Brokered Trading in Fast Markets under Heterogeneous Private Values and Information

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ABSTRACT

The paper develops a model of an order-driven market with very volatile motives for trade and a large number of (nearly) simultaneous limit and market order submissions. This corresponds to an electronic brokerage for very frequently traded securities such as high-grade bonds or FX. I consider small investors with a non-trivial distribution of private values for the traded asset as well as heterogeneous information about the parameters of this distribution across traders. The aim is to describe the price formation mechanism in this market. I study the properties of the mapping from the histogram of private asset values and private information endowments to the inside bid and ask price. Basic relationships between the limit order book, market sell and market buy order flow distributions, expected market order execution prices and the probabilities of a limit order execution at a given price, all as functions of private information, are derived. Traders are risk neutral as long as the transacted quantities are small, so that the limit and the market order decisions within one trading round do not feed back into private values. We formulate the equations that characterize the best ask and bid prices in this environment, as well as establish a number of properties of the equilibrium order book.

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1. Introduction

This paper models asset pricing mechanism for the market characterized by three interrelated features:

- a large number of participants prepared to trade within a narrow time span
- very short-lived motives for trade (frequent changes of desired positions) as well as very frequent changes of the aggregate supply/demand patterns implied by those motives
- uncertainty regarding the terms of individual trade for any given participant due to a high concentration in time of other participants' actions.

All three elements are typical for markets in many top-end securities. Since these are being increasingly traded through electronic brokerages, it is logical and even unavoidable to study price formation in a brokered market with a large number of participants who place limit and market orders at high frequencies. Henceforth, we will talk about a "fast market". The importance of (especially electronically) brokered trading for many securities has been growing in recent years. So, nowadays, the general question about the reasons of a market-clearing security price taking a particular value is likely to be specialized to: "Why the security price shown by a broker is on the observed level?" Therefore one needs a theory of price formation in an order driven market. When the market of interest is fast in the sense defined above, such a theory turns out to be technically difficult. This is probably why there does not exist much literature addressing the subject.

The paper aims at reducing the said deficit by constructing a formal tool for interpreting orders and trades in a fast order driven market endowed with a stylized representation of the three features as named above. The proposed formal setup can be applied to any security traded through a broker. However, some features of the model make it particularly appropriate to model markets for highly liquid securities with transaction and store-of-value roles, such as gilt bonds and FX. Firstly, the high frequency of order arrival and quote revisions is typical for the named security class. Secondly, as opposed to stocks, the economic information available on currencies and top-end bonds is usually hard to interpret in terms of conventional risk/return criteria, it is also hard to give a satisfactory definition of fundamentals or private information for them. Quite often, investors in those instruments are unable to agree on pricing implications of news traditionally tagged as "fundamental". Incoming information is practically impossible to interpret unambiguously even at the qualitative level, let alone in terms of quantifiable factors. Therefore, we have developed a model without an exogenous benchmark value. Thirdly, the short-lived nature of motives for trade of most participants in cash and gilt bonds, as well as the quickly evolving environment in which they have to make decisions, makes "learning" in the conventional microeconomic theory meaning irrelevant. There may simply not exist any stable parameter of the security to survive and be learned, a minute from now. Therefore, we consider a one-shot game an appropriate approach to analyze a brokerage in one of the named assets.

Although it is widely believed that the brokered (order-driven) market mechanism is more transparent than the decentralized dealership (quote-driven) one, the information available to fast brokered market users is rather limited. A trader is unable to benefit in full from the partial transparency offered by the broker. The best bid and offer prices I will call them "inside quotes") can change before he decides to submit a market order. For the same reason, under high order arrival rates, the observed first-best and the second-best limit order quantities are of limited usefulness, because the trader may not have enough time to hit any of them: others might attain service priority by pure luck if their orders are registered a splitsecond earlier. The degree of imperfection in the terms of trade observability can vary across market segments, but also with time within a given segment. In this paper, I define two stylized cases that differ by the severity of the terms-of-trade uncertainty problem faced by a market order submitter. The first one, to be called "hidden liquidity" (HL), applies when it is necessary to submit market orders and limit orders simultaneously, based only on imperfect knowledge of the limit order-submitters' preferences. The second one, which I dub "hidden priority" (HP), refers to a known state of the book at the moment market orders are to be submitted. Therefore, in principle, after having seen the book prior to ordering or not ordering, the trader can reconstruct the true distribution of private values in the market (for parametric distributions, I will call the corresponding vector of parameters *market state*). However, given that there are many other MO-submitters on the same side of the market with no trader having an ex ante time priority over the others, he cannot be sure how far away from the best quote his order will be executed. When I talk about fast markets, at least the HP-property must hold. HL, which contains HP as a sub-condition, means an even faster market with more terms-of-trade uncertainty. Under HL, even if the book is being displayed, the physical time is too scarce to make the market state-extracting calculations. In other words, the limit order book is formally "open", but for time reasons it is impossible for the traders to exploit this openness.

In the HL case, there is nothing better a trader can do than use private prior information to form beliefs about the complete state of the book, the probabilities of different market sell and market buy order flow realizations, the probabilities of a limit order execution depending on price, and the expected execution prices for a market sell and buy.

In this paper, the formal equilibrium conditions for both HL and HP cases are formulated, but numerical solutions are discussed only for an HP model. Due to analytic and numeric complexities of the HL model, its study is relegated to future research.

In the proposed model, the risk-neutral surplus maximizing investors that trade through the broker are assumed to have heterogeneous private values for the traded asset. They are allowed to submit limit and market orders for a pre-defined small standard quantity. A histogram of the traders' private values is not observed, but each market participant has some information about its parameters. This information is also heterogeneous across traders. We do not need to introduce noise traders to support an equilibrium. In short, the model has been constructed to explain how a collection of private values is transformed by a broker into a public price. In this sense, trades processed by the broker contain price-relevant information by construction: there are no exogenous price benchmarks. "Price" means the inside bid, inside ask pair that comes out of optimal reactions to the rationally anticipated flow of market orders.

In equilibrium, the broker can support a non-trivial trading pattern thanks to the two key properties of trader population:

(a) non-trivial and non-atomic distribution of private values for the asset among the investors who are only able to trade a small amount within one trading round
(b) limited and heterogeneous knowledge about the parameters of this distribution by the traders.

I state the equations that fully characterize the equilibrium inside bid and ask price, and derive a number of properties of the mapping from the preferences and beliefs histogram to these inside quotes. At the same time, I infer the equilibrium quantities of market buy and sell orders and the shape of the limit buy and sell books. The equilibrium prices and order quantities can be computed by an iterative procedure, taking any histogram of private asset values and beliefs from a sufficiently broad class as a primitive. In this equilibrium,

A. there is a non-trivial distribution of limit orders outside the best bid and ask quotes and there is no clustering of limit orders at the inside quotes

B. all market orders are executed, whereas the execution probability of almost every limit order is less than one

C. the highest individual trader surplus is not associated with the most precise information but with the right relation between the information bias and the trader's own private value.

I discuss a calibration procedure which is able to approximate typical order book configurations. However, independently of calibration, there exists an easily computable mapping from equilibrium market order realizations to the position and relative mode size of an arbitrary bimodal Gaussian private asset value histogram. The model is able to illustrate, for a given private value, the dependence of trader welfare on the quality of information. Finally, the model can incorporate such a “soft” factor as market sentiment into the traditional trade/price equivalence concept of microstructure finance.

More generally, although the model contains the usual attribute of microstructure models - a map from trades into price - it also provides an important qualifier. Namely, it shows that orders (and, consequently, trades, since every order is a trade at least potentially) are transmitted into price differently depending on informational parameters of the investor population. That is, any common element in the trader beliefs co-determines the price. This common element can be just the sentiment in conventional sense, but it can also reflect attitudes with a fundamental background. The brokerage mechanism that we model indeed channels order flow into the pair of inside prices. However, the buy and the sell market order flows alone are insufficient to fully explain those prices. This can be considered an innovative contribution of this paper to the “flow-centrist” vs. “fundamentalist” debate of the present day finance literature (a recent summary of this debate with the stress on FX markets is given in Froot and Ramadorai, 2005).

No special assumptions are made about the exclusivity of the modeled broker as a trading venue. Moreover, one can analyze quote-setting behavior and the processing of order flow information by a dealer acting in parallel with the discussed limit order book. Indeed, by observing the buy and sell order flows of the same trader population at her initial quotes, the dealer can form a posterior belief about the private value/information mass distribution parameters. Then, by offering traders a pair of adjusted quotes based on this knowledge, she is able to unload the risky position and derive positive utility from the second pair of order flows. In the model of this paper, we consider traders who only trade a small standard quantity both through the broker and the dealer, and remain risk-neutral within the range of these position shifts. This shift can be no higher than four standard quantities in either direction: at most two through the broker (e.g. as a result of a market sell and an executed limit sell) and at most two more with the dealer (before and after the brokered trading round). However, the model can be adjusted to reflect the possible impact of effectuated trades on the new private value, without changing the substance of the results. Also, in this paper we discuss only the easiest case when the private value/information distribution has just two unknown scalar parameters that can be reconstructed by the dealer exactly from the two observed order flows. Generalizations involving higher parameter dimensions and their imperfect extraction by the dealer are possible.

The model belongs to the simultaneous trade category. It seems that simultaneous trading in order-driven markets is a much harder case for formal treatment than that in quote-driven (dealership) ones, for which the models by Kyle, 1985, Glosten, 1989, or Evans and Lyons, 2002, are widely known. So, regardless of the degree of realism with which the present model captures the specifics of brokered trading in fast markets, it contributes to the microstructure finance literature by providing an order-driven counterpart to simultaneous trade dealership models. Accordingly, my conclusions on the (in)sufficiency of order flow statistics for price determination by a broker may cast a new light on the similar proposition discussed in the context of dealer markets. Specifically, if a limit order book looks differently under the same values but different common biases in beliefs, could it be that also dealer quotes react differently to the same incoming order flow under different opinions the dealer might share with her customers? The answer appears to be positive at least for a dealer whom one is able to incorporate in the present setting as a competitor to the broker.

In principle, application to real quote data at practically any frequency is possible, although the one-shot game that we analyze is more suitable as an instrument for an instantaneous capture of market conditions than a framework for generalizations over longer periods. In the former capacity, the model offers a potential policy tool for traders and risk managers (including ones at a central bank) in real time, suited to assess the unobserved distribution of the actual market valuations across traders.

1.1 Relation to existing literature

An electronic broker is an extremely complex object for formal analysis. So, it is not surprising that there exists unsatisfied demand for theoretical modeling of brokerage in fast markets. Most limit order book models have been created to address stock markets (Glosten, 1994, Parlour, 1998, Foucault, 1999, Handa, Schwartz and Tiwari, 2003). These models naturally reflect the institutions for trading in equities that have gradually evolved over the course of time, starting with the infrequent trading of a limited number of titles. This may be the explanation for the dominant use of the sequential trade approach. Similarly, the best-known empirical work on limit order books is based on stock exchange data (e.g. Biais, Hillion and Spatt, 1995, Handa and Schwartz, 1996). So far, FX brokers have been mainly analyzed empirically (Daníelsson and Payne, 2002, Rime, 2003, and some others). So, the missing theoretical foundation is probably due to the fact that the limit order book literature is captured by the mentioned sequential trade/external liquidation value discovery paradigm (this is the case of Parlour, 1998, Foucault, 1999, Handa, Schwartz and Tiwari, 2003, and other papers based on their models). As was argued earlier, this paradigm is inept at reflecting a number of prominent features of brokered trade in fast markets. Compared to this, the earlier model of Glosten, 1994, gets much closer to the objectives of our analysis. That is, Glosten models simultaneous trade in a continuous price and quantity space of orders and he does not rely on noise traders to derive the equilibrium. Market orders are executed according to a discriminatory pricing rule (they "walk the book"). The resulting inside bid and ask prices are endogenous outcomes of limit order decisions that rationally anticipate market orders, in an adverse selection environment. However, this model also relies on an exogenous liquidation value, made stochastic by the introduction of a random liquidation (stopping) time. Glosten's market order submitter and limit order submitter sets are disjoint, the former consists of one representative risk-averse agent, the latter – of many risk neutral agents. Most importantly, the limit order competition is reduced to a zero profit constraint. Limit order users have no private information. Their decision sets do not leave space for undercutting/overbidding the competitors. Consequently, the Pareto equilibrium derived by Glosten corresponds to a representative liquidity supplier who generates the limit order book single-handedly under the zero profit condition. It is not a non-cooperative game equilibrium. The present model overcomes this handicap, since imperfectly and heterogeneously informed traders compete against each other by means of limit orders that take into consideration their execution probabilities. That is, the equilibrium order book in my model is undercutting/overbidding-proof.

Beside the sequential trade and exogenous asset value assumptions, the other two mentioned models of limit order trading (Parlour, 1998, and Foucault, 1999) differ from ours by the imposition of full terms of trade observability to market order submitters. The same is done in Handa, Schwartz and Tiwari, 2003, who extend the model of Parlour, 1998. In addition, Foucault requires from every trader at every moment a choice between a market and a limit order, whereas Parlour pre-assigns a type (buyer or seller) to every arriving trader. Altogether, both models are concerned by trade timing and costs rather than price formation. What we share with both is the assumption of private value distribution as a primitive of the model.

The complexity and poor analytical tractability of broker models have resulted in efforts to apply numeric algorithms to equilibrium computation (Goettler, Parlour and Rajan, 2003). These authors, employing the very same modeling features of Parlour, 1998, as discussed above, compute an equilibrium on a discrete price grid under continuous order sizes. Again, in view of the existence of the consensus asset value, price discovery or, more generally, price formation is not an issue in that paper. My paper takes a similar path, as we prepare the ground for a numerical solution of the best ask and best bid equations. One difference of our work in addition to those already mentioned, is that Goettler, Parlour and Rajan, 2003, need to apply a numerical procedure directly to the decisions of the agents. In my paper, although implicit, all the necessary analytical expressions for equilibrium prices, the order book, etc., and their dependence on parameters (for the purpose of comparative static exercises) are available, so that numerical methods are only required to get to explicit numbers.

Here, I have chosen to model small traders who place orders for standard amounts only. Therefore, it is natural to assume that portfolio change as a result of the limited number of such infinitesimal trades within one infinitesimal period (simultaneous trading round) only causes an infinitesimal change in the trader's utility level. It is then admissible to consider traders risk-neutral. (On the other hand, a dealer, attracting a positive mass of trader orders, is likely to experience a shift in utility.) An opposite view is taken in the model by Derviz, 2003. This, like the present one, is a one-shot simultaneous trading model, in which limit orders take the form of pricing schedules, as in Kyle, 1985, or Glosten, 1994, whereas market orders are of arbitrary size. Market participants are risk averse. The difference between order- and quote-driven markets results from an order batching procedure used by the broker (making the mechanism similar to a uniform-price auction). The existence of a solution is established by means of differential game theory methods. A special case (two representative investors with different endowments who submit limit orders to a broker plus another investor submitting market orders, as opposed to the same two investors acting as market makers plus the third investor as a market user) is solved numerically. There are some qualitative differences in the equilibrium price, order flow and welfare between the two trading mechanisms, although quantitatively, those differences are small. Modeling a hybrid market with both a set of dealers and a broker within the same approach would require even more advanced numerical techniques, and is not considered. In the present paper, the choice between the broker and a dealer is not an issue (traders can trade with both and the decisions are mutually independent thanks to risk-neutrality), and a simplified treatment of optimal trader decisions leads to more advanced analytical results.

The rest of the paper is organized as follows: Section 2 presents the main attributes of the fast brokered trading model in full generality, applicable to a wide range of asset value distributions, and derives the first set of equilibrium conditions. Then I refine the general model in such a way that the equilibrium limit orders become represented in relation to the best bid and ask prices. This allows me to derive a more tractable set of equilibrium conditions for the latter. Section 3 completes the parametrization of the model, by assuming Gaussian functional forms for asset value and information parameter distributions. In that section, we are able to formulate the equilibrium conditions for the inside bid and ask prices in a numerically solvable form. Moreover, we analytically derive a number of basic properties of the equilibrium limit order book. Section 4 discusses some implications of the model in terms of common prior and trader-specific prior and posterior information for price determination and trader welfare. Section 5 concludes. Proofs of technical statements are collected in the Appendix.

2. The general model

2.1 Basic relationships

For a given trader, let x be the private asset value logarithm and y – a parameter characterizing his information (to be specified later). The number of traders who have private value x and have received signal y is equal to $z(x,y;c)\geq 0$. Here, c is a (vector) parameter whose exact realization c_0 is unknown to the traders. We will call c the *market state*. A trader with signal y believes that c has a continuous non-atomic distribution with p.d.f. $c \mapsto f(y;c)$, which can be biased. Intuitively, the quality of information corresponds to the bias size and the distance of $f(y;\cdot)$ from the atomic density $\delta_{c_0}(\cdot)$ in an appropriately chosen metric.

The traders are first allowed to submit limit orders and then, after the limit order book has been formed, can submit market orders if they wish.

The trading round is fully described by the state of the book and the pair of market order numbers. For a trader with information y, the market order flow pair, i.e. m^B market buys and m^S market sells, is a random variable distributed with joint density $m = \left(m^B, m^S\right) \mapsto h(y;m)$. We will denote by $h^B(y;\cdot)$, $h^S(y;\cdot)$ the corresponding marginal densities and by $H^B(y;\cdot)$, $H^S(y;\cdot)$ the corresponding cumulative distribution functions.

Throughout this paper, the buyer and the seller market sides will be distinguished by superscripts B and S based on the market order direction, even when the limit order variables are involved. So, superscript B will also service the limit sell order parameters, whereas superscript S – the limit buy order parameters.

Limit order submission Let us fix market state c. The equilibrium trade pattern that obtains under c includes the inside ask p^{ia} (the lowest ask price for which there are limit sells) and the inside bid p^{ib} (the highest bid price for which there are limit buys). They are both functions of market state c. Naturally, one expects to see $p^{ib}\leq p^{ia}$. Denote by $q^B(c;p)$ the number of limit sells placed at price p under market state c, and by $Q^B(c;p)$ - the number of limit sells that are, under market state c, placed at prices between p^{ia} and p. Analogously, we denote by $q^S(c;p)$ the number of limit buys at price $p<p^{ib}$. For cumulative limit order (LO) quantities we get

$$Q^B(c;p) = \int_{\log p^{ia}}^{\log p} q^B(c;e^\rho)d\rho\,, \quad Q^S(c;p) = \int_{\log p}^{\log p^{ib}} q^S(c;e^\rho)d\rho\,. \tag{1}$$

The true limit order quantities are obtained for $c=c_0$. Recall that the latter value is not fully observed. (For example, this is so in the case of the major FX brokers, both voice and electronic, if we disregard the knowledge of the quantities at the first-best and the second-best prices, as in Reuters Dealing 3000. However, as argued above, under frequent trading, this latter information, which is highly volatile, is not very useful. Actually, even p^{ia} and p^{ib} are volatile and usually change before a trader has time to submit a market order.)

To characterize the optimal limit order behavior, we assume the existence of continuous smooth strictly monotone functions $p \mapsto B(y;p)$, $p \mapsto S(y;p)$ giving subjective probabilities of execution of a limit sell, respectively, buy at price p, of a trader endowed with information y. The basic properties of these functions are (subscripts stand for partial derivatives): $B_p<0$, $S_p>0$, $\lim_{p\to+\infty} B(y;p)=0$, $\lim_{p\to 0} S(y;p)=0$ for every y. In addition, in the following sections we will restrict attention to trading patterns in which a limit sell/buy at the inside ask/bid is executed with certainty: $\lim_{p\to p^{ia}+0} B(y;p)=1$, $\lim_{p\to p^{ib}-0} S(y;p)=1$.

In the sequel, logs of the inside limit prices will be denoted by i^a and i^b.

A risk-neutral trader with private asset log-value x and information y will choose price p at which to place a limit sell so as to maximize the expected surplus $B(y;p)(p-e^x)$. That is, p will be chosen so that

$$e^x = p + \frac{B(y;p)}{B_p(y;p)} = T^B(y,p)\,. \tag{2}$$

Similarly, the optimal limit buy price for the same trader must satisfy

$$e^x = p + \frac{S(y;p)}{S_p(y;p)} = T^S(y,p)\,. \tag{3}$$

Quantities $-\dfrac{B(y;p)}{B_p(y;p)}$ and $\dfrac{S(y;p)}{S_p(y;p)}$ are the net welfare gains resulting to the trader from the limit order executions at price p. Since we will use the logarithmic scale to express prices, it is convenient to introduce

$$\xi^{B,S}(y,\rho) = \log T^{B,S}(y,e^\rho)\,.$$

This is the private log-asset value of a trader who has beliefs y and has optimally submitted a limit buy/sell at $p=e^\rho$.

In the model of this paper, functions $\rho \mapsto \xi^{B,S}(y,\rho)$ will be assumed strictly increasing with range equal to the whole real line, for every information parameter *y*. Indeed, if there existed values of *x* such that no log-price ρ could be chosen to satisfy (2) or (3), that would mean such a trader could not place an optimal limit buy or sell order with a finite price. But then, such traders would be irrelevant for the analysis, so that we don't take them into consideration. As regards the strict growth feature of $\xi^{B,S}$ in ρ, this is a generic property of the cumulative distribution functions that generate *B* and *S*. In Section 3, we will restrict attention to the class of subjective execution probabilities that satisfy it.

Under the given notation and in view of the ρ-monotonicity of ξ^{B}, if the market state is *c*, the mass of limit sells with the price not exceeding $p=e^{\rho}$ is the mass of traders whose private value and information satisfy the inequality $x\leq \xi^{B}(y,\rho)$. Similarly, the mass of limit buys with the price not lower than $p=e^{\rho}$ is the mass of traders whose private value and information satisfy the inequality $x\geq \xi^{S}(y,\rho)$. In other words,

$$Q^{B}(c,p)=\int \int_{\xi^{B}(y,i^{a})}^{\xi^{B}(y,\log p)} z(x,y;c)dxdy\,,\ Q^{S}(c,p)=\int \int_{\xi^{S}(y,\log p)}^{\xi^{S}(y,i^{b})} z(x,y;c)dxdy\,. \tag{4}$$

Comparing (4) with (1), we see that

$$q^{B}(c;e^{\rho})=\int z\left(\xi^{B}(y,\rho),y;c\right)\xi^{B}_{\rho}(y,\rho)dy\,,\ q^{S}(c;e^{\rho})=\int z\left(\xi^{S}(y,\rho),y;c\right)\xi^{S}_{\rho}(y,\rho)dy\,. \tag{5}$$

Now observe that *B*(*y*;*p*) (*S*(*y*;*p*)) is the *y*-probability that the number of market buys (sells) exceeds Q^{B} (Q^{S}). To shorten the notations, set $G^{B,S}=1-H^{B,S}$. Then

$$B(y;p)=\int G^{B}\left(y;Q^{B}(c,p)\right)f(y;c)dc\,,\ S(y;p)=\int G^{S}\left(y;Q^{S}(c,p)\right)f(y;c)dc\,. \tag{6}$$

By the same token,

$$B_{p}(y;p)=-\int h^{B}\left(y;Q^{B}(c,p)\right)q^{B}(c;p)f(y;c)dc\,, \tag{7a}$$

$$S_{p}(y;p)=\int h^{S}\left(y;Q^{S}(c,p)\right)q^{S}(c;p)f(y;c)dc\,. \tag{7b}$$

Jointly, (2), (3), (6) and (7) pin down the private log-asset values ξ. However, the mathematics of the corresponding nonlinear integro-differential equation system is too complex to be analyzed in full generality. Therefore, below I will introduce a sequence of specializing assumptions that will allow one to get nearer a tangible solution.

Limit order crossing A priori, one cannot exclude the case when, after all limit orders have been submitted, the lowest limit ask price happens to be below the highest limit bid price: $\log p^{ia}=\rho^{ia}<\rho^{ib}=\log p^{ib}$. In this case, the inside quotes and the inside spread are not well-defined. To deal with this eventuality, we define a limit order crossing procedure derived from the widely accepted principle of brokerage: a limit buy/sell with a price above/below the best ask/bid is treated like a market order. In the present model, this will be an off-the-equilibrium-path rule, since, as we will now demonstrate, individually optimal behavior of limit order submitters implies $\rho^{ib}\leq\rho^{ia}$.

Specifically, if $\rho^{ia}<\rho^{ib}$, there must exist such a ρ^m between ρ^{ia} and ρ^{ib} that the mass of limit sells at prices between ρ^{ia} and ρ^m equals the mass of limit buys at prices between ρ^m and ρ^{ib}:

$$Q^B\left(c,e^{\rho^m}\right)=\int\int_{\xi^B(y,\rho^{ia})}^{\xi^B(y,\rho^m)} z(x,y;c)dxdy=\int\int_{\xi^S(y,\rho^m)}^{\xi^S(y,\rho^{ib})} z(x,y;c)dxdy=Q^S\left(c,e^{\rho^m}\right).$$

Given these two subsets of limit orders on both sides of the market with the same mass, the broker is assumed to use any randomization procedure he prefers, to match each of the so selected limit buys with a single one of the selected limit sells. This means, among other things, that limit sells at prices between ρ^{ia} and ρ^m and limit buys at prices between ρ^m and ρ^{ib} are executed with certainty (and even generate a higher surplus than the one targeted by each involved trader). After having matched the extreme limit orders in the described fashion, the broker updates the book by eliminating them. Thus, the inside quotes are now well-defined: $i^a=i^b=\rho^m$, with the inside spread equal to zero. Market orders are then executed against this updated book.

However, the involved limit order submitters, knowing their order execution is 100 per cent certain, will prefer to move their quotes closer to ρ^m, since this will increase the surplus. As a result, among the selected limit orders, there will only remain ones with prices infinitely close to ρ^m. This contradicts the assumption that $\rho^{ia}<\rho^{ib}$ is a part of an equilibrium trade pattern. Therefore, $\rho^{ib}\leq\rho^{ia}$ must prevail in equilibrium and the inside quotes i^a, i^b are well-defined.

Market order execution rules The market state should uniquely determine the traders' decisions about market order (MO) placement. In particular, the numbers of market buys and sells, M^B and M^S, are functions of c. Here, we recall the distinction between the hidden liquidity (HL) and hidden priority (HP) cases defined in the introduction. In the sequel, both cases are treated in parallel.

We note that in the current setting, all market orders must be executed in equilibrium. If it were not the case, e.g. $Q^B(c,\infty)<M^B(c)$, then there would exist a finite limit sell price p such that $q^B(c,p)=0$ and $B(y,p)=1$ for a set of signals y of positive Lebesgue measure. However, this cannot be a part of an equilibrium trade pattern, since traders with signals y from the said subset and private asset values below p would be better off placing limit sells at prices above p – a contradiction. A similar argument shows that one cannot have $Q^S(c,\infty)<M^S(c)$.

Accordingly, there must exist such a finite limit ask price $K^B(c)$ for which, under market state c, the number of limit sells with prices between p^{ia} and $K^B(c)$ exactly equals M^B, the true number of market buys under c:

$$Q^B(c;K^B(c))=\int_{i^a}^{\log K^B(c)} q^B(c;e^{\rho})d\rho=M^B(c)\,.$$

In other words, $K^B(c)$ is the highest ask price the market buyer may have to pay under c. Price $K^S(c)$ is defined analogously.

The broker's rule of assigning limit prices to incoming market orders will be assumed to use the logarithmic scale. Then the infinitesimal probability of executing one's market buy order at price $p \in \left(p^{ia}, K^B(c)\right)$ under market state c is $\frac{q^B(c;p)}{M^B(c)}\frac{dp}{p}$ and the infinitesimal probability of execution of a market sell at $p \in \left(K^S(c), p^{ib}\right)$ is $\frac{q^S(c;p)}{M^S(c)}\frac{dp}{p}$. Taking the HP-case first, we recall that c is known to the market buy submitter. The uncertainty only involves the choice of execution price from the known interval. In expectation, the logs of the market buy and sell prices equal

$$\hat{a} = \int_{i^a}^{\log K^B(c)} \rho \frac{q^B\left(c;e^\rho\right)}{M^B(c)} d\rho,\ \hat{b} = \int_{\log K^S(c)}^{i^b} \rho \frac{q^S(c;e^\rho)}{M^S(c)} d\rho. \tag{8}$$

In the HL-case, the market order submitter is uncertain about the market state as well. Accordingly, the logarithm of the subjectively expected execution price by the market buy submitter is equal to

$$a(y) = \log P^a(y) = \int \int_{i^a}^{\log K^B(c)} \rho \frac{q^B(c;e^\rho)}{M^B(c)} d\rho f(y;c)dc, \tag{9a}$$

whereas for a market sell the expected execution log-price is

$$b(y) = \log P^b(y) = \int \int_{\log K^S(c)}^{i^b} \rho \frac{q^S(c;e^\rho)}{M^S(c)} d\rho f(y;c)dc. \tag{9b}$$

Due to the assumed risk neutrality, a trader in an HL-market with a private asset value e^x and information y places a market buy if and only if $x>a(y)$ and a market sell – if and only if $x<b(y)$. In a HP-market, these conditions simplify to $x>\hat{a}$ and $x<\hat{b}$, respectively. Traders with x between $a(y)$ and $b(y)$ ($\hat{a}$ and $\hat{b}$ in the HP-case) only place limit orders.

The relation between the market state and subjective order flow distributions When the trader private value and information histogram is given by $z(\cdot,\cdot;c)$ with parameter c, the numbers of market buys and sells in an HL-market are given by

$$M^B(c) = \int \int_{a(y)}^{\infty} z(x,y;c)dxdy,\ \ M^S(c) = \int \int_{0}^{b(y)} z(x,y;c)dxdy. \tag{10}$$

In an HP-market, $a(y)$ and $b(y)$ in (10) must be replaced by $\hat{a}$ and $\hat{b}$.

Together, (9) and (10) ((8) and (10) for an HP-market) form a system of equations that should uniquely determine P^a, P^b or, equivalently, M^B, M^S in an HL market for any state of the limit order book specified by q^B, q^S.

The existence proof for the solution of the model in the general form exceeds the scope of the present paper. More can be said in a specialized setting of Section 3, where the private asset values and signals about market state are assumed to be Gaussian. For the moment, we will reduce the number of "degrees of freedom" of the general model by fixing the correspondence between the market state space and the space of market order realizations.

The true market order flows are obtained when c=c_0. Let us assume that c consists of two components, c^H and c^L, which we informally associate with the two modes, high and low, of the private value histogram. In the HL-case, we need the following technical assumption.

Assumption 1 *The trader histogram z is such that for every non-singular functions a and b of the signals, the map* $(c^H, c^L) = c \mapsto M(c) = \left(M^B(c), M^S(c)\right)$ *with M defined in* (11), *is a one-to-one, non-singular, and monotonous, with a well-defined inverse* $n \mapsto \Gamma(n)$.

In Section 3, we construct a special parametric version of the model for which the above assumption is easily verified. Note that in the HP-case, an analogue of Assumption 1 is trivially satisfied for any histogram z with a non-degenerate dependence on c (in the opposite case, we would have to redefine the range of market states to establish a one-to-one property), so that we can omit it.

Now, from the standard change of the variable result it follows that the subjective market order flow density h is uniquely determined by the subjective market state density f and the mapping M (or Γ, $|DM|$ denotes the determinant of the map M differential):

Lemma 1 *For every information parameter y,* $f(y;c) = h(y; M(c))|DM|(c)$.

In principle, the model is exhaustively described by the system of equations (9)-(11) which show that the equilibrium trade pattern becomes a function of the trader histogram z and the market state c_0. Indeed, for every ξ, (5)-(7) determine $q^{B,S}$, and then (9)-(11) determine M and $P^{a,b}$. By Lemma 1, M determines the market order flow density h, and then (3), (4) and (8) definitively pin down ξ.

Dealers competing with the broker If the traders are risk-neutral, they may accept a transaction with a dealer regardless of what they do in the brokered market segment. So, if a dealer offers a pair of log-price quotes p^b<p^a, traders with x<p^b will sell to, and traders with p^a<x – buy from, the dealer. Therefore, in market state c, the dealer will receive the purchasing and the selling order flows equal to

$$\omega^B(c) = \int \int_{\log p^a}^{\infty} z(x, y; c)dxdy\,, \quad \omega^S(c) = \int \int_{-\infty}^{\log p^b} z(x, y; c)dxdy\,. \qquad (11)$$

As long as the map $c \mapsto \omega(c)$ is non-singular, the true market state c_0 can be derived by the dealer from the pair of received order flows. Then, with this knowledge, she can offer an adjusted pair of quotes in order to extract the maximum surplus from another round of trades. The dealer advantage is the ability to attract a representative – hence informative - order flow, a thing that a limit order submitter in the brokered market cannot do.

2.2 Individual quotes, inside quotes and equilibrium orders

The general model of the previous subsection does not have an explicit closed-form solution. To facilitate the numerical solution of important special cases, we shall now restate the basic equations in terms of the inside quotes and the limit price distances from those quotes. We shall also simplify the relation between the newly defined working variables of the model and the market state space.

In the sequel, a limit sell price p will be represented as $e^{i^a+v} = p^{ia}e^v$, $v \geq 0$. Analogously, a limit buy price will be represented as $e^{i^b+v} = p^{ib}e^v$, $v \leq 0$. That is, the (absolute value of the) new variable v stands for the log-distance of the limit price from the best quote on the same side of the market. Note that the best quotes and their logs i^a, i^b are functions of the market state. As such, they are unobserved as long as the market state itself is unobserved. It is worth noting that unobservability of best quotes is not the same thing as complete ignorance of the traders regarding the terms of trade to be faced. More precisely, even if traders do not know $i^{a,b}$ in the form of numbers (the HL-case), they do know that these numbers are always *well-defined and exist* in the given trading mechanism. This inside quote existence is, essentially, what makes a broker a broker in our model (cf. the market maker, for whom the question is moot). The knowledge that all other limit orders lie outside the inside quotes are indispensable for the limit order decisions of each individual trader to look the way they do.

Recalling the definition of the maximal executed ask price K^B and the minimal executed bid price K^S, we write

$$K^B(c) = p^{ia}(c)e^{k^B(c)} = e^{i^a(c)+k^B(c)},\ K^S(c) = p^{ib}(c)e^{k^S(c)} = e^{i^b(c)+k^S(c)}.$$

Given these notations, the equilibrium limit order values will be sought in the form

$$q^B\left(c, e^{i^a(c)+v}\right) = \tilde{q}^B(c; i^a(c), v),\ v \geq 0,\ q^S\left(c, e^{i^b(c)+v}\right) = \tilde{q}^S(c; i^b(c), v),\ v \leq 0,$$

for some functions $\tilde{q}^B$, $\tilde{q}^S$.

Consequently, one must have

$$Q^B(c;p) = \int_0^v \tilde{q}^B(c; i^a(c), v')dv' = \tilde{Q}^B\left(c; i^a(c), v\right),$$

$$Q^S(c;p) = \int_v^0 \tilde{q}^S(c; i^b(c), v')dv = \tilde{Q}^S\left(c; i^b(c), v\right).$$

The probabilities $B(y,p)$ and $S(y,p)$ of execution of a limit order at price p under information y are the probabilities of events $\{v < k^B(c)\}$ and $\{k^S(c) < v\}$, respectively. Accordingly, we can write

$$B(y,p) = \tilde{B}(y;v) = \iint_{\{v<k^B(c)\}} f(y;c)dc,\ S(y,p) = \tilde{S}(y;v) = \iint_{\{k^S(c)<v\}} f(y;c)dc. \qquad (12)$$

In the new notation,

$$\xi^B = i^a + v + \log\left(1 + \frac{\widetilde{B}}{\widetilde{B}_v}\right) = i^a + v + \widetilde{\xi}^B,\ \xi^S = i^b + v + \log\left(1 + \frac{\widetilde{S}}{\widetilde{S}_v}\right) = i^b + v + \widetilde{\xi}^S.$$

Observe that $\widetilde{B}$, $\widetilde{S}$, $\widetilde{\xi}^B$, $\widetilde{\xi}^S$ only depend on the distance from the best same-side quote but not on that quote itself. Also, in the case of $\widetilde{\xi}^B$, since $\widetilde{B}_v < 0$, only such combinations of y and v>0 are acceptable for which $-1 < \frac{\widetilde{B}(y,v)}{\widetilde{B}_v(y,v)}$. Other limit sells will not be submitted by a trader with information y. Given our assumptions on the monotonicity of ξ^B in the price variable, we conclude that, as v declines from $k^B(c)>0$ towards zero, the value of ξ^B must fall to minus infinity. The exact limit value under which v cannot fall without violating the $-1 < \frac{\widetilde{B}(y,v)}{\widetilde{B}_v(y,v)}$-condition, is y-dependent (see the next section for a more specific formulation). Notwithstanding this lower limit on the admissible limit sell prices, the first expression in (4) is still valid for all admissible values of v. On the market sell side, as v rises from $k^S(c)$ towards zero, the monotonicity of ξ^S in the price variable implies that it must increase to plus infinity. Accordingly, (4) can be rewritten in the new notation as

$$\widetilde{Q}^B(c, i^a(c), v) = \int \int_{-\infty}^{i^a(c)+v+\widetilde{\xi}^B(y,v)} z(x, y; c)\,dxdy,\ \widetilde{Q}^S(c, i^b(c), v) = \int \int_{i^b(c)+v+\widetilde{\xi}^S(y,v)}^{+\infty} z(x, y; c)\,dxdy. \quad (13)$$

The limit orders at individual prices are now described by the equalities

$$\widetilde{q}^B\left(c; i^a(c), v\right) = \int z\left(i^a(c) + v + \widetilde{\xi}^B(y,v), y; c\right)\left(1 + \widetilde{\xi}_v^B(y,v)\right)dy, \quad (14a)$$

$$\widetilde{q}^S\left(c; i^b(c), v\right) = \int z\left(i^b(c) + v + \widetilde{\xi}^S(y,v), y; c\right)\left(1 + \widetilde{\xi}_v^S(y,v)\right)dy. \quad (14b)$$

In (14a), the integration is over such signals y that satisfy $-1 < \frac{\widetilde{B}(y,v)}{\widetilde{B}_v(y,v)}$.

To proceed, we denote by k the $\mathbf{R}^2 \to \mathbf{R}^2$ map with components k^B, k^S and assume that this map is globally non-singular with inverse l. Then, applying the change of variables $c=l(\chi)$, we can rewrite (12) as

$$\widetilde{B}(y;v) = \int_{-\infty}^{\infty} \int_{v}^{\infty} f\left(y; l\left(\chi^B, \chi^S\right)\right)\left|Dl(\chi)\right| d\chi^B d\chi^S,\ \widetilde{S}(y;v) = \int_{-\infty}^{\infty} \int_{-\infty}^{v} f\left(y; l\left(\chi^B, \chi^S\right)\right)\left|Dl(\chi)\right| d\chi^S d\chi^B.$$

Finally, for HL-markets, we need to restate the expressions (9) for the logs of the subjectively expected execution prices $P^{a,b}(y)$ in the new notation, which will allow us to refine the MO flow formulae (10). One can easily check that

$$a(y)=\int \hat{a}(c')f(y;c')dc'=\int\left[i^a(c')+\int_0^{k^B(c')}\frac{\tilde{q}^B(c';i^a(c'),v)}{\tilde{Q}^B(c';i^a(c'),k^B(c'))}vdv\right]f(y;c')dc', \quad (15a)$$

$$b(y)=\int \hat{b}(c')f(y;c')dc'=\int\left[i^b(c')+\int_{k^S(c')}^{0}\frac{\tilde{q}^S(c';i^b(c'),v)}{\tilde{Q}^S(c';i^b(c'),k^S(c'))}vdv\right]f(y;c')dc'. \quad (15b)$$

Obviously, in an HP-market, the expressions for $\hat{a}$, $\hat{b}$ are those appearing in the square brackets on the right hand side of (15).

The following assumption, to be maintained in the remaining part of this paper, will be useful to simplify the calculations.

Assumption 2 (A) *The map* $c\mapsto\left(k^B(c),k^S(c)\right)$ *from the market state to the pair of highest executed limit ask-lowest executed limit bid is separable in* (c^H,c^L), *i.e.* $k^B=k^B(c^H)$, $k^S=k^S(c^L)$ *with* $k^{B,S}$ *being strictly increasing, continuously differentiable functions, with* k^B *mapping* **R** *onto* $(0,\infty)$ *and* k^S *mapping* **R** *onto* $(0,-\infty)$, *with inverses* $r_{H,L}$. *That is, the market state is fully characterized directly by the ultimate executable limit orders.*
(B) *Both the prior distributions of the components of c and the private signals about their values are mutually independent. That is,* $f(y;c)=f^H(y^H;c^H)f^L(y^L;c^L)$.

Although Assumption 2 may seem to overstretch particularities in the information structure of the traders, one would do best by regarding it merely as a special parametrization of the problem. We should think of the traders getting signals about the depth of both market sides. The formal quantification of the depth signals is non-unique: one can cast private information in the form of the two modes in the private asset value histogram, the two ultimate limit order prices, or anything else with the ability to describe the composition of private values exhaustively.

We denote by $F^H(y^H;\cdot)$, $F^L(y^L;\cdot)$ the cumulative distribution functions corresponding to $f^H(y^H;\cdot)$, $f^L(y^L;\cdot)$. Under Assumption 2 and with a slight abuse of notation, the expressions for $\tilde{\xi}^B$, $\tilde{\xi}^S$ simplify to

$$\tilde{\xi}^B(y^H,v)=\log\left(1-\frac{1-F^H(y^H;r_H(v))}{r_H'(v)f^H(y^H;r_H(v))}\right),\ \tilde{\xi}^S(y^L,v)=\log\left(1+\frac{F^L(y^L;r_L(v))}{r_L'(v)f^L(y^L;r_L(v))}\right). \quad (16)$$

Definition *Equilibrium in the order-driven market described so far is a map* $c\mapsto\left(i^a(c),i^b(c)\right)$ *from the space of market states to the pair of inside prices, such that*

(a) $i^b(c)\le i^a(c)$ *for all c*;
(b) *traders place limit orders optimally in accordance with* (2), (3), *cumulative LO-quantities are given by* (13);
(c) *market orders are optimally submitted by those traders who expect a positive surplus, the total MO-quantities satisfy* (10);
(d) *the executed LO-quantities are linked to the total MO-quantities by*

$$M^B(c)=\tilde{Q}^B(c;i^a(c),k^B(c^H)),\ M^S(c)=\tilde{Q}^S(c;i^b(c),k^S(c^L)). \quad (17)$$

In the HL-case, (17) is a pair of non-linear integral equations for the vector function (i^a, i^b). In the HP-case, integration is not involved and one gets a 2-dimensional algebraic equation system for each value of c. Equations (17) reflect the definition of $k^B(c^H)$, $k^S(c^L)$ as those log-distances from the best ask and bid for which the number of market orders exactly matches the number of limit orders in the book up to the corresponding limit price. In other words, the limit order is executed if and only if its log-distance from p^{ia} (p^{ib}) is between zero and $k^B(c^H)$ (between $k^S(c^L)$ and zero).

From (13) it follows that functions $g \mapsto L^B(c,g) = \widetilde{Q}^B(c,g,k^B(c^H))$ and $g \mapsto L^S(c,g) = \widetilde{Q}^S(c,g,k^S(c^L))$ are strictly monotonic (the first one - strictly increasing and the second one – strictly decreasing) for every market state c. This means that the right hand sides of (17) can be inverted with respect to i^a and i^b, making them amenable to iterative numerical solutions for individual parametric specifications. In addition, for the class of Gaussian asset value and belief histograms to be considered in the next section, the existence of solutions can be reduced to the fixed point properties of the functional on the space of vector functions (i^a, i^b) that one obtains from (17) by inverting $\left(L^B(c,g^a), L^S(c,g^b)\right)$ with respect to (g^a,g^b).

3. Gaussian trader histograms and Bayesian belief structure
3.1 Definitions

We will consider the information structure based on a Bayesian update of a prior Gaussian density of the market state. Let the prior beliefs of all traders about c^H and c^L be independently normally distributed with means c_p^H, c_p^L and standard deviations ζ_H, ζ_L. Let the traders receive signals s represented by a pair s=(s^H,s^L) of random variables normally distributed around the true values c_0^H, c_0^L with standard deviations δ_H, δ_L. As is well known, the posterior beliefs of a trader who receives signal s are then Gaussian with means $\kappa_H s^H + (1-\kappa_H)c_p^H, \kappa_L s^L + (1-\kappa_L)c_p^H$ and precisions $\frac{1}{\eta_H^2} = \frac{1}{\zeta_H^2} + \frac{1}{\delta_H^2}$, $\frac{1}{\eta_L^2} = \frac{1}{\zeta_L^2} + \frac{1}{\delta_L^2}$, where $\kappa_{H,L} = \frac{\zeta_{H,L}^2}{\zeta_{H,L}^2 + \delta_{H,L}^2}$. By the law of large numbers, the number of traders with signal s is equal to $f(c\text{-}s)$. We will identify the information available to a trader with signal s with the pair $y = \left(y^H, y^L\right) = \left(\kappa_H s^H + (1-\kappa_H)c_p^H, \kappa_L s^L + (1-\kappa_L)c_p^L\right)$ of mean values of his ex post subjective p.d.f. for c^H and c^L, which is given by

$$\gamma \mapsto f(\gamma - y) = f^H\left(\gamma^H - y^H\right) f^L\left(\gamma^L - y^L\right) = \frac{1}{\eta_H \eta_L} n\left(\frac{\gamma^H - y^H}{\eta_H}\right) n\left(\frac{\gamma^L - y^L}{\eta_L}\right), \quad \text{(18pdf)}$$

where n is the standard normal density.

For Z=H,L, put $\bar{c}^Z = \kappa_Z c^Z + (1-\kappa_Z)c_p^Z$ and $\bar{\eta}_Z = \kappa_Z \eta_Z$. If taken as a function of y, the informational histogram (i.e. the number of traders with information y) has the form

$$f(y;c) = f(c-s) = \bar{f}^{H}\left(\bar{c}^{H} - y^{H}\right)\bar{f}^{L}\left(\bar{c}^{L} - y^{L}\right)\frac{ds^{H}}{sy^{H}}\frac{ds^{L}}{sy^{L}} = \frac{1}{\eta_H \eta_L} n\left(\frac{\bar{c}^{H} - y^{H}}{\eta_H}\right) n\left(\frac{\bar{c}^{L} - y^{L}}{\eta_L}\right), \tag{18hist}$$

Note the formal difference in the expressions (18pdf) and (18hist) due to the initial common bias c_p and the precision improvement factor $\kappa \in (0,1)$. We will return to these distinctions in Section 4.

The probabilities of limit order execution are now given by

$$\widetilde{B}(y^{H}, v^{a}) = N\left(\frac{y^{H} - r_H(v^{a})}{\eta_H}\right),\ \widetilde{S}(y^{L}, v^{b}) = N\left(\frac{r_L(v^{b}) - y^{L}}{\eta_L}\right),\ v^{a}>0,\ v^{b}<0.$$

(N denotes the standard normal cumulative distribution function.) Now, setting $k^{B,S}\left(c^{a,b}\right) = v^{a,b}$ and introducing an auxiliary function $s \mapsto \varphi(s) = \dfrac{N(s)}{n(s)}$, $s \in \mathbf{R}$ (its properties will be discussed in Subsection 3.3), we can reduce (18) to

$$\xi^{B,S}\left(y, k^{B,S}\left(c^{a,b}\right)\right) = \log\left[1 \mp \eta_{H,L} k^{B,S\,\prime}\left(c^{a,b}\right)\varphi\left(\mp \frac{c^{a,b} - y}{\eta_{H,L}}\right)\right].$$

Based on this representation, it is easy to formulate a relatively simple sufficient condition guaranteeing that functions $\xi^{B,S}$, as defined in Section 2, are strictly increasing in the price argument for every value of the signal argument. Namely, we make the following technical assumption.

Assumption 3 *For every y, the parameter values of functions $k^{B,S}$ are such that functions*

$$s \mapsto 1 + \left(s \mp \frac{\eta_{H,L} k^{B,S\,\prime\prime}(y \mp \eta_{H,L} s)}{k^{B,S\,\prime}(y \mp \eta_{H,L} s)}\varphi(s)\right)$$

are both strictly positive on the whole real line.

The above assumption, indeed, guarantees that $\xi^{B,S}$ are strictly increasing functions of the second argument, which can be easily seen by making the change of variables $c^{a,b} = y \mp \eta_{H,L} s$. Note that our leading example of exponential k ($k^{B,S}(c) = \pm e^{\pm c/\lambda_{H,L}}$) implies that the ratios $\dfrac{k''}{k'}$ are constant (equal to $\pm 1/\lambda_{H,L}$). Then, for the Assumption 3 to be satisfied it is sufficient that both ratios η/λ lie below 0.3.

Next, let μ^{H} and μ^{L} be two functions of c, such that $\mu^{H}(c) > \mu^{L}(c')$ for any pair of arguments c, c', and let α_H, α_L, σ_H and σ_L be strictly positive constants. Under market state c, the number of traders with private value x and information y will, in accordance with (2) be assumed to equal

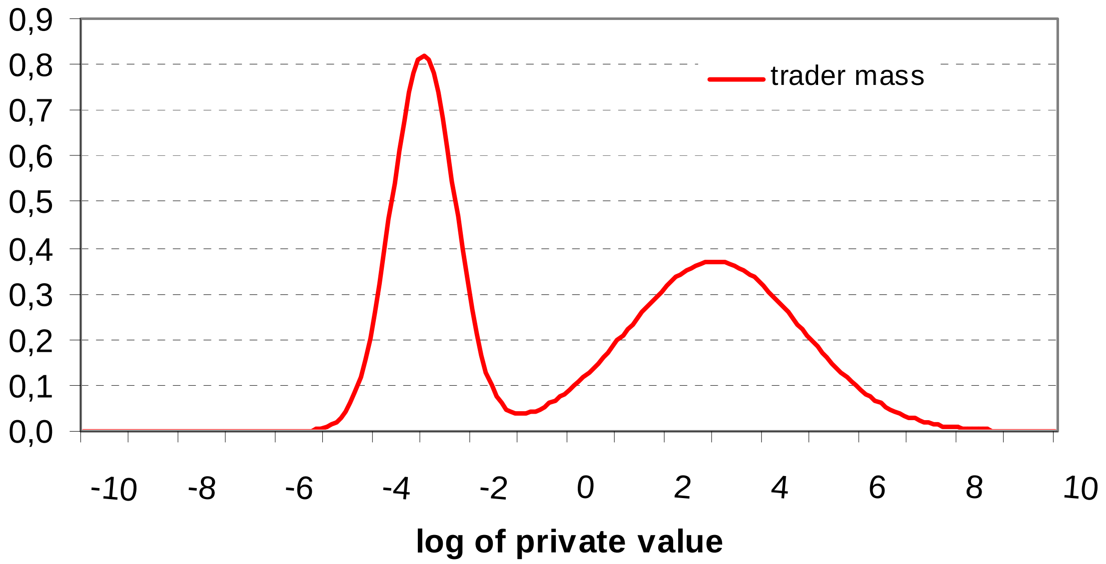
$$z(x,y;c)=\left\{\frac{\alpha_H}{\sigma_H}n\left(\frac{x-\mu^H(c)}{\sigma_H}\right)+\frac{\alpha_L}{\sigma_L}n\left(\frac{x-\mu^L(c)}{\sigma_L}\right)\right\}f(y;c)$$

$$=\Phi(x,c)\,f(c-s)=\Phi(x,c)\,\bar{f}(\bar{c}-y).$$

That is, for every market state c and any given signal value y, the private value histogram has two modes, high mode $\mu^H(c)$ and low mode $\mu^L(c)$. This is the result of mixing a $\mu^H(c)$-unimodal histogram and a $\mu^L(c)$-unimodal histogram with weights α_H, α_L. In the sequel, we shall take α_L=1-α_H and α_H=$\alpha(c)\in(0,1)$ – a function of the market state. Parameter α determines the relative weight of the high mode in z.

A typical histogram of private asset values for a fixed value of signal is shown in Fig. 1.

Fig. 1 Histogram of private asset log-values



To simplify the resulting formulae in the Gaussian histogram case,, we introduce more auxiliary notations:

$$\Psi^+(c,\rho)=\alpha_H N\left(\frac{\mu^H(c)-\rho}{\sigma_H}\right)+\alpha_L N\left(\frac{\mu^L(c)-\rho}{\sigma_L}\right), \tag{19a}$$

$$\Psi^-(c,\rho)=\alpha_H N\left(\frac{\rho-\mu^H(c)}{\sigma_H}\right)+\alpha_L N\left(\frac{\rho-\mu^L(c)}{\sigma_L}\right), \tag{19b}$$

so that $\Psi^-(c,\rho)$+ $\Psi^+(c,\rho)$=α_H+α_L=1. Specification (19) generates a private asset value histogram with two modes, $\mu^H(c)$ and $\mu^L(c)$. One should think of those modes as standing for a high (bullish) valuation typical for potential buyers and a low (bearish) valuation typical for potential sellers. Constants α_H and α_L are weights with which potential buyers and sellers are represented in the trader population. The signals about the modes are distributed across traders with a given private value according to the earlier defined Gaussian densities with variances η_H^2, η_L^2.

In the new notation, the cumulative limit order quantities can be written as

$$\tilde{Q}^B(c, i^a(c), v) = \int_{-\infty}^{\theta(v)} \Psi^-\left(c, i^a(c) + v + \tilde{\xi}^B(y^H, v)\right) \bar{f}^H(\bar{c}^H - y^H) dy^H, \; v>0, \tag{20a}$$

$$\tilde{Q}^S(c, i^b(c), v) = \int_{-\infty}^{+\infty} \Psi^+\left(c, i^b(c) + v + \tilde{\xi}^S(y^L, v)\right) \bar{f}^L(\bar{c}^L - y^L) dy^L, \; v<0. \tag{20b}$$

Note that integration in (20a) is over the interval $(-\infty, \theta(v))$ of information parameters y^H for which the quantity $\tilde{\xi}^H(y^H, v)$ is well defined (cf. the discussion preceding equation (13) in Subsection 2.2). Technical details on the value $\theta(v)$ will be given in Subsection 3.3.

The market order quantities under market state c are given by

$$M^B(c) = \iint f(c-y)\left[\alpha_H N\left(\frac{\mu^H(c) - a(y)}{\sigma_H}\right) + \alpha_L N\left(\frac{\mu^L(c) - a(y)}{\sigma_L}\right)\right] dy^H dy^L$$

$$= \iint \Psi^+\left(c, a(y)\right) f(c-y) dy \tag{21a}$$

in the HL-case and by

$$M^B(c) = \Psi^+(c, \hat{a}(c)) \tag{21b}$$

in the HP-case. The corresponding quantities of market sells are

$$M^S(c) = \iint f(c-y)\left[\alpha_H N\left(\frac{b(y) - \mu^H(c)}{\sigma_H}\right) + \alpha_L N\left(\frac{b(y) - \mu^L(c)}{\sigma_L}\right)\right] dy^H dy^L$$

$$= \iint \Psi^-\left(c, b(y)\right) f(c-y) dy \tag{22a}$$

in the HL-case and

$$M^S(c) = \Psi^-(c, \hat{b}(c)) \tag{22b}$$

in the HP-case. Note that the mapping $c \mapsto M(c) = \left(M^B(c), M^S(c)\right)$ defined in both variants of (21), (22) satisfies Assumption 1 from Section 2.

We conclude this subsection by observing that in market state c, a dealer with log-quotes $\rho^{a,b}$ receives the order flows equal to (cf. (11) in Subsection 2.1)

$$\omega^B(c) = \alpha_H N\left(\frac{\mu(c)-\rho^a}{\sigma_H}\right) + \alpha_L N\left(\frac{\mu(c)-\rho^a}{\sigma_L}\right) = \Psi^+\left(c,\rho^a\right),$$

$$\omega^S(c) = \alpha_H N\left(\frac{\rho^b-\mu^H(c)}{\sigma_H}\right) + \alpha_L N\left(\frac{\rho^b-\mu^L(c)}{\sigma_L}\right) = \Psi^-\left(c,\rho^b\right).$$

3.2 Equilibrium existence in an HP-economy

This subsection discusses the case of an HP-market only, since dealing with fixed point theorems for functionals in an infinite dimensional vector space in an HL-environment (cf. the end of Subsection 2.2) would exceed the scope of this paper. Moreover, although algorithms fro equilibrium calculations in the HL-case would be easy to design on paper, their software implementation constitutes a separate problem. On the contrary, the HP-case at least allows for some simple numerical experiments with the model, the results of which will be commented in the next section.

In a market with hidden priority of MO-execution (i.e. the market state c is known to the MO-submitters), the abstract equilibrium conditions (17) are reduced to a pair of scalar equations for the inside ask $g^a=i^a(c)$ and inside bid $g^b=i^b(c)$.

Due to the fact that $\hat{a}$, $\hat{b}$ happen to be increasing functions of g (see Lemma 2 below), it is easy to prove that a solution to each one of the equations (17) exists for every value of c. Naturally, an equilibrium only makes sense if these solutions satisfy the inequality $i^b(c) \leq i^a(c)$ everywhere. This is a restriction on α and $\mu^{H,L}$ as functions of c.

The quantities of executed limit orders, considered functions of market state and the inside quote on the same side of the market, are equal to

$$L^B(c,g) = \widetilde{Q}^B\left(c,g,k^B(c^H)\right),\ L^S(c,g) = \widetilde{Q}^S\left(c,g,k^S(c^L)\right).$$

It is necessary to note here that the market buy side shows a quantitative difference compared to the market sell side, due to the upper limit $\theta(v)$ for the information parameters leading the trader to place a limit sell at v (see the next subsection for details). Specifically, for $v=k^B(c^H)$ we find that limit sells between g^a and $g^a + k^B(c^H)$ will be submitted by a strict subset of the trader set even for very large values of g^a. Therefore, $L^B(c,g)$ converges to a value below unity as g goes to plus infinity. On the contrary, the limit of $L^S(c,g)$ as $g \to -\infty$ is exactly 1.

The realized market order quantities (i.e. the buy and the sell order flows) are[2]

[2] Observe that these are the order flows that the dealer would receive if she quoted ask and bid prices at levels $\hat{a}$, $\hat{b}$. In an HP-market, provided the dealer had the same initial information as the traders, she could reconstruct the true market state by just watching the LO book. In an LP-market, i.e. when the time to analyze the book is too short for a trader, the dealer's advantage is even bigger, since c-extraction based on her order flow is easier.

$$M^B\left(c,g^a\right)=\Psi^+\left(c,\hat{a}(c,g^a)\right)=\Psi^+\left(c,g^a+\int_0^{k^B(c^H)}\frac{\tilde{q}^B\left(c,g^a,v\right)dv}{L^B\left(c,g^a\right)}\right),$$

$$M^S\left(c,g^b\right)=\Psi^-\left(c,\hat{b}(c,g^b)\right)=\Psi^-\left(c,g^b+\int_{k^S(c^L)}^{0}\frac{\tilde{q}^S\left(c,g^b,v\right)dv}{L^S\left(c,g^b\right)}\right).$$

These expressions, combined with (20), allow one to formulate the equilibrium conditions on g^a, g^b which are a special case of the general market clearing formula (17):

$$L^B\left(c,g^a\right)=\Psi^+\left(c,g^a+\int_0^{k^B(c^H)}\frac{\tilde{q}^B\left(c,g^a,v\right)dv}{L^B\left(c,g^a\right)}\right), \tag{23a}$$

$$L^S\left(c,g^b\right)=\Psi^-\left(c,g^b+\int_{k^S(c^L)}^{0}\frac{\tilde{q}^S\left(c,g^b,v\right)dv}{L^S\left(c,g^b\right)}\right). \tag{23b}$$

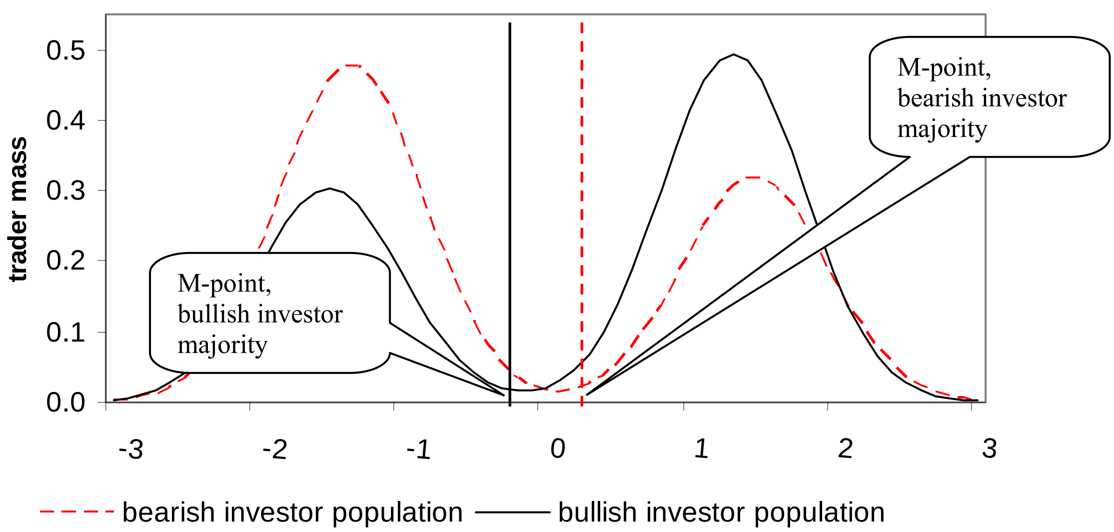
Equations (23) have a solution with respect to g^a, g^b for every c. The exact result can be expressed in the form of a technical lemma (proved in the Appendix).

Lemma 2 *For any market state c, function $J^B=L^B-M^B$ (the difference between the left and the right hand side of* (23a)*) is strictly increasing in g, converges to -1 when $g\to-\infty$ and has a positive limit below 1 when $g\to\infty$. For any market state c, function $J^S=L^S-M^S$ (the difference between the left and the right hand side of* (23b)*) is strictly decreasing in g, has limit 1 when $g\to-\infty$ and limit -1 when $g\to\infty$.*

Now, g^a, g^b can be found as zeros of the 2-dimensional map (J^B,J^S) which exist according to the above lemma. Of course, only those solutions qualify as equilibrium for which g^b does not exceed g^a. That is, one needs to characterize α and $\mu^{H,L}$ as functions of c that generate correctly positioned inside quotes. Since there are many functions satisfying this condition, it would make sense to calibrate the model based on some really observed limit order book. For instance, let $g^a=g^m+h$, $g^b=g^m-h$, $\mu^H=m+\beta$, $\mu^L=m-\beta$, both m and $\beta>0$ being functions of c. Observe that the equilibrium prices only depend on the difference $\gamma=g^m-m$. It can be shown that for any given mid-quote g^m and half-spread h, there is a continuum of values of β for which there exists a smooth map from c to (α,γ) satisfying (23). Then, a map from c to β can be chosen to approximate any specific limit order book generating (g^m,h). Empirically based choice of $\alpha(c)$, $\gamma(c)$ is necessary since, as the model shows, there are always two possibilities for every c, h and β. Namely, there always exists one solution to (23) with $\alpha(c)>0.5$, $\gamma(c)>0$, and another with $\alpha(c)<0.5$, $\gamma(c)<0$. Intuitively, there can be more traders with a high asset private value ($\alpha(c)>0.5$) or more traders with a low private asset value ($\alpha(c)<0.5$). In the first case, the equilibrium mid-quote g^m shifts upward from the level of the average point m between the private value histogram modes ($\gamma>0$), in the second – it shifts downward ($\gamma<0$). Viewed inversely, the same price can be generated by two different private value histograms, one imbalanced toward potential buyers but positioned below g^m and another – imbalanced toward potential sellers but positioned above it. Fig. 2 depicts this situation for the (log) mid-price g^m equal to zero.

The argument illustrