i} \upharpoonright F_2$, for all $i \leq n$. Then there is an isometry $\bar{\phi} \supseteq \phi$ of the whole space X extending ϕ and a bijection $\pi : \mathbb{N} \to \mathbb{N}$, such that $\pi(k_i) = l_i$ for $i \leq n$, such that $\bar{\phi}$ maps O_m into $O_{\pi(m)}$ for all $m \in \mathbb{N}$.

We remark that in all cases the underlying Polish metric space X for a given structure is isometric to the Urysohn universal space $\mathbb{U}$, thus from now on we will always denote it as $\mathbb{U}$. We will comment on this in Remark 5.1.7 after the proof of Theorem 5.1.2.

Notational convention. For any metric space X we will denote the metric as either d_X but more often, when there is no danger of confusion, just as d. When working with a metric on a product of metric spaces we always consider the sum metric, i.e. $d((x_1, x_2), (y_1, y_2)) = d(x_1, y_1) + d(x_2, y_2)$.

We usually denote tuples $(x_1, \ldots, x_m)$, for an arbitrary $m \in \mathbb{N}$ clear from the context, by $\vec{x}$. When ϕ is some mapping we denote $(\phi(x_1), \ldots, \phi(x_m))$ by $\phi^m(\vec{x})$.

5.1 Universal closed relations

Theorem 5.1.1. *Let $n_1 \leq \ldots \leq n_m$ be an arbitrary non-decreasing sequence of natural numbers. Then there exist closed relations (subsets) $F_{n_i} \subseteq \mathbb{U}^{n_i}$, for $i \leq m$, such that the structure $(\mathbb{U}, F_{n_1}, \ldots, F_{n_m})$ is universal and ultrahomogeneous and it is unique (up to isometry preserving the relations) with this property.*

Instead of giving a proof of this theorem we prove the theorem below which is "almost" more general. In remarks after the proof of Theorem 5.1.2 we indicate how to modify the proof so that it works also for Theorem 5.1.1.

Theorem 5.1.2. *There exists an almost universal and ultrahomogeneous structure $(\mathbb{U}, (F^n_m)_{n,m\in\mathbb{N}})$ where $F^n_m \subseteq \mathbb{U}^n$ is a closed n-ary relation (i.e. a closed subset of the n-th power of $\mathbb{U}$). It is also unique with this property (up to permutation of the set of n-ary relations for each n and isometry preserving the relations).*

Remark 5.1.3. Let us elaborate more on the statement of the theorem. Let (X,d) be a Polish metric space equipped with closed sets G^n_m, for all $m,n \in \mathbb{N}$, where $G^n_m \subseteq X^n$. Then there exist an isometric embedding $\psi : X \hookrightarrow \mathbb{U}$ and injections $\pi_n : \mathbb{N} \to \mathbb{N}$, for each $n \in \mathbb{N}$, such that $\forall n,m \in \mathbb{N}(\psi(X)^n \cap F^n_{\pi_n(m)} = \psi^n(G^n_m))$, or in other words $\forall n,m \in \mathbb{N} \forall \vec{x} \in X^n(\vec{x} \in G^n_m \Leftrightarrow \psi^n(\vec{x}) \in F^n_{\pi_n(m)})$.

In particular, $\psi^n : G^n_m \hookrightarrow F^n_{\pi_n(m)}$ is an isometric embedding.

Moreover, let $M_1, M_2 \subseteq \mathbb{U}$ be two finite metric subspaces, some $n_M \leq |M_1| = |M_2|$, for each $n \leq n_M$ there are finite sets of indices $I^{M_1}_n, I^{M_2}_n \subseteq \mathbb{N}$ such that $|I^{M_i}_n| = n_M - n + 1$, for $i \in \{1,2\}$, and for each $n \leq n_M$ there are bijections $\pi_n : I^{M_1}_n \to I^{M_2}_n$ and an isometry $\psi : M_1 \to M_2$ such that $\forall n \leq n_M \forall m \in I^{M_1}_n \forall \vec{x} \in M^n_1(\vec{x} \in F^n_m \Leftrightarrow \psi^n(\vec{x}) \in F^n_{\pi_n(m)})$; i.e. ψ reduces the closed relation F^n_m into $F^n_{\pi_n(m)}$. Then there are an isometry $\bar{\psi} : \mathbb{U} \to \mathbb{U}$ and bijections $\bar{\pi_n} : \mathbb{N} \to \mathbb{N}$, for each $i \in \mathbb{N}$, such that $\forall n,m \in \mathbb{N} \forall \vec{x} \in \mathbb{U}^n(\vec{x} \in F^n_m \Leftrightarrow \bar{\psi}^n(\vec{x}) \in F^n_{\bar{\pi_n}(m)})$, and $\bar{\psi}$ extends ψ and $\bar{\pi_n}$ extends π_n for each $n \leq n_M$.

We will construct these sets along with the underlying metric space (universal Urysohn space) as a Fraïssé limit of a certain countable class $\mathcal{K}$ of finite structures. This is basically also an original method of construction of the Urysohn universal space eventhough the general Fraïssé theory did not exist at that time! We note that there is another construction of the Urysohn space due to M. Katětov ([19]).

Let us make another notational convention here. In the languages of structures that we will use there will always be defined (partial) functions into some fixed countable set, e.g. a function with rational values. It is clear that each such function can be replaced by countably many predicates; for example, a rational function f can be replaced by predicates f_q, for each $q \in \mathbb{Q}$, and then we could demand that for each element a of our structure there is precisely (or at most) one $q \in \mathbb{Q}$ such that $f_q(a)$ holds. We will always implicitly assume this.

Let L be a countable language consisting of n-ary p^n_m functions with values in nonnegative rationals for every pair $m,n \in \mathbb{N}$ and binary function d with values in nonnegative rationals. For any structure A we will usually write just p^n_m (or d) on A instead of $(p^A)^n_m$ (or d^A). However, we may use the latter in few cases where there is a possibility of confusion.

Definition 5.1.4 (The class $\mathcal{K}$)**.** A finite structure A (we will not notationally distinguish a structure and its underlying set) for the language L of cardinality $k > 0$ belongs to $\mathcal{K}$ if the following conditions are satisfied:

1. A is a rational metric space; i.e.

   - d is a total function (defined on all pairs) on A
   - $\forall x, y \in A(d(x, y) = d(y, x))$
   - $\forall x, y \in A(d(x, y) = 0 \Leftrightarrow x = y)$
   - $\forall x, y, z \in A(d(x, y) \leq d(x, z) + d(z, y))$

   Thus we will interpret d as a metric.

2. There is some $n_A \leq k$ (recall that k is the cardinality of A) such that for every $n \leq n_A$ and $m \leq n_A + 1 - n$ p^n_m is a total function on A; on the other hand, for $n > n_A$ or $m > n_A + 1 - n$ p^n_m is defined on no n-tuple from A; i.e.

   $\exists n_A \leq k$

   - $\forall n, m \in \mathbb{N} \forall \vec{a} \in A^n (p^n_m(\vec{a})$ is defined $\Leftrightarrow n \leq n_A \wedge m \leq n_A + 1 - n)$

   We consider p^n_m as a function to rationals with an interpretation that it gives a rational distance (in a "sum" metric on A^n) of an n-tuple from one of the desired set $F^n_{m'}$. We note that m and m' will not necessarily be equal.

3. In order to satisfy the joint embedding property and the amalgamation property we must put some additional restrictions on these structures.

   - $\forall m, n \in \mathbb{N}(n \leq n_A \wedge m \leq n_A + 1 - n \Rightarrow \forall \vec{a}, \vec{b} \in A^n (p^n_m(\vec{a}) \leq p^n_m(\vec{b}) + d(\vec{a}, \vec{b}))$

   The previous formula is interpreted as follows. Consider the "sum" metric d on A^n, i.e. $d(\vec{a}, \vec{b}) = d(a_1, b_1) + \ldots + d(a_n, b_n)$. The function p^n_m assigns to each n-tuple a non-negative rational. We interpret this function as a distance function from a fixed closed set in the sum metric. The previous formula says that a distance of some n-tuple from this closed set must be less or equal to the sum of a distance of another n-tuple from the same closed set and the distance between these two n-tuples. In particular, if this distance is 0 for some n-tuple $\vec{a}$, i.e. we demand it will lie in the closed set, then this distance for some other n-tuple $\vec{b}$ must be less or equal to the distance between $\vec{a}$ and $\vec{b}$.

There is still one more condition which we must demand on these structures in order to satisfy the amalgamation property and to have only countably many isomorphism types of finite structures. We specify when we consider two structures to be isomorphic and what an embedding of one structure into another is. Informally, an isomorphism between two structures does not respect the enumeration of the rational functions p^n_m for every power n, i.e. for example we consider structures $A = \{a_1, a_2\}$ and $B = \{b_1, b_2\}$ such that $p^1_1(a_1) = q$, $p^1_1(a_2) = 0$, $p^1_2(a_1) = h$, $p^1_2(a_2) = 0$ and p^2_1 is equal to 0 on all pairs, and $p^1_1(b_1) = h$, $p^1_1(b_2) = 0$, $p^1_2(b_1) = q$, $p^1_2(b_2) = 0$ and p^2_1 is equal to 0 on all pairs to be isomorphic although the roles of p^1_1 and p^1_2 are switched in these two structures.

The precise definition follows.

Definition 5.1.5 (Isomorphism and embedding)**.** An isomorphism between two finite structures A, B in the language L is a pair $\phi, (\pi^\phi_n))$ where ϕ is an isometry between A and B for every $n \leq n_A (= n_B)$ $\pi^\phi_n : \{1, \ldots, n_A+1-n\} \to \{1, \ldots, n_B+1-n\}$ is a permutation such that

$$\forall n \leq n_A \forall m \leq n_A + 1 - n \forall \vec{a} \in A^n$$

$$(p^n_m(\vec{a}) = q \Leftrightarrow p^n_{\pi_n(m)}(\phi^n(\vec{a})) = q)$$

Two structures are isomorphic if there exists an isomorphism (pair) between them.

Similarly, an embedding of a structure A into a structure B is a pair $(\phi, (\pi_n))$ such that $\phi : A \hookrightarrow B$ is an isometric embedding and for every $n \leq n_A$ $(\leq n_B)$ $\pi_n : \{1, \ldots, n_A + 1 - n\} \to \{1, \ldots, n_B + 1 - n\}$ is an injection such that

$$\forall n \leq n_A \forall m \leq n_A + 1 - n \forall \vec{a} \in A^n$$

$$(p^n_m(\vec{a}) = q \Leftrightarrow p^n_{\pi_n(m)}(\phi^n(\vec{a})) = q)$$

Now we must prove that $\mathcal{K}$ is countable, satisfies the hereditary, joint embedding and amalgamation property.

Lemma 5.1.6. *$\mathcal{K}$ is a Fraïssé class.*

Proof. We will prove that $\mathcal{K}$ is countable, satisfies the hereditary property, joint embedding property and amalgamation property.

For the cardinality, there are only countably many finite rational metric spaces. For each finite rational metric space A of cardinality n there are $n+1$ choices for n_A (recall that $n_A \leq n = |A|$) and for each such a choice only finitely many rational functions p^n_m can be defined, hence the claim follows.

The hereditary property is obvious. To check the joint embedding property, consider two structures $A, B \in \mathcal{K}$. Let $m_A = \max\{q : (A \vDash p_q(\vec{a})) \wedge p_q$ is either d_q or $p^n_{q,m}$ for some $n \leq n_A, m \leq n_A + 1 - n, \vec{a} \in A^n\}$; m_B is defined similarly for B. Let $m = \max\{m_A, m_B\}$. Let $C = A \coprod B$ be the disjoint union of A and B. For $a \in A$ and $b \in B$ we may set $d(a,b) = 2m$, so we extend the metric on the whole C. To extend other predicates, we set $n_C = \max\{n_A, n_B\}$ and it is easy to see that for every $n \leq n_C$ and $m \leq n_C + 1 - n$ and every n-tuple $(c_1, \ldots, c_n) \in C^n$ on which p^n_m has not been already defined there is always a choice which is consistent. For instance, for any $n \leq n_C$ and $m \leq n_C + 1 - n$ and $(c_1, \ldots, c_n) \in C^n$ for which p^n_m has not been yet defined we may set $p^n_m(c_1, \ldots, c_n) = 0$; this is consistent.

Finally, we need to check the amalgamation property. Let $A, B, C \in \mathcal{K}$ be structures, we can assume WLOG that A is a substructure of both B and C and for all $n \leq n_A$ and $m \leq n_A - n$ $(p^B)^n_m = (p^C)^n_m$. Let $D = A \coprod (B \setminus A) \coprod (C \setminus A)$. The metric is extended in a standard way, i.e. for $b \in B$ and $c \in C$ we set $d(b,c) = \min\{d(b,a) + d(a,c) : a \in A\}$.

Let us set $n_D = n_B + (n_C - n_A)$ (note that $n_A \leq \min\{n_B, n_C\}$). We reenumerate some rational functions on D (see Definition 5.1.5).

- For all $n \leq n_B$, $m \leq n_B + 1 - n$ and $\vec{b} \in B^n \subseteq D^n$ we let $(p^D)^n_m(\vec{b}) = (p^B)^n_m(\vec{b})$, i.e. we keep the enumeration from the original one in B.

- For all $n \leq n_A$, $m \leq n_A + 1 - n$ and $\vec{c} \in C^n \subseteq D^n$ we again let $(p^D)^n_m(\vec{c}) = (p^C)^n_m(\vec{c})$, i.e. keep the previous enumeration.
- For $n \leq n_A$ and $n_A + 1 - n < m \leq n_C + 1 - n$ or for $n_A < n \leq n_C$ and any $m \leq n_C + 1 - n$ and $\vec{c} \in C^n$ we set $(p^D)^n_{m+(n_B-n_A)}(\vec{c}) = (p^C)^n_m(\vec{c})$, i.e. we change the enumeration by adding $n_B - n_A$.

We need to check that this metric extension along with the reenumeration of some predicates is consistent.

Specifically, we need to check that the following formula still holds true whenever the function p^n_m is defined on both $\vec{b}$ and $\vec{c}$:

$$\forall \vec{b}, \vec{c} \in D^n (p^n_m(b_1, \dots, b_n) \leq p^n_m(\vec{c}) + d(b_1, c_1) + \dots + d(b_n, c_n))$$

Let some n, m, $\vec{b}$ and $\vec{c}$ in D^n such that both $p^n_m(\vec{b})$ and $p^n_m(\vec{c})$ are defined be given.

- If $n_A < n \leq n_B$ and $m \leq n_B + 1 - n$ then it follows that both $\vec{b}$ and $\vec{c}$ are from B^n and the formula holds in D since it holds in B.
- For any n if $m > n_B + 1 - n$ then it follows that both $\vec{b}$ and $\vec{c}$ are from C^n and the formula holds in D since it holds in B with $m' = m - (n_B - n_A)$.
- Finally, assume that $n \leq n_A$ and $m \leq n_A + 1 - n$. If $\vec{b}$ and $\vec{c}$ are either both from B^n or both from C^n then the formula holds in D since it holds in B, resp. in C. So let us assume that $\vec{b}$ is originally from B^n and $\vec{c}$ is originally from C^n (the opposite case is the same of course). From definition, for every $i \leq n$ there is some $a_i \in A$ such that $d(b_i, c_i) = d(b_i, a_i) + d(a_i, c_i)$. We have

$$p^n_m(\vec{b}) \leq p^n_m(\vec{a}) + d(b_1, a_1) + \dots + d(b_n, a_n)$$

since this formula holds true in B. Similarly, we have

$$p^n_m(\vec{a}) \leq p^n_m \vec{c}) + d(a_1, c_1) + \dots + d(a_n, c_n)$$

since this formula holds true in C. Putting together, we obtain

$$p_m^n(\vec{b}) \leq p_m^n(\vec{c}) + d(b_1, c_1) + \ldots + d(b_n, c_n)$$

which is what we wanted to prove.

For any other m, n that were not listed above the functions p_m^n were not yet defined. So for $n \leq n_D$ and $m \leq n_D + 1 - n$ that were not listed above we may set $p_m^n(\vec{d}) = 0$ for all $\vec{d} \in D^n$ for instance. For any fixed pair $n \leq n_D$ and $m \leq n_D + 1 - n$ that was listed above but p_m^n was not yet defined on some $\vec{d} \in D^n$ we define it canonically as follows (but there are other possible definitions too): We set $p_m^n(\vec{d}) = \max\{0, \max\{p_m^n(\vec{d'}) - d(\vec{d'}, \vec{d}) : p_m^n \text{ was defined on } \vec{d'}\}\}$. Note that even if we used this definition on some n-tuple on which p_m^n had been already defined then it would get the same value. That is why we call it canonical. To check it is consistent let $\vec{d_0}, \vec{d_1} \in D^n$ be some n-tuples. Assume at first that $p_m^n(\vec{d_0}) > 0$ and $p_m^n(\vec{d_1}) > 0$ and let $\vec{d'_0}, \vec{d'_1} \in D^n$ be such that $p_m^n(\vec{d_0}) = p_m^n(\vec{d'_0}) - d(\vec{d'_0}, \vec{d_0})$ and $p_m^n(\vec{d_1}) = p_m^n(\vec{d'_1}) - d(\vec{d'_1}, \vec{d_1})$. Then $p_m^n(d_0) = p_m^n(\vec{d'_0}) - d(\vec{d'_0}, \vec{d_0}) \leq p_m^n(\vec{d'_0}) - d(\vec{d'_0}, \vec{d_1}) + d(\vec{d_0}, \vec{d_1}) \leq p_m^n(\vec{d'_1}) + d(\vec{d_0}, \vec{d_1})$. The case when $p_m^n(\vec{d_0}) = 0$ or $p_m^n(\vec{d_1}) = 0$ is similar and the proof is left to the reader. □

Since $\mathcal{K}$ is a Fraïssé class it has a Fraïssé limit which we denote U. Besides other things it is a metric space. In fact it is a countable universal homogeneous rational metric space (see Remark 5.1.7). By $\mathbb{U}$ we denote its metric completion which is the universal Urysohn space. For every natural n we also have the set $\mathcal{F}_n$ of countably many rational functions on U^n without an enumeration arising from the Fraïssé limit though. We choose some enumeration and denote the set $\mathcal{F}_n$ as $\{f_m^n : m \in \mathbb{N}\}$ for every n. For every $m, n \in \mathbb{N}$ the set $\tilde{F}_m^n$ of all n-tuples $\vec{u}$ from U such that $f_m^n(\vec{u}) = 0$ is a closed subset of U^n. By F_m^n we denote the closure of $\tilde{F}_m^n$ in the completion $\mathbb{U}$ (thus we have $F_m^n \cap U = \tilde{F}_m^n$). This finishes the construction of the sets from the statement of Theorem 5.1.2. We must now prove the almost universality and ultrahomogeneity of these sets which we do in the following section.

5.1.1 The one-point extension property

When constructing the Fraïssé limit we had to work only with rational metric spaces and rational functions p_m^n on them in order to have the class $\mathcal{K}$ countable and to have the limit U. The one-point extension property holds for substructures of U (see 1.3.9). We formulate it here for convenience again. We call it here "rational one-point extension property".

Rational one-point extension property. Let $A \in \mathcal{K}$ be a finite rational metric space such that the rational functions f_m^n, for $n \leq n_A \leq |A|$ and $m \leq n_A + 1 - n$, are defined on it. Let $B \in \mathcal{K}$ be a one point extension of A, i.e. $|B| - |A| = 1$ and there is an embedding $(\iota, (\pi_n^\iota)) : A \hookrightarrow B$. Assume that there is an embedding $(\phi, (\pi_n^\phi)) : A \hookrightarrow U$. Then there is an embedinng $(\psi, (\pi_n^\psi)) : B \hookrightarrow U$ extending $(\phi, (\pi_n^\phi))$, i.e. $(\phi, (\pi_n^\phi)) = (\psi, (\pi_n^\psi)) \circ (\iota, (\pi_n^\iota))$.

Before we proceed further we use this place for the following remark.

Remark 5.1.7. We still owe the explanation that the underlying metric space of our (almost) universal and ultrahomogeneous structure is isometric to the Urysohn universal metric space. To prove it it suffices to check that the underlying metric structure U of the countable Fraïssé limit is isometric to the universal rational metric space (as its completion is isometric to the Urysohn space). However, realize that a countable rational metric space X is isometric to the universal rational metric space if and only if it has the rational one-point metric extension property: for any finite metric subspace $F \subseteq X$ and any one-point extension $G \supseteq F$ which is still a rational metric space, there is an isometric embedding $\iota : G \hookrightarrow X$ such that $\iota \restriction F = \mathrm{id}$.

However, U has this rational one-point metric extension property. Here, and also in the next section, its rational one-point extension property is always stronger.

However, since we made the completion $\mathbb{U}$ we want to have this kind of one-point extension property for all finite substructures of $\mathbb{U}$, not just for those that are actually substructures of U. In this section we prove this full one-point extension property. The almost universality and homogeneity, and uniqueness will follow by a standard argument. We define a generalized class $\bar{\mathcal{K}}$ of structures (that correspond to finite substructures of $(\mathbb{U}, (F^n_m))$).

Definition 5.1.8. A substructure $A \in \bar{\mathcal{K}}$ is a finite metric space, moreover there is some $n_A \leq |A|$ and for each $n \leq n_A$ there is a finite set of indices $I^A_n \subseteq \mathbb{N}$ such that $|I^A_n| = n_A - n + 1$. For each $n \leq n_A$ and $m \in I^A_n$ there is a closed subset $G^n_m \subseteq A^n$. By p^n_m we shall again denote the distance function from the set G^n_m.

An embedding of a substructure A into a substructure B is a pair $(\phi, (\pi_n))$ where $\phi : A \hookrightarrow B$ is an isometric embedding and for each $n \leq n_A$ $\pi_n : I^A_n \to I^B_n$ is an injection such that $\forall n \leq n_A \forall m \in I^A_n \forall \vec{x} \in A^n (p^n_m(\vec{x}) = p^n_{\pi_n(m)}(\phi^n(\vec{x})))$.

Thus we just drop the condition that the metric and functions p^n_m have to have rational values.

Proposition 5.1.9 (One-point extension property)**.** *Let A be a finite substructure of $(\mathbb{U}, (F^n_m))$ and let $B \in \bar{\mathcal{K}}$ be such that $|B| = |A| + 1$ and there is an embedding $(\phi, (\pi^\phi_n))$ of A into B. Then there exists an embedding $(\psi, (\pi^\psi_n))$ of B into $(\mathbb{U}, (F^n_m))$ such that* $\mathrm{id} = (\psi, (\pi^\psi_n)) \circ (\phi, (\pi^\phi_n))$.

Before we provide a proof we show that the almost universality and homogeneity and also the uniqueness follow from Proposition 5.1.9.

Claim 5.1.10 (Almost universality)**.** $(\mathbb{U}, (F^n_m))$ *is almost universal.*

Proof of the Claim. Let (X, d) be a Polish metric space (in fact, it can be just separable metric) equipped with sets $(G^n_m)_{m,n \in \mathbb{N}}$ where for each n and m $G^n_m \subseteq X^n$ is a closed subset of the n-th power of X. Let $D \subseteq X$ be a countable subset with the following properties:

- D is a dense subset of X
- For every m and n $D^n \cap G^n_m$ is a dense subset of G^n_m.

We prove that there exist an isometric copy D' of D in $\mathbb{U}$ and injections π_i from $\mathbb{N}$ to $\mathbb{N}$ for all i such that for every m and n and $\vec{d} \in D^n$ we have $\vec{d} \in G^n_m \Leftrightarrow \vec{d'} \in F^n_{\pi_n(m)}$ and if $\vec{d} \notin G^n_m$ then $d(\vec{d}, G^n_m\} = d_{\mathbb{U}}(\vec{d'}, F^n_{\pi_n(m)})$, where $\vec{d'}$ corresponds to $\vec{d}$ in the copy. Then we will extend the isometry to the closure of D which is the whole space X and we will be done. To see that, let $m, n \in \mathbb{N}$ and $\vec{x} \in X^n$ be arbitrary.

If $\vec{x} \in G^n_m$ then there is a sequence $(d^j_1, \ldots, d^j_n)_j \subseteq D^n$ converging to $\vec{x}$ such that $\vec{d^j} \in G^n_m$ for every j. From our assumption, $\vec{d'^j} \in F^n_{\pi_n(m)}$ and since $F^n_{\pi_n(m)}$ is closed the image of $\vec{x}$ also lies in $F^n_{\pi_n(m)}$.

If $\vec{x} \notin G^n_m$ and $\varepsilon = d(\vec{x}, G^n_m)$ then there is $\vec{d} \in D^n$ such that $d(\vec{x}, \vec{d}) < \varepsilon/3$. It follows that $d(\vec{d}, G^n_m) > 2\varepsilon/3$, thus $d_{\mathbb{U}}(\vec{d'}, F^n_{\pi_n(m)}) > 2\varepsilon/3$ and thus the image of $\vec{x}$ also does not lie in $F^n_{\pi_n(m)}$.

Let us enumerate the set D as $\{d_1, d_2, \ldots\}$. The construction of D' is by induction, just a series of applications of Proposition 5.1.9. Let B_1 be a one-point structure containing d_1, $n_{B_1} = 1$ and $I^{B_1}_1 = \{1\}$. Let A_1 be an empty structure and use Proposition 5.1.9 to get an embedding of B_1 into $\mathbb{U}$. The embedding determines a point $u_1 \in \mathbb{U}$ and also an injection $\pi_1 : I^{B_1}_1 \to \mathbb{N}$. We have $d(d_1, G^1_1) = p^1_{\pi_1(1)}(u_1) = d_{\mathbb{U}}(u_1, F^1_{\pi_1(1)})$.

Assume we have found $u_1, \ldots, u_{k-1}$. Consider a structure B_k containing $\{d_1, \ldots, d_k\}$, $n_{B_k} = k$, for $i \leq k$ $I^{B_k}_i = \{1, \ldots, k - i + 1\}$. Let A_k be a substructure of $(\mathbb{U}, (F^n_m))$ containing $\{u_1, \ldots, u_{k-1}\}$, $n_{A_k} = k - 1$ and for $i \leq k - 1$ $I^{A_k}_i = \{\pi_i(1), \ldots, \pi_i(k - i)\}$. There is an obvious embedding of A_k into B_k so we can use Proposition 5.1.9 to extend A_k by some new point u_k. We also extend the domain of π_i, for $i \leq k - 1$, by $k - i + 1$ and obtain a new injection π_k with domain $\{1\}$. This finishes the induction. □

Claim 5.1.11 (Almost ultrahomogeneity). *$(\mathbb{U}, (F^n_m))$ is almost ultrahomogeneous.*

Sketch of the proof. Let A and B be two isomorphic substructures (witnessed by $(\phi, (\pi^\phi_n))$) of $(\mathbb{U}, (F^n_m))$. WLOG assume that for every $n \leq n_A = n_B$ we have $I^A_n = I^B_n = \{1, \ldots, n_A - n + 1\}$ and π^ϕ_n is the identity on I^A_n. Let $D = \{u_n : n \in \mathbb{N}\} \subseteq U$ be a countable dense subset such that for every m, n $D^n \cap F^n_m$ is dense in F^n_m. By a back-and-forth series of use of the one-point extension property (Proposition 5.1.9) we shall be extending the isomorphism $(\phi, (\pi^\phi_n))$ into a chain $(\phi, (\pi^\phi_n)) \subseteq (\phi_1, (\pi^{\phi_1}_{n,1})) \subseteq (\phi_2, (\pi^{\phi_2}_{n,2})) \subseteq \ldots$ so that for every $m \in \mathbb{N}$ u_m is both in the domain and range of ϕ_m and m is in the domain and range of $\pi_{m,1}$. $\bigcup_m (\phi_m, (\pi^{\phi_m}_{n,m}))$ is the desired isomorphism of $(\mathbb{U}, (F^n_m))$. □

Claim 5.1.12 (Uniqueness). $(\mathbb{U}, (F^n_m))$ *is unique with the almost universality and ultrahomogeneity property.*

This is again done by a standard back-and-forth argument using Proposition 5.1.9.

Before we prove Proposition 5.1.9 we need the following lemma that will be useful in the next section too.

Lemma 5.1.13. *Let $M = \{d_1, \ldots, d_k\}$ be a given finite metric space. Also, for every $i < k$ let $(u^j_i)_j \subseteq U$ be a given rational Cauchy sequence from the rational Urysohn space such that $d(u^j_i, u^{j+1}_i) \leq 1/2^{j+1}$ for all j and moreover, $d_{\mathbb{U}}(\lim_n u^n_i, \lim_n u^n_j) = d_M(d_i, d_j)$ for every $i, j < k$.*

Moreover, let $l \in \mathbb{N}$ be given and let $\{u^1_k, \ldots, u^{l-1}_k\} \subseteq U$ (if $l = 1$ then it is an empty sequence) be a given finite rational sequence with the following property: for every $j < l$ and every $i < k$ we have $d_M(d_k, d_i) + 1/(k \cdot 2^{j+1}) \leq d(u^j_k, u^{j+k+2}_i) \leq d_M(d_k, d_i) + 1/2^j$.

Then if we consider the space $A_k = \{u^{l+k+2}_1, \ldots, u^{l+k+2}_{k-1}, u^{l-1}_k\}$ (resp. $A_k = \{u^{l+k+2}_1, \ldots, u^{l+k+2}_{k-1}\}$ if $l = 1$) then there exists a rational metric extension $U \supseteq M_k = A_k \cup \{g_k\}$ such that $d_M(d_k, d_i) + (2i - 1)/(k \cdot 2^{l+1}) \leq d(g_k, u^{l+k+2}_i) \leq d_M(d_k, d_i) + (2i)/(k \cdot 2^{l+1}$ for all $i < k$ and if $l > 1$ then also $d(g_k, u^{l-1}_k) = 1/2^l$.

Proof of the lemma.
We will treat separately two cases. Case 1 is when $l = 1$ and Case 2 is when we are moreover given a non-empty finite sequence $\{u^1_k, \ldots, u^{l-1}_k\}$, i.e. $l > 1$.

Case 1: $l = 1$.

Let $i_1, \ldots, i_{k-1}$ be a permutation of $\{1, \ldots, k-1\}$ such that we have $d(d_k, d_{i_1}) \geq d(d_k, d_{i_2}) \geq \ldots \geq d(d_k, d_{i_{k-1}})$. For each $j < k$ we shall denote v_j the element u^{l+k+2}_j. We have that $d_{\mathbb{U}}(v_j, u_j) \leq 1/2^{l+k+2}$. We now work with $\{v_1, \ldots, v_{k-1}\}$. For $j < k$ let $\gamma_j \in \mathbb{R}^+$ be arbitrary positive real numbers such that $(2j - 1)/(k \cdot 2^{l+1}) \leq \gamma_j \leq (2j)/(k \cdot 2^{l+1})$ and $\eta_j = d(d_k, d_{i_j}) + \gamma_j \in \mathbb{Q}$. We claim there exists $g_k \in U$ such that $d_{\mathbb{U}}(g_k, v_j) = \eta_j$. We just need to check that the triangle inequalities are satisfied, then it will follow that such an element g_k does exist from the one-point (metric) extension property of U.

Let $i < j < k$, we shall check that $\eta_i - \eta_j \leq d(v_{i_i}, v_{i_j}) \leq \eta_i + \eta_j$. We have $|d(v_{i_i}, v_{i_j}) - d(d_{i_i}, d_{i_j})| < 1/2^{l+k+1} \leq 1/(k \cdot 2^l)$. Since $\eta_i - \eta_j \leq d(d_k, d_{i_i}) - d(d_k, d_{i_j}) - 1/(k \cdot 2^{l+1}) \leq d(d_k, d_{i_i}) - d(d_k, d_{i_j}) - 1/2^{l+k+1}$, thus $\eta_i - \eta_j \leq d(v_{i_i}, v_{i_j})$. Since $\eta_i + \eta_j \geq d(d_k, d_{i_i}) + d(d_k, d_{i_j}) + 1/(k \cdot 2^{l+1}) \geq d(d_k, d_{i_i}) + d(d_k, d_{i_j}) + 1/2^{l+k+1}$, thus also $d(v_{i_i}, v_{i_j}) \leq \eta_i + \eta_j$.

So by the one-point extension there exists such $g_k \in U$.

Case 2: $l > 1$. We proceed identically as in Case 1, we just need to care about the element u_k^{l-1}. Let again $i_1, \ldots, i_{k-1}$ be a permutation of $\{1, \ldots, k-1\}$ such that we have $d(d_k, d_{i_1}) \geq d(d_k, d_{i_2}) \geq \ldots \geq d(d_k, d_{i_{k-1}})$. For each $j < k$ we shall denote v_j the element u_j^{l+k+2}. We work with the space $\{u_k^{l-1}, v_1, \ldots, v_{k-1}\}$. For $j < k$ let $\gamma_j \in \mathbb{R}^+$ be arbitrary positive real numbers such that $(2j-1)/(k \cdot 2^{l+1}) \leq \gamma_j \leq (2j)/(k \cdot 2^{l+1})$ and $\eta_j = d(d_k, d_{i_j}) + \gamma_j \in \mathbb{Q}$. We claim there exists $g_k \in U$ such that $d_{\mathbb{U}}(g_k, v_j) = \eta_j$ and moreover $d_{\mathbb{U}}(g_k, u_k^{l-1}) = 1/2^l$. We again just need to check that the triangle inequalities are satisfied, then it will follow that such an element g_k does exist.

For $i < j < k$ the verification that $\eta_i - \eta_j \leq d(v_{i_i}, v_{i_j}) \leq \eta_i + \eta_j$ holds is the same as in Case 1.

Now let $j < k$ be given. We need to check that $\eta_j - 1/2^l \leq d(v_{i_j}, u_{k+1}^{l-1}) \leq \eta_j + 1/2^l$. Note that

$$d(u_k^{l-1}, u_{i_j}^{k+l+1}) - d(u_{i_j}^{k+l+1}, v_{i_j}) \leq d(v_{i_j}, u_k^{l-1})$$

and

$$d(v_{i_j}, u_k^{l-1}) \leq d(u_k^{l-1}, u_{i_j}^{k+l+1}) + d(u_{i_j}^{k+l+1}, v_{i_j})$$

The following estimates on $d(u_k^{l-1}, u_{i_j}^{k+l+1})$ follow from the assumption from the statement of the lemma. We have

$$d(d_{i_j}, d_k) + (2j-1)/(k \cdot 2^l) \leq d(u_k^{l-1}, u_{i_j}^{k+l+1}) \leq d(d_{i_j}, d_k) + (2j)/(k \cdot 2^l)$$

Similarly, we have the following estimates on η_j:

$$d(d_{i_j}, d_k) + (2j-1)/(k \cdot 2^{l+1}) \leq \eta_j \leq d(d_{i_j}, d_k) + (2j)/(k \cdot 2^{l+1})$$

We check the inequality $\eta_j - 1/2^l \leq d(v_{i_j}, u_k^{l-1})$. Putting the previous inequalities together it suffices to check that

$$d(d_{i_j}, d_k) + (2j)/(k \cdot 2^{l+1}) - 1/2^l \leq d(d_{i_j}, d_k) + (2j-1)/(k \cdot 2^l) - 1/2^{k+l+2}$$

By subtracting from both sides we get

$$(-2j+2)/(k \cdot 2^{l+1}) - 1/2^l \leq -1/2^{k+l+2}$$

which clearly holds.

To check the other inequality $d(v_{i_j}, u_k^{l-1}) \leq \eta_j + 1/2^l$ using the previous inequalities it suffices to check that

$$d(d_{i_j}, d_k) + (2j)/(k \cdot 2^l) + 1/2^{k+l+2} \leq d(d_{i_j}, d_k) + (2j-1)/(k \cdot 2^{l+1}) + 1/2^l$$

By subtracting from both sides we get

$$(2j+1)/(k \cdot 2^{l+1}) + 1/2^{k+l+2} \leq 1/2^l$$

Since $j \leq k-1$ we have

$$(2j+1)/(k \cdot 2^{l+1}) + 1/2^{k+l+2} \leq (2k-1)/(k \cdot 2^{l+1}) + 1/2^{k+l+2}$$

and the following equality holds

$$(2k-1)/(k \cdot 2^{l+1}) + 1/2^{k+l+2} = 1/2^l - 1/(k \cdot 2^{l+1}) + 1/2^{k+l+2}$$

The right hand side is clearly less or equal to $1/2^l$ so we are done.

So again by the one-point (metric) extension property there exists such $g_k \in U$. □

Proof of Proposition 5.1.9. Let us at first treat the case when A is empty and B is a one-point structure $\{b_1\}$. We have $n_B = 1$ and WLOG assume that $I_1^B = \{1\}$. Thus we only need to find some $a_1 \in \mathbb{U}$ and $m \in \mathbb{N}$ such that $p_m^1(a_1) = p_1^1(b_1)$. For every $n \in \mathbb{N}$ let $\delta_n \in \mathbb{Q}_0^+$ be any non-negative rational number such that

$p^1_1(b_1) \leq \delta_n \leq p^1_1(b_1) + 1/2^{l+2}$. We use the rational one-point extension property to define a sequence $(u^j_1)_j \subseteq U$ and to obtain $m \in \mathbb{N}$ such that for every $j \in \mathbb{N}$ $p^1_m(u^j_1) = \delta_j$ and $d_{\mathbb{U}}(u^j_1, u^{j+1}_1) = 1/2^{j+1}$. It is straightforward to check that we have $p^1_m(a_1) = p^1_1(b_1)$ where a_1 is the limit of the sequence $(u^j_1)_j$.

We now assume that A is non-empty. Let us enumerate A as $\{a_1, \ldots, a_{k-1}\}$ and B as $\{b_1, \ldots, b_k\}$ so that the embedding $((\phi, (\pi^\phi_n))$ of A into B sends a_i to b_i for every $i < k$. We extend A by adding a point a_k. We will find a Cauchy sequence of elements from U such that the limit will be this desired point a_k. For each $l < k$ let us choose a converging sequence $(u^j_l)_j \subseteq U$ of elements from the Fraïssé limit such that $\lim_j u^j_l = a_l$, $d_{\mathbb{U}}(u^j_l, a_l) < 1/2^j$ and for $i < j$ we have $d_{\mathbb{U}}(u^j_l, a) < d_{\mathbb{U}}(u^i_l, a)$.

In order to simplify the notation we assume that $n_B = n_A + 1$ and for each $n \leq n_A$ $I^A_n = \{1, \ldots, n_A - n + 1\}$ and also for each $n \leq n_B$ $I^B_n = \{1, \ldots, n_B - n + 1\}$ and the injections π^ϕ_n are the identities. Consider a structure $S_1 = \{u^{k+3}_1, \ldots, u^{k+3}_{k-1}\}$ with $n_{S_1} = n_A$ and for every $n \leq n_{S_1}$, $m \leq n_{S_1} - n + 1$ and $\vec{x} \in S^n_1$ $p^n_m(\vec{x}) = d_{\mathbb{U}}(\vec{x}, F^n_m)$. Thus $S_1 \in \mathcal{K}$ and for any $i, j < k$ we have $|d_{\mathbb{U}}(u^{k+3}_i, u^{k+3}_j) - d(b_i, b_j)| < 1/2^{k+2}$. We use Lemma 5.1.13 to define a metric one-point extension $M_1 = \{u^{k+3}_1, \ldots, u^{k+3}_{k-1}, g\}$ of S_1 such that for all $i < k$ we have $d(b_i, b_k) \leq d_{\mathbb{U}}(u^{k+3}_i, g) \leq d(b_i, b_k) + 1/2$. We define a structure V_1 with $n_{V_1} = n_{S_1} + 1 = n_B$ such that M_1 is its underlying (rational) metric space. We need to define (rational) p^n_m on all n-tuples containing g for all $n \leq n_B$ and $m \leq n_B - n + 1$ and also on all n-tuples (not necessarily containing g) for $n \leq n_B$ and $m = n_B - n + 1$ to obtain a one-point extension V_1 of S_1.

Fix such a pair n, m. Let us enumerate all n-tuples $\vec{x} \in M^n_1$ as $(\vec{x}^1_j)_{j<J}$ so that all n-tuples not containing g precede every n-tuple containing g. Also, for any n-tuple $\vec{x} \in M^n_1$ let $\vec{b}_{\vec{x}}$ denote the corresponding n-tuple $\vec{y}$ from B^n (via the function sending u^{k+3}_i to b_i for $i < k$ and g to b_k). We inductively define p^n_m on $\vec{x}^1_j$'s. Let $\vec{x}^1_j$, for some $j < J$, be given. Let $\varepsilon^1_j = p^n_m(\vec{b}_{\vec{x}^1_j})$. It is not necessarily a rational number. Let $r^1_j \in \mathbb{Q}$ be an arbitrary rational number such that $\varepsilon^1_j \leq r^1_j \leq \varepsilon^1_j + n/2^{k+3}$. Also, let $m^1_j = \max\{p^n_m(\vec{x}) - d(\vec{x}^1_j, \vec{x}) : \vec{x} \in M^n_1 \wedge p^n_m \text{ has been already defined on } \vec{x}\}$ and $M^1_j = \min\{p^n_m(\vec{x}) + d(\vec{x}^1_j, \vec{x}) : \vec{x} \in M^n_1 \wedge p^n_m \text{ has been already defined on } \vec{x}\}$.

If $m_j^1 \leq r_j^1 \leq M_j^1$ then we set $p_m^n(\vec{x}_j^1) = r_j^1$. If $r_j^1 < m_j^1$, resp. $r_j^1 > M_j^1$ then we set $p_m^n(\vec{x}_j^1) = m_j^1$, resp. $p_m^n(\vec{x}_j^1) = M_j^1$. Note that if $n \leq n_A$ and $m \leq n_A - n + 1$ and $\vec{x}_j^1 \in S_1^n$ then $m_j^1 = M_j^1 = p_m^n(\vec{x}_j^1)$, thus by our assigning we really obtain an extension of S_1. Thus by a weak one-point extension property we obtain some $u_k^1 \in U$ playing the role of g.

Assume we have already constructed $u_k^1, \ldots, u_k^{l-1} \subseteq U$ such that $d_{\mathbb{U}}(u_k^i, u_k^{i+1}) = 1/2^{i+1}$ for $0 \leq i < l-1$. Consider a structure $S_l = \{u_1^{k+l+2}, u_{k-1}^{k+l+2}, u_k^{l-1}\}$ with $n_{S_l} = n_B$ and for every $n \leq n_{S_l}$, $m \leq n_{S_l} - n + 1$ and $\vec{x} \in S_l^n$ $p_m^n(\vec{x}) = d_{\mathbb{U}}(\vec{x}, F_m^n)$. Thus $S_l \in \mathcal{K}$ and for any $i, j < k$ we have $|d_{\mathbb{U}}(u_i^{k+l+2}, u_j^{k+l+2}) - d(b_i, b_j)| < 1/2^{k+l+1}$. We again use Lemma 5.1.13 to obtain a metric one-point extension $M_l = \{u_1^{k+l+2}, u_{k-1}^{k+l+2}, u_k^{l-1}, g\}$ of S_l such that such that for all $i < k$ we have $d(b_i, b_k) \leq d_{\mathbb{U}}(u_i^{k+l+2}, g) \leq d(b_i, b_k) + 1/2^l$.

For $n \leq n_B$ and $m \leq n_B - n + 1$ we need to define p_m^n on all n-tuples from M_l^n containing the new element g. We do it as before: Fix such a pair n, m. Let us again enumerate all n-tuples $\vec{x} \in M_l^n$ as $(\vec{x}_j^l)_{j<K}$ so that all n-tuples not containing g precede any n-tuple containing g. Also, for any n-tuple $\vec{x} \in M_l^n$ let again $\vec{b}_{\vec{x}}$ denote the corresponding n-tuple $\vec{y}$ from B^n (via the function sending u_i^{k+l+2} to b_i for $i < k$ and u_k^{l-1} and g to b_k). We inductively define p_m^n on $\vec{x}_j^l$'s. Let $\vec{x}_j^l$, for some $j < K$, be given. Let $\varepsilon_j^l = p_m^n(\vec{b}_{\vec{x}_j^l})$. It is not necessarily a rational number. Let $r_j^l \in \mathbb{Q}$ be an arbitrary rational number such that $\varepsilon_j^l \leq r_j^l \leq \varepsilon_j^l + n/2^{k+l+2}$. Also, let $m_j^l = \max\{p_m^n(\vec{x}) - d(\vec{x}_j^l, \vec{x}) : \vec{x} \in M_l^n \wedge p_m^n$ has been already defined on $\vec{x}\}$ and $M_j^l = \min\{p_m^n(\vec{x}) + d(\vec{x}_j^l, \vec{x}) : \vec{x} \in M_l^n \wedge p_m^n$ has been already defined on $\vec{x}\}$. If $m_j^l \leq r_j^l \leq M_j^l$ then we set $p_m^n(\vec{x}_j^l) = r_j^l$. If $r_j^l < m_j^l$, resp. $r_j^l > M_j^l$ then we set $p_m^n(\vec{x}_j^l) = m_j^l$, resp. $p_m^n(\vec{x}_j^l) = M_j^l$.This is again a consistent extension of S_l. Thus by a weak one-point extension property we obtain some $u_k^l \in U$ playing the role of g.

Assume the induction is finished. We have found a sequence $(u_k^j)_j$. Moreover, realize that for every $n \leq n_B$ and $m = n_B - n + 1$ there is some $\varpi_n \in \mathbb{N}$ such that for every $\vec{x} \in \{u_i^j : i \leq k, j \in \mathbb{N}\}^n$ we have $p_m^n(\vec{x}) = d_{\mathbb{U}}(\vec{x}, F_{\varpi_n}^n)$. Since for any $j \in \mathbb{N}$ we have $d_{\mathbb{U}}(u_k^j, u_k^{j+1}) = 1/2^{j+1}$, this sequence is Cauchy with a limit that we denote a_k. We define an embedding $(\psi, (\pi_n^\psi))$ of B into $(\mathbb{U}, F_m^n)$ as follows: $\psi(b_i) = a_i$ for every $i \leq k$ and for $n \leq n_B$ we set $\pi_n^\psi(i) = i$ if $i < n_B - n + 1$

and $\pi_n^\psi(i) = \varpi_n$ if $i = n_B - n + 1$. It follows from the use of Lemma 5.1.13 that $d_{\mathbb{U}}(a_i, a_k) = d(b_i, b_k)$ for every $i < k$. We must check that $p_n^m(\vec{x}) = p^n_{\pi_n^\psi(m)}(\psi^n(\vec{x}))$ for all $n \leq n_B$, $m \leq n_B - n + 1$ and $\vec{x} \in B^n$.

Claim 5.1.14. *For every $j < K$ and $l \in \mathbb{N}$ we have $\varepsilon_j^l - n/2^{k+l+2} \leq p_m^n(\vec{x}_j^l) \leq \varepsilon_j^l + n/2^{k+l+2}$.*

Once the claim is proved the assertion follows. So it remains to prove the claim.

Proof of the Claim. We prove it for every $j < K$ by induction on l.

Step 1.

Suppose $l = 1$. Let us prove that $p_m^n(\vec{x}_j^1) \leq \varepsilon_j^1 + n/2^{k+3}$. We have $p_m^n(\vec{x}_j^1) = \max\{r_j^1, m_j^1\}$. Since $r_j^1 \leq \varepsilon_j^1 + 1/2^{1+1}$ it suffices to prove that $m_j^1 \leq \varepsilon_j^1 + (2n + 1)/2^{1+1}$.

Realize that $m_j^1 = p_m^n(\vec{x}_p^1) - d(\vec{x}_j^1, \vec{x}_p^1)$ for some $\vec{x}_p^1$.

1. There exists $\vec{x}_p^1 \in S_1^n$ such that $m_j^1 = p_m^n(\vec{x}_p^1) - d(\vec{x}_j^1, \vec{x}_p^1)$. Let $\vec{x}_p^1 = (u_{i_1}^{k+3}, \ldots, u_{i_n}^{k+3})$. Since for every $m \leq n$ we have $d(u_{i_m}^{k+3}, a_{i_m}) \leq 1/2^{k+3}$, we have that $d(\vec{x}_p^1, (a_{i_1}, \ldots, a_{i_n})) \leq n/2^{k+3}$, thus $p_m^n(\vec{x}_p^1) \leq \varepsilon_p^1 + n/2^{k+3}$. We also have that $d(\vec{b}_{\vec{x}_p^1}, \vec{b}_{\vec{x}_j^1}) \leq d(\vec{x}_p^1, \vec{x}_j^1)$. Finally, since $\varepsilon_p^1 \leq \varepsilon_j^1 + d(\vec{b}_{\vec{x}_p^1}, \vec{b}_{\vec{x}_j^1})$, putting the inequalities together we obtain $m_j^1 \leq \varepsilon_j^1 + n/2^{k+3}$.

2. There does not exist such $\vec{x}_p^1 \in S_1^n$. We claim that then $m_j^1 = p_m^n(\vec{x}_p^1) - d(\vec{x}_j^1, \vec{x}_p^1)$ where $p_m^n(\vec{x}_p^1) = r_p^1$. Once we prove this is true then from the same series of inequalities as in the item above we prove the desired inequality. Suppose it is not true. Then $m_j^1 = p_m^n(\vec{x}_p^1) - d(\vec{x}_j^1, \vec{x}_p^1)$ and $p_m^n(\vec{x}_p^1) = p_m^n(\vec{x}_{q_1}^1) - d(\vec{x}_p^1, \vec{x}_{q_1}^1)$ for some $\vec{x}_{q_1}^1$. If still $p_m^n(\vec{x}_{q_1}^1) \neq r_{q_1}^1$ then $p_m^n(\vec{x}_{q_1}^1) = p_m^n(\vec{x}_{q_2}^1) - d(\vec{x}_{q_1}^1, \vec{x}_{q_2}^1)$ for some $\vec{x}_{q_2}^1$. We continue until after finitely many steps we reach $\vec{x}_{q_n}^1$ such that $p_m^n(\vec{x}_{q_n}^1) = r_{q_n}^1$. However, observe that it follows from the series of triangle inequalities that $p_m^n(\vec{x}_j^1) = m_j^1 = p_m^n(\vec{x}_{q_n}^1) - d(\vec{x}_j^1, \vec{x}_{q_n}^1)$ and we are done.

Let us now prove that $\varepsilon_j^1 - n/2^{k+3} \leq p_m^n(\vec{x}_j^1)$. Since we have $p_m^n(\vec{x}_j^1) = \min\{r_j^1, M_j^1\}$ it suffices to prove that $M_j^1 \geq \varepsilon_j^1 - n/2^{k+3}$. Again realize that

$M_j^1 = p_m^n(\vec{x}_p^1) + d(\vec{x}_j^1, \vec{x}_p^1)$ for some $\vec{x}_p^1$. There are again two cases:

1. There exists $\vec{x}_p^1 \in S_1^n$ such that $M_j^1 = p_m^n(\vec{x}_p^1) + d(\vec{x}_j^1, \vec{x}_p^1)$. Then since $\varepsilon_j^1 \leq \varepsilon_p^1 + d(\vec{b}_{\vec{x}_p^1}, \vec{b}_{\vec{x}_j^1})$ we get from the inequalities above that $\varepsilon_j^1 - n/2^{k+3} \leq M_j^1$.

2. If there is no such $\vec{x}_p^1 \in S_1^n$ then as in item (2) above we can find $\vec{x}_p^1$ such that $M_j^1 = p_m^n(\vec{x}_p^1) + d(\vec{x}_j^1, \vec{x}_p^1)$ and $p_m^n(\vec{x}_p^1) = r_p^1$. Then the verification is again analogous.

Step 2. Now we assume that $l > 1$ and for all $m < l$ the claim has been proved.

If $p_m^n(\vec{x}_j^l) = r_j^l$ then it is clear. So we only have to prove that $m_j^l \leq \varepsilon_j^l + n/2^{k+l+2}$ and $\varepsilon_j^l - n/2^{k+l+2} \leq M_j^l$. We only prove the former, the latter is completely analogous.

We have $m_j^l = p_m^n(\vec{x}_p^l) - d(\vec{x}_j^l, \vec{x}_p^l)$ for some $\vec{x}_p^l$. As in Step 1 we find out that there (now) three possibilities (the verification that there precisely one of these three possibilities happens is similar to the verification that precisely one of those two possibilities in Step 1 happens).

1. There exists $\vec{x}_p^l \in (S_l \setminus \{u_k^{l-1}\})^n$ such that $m_j^l = p_m^n(\vec{x}_p^l) - d(\vec{x}_j^l, \vec{x}_p^l)$. Then it is analogous to the item (1) in Step 1.

2. There exists $\vec{x}_p^l$ such that $m_j^l = p_m^n(\vec{x}_p^l) - d(\vec{x}_j^l, \vec{x}_p^l)$ and $p_m^n(\vec{x}_p^l) = r_p^l$. This is analogous to the item (2) from Step 1.

3. There exists $\vec{x}_p^l$ such that $m_j^l = p_m^n(\vec{x}_p^l) - d(\vec{x}_j^l, \vec{x}_p^l)$ and $\vec{x}_p^l$ is an n-tuple obtained from $\vec{x}_j^l$ by replacing all occurences of g by u_k^{l-1}, thus $\vec{x}_p^l$ is in fact equal to some $\vec{x}_q^{l-1}$ and $\varepsilon_j^l = \varepsilon_q^{l-1}$. By induction hypothesis we have that $p_m^n(\vec{x}_q^{l-1}) \leq \varepsilon_j^l + (2n+1)/2^l$. Since $d(u_k^{l-1}, g) = 1/2^l$ we have that $d(\vec{x}_q^{l-1}, \vec{x}_j^l) \geq 1/2^l \geq n/2^{k+l+2}$, thus $m_j^l = p_m^n(\vec{x}_q^{l-1}) - d(\vec{x}_q^{l-1}, \vec{x}_j^l) \leq \varepsilon_j^l + n/2^{k+l+2}$ as desired.

□

Remark 5.1.15. The previous proof can be slightly modified so that it proves Theorem 5.1.1. We consider a language containing a symbol for rational metric and for every n_i, $i \leq m$, a symbol for rational n_i-ary function p_{n_i}. These functions are interpreted as distance functions from the desired closed sets F_{n_i}. Since there are only finitely many such rational functions they are all defined on all finite structures from $\mathcal{K}$. The restrictions are the same, i.e. for any finite structure $A \in \mathcal{K}$ we have for all $i \leq m$ that $\forall \vec{a}, \vec{b} \in A^{n_i}(p_{n_i}(\vec{a}) \leq p_{n_i}(\vec{a}) + d(a_1, b_1) + \ldots + d(a_{n_i}, b_{n_i}))$. The verification that such $\mathcal{K}$ is a Fraïssé class is similar (only easier) as in Lemma 5.1.6. Similarly, the proof one-point extension property is similar, just easier, as in the proof of Proposition 5.1.9.

Observation 5.1.16. The method used in the proof of Theorem 5.1.2 to obtain countably many almost universal closed sets can be repeated in other instances. What we describe below is a general scheme. Note that we are very informal there and we refer to the proof of Theorem 5.1.2 for an example with details.

Suppose we have a proof of universality and ultrahomogeneity of some metric structure using a Fraïssé limit of some class $\mathcal{K}$ of structures in some language L consisting of rational metric and some other predicates or functions $p_1, \ldots, p_n$ with values in some fixed countable set. We may consider a new language consisting of the rational metric and predicates or functions $p_1^i, \ldots, p_n^i$ with values in the same fixed countable set for each $i \leq \mathbb{N}$. A structure A belongs to this new class of structures $\tilde{\mathcal{K}}$ if there is some n_A (e.g. $|A|$) such that for all $i \leq n_A$ the functions (or predicates) $p_1^i, \ldots, p_n^i$ are defined on A with the same restrictions for each i as in $\mathcal{K}$ for a single set of these functions (or predicates). The isomorphism and embedding relation between structures in $\tilde{\mathcal{K}}$ is as in Definition 5.1.5. The verification that $\tilde{\mathcal{K}}$ is a Fraïssé class is similar as in Lemma 5.1.6. The one-point extension property is also similar as in the proof of Proposition 5.1.9.

5.2 Universal and ultrahomogeneous closed subsets of $\mathbb{U} \times K$ and Lipschitz functions from $\mathbb{U}$ to Z

In this section we consider a universal closed subset of $\mathbb{U} \times K$, where K is an arbitrary fixed compact metric space, and a universal L-Lipschitz function from $\mathbb{U}$ to Z, where L is an arbitrary fixed positive real number and Z is an arbitrary fixed Polish metric space.

Theorem 5.2.1. *Let K be an arbitrary compact metric space, Z an arbitrary Polish metric space and $L \in \mathbb{R}^+$. Then the structure $(\mathbb{U}, C, F)$ is universal and ultrahomogeneous and unique with this property, where $C \subseteq \mathbb{U} \times K$ is a closed subset and $F : \mathbb{U} \to Z$ is an L-Lipschitz function.*

See the third and fourth example.

Proof. We split the proof into two parts. In order to increase transparency of the proof we separately prove that there is such a universal closed set $C \subseteq \mathbb{U} \times K$ and then that there is such a universal L-Lipschitz function $F : \mathbb{U} \to Z$. It will be a routine modification to prove that are "simultaneously" universal and ultrahomogeneous. We will again use Fraïssé theory.

<u>The closed set C</u>.

Let $Q = \{q_n : q \in \mathbb{N}\}$ be an enumeration of a countable dense subset of K. We define the set $\mathcal{F} \subseteq (\mathbb{Q}_0^+)^{\mathbb{N}}$ of all suitable functions. A function $f : \mathbb{N} \to \mathbb{Q}_0^+$ belongs to $\mathcal{F}$ if there is a finite set $F \subseteq \mathbb{N}$ and non-negative rationals $r_i \geq 0$ for $i \in F$ such that $f(j) = \max\{0, \max\{r_i - d_K(q_j, q_i) : i \in F\}\}$ and it is always the case that $f(i) = r_i$ for every $i \in F$; i.e. f has the domain $\mathbb{N}$, however it is uniquely determined only by values on the finite set F. For $f \in \mathcal{F}$ we will denote such a finite set as F_f (it is not unique, however there is unique such a set F_f that is minimal in inclusion). Note that $\mathcal{F}$ is countable.

Let p be an unary function with values in the set $\mathcal{F}$. Also, we again consider the binary rational function d for metric. Let L be a language consisting precisely of these functions.

We now define the new class $\mathcal{K}$ of finite structures of the language L.

Definition 5.2.2. A finite structure A for the language L of cardinality $k > 0$ belongs to $\mathcal{K}$ if the following conditions are satisfied.

1. A is a finite rational metric space; i.e. it satisfies the same requirements as in the definition 5.1.4.

2. The function p is a total function, i.e. defined on all elements of A.

The interpretation of this functions is as follows: if $p(a)(n) = q > 0$ then the distance between (a, q_n) and C is at least (in fact precisely) q; on the other hand, if $p(a)(n) = 0$ then $(a, q_n) \in C$.

3. Here we describe the restriction that we must put on these functions.

   - $\forall a, b \in A \forall n, m \in \mathbb{N}(p(a)(n) \leq p(b)(m) + d_K(q_n, q_m) + d(a, b)$

   This requirement resembles the restriction from the proof of Theorem 5.1.2. The value $p(a)(n)$ determines a rational distance of (a, q_n) in the sum metric from the set C. Thus in case that for example $p(a)(n) = 0$, i.e. $(a, q_n) \in C$ and $p(b)(m) = q$, i.e. the distance in the sum metric of (b, q_m) from C is at least q, then necessarily the distance between (a, q_n) and (b, q_m) is at least q, i.e. $d(a, b) + d_K(q_n, q_m) \geq q$.

We must check that $\mathcal{K}$ is again countable (up to isomorphism classes), satisfies hereditary, joint embedding and amalgamation property. The first two properties are clear. The verification of the third one is similar as in Theorem 5.1.2, we can just put the structures sufficiently far apart from each other. To check the amalgamation property, suppose we have structures A, B, C such that A is a substructure of both B and C. We can again define D with underlying set $A \coprod (B \setminus A) \coprod (C \setminus A)$ with metric extended so that $d(b, c) = \min\{d(b, a) + d(a, c) : a \in A\}$ for $b \in B$ and $c \in C$, and $p^D(b) = p^B(b)$, resp. $p^D(c) = p^C(c)$, for $b \in B$, resp. $c \in C$ of course. Let us check that this works. Let $b \in B$, $c \in C$ and $n, m \in \mathbb{N}$. We check that $p(b)(n) \leq p(c)(m) + d(b, c) + d_K(q_n, q_m)$. Let $a \in A$ be such that $d(b, c) = d(b, a) + d(a, c)$. Then we have $p(b)(n) \leq p(a)(m) + d(b, a) + d_K(q_n, q_m) \leq p(c)(m) + d(a, c) + d(b, a) + d_K(q_n, q_m) = p(c)(m) + d(b, c) + d_K(q_n, q_m)$.

We again denote by U the Fraïssé limit which is besides other things again a rational Urysohn space. We define the set $C \subseteq \mathbb{U} \times K$ in the completion $\mathbb{U}$ as follows: $(a, r) \in C \equiv \neg \exists (u, g) \in U \times Q \exists n \in \mathbb{N}(g = q_n \wedge d(a, u) + d_K(r, g) < p(u)(n))$. It is obviously closed.

Let us now state and prove the following useful claim that confirms that p is really the distance function from the closed set.

Claim 5.2.3. *For any $u \in U$ and $n \in \mathbb{N}$ we have $p(u)(n) = d((u, q_n), C) = q$.*

Proof of the claim. We prove that for an arbitrary $\varepsilon > 0$ there exists $v \in U$ such that $d(u,v) < q + \varepsilon$ and $p(v)(n) = 0$. It follows that $q \leq d(u,q_n),C) \leq q + \varepsilon$ for an arbitrary $\varepsilon > 0$, thus $p(u)(n) = d((u,q_n),C) = q$. So let $\varepsilon > 0$ be given. Let $d_\varepsilon \in \mathbb{Q}^+$ be an arbitrary positive rational number so that $q \leq d_\varepsilon < q + \varepsilon$. Moreover, let $q_\varepsilon \in \mathbb{Q}_0^+$ be an arbitrary nonnegative rational number smaller or equal to $d_q - q$. Let $F = \{i \in F_{p(u)} : p(u)(i) > d_\varepsilon\}$. We define $f \in \mathcal{F}$ such that $F_f = F$, and for $i \in F$ we set $f(i) = p(u)(i) - d_q + q_\varepsilon$. We define a one-point extension of $\{u\}$ as follows: the underlying set is $\{u,v\}$, we set $d(u,v) = d_\varepsilon$ and $p(v) = f$. We claim it belongs to $\mathcal{K}$. Then by one-point extension property we can find such v in U and it is as desired: we need to prove that $p(v)(n) = 0$. Suppose not, then there is $i \in F_f$ such that $p(v)(i) - d_K(q_i,q_n) > 0$. However, since $p(u)(i) = p(v)(i) - q_\varepsilon + d_q$ we would have $p(u)(n) \geq p(u)(i) - d_K(q_i,q_n) = p(v)(i) - q_\varepsilon + d_q - d_K(q_i,q_n) > q$, a contradiction.

It remains to prove that $\{u,v\} \in \mathcal{K}$. Let $n,m \in \mathbb{N}$ be given. We prove that $p(u)(n) \leq p(v)(m) + d(u,v) + d_K(q_n,q_m)$. If $p(u)(n) \leq d(u,v)$ then it is clear, so let us suppose that $p(u)(n) > d(u,v)$ and let $i \in F$ be such that $p(u)(n) = p(u)(i) - d_K(q_n,q_i)$. Then $p(v)(m) \geq p(v)(i) - d_K(q_m,q_i) \geq p(u)(i) - d(u,v) + q_\varepsilon - d_K(q_n,q_i) - d_K(q_n,q_m)$. It follows that $p(v)(m) + d(u,v) + d_K(q_n,q_m) \geq p(u)(i) - d_K(q_n,q_i) + q_\varepsilon = p(u)(n) + q_\varepsilon$.

Now we prove that also $p(v)(m) \leq p(u)(n) + d(u,v) + d_K(q_n,q_m)$. If $p(v)(m) = 0$ then it is trivial, so let us suppose that $p(v)(m) > 0$ and let $i \in F$ be such that $p(v)(m) = p(v)(i) - d_K(q_m,q_i) = p(u)(i) - d(u,v) + q_\varepsilon - d_K(q_m,q_i)$. Then $p(u)(n) \geq p(u)(i) - d_K(q_i,q_m) - d_K(q_n,q_m)$, thus $p(u)(n) + d(u,v) + d_K(q_n,q_m) \geq p(u)(i) - d_K(q_i,q_m) + d(u,v) \geq p(u)(i) - d(u,v) + q_\varepsilon - d_K(q_m,q_i)$ and we are done. Note that the last inequality follows from $d(u,v) \geq -d(u,v) + q_\varepsilon$ which is immediate from the definition of $d(u,v)$ and q_ε. $\square$

5.2.1 The one-point extension property for $(\mathbb{U}, C)$

Let $\bar{\mathcal{K}}$ be again the "real" variant of $\mathcal{K}$, i.e. a structure A belongs to $\bar{\mathcal{K}}$ if it is a finite metric space equipped with a closed subset C_A of $A \times Z$, where C_A need not to be finite. For each $n \in \mathbb{N}$ and $a \in A$ we denote by $p(a)(n)$ the distance of (a,q_n) from C_A; $p(a)(n)$ in this case need not to be rational. The notions of embedding and isomorphism are obvious.

We again prove the one-point extension property for $\bar{\mathcal{K}}$ which simplifies the proofs of universality, ultrahomogeneity and uniqueness of $(\mathbb{U}, C)$. By "rational one-point extension property" we again mean the one-point extension property for structures from $\mathcal{K}$.

Proposition 5.2.4 (The one-point extension property)**.** *Let (A, C_A) be a finite substructure of $(\mathbb{U}, C)$ and let $(B, C_B) \in \bar{\mathcal{K}}$ be a one-point extension, i.e. $|B| = |A| + 1$ and there is an embedding $\phi : A \hookrightarrow B$. Then there exists an embedding $\psi : (B, C_B) \hookrightarrow (\mathbb{U}, C)$ such that $\mathrm{id} = \psi \circ \phi$.*

Before we provide the proof we again begin by showing how universality, ultrahomogeneity and uniqueness follow.

Proposition 5.2.5. *The Polish metric structure $(\mathbb{U}, C)$ is universal.*

Proof. Let (X, d) be a Polish metric space and $B \subseteq X \times K$ a closed set. Let again $D = \{d_n : n \in \mathbb{N}\} \subseteq X$ be a countable dense set. We will find an isometric copy D' of D in $\mathbb{U}$ such that for any $d_n \in D$ and $q_m \in Q$ we have $d((d_n, q_m), B) = d((d'_n, q_m), C)$. This suffices. We can then extend the isometry, let us call it ϕ, to the closure of D which is the whole space X. Let $(x, r) \in X \times K$ be arbitrary. Assume at first that $(x, r) \notin B$. Let $\varepsilon = d((x, r), B)$. Then there exist $i, n \in \mathbb{N}$ such that $d((d_i, q_n), B) \geq 2\varepsilon/3$ and $d((d_i, q_n), (x, r)) < \varepsilon/3$, thus $d((d'_i, q_n), C) \geq 2\varepsilon/3$, so $(\phi(x), r) \notin C$. On the other hand, assume that $(x, r) \in B$. Then there exists a sequence $(d_n, q_n)_n \subseteq D \times Q$ such that $(d_n, q_n) \to (x, r)$ and $(d(d_n, q_n), B) \to 0$. Thus also $(d'_n, q_n) \to (\phi(x), r)$ and since $d((d'_n, q_n), C) \to 0$ we have $d((\phi(x), r), C) = 0$.

The construction of D' is again just a series of applications of Proposition 5.2.4. ∎

Claim 5.2.6. *The structure $(\mathbb{U}, C)$ is ultrahomogeneous and a unique structure having this kind of one-point extension property.*

Proofs are completely analogous to those in the first section.

Proof of Proposition 5.2.4. Let us again at first treat the case when A is empty and $B = \{b_1\}$. We just need to find some $a_1 \in \mathbb{U}$ such that for every $n \in \mathbb{N}$ we have $p(a_1)(n) = p(b_1)(n)$. For every $l \in \mathbb{N}$ we define $f_l \in \mathcal{F}$ such that for every n we shall have $|f_l(n) - p(b_1)(n)| < 1/2^l$.

For every $n \in \mathbb{N}$ let $\varepsilon_n = d((b_k, q_n), C_B)$ $(= p(b_k)(n))$. Let $N \subseteq Q$ be a $1/2^{l+2}$-net in K, i.e. $\forall y \in K \exists x \in N (d_K(y, x) < 1/2^{l+2})$. N can be supposed to be finite since K is totally bounded (this is the place where we need K to be compact). Let F be the set of indices of elements from N, i.e. $N = \{q_i : i \in F\}$. For every $i \in F$ let $\gamma_i^l \in \mathbb{Q}_0^+$ be any non-negative rational number such that $0 \leq \gamma_i^l - \varepsilon_i < 1/2^{l+2}$.

We define $f_l \in \mathcal{F}$. It suffices to define f_l on a finite set F. Let F be equal to the set $\{i_1, \ldots, i_m\}$. WLOG we assume that $\gamma_{i_j}^l \geq \gamma_{i_l}^l$ for $j \leq l \leq m$.

We define f_l inductively as follows: at step 1 we set $f_l(i_1) = \gamma_{i_1}^l$. Suppose we are at step $n \leq m$. If $\eta_{i_n}^l = \max\{\gamma_i^l - d_K(q_i, q_{i_n}) : i \in \{i_1, \ldots, i_{n-1}\}\} > \gamma_{i_n}^l$ then we set $f_l(i_n) = \eta_{i_n}^l$; otherwise, we set $f_l(i_n) = \gamma_{i_n}^l$. If we have finished then we have defined f_l on F $(= F_{f_l})$ which uniquely determines the values of f_l on $\mathbb{N}$. We now check that for every $l, n \in \mathbb{N}$ we have $|f_l(n) - p(b_1)(n)| < 1/2^l$. Let $n \in \mathbb{N}$ be arbitrary. There exists $i \in F$ such that $d_K(q_i, q_n) < 1/2^{l+2}$. Since it follows $|\varepsilon_i - \varepsilon_n| < 1/2^{l+2}$ and $|p(u_1^l)(n) - p(u_1^l)(i)| < 1/2^{l+2}$ (the functions $q_i \to \varepsilon_i$ and $q_i \to p(u_1^l)(i)$ are 1-Lipschitz) it suffices to check that for any $n \in F_{f_l}$ we have $|p(u_1^l)(n) - \varepsilon_n| \leq 1/2^{l+1}$. For $n \in F_{f_l}$ we either have that $p(u_1^l)(n) = \gamma_n^l$ or that $p(u_1^l)(n) = \eta_n^l$. If the former case holds then it is clear from the choose of γ_n^l. If $p(u_1^l)(n) = \eta_n^l$ then from the definition of η_n^l we have $\eta_n^l > \gamma_n^l$ and there exists $i \in F$ such that $p(u_1^l)(i) = \gamma_i^l$ and $p(u_1^l)(n) = \eta_n^l = p(u_1^l)(i) - d_K(q_i, q_n)$. Since $\eta_n^l > \gamma_n^l \geq \varepsilon_n - 1/2^{l+1}$ it suffices to check that $\eta_n^l \leq \varepsilon_n + 1/2^{l+1}$. However since $\varepsilon_i \leq \varepsilon_n + d_K(q_i, q_n)$ and $|\gamma_i^l - \varepsilon_i| \leq 1/2^{l+1}$ this follows.

Now we use the rational one-point extension property to define a sequence $(u_1^j)_j \subseteq U$ such that for every $j \in \mathbb{N}$ we have $p(u_1^j) = f_j$ and $d_{\mathbb{U}}(u_1^j, u_1^{j+1}) = 1/2^{j+1}$. It is straightforward to check that this is possible and since for every $j, n \in \mathbb{N}$ we have $|p(u_1^l)(n) - p(b_1)(n)| < 1/2^l$ it follows that $p(a_1)(n) = p(b_1)(n)$, for every $n \in \mathbb{N}$, where $a_1 = \lim_l u_1^l$.

We now assume that A is non-empty. Let us enumerate A as $\{a_1, \ldots, a_{k-1}\}$

and B as $\{b_1, \dots, b_k\}$ in such a way that for every $i < k$ we have $\phi(a_i) = b_i$. We shall find a new point $a_k \in \mathbb{U}$ such that the structures $A \cup \{a_k\}$ and B will be isomorphic. For every $i < k$ let $(u_i^j)_j \subseteq U$ be a sequence from the rational space U converging to a_i such that $d_{\mathbb{U}}(u_i^j, a_i) < 1/2^i$, for every $n \in \mathbb{N}$ $|p(a_i)(n) - p(u_i^j)(n)| < 1/2^{j+1}$ and for any pair $j < l$ we have $d_{\mathbb{U}}(u_i^j, a_i) > d_{\mathbb{U}}(u_i^l, a_i)$. We shall find a new sequence from U converging to the desired point a_k. This is done by induction.

Consider a structure $S_1 = \{u_1^{k+3}, \dots, u_{k-1}^{k+3}\}$ such that for every $i < k$ and $n \in \mathbb{N}$ we have $p(u_i^{k+3})(n) = d((u_i^{k+3}, q_n), C)$. Thus $S_1 \in \mathcal{K}$. We use Lemma 5.1.13 to define a metric one-point extension $M_1 = \{u_1^{k+3}, \dots, u_{k-1}^{k+3}, g\}$ such that for all $i < k$ we have $d(b_i, b_k) \leq d_{\mathbb{U}}(u_i^{k+3}, g) \leq d(b_i, b_k) + 1/2$. We define a structure V_1 such that M_1 is its underlying (rational) metric space. We need to define p on g. This will be similar to the definition of p on u_1^l's (from case when A was empty) but more complicated.

For every $n \in \mathbb{N}$ let $\varepsilon_n = d((b_k, q_n), C_B)$ $(= p(b_k)(n))$. Let $N \subseteq Q$ be a $1/2^3$-net in K, i.e. $\forall y \in K \exists x \in N (d_K(y, x) < 1/2^3)$. N can be supposed to be finite since K is totally bounded. Let F' be the set of indices of elements from N, i.e. $N = \{q_i : i \in F'\}$. We set $F = F' \cup \bigcup_{i<k} F_{p(u_i^{k+3})}$. For every $i \in F$ let $\delta_i^1 \in \mathbb{Q}_0^+$ be any non-negative rational number such that $0 \leq \delta_i^1 - \varepsilon_i < 1/2^2$. Also, we define m_i^1 to be $\max\{p(u_j^{k+3})(i) - d_{\mathbb{U}}(g, u_j^{k+3}) : j < k\}$ and M_i^1 to be $\min\{p(u_j^{k+3})(i) + d_{\mathbb{U}}(g, u_j^{k+3}) : j < k\}$. For every $i \in F$ if $m_i^1 \leq \delta_i^1 \leq M_i^1$ then we set $\gamma_i^1 = \delta_i^1$. If $\delta_i^1 < m_i^1$, resp. $M_i^1 < \delta_i^1$ then we set $\gamma_i^1 = m_i^1$, resp. $\gamma_i^1 = M_i^1$.

We define $f \in \mathcal{F}$. It suffices to define f on a finite set F. Let F be equal to the set $\{i_1, \dots, i_m\}$. WLOG we assume that $\gamma_{i_j}^1 \geq \gamma_{i_l}^1$ for $j \leq l \leq m$.

We define f inductively as follows: at step 1 we set $f(i_1) = \gamma_{i_1}^1$. Suppose we are at step $n \leq m$. If $\eta_{i_n}^1 = \max\{\gamma_i^1 - d_K(q_i, q_{i_n}) : i \in \{i_1, \dots, i_{n-1}\}\} > \gamma_{i_n}^1$ then we set $f(i_n) = \eta_{i_n}^1$; otherwise, we set $f(i_n) = \gamma_{i_n}^1$. If we have finished then we have defined f on F $(= F_f)$ which uniquely determines the values of f on $\mathbb{N}$.

We now put $p(g) = f$. It is straightforward to check it is consistent. We defined an extension $V_1 \in \mathcal{K}$ of S_1 and thus there is some $u_k^1 \in U$ playing the role of g.

Suppose we have already constructed $u_k^1, \dots, u_k^{l-1} \subseteq U$ such that $d_{\mathbb{U}}(u_k^i, u_k^{i+1}) =$

$1/2^{i+1}$ for any $i < l-1$. Consider a structure $S_l = \{u_1^{k+l+2}, \ldots, u_{k-1}^{k+l+2}, u_k^{l-1}\}$ with $p(u)(n) = d((u, q_n), C)$ for every $u \in S_l$ and $n \in \mathbb{N}$. We use Lemma 5.1.13 to obtain a metric extension $M_l = \{u_1^{k+l+2}, \ldots, u_{k-1}^{k+l+2}, u_k^{l-1}, g\}$ such that such that for all $i < k$ we have $d(b_i, b_k) \leq d_{\mathbb{U}}(u_i^{k+l+2}, g) \leq d(b_i, b_k) + 1/2^l$. We need to define p on g. This is done in the same way as in the first induction step: For every $n \in \mathbb{N}$ let $\varepsilon_n = d((b_k, q_n), C_B)$ $(= p(b_k)(n))$. Let $N \subseteq Q$ be a $1/2^{l+2}$-net in K, i.e. $\forall y \in K \exists x \in N(d_K(y, x) < 1/2^{l+2})$. N can be supposed to be finite since K is totally bounded. Let F' be the set of indices of elements from N, i.e. $N = \{q_i : i \in F'\}$. We set $F = F' \cup \bigcup_{i<k} F_{p(u_i^{k+l+2})}$. For every $i \in F$ let $\delta_i^l \in \mathbb{Q}_0^+$ be any non-negative rational number such that $0 \leq \delta_i^l - \varepsilon_i < 1/2^{l+2}$. Also, we define m_i^l to be $\max\{p(u)(i) - d_{\mathbb{U}}(g, u) : u \in \{u_j^{k+l+2} : j < k\} \cup \{g\}\}$ and M_i^l to be $\min\{p(u)(i) + d_{\mathbb{U}}(g, u) : u \in \{u_j^{k+l+2} : j < k\} \cup \{g\}\}$. For every $i \in F$ if $m_i^l \leq \delta_i^l \leq M_i^l$ then we set $\gamma_i^l = \delta_i^l$. If $\delta_i^l < m_i^l$, resp. $M_i^l < \delta_i^l$ then we set $\gamma_i^l = m_i^l$, resp. $\gamma_i^l = M_i^l$.

We define $f \in \mathcal{F}$. It suffices to define f on a finite set F. Let F be equal to the set $\{i_1, \ldots, i_m\}$. WLOG we assume that $\gamma_{i_j}^l \geq \gamma_{i_l}^l$ for $j \leq l \leq m$.

We define f inductively as follows: at step 1 we set $f(i_1) = \gamma_{i_1}^l$. Suppose we are at step $n \leq m$. If $\eta_{i_n}^l = \max\{\gamma_i^l - d_K(q_i, q_{i_n}) : i \in \{i_1, \ldots, i_{n-1}\}\} > \gamma_{i_n}^l$ then we set $f(i_n) = \eta_{i_n}^l$; otherwise, we set $f(i_n) = \gamma_{i_n}^l$. If we have finished then we have defined f on F $(= F_f)$ which uniquely determines the values of f on $\mathbb{N}$.

We now put $p(g) = f$. It is straightforward to check it is consistent. We defined an extension $V_l \in \mathcal{K}$ of S_l and thus there is some $u_k^l \in U$ playing the role of g.

Assume the induction is finished. We have produced a sequence $(u_k^j)_j \subseteq U$ such that for every $i \in \mathbb{N}$ we have $d_{\mathbb{U}}(u_k^i, u_k^{i+1}) = 1/2^{i+1}$ thus the sequence is Cauchy and we denote a_k its limit point. It immediately follows from the construction that $d_{\mathbb{U}}(a_i, a_k) = d(b_i, b_k)$ for every $i < k$. It remains to check that for every $y \in K$ $d((a_k, y), C) = d((b_k, y), C_B)$. It obviously suffices to check that for every $n \in \mathbb{N}$ $d((a_k, q_n), C) = d((b_k, q_n), C_B)$.

Claim 5.2.7. *Let $l, n \in \mathbb{N}$ be arbitrary. Then $|p(u_k^l)(n) - \varepsilon_n| \leq 1/2^l$.*

Once the claim is proved the previous assertion is clear so it remains to prove the claim.

Proof of the claim. As in the proof of the analogous Claim 5.1.14 we prove it by induction on l.

Step 1.

Suppose $l = 1$ (in some places where it may be confusing we shall still use the symbol l eventhough it is equal to 1 in Step 1). Let $n \in \mathbb{N}$ be arbitrary. There exists $i \in F$ such that $d_K(q_i, q_n) < 1/2^{l+2} = 1/2^2$. Since it follows $|\varepsilon_i - \varepsilon_n| < 1/2^{l+2}$ and $|p(u_k^1)(n) - p(u_k^1)(i)| < 1/2^{l+2}$ (the functions $q_i \to \varepsilon_i$ and $q_i \to p(u_k^1)(i)$ are 1-Lipschitz) it suffices to check that for any $n \in F$ we have $|p(u_k^1)(n) - \varepsilon_n| \leq 1/2^{l+1}$.

From the definition of $p(u_k^1)$ above we have two cases:

- $p(u_k^1)(n) = \gamma_n^1$. This splits into three subcases:

  1. $p(u_k^1)(n) = \delta_n^1$. However we defined that $0 \leq \delta_n^1 - \varepsilon_n \leq 1/2^{l+1} = 1/2^2$ so we are done.

  2. $p(u_k^1)(n) = m_n^1$. In that case $m_n^1 > \delta_n^1$ and we must check that $m_n^1 \leq \varepsilon_n + 1/2^{l+1}$.

     From the definition there is some $i < k$ such that $m_n^1 = p(u_i^{k+l+2})(n) - d(u_i^{k+l+2}, u_k^1)$. However, from the assumption we have $|p(u_i^{k+l+2})(n) - p(a_i)(n)| < 1/2^{k+l+2}$ and recall that $d(b_i, b_k) + 1/(k \cdot 2^{l+1}) \leq d(u_i^{k+l+2}, u_k^1)$. Since $p(b_i)(n) \leq \varepsilon_n + d(b_i, b_k)$ (recall that $p(b_i)(n) = p(a_i)(n)$ and $\varepsilon_n = p(b_k)(n)$), putting these three inequalities together the inequality $m_n^1 \leq \varepsilon_n + 1/2^{l+1}$ follows.

  3. $p(u_k^1)(n) = M_n^1$. In that case $M_n^1 < \delta_n^1$ and we must check that $M_n^1 \geq \varepsilon_n^1 - 1/2^{l+1}$. From the definition there is some $i < k$ such that $M_n^1 = p(u_i^{k+l+2})(n) + d(u_i^{k+l+2}, u_k^1)$. We again use the inequalities from the previous item, i.e. $|p(u_i^{k+l+2})(n) - p(a_i)(n)| < 1/2^{k+l+2}$ and $d(b_i, b_k) + 1/(k \cdot 2^{l+1}) \leq d(u_i^{k+l+2}, u_k^1)$. Moreover, since $\varepsilon_n (= p(b_k)(n)) \leq p(b_i)(n) + d(b_i, b_k)$, putting these three inequalities together the inequality $M_n^1 \geq \varepsilon_n - 1/2^{l+1}$ follows.

- $p(u_k^1)(n) = \eta_n^1$. Then it follows from the definition that there exists some

$i \in F$ such that $p(u_k^1)(i) = \gamma_i^1$ and $\eta_n^1 = \gamma_i^1 - d_K(q_n, q_i) > \delta_n^1$. Since we already know from the previous item that $\delta_n^1 \geq \varepsilon_n - 1/2^{l+1}$ and we know that $\eta_n^1 > \delta_n^1$ we have that $\eta_n^1 > \varepsilon_n - 1/2^{l+1}$. Thus it suffices to check that $\eta_n^1 \leq \varepsilon_n + 1/2^{l+1}$. We again have three subcases:

1. $\gamma_i^1 = \delta_i^1$. We have that $p(b_k)(i)(= \varepsilon_i) \leq p(b_k)(n)(= \varepsilon_n) + d_K(q_i, q_n)$. Since we know from the previous item that $\delta_i^1 \leq \varepsilon_i + 1/2^{l+1}$ and since $\eta_n^1 = \delta_i^1 - d_K(q_n, q_i)$ we get that $\eta_n^1 \leq \varepsilon_n + 1/2^{l+1}$.
2. $\gamma_i^1 = M_i^1$. In that case we have that $M_i^1 < \delta_i^1$ thus the inequality $\eta_n^1 \leq \varepsilon_n + 1/2^{l+1}$ follows from (1) immediately above.
3. $\gamma_i^1 = m_i^1$. In that case there is some $j < k$ such that $\gamma_i^1 = m_i^1 = p(u_j^{k+l+2})(i) + d(u_j^{k+l+2}, u_k^1)$. Since $p(b_j)(i) \leq \varepsilon_n(= p(b_k)(n)) + d_K(q_i, q_n) + d(b_i, b_k)$, using the inequalities from (2) and (3) from the previous item we get that $\eta_n^1 \leq \varepsilon_n + 1/2^{l+1}$.

Step 2.

Now we assume that $l > 1$ and for all $i < l$ the claim has been proved. Let again $n \in \mathbb{N}$ be arbitrary. Then there exists $i \in F$ such that $d_K(q_i, q_n) < 1/2^{l+2}$. Thus it again suffices to check that for any $n \in F$ we have $|p(u_k^1)(n) - \varepsilon_n| \leq 1/2^{l+1}$. There are again two cases: either $p(u_k^l)(n) = \gamma_n^l$ or $p(u_k^l)(n) = \eta_n^l$. Both of them are treated similarly as in Step 1; let us illustrate it only on the former. We again have three subcases:

1. $p(u_k^l)(n) = \delta_n^l$. However we defined that $0 \leq \delta_n^l - \varepsilon_n \leq 1/2^{l+1}$ so we are done.

2. $p(u_k^l)(n) = m_n^l$. In that case $m_n^l > \delta_n^l$ and we must check that $m_n^l \leq \varepsilon_n + 1/2^{l+1}$.

   From the definition there is some $u \in \{u_i^{k+l+2} : i < k\} \cup \{u_k^{l-1}\}$ such that $m_n^1 = p(u)(n) - d(u, u_k^l)$. If $u \in \{u_i^{k+l+2} : i < k\}$ then the proof is completely analogous to the corresponding item in Step 1. So we assume that $u = u_k^{l-1}$. However, we have from the induction hypothesis that $|p(u_k^{l-1})(n) - \varepsilon_n| < 1/2^l$ and since $d(u_k^{l-1}, u_k^l) = 1/2^l$ we have that $m_n^l \leq \varepsilon_n + 1/2^{l+1}$.

3. $p(u_k^l)(n) = M_n^l$. In that case $M_n^l < \delta_n^l$ and we must check that $M_n^l \geq \varepsilon_n^1 - 1/2^{l+1}$. From the definition there is some $u \in \{u_i^{k+l+2} : i < k\} \cup \{u_k^{l-1}\}$ such that $M_n^l = p(u)(n) - d(u, u_k^l)$. Again as in (2) above, if $u \in \{u_i^{k+l+2} : i < k\}$ then the proof is completely analogous to the corresponding item in Step 1, so we assume that $u = u_k^{l-1}$. However, we again use the induction hypothesis that $|p(u_k^{l-1})(n) - \varepsilon_n| < 1/2^l$ and since $d(u_k^{l-1}, u_k^l) = 1/2^l$ we have that $M_n^l \geq \varepsilon_n^1 - 1/2^{l+1}$.

This finishes the proof of the claim and also of Proposition 5.2.4.

The Lipschitz function F

Let a Lipschitz constant $L \in \mathbb{R}^+$ be fixed. Let $Q = \{q_n : n \in \mathbb{N}\}$ be an enumeration of some fixed countable dense subset Q of the Polish metric space Z.Let p be an unary function with values in $\mathbb{N}$ and d again a binary rational function. Let L be a language consisting of these functions.

We again define the (new) class $\mathcal{K}$ of structures in the language L and then prove it satisfies the required properties of the Fraïssé theory.

Definition 5.2.8. A finite structure A for the language L of cardinality k belongs to $\mathcal{K}$ if the following conditions are satisified

1. A is again a finite rational metric space, i.e. it satisfies the same requirements as in definitions before. We will again interpret d as a metric.

2. The function p is a total function.

   The intended interpretation of this function is that the value $p(a)$ determines the value of the universal continuous function F at a as follows: $F(a) = q_{p(a)}$.

3. Here we put the restrictions on these structures which is just the demand that the desired function F is L-Lipschitz. For every a and b from A $d_Z(q_{p(a)}, q_{p(b)}) \leq L \cdot d(a, b)$.

Now we verify that $\mathcal{K}$ satisfies all properties needed to have a Fraï ssé limit. The countability and hereditary property are clear. To check joint embedding property, consider two structures A and B. Consider again m_A defined as $\max\{d(a,b) : a,b \in A\}$, m_B defined analogously for B and moreover, $m_F = \max\{L \cdot d_Z(q_{p(a)}, q_{p(b)}) : a \in A, b \in B\}$. Set $m = \max\{m_A, m_B, m_F\}$ and define the metric on $A \coprod B$ as follows: for $a \in A$, $b \in B$, $d(a,b) = 2m$. This again works.

Finally, we need to check the amalgamation property. So let $A, B, C \in \mathcal{K}$ be structures and we assume that A is a substructure of both B and C. We set $D = A \coprod (B \setminus A) \coprod (C \setminus A)$. The metric is again extended in the standard way, i.e. for $b \in B$ and $c \in C$ we set $d(b,c) = \min\{d(b,a) + d(a,c) : a \in A\}$.

We need to check that for any $b \in B$ and $c \in C$ we still have $d_Z(q_{p(b)}, q_{p(c)}) \leq L \cdot d(b,c)$. Let $a \in A$ be such that $d(b,c) = d(b,a) + d(a,c)$. We have $d_Z(q_{p(b)}, q_{p(c)}) \leq d_Z(q_{p(b)}, q_{p(a)}) + d_Z(q_{p(a)}, q_{p(c)}) \leq L \cdot d(b,a) + L \cdot d(a,c) = L \cdot d(b,c)$.

We again denote the Fraïssé limit as U. We define a function $\tilde{F}$ on U to Z as follows: $\tilde{F}(u) = q_{p(u)}$. It follows from our construction that $\tilde{F}$ is L-Lipschitz, thus we may extend $\tilde{F}$ to the completion $\mathbb{U}$; we denote F this unique L-Lipschitz extension and claim that this is the desired universal L-Lipschitz function to the Polish metric space Z.

5.2.2 The one-point extension property for $(\mathbb{U}, F)$

We again prove a particular version of one-point extension property. The method how to use it to derive the universality, ultrahomogeneity and uniqueness is the same as before. By $\bar{\mathcal{K}}$ we denote the class of all finite metric spaces equipped with an L-Lipschitz function into Z. Recall that we have the rational one-point extension property concerning structures from $\mathcal{K}$.

Proposition 5.2.9 (One-point extension property)**.** *Let A be a finite substructure of $(\mathbb{U}, F)$ and let $B \in \bar{\mathcal{K}}$ be such that $|B| = |A| + 1$ and there is an embedding ϕ of A into B. Then there exists an embedding ψ of B into $(\mathbb{U}, F)$ such that* $\mathrm{id} = \psi \circ \phi$.

Proof of the proposition. We again start with the case when A is empty and $B = \{b_1\}$. We just need to find some $a_1 \in \mathbb{U}$ such that $F(a_1) = F(b_1)$. Choose some sequence $(f_1^l)_l \subseteq \mathbb{N}$ such that for every $n \in \mathbb{N}$ $d_Z(q_{f_1^n}, q_{f_1^{n+1}}) \leq L/2^{n+1}$ and

$q_{f_1^l} \to F(b_1)$. Using the rational one-point extension property we find a sequence $(u_1^j)_j \subseteq$ such that for every $n \in \mathbb{N}$ we have $p(u_1^n) = f_1^n$ and $d_{\mathbb{U}}(u_1^n, u_1^{n+1}) = 1/2^{n+1}$. This is possible and we have that $F(a_1) = F(b_1)$ where $a_1 = \lim_n u_1^n$.

We now assume that A is non-empty. Let us enumerate A as $\{a_1, \ldots, a_{k-1}\}$ and B as $\{b_1, \ldots, b_k\}$ so that the embedding ϕ of A into B sends a_i to b_i for every $i < k$. We shall find a new point $a_k \in \mathbb{U}$ and define an embedding $\psi : B \hookrightarrow (\mathbb{U}, F)$ sending b_i to a_i for every $i \leq k$. We will find a Cauchy sequence of elements from U such that the limit will be this desired point a_k. For each $l < k$ let us choose a converging sequence $(u_l^j)_j \subseteq U$ of elements from the Fraïssé limit such that $\lim_j u_l^j = a_l$, $d_{\mathbb{U}}(u_l^j, a_l) < 1/2^j$, for $i < j$ we have $d_{\mathbb{U}}(u_l^j, a) < d_{\mathbb{U}}(u_l^i, a)$, and moreover for every natural numbers $i > j$ we have $d_Z(F(u_l^j), F(u_l^i)) < L/(k \cdot 2^{j+2})$. For every $l < k$ and $i \in \mathbb{N}$ let $f_l^i \in \mathbb{N}$ be such that $F(u_l^i) = q_{f_l^i}$.

Now, let us a choose a sequence $(f_k^j)_j \subseteq \mathbb{N}$ of natural numbers such that $\forall j \in \mathbb{N} \forall i > j(d_Z(q_{f_k^j}, q_{f_k^i}) < L/(k \cdot 2^{j+2}))$ and $q_{f_k^j} \to F(b_k)$. Consider a structure $S_1 = \{u_1^{k+3}, \ldots, u_{k-1}^{k+3}\}$. For every $a \in S_1$ we set $p(a) = n$ iff $F(a) = q_n$, thus $S_1 \in \mathcal{K}$. We use Lemma 5.1.13 to find a metric extension $M_1 = \{u_1^{k+3}, \ldots, u_{k-1}^{k+3}, g\}$ such that for all $i < k$ we have $d(b_i, b_k) + 1/(k \cdot 2^2) \leq d_{\mathbb{U}}(u_i^{k+3}, g) \leq d(b_i, b_k) + 1/2$. We extend M_1 into a structure V_1 from $\mathcal{K}$. We just need to define p on g. We set $p(g) = f_k^1$. To check that this is consistent we need to verify $d_Z(q_{f_k^1}, q_{f_j^{k+3}}) \leq L \cdot d(u_j^{k+3}, g)$ for all $j < k$. However, since $d_Z(F(b_i), f_j^{k+3}) \leq L/(k \cdot 2^{k+5}))$ and $d(b_j, b_k) + 1/(k \cdot 2^2) \leq d_{\mathbb{U}}(u_j^{k+3}, g)$ it follows that

$$d_Z(q_{f_k^1}, q_{f_j^{k+3}}) \leq d_Z(F(b_j), F(b_k)) + L/(k \cdot 2^{k+5}) + L/(k \cdot 2^3) \leq L \cdot d(b_j, b_k)$$

$$+L/(k \cdot 2^2)) \leq L \cdot d(u_j^{k+3}, g) - L/(k \cdot 2^2) + L/(k \cdot 2^2)) \leq L \cdot d(u_j^{k+3}, g)$$

Thus $V_1 \in \mathcal{K}$ and there is some $u_k^1 \in U$ playing the role of g.

Suppose we have already constructed $u_k^1, \ldots, u_k^{l-1} \subseteq U$. We consider a structure $S_l = \{u_1^{k+l+2}, \ldots, u_{k-1}^{k+l+2}, u_k^{l-1}\}$ with an obvious definition of p on elements of S_l. We use again Lemma 5.1.13 to obtain a metric extension $M_l = \{u_1^{k+l+2}, \ldots, u_{k-1}^{k+l+2}, u_k^{l-1}, g\}$ such that for all i, k we have $d(b_i, b_k) + 1/(k \cdot$

$2^{l+1}) \leq d_{\mathbb{U}}(u_i^{k+l+2}, g) \leq d(b_i, b_k)+1/2^l$. We need to define p on g; we set $p(g) = f_k^l$. The verification that it is consistent is the same as above. So we obtain some $u_k^l \in U$ playing the role of g. This finishes the induction and the proof. $\square$

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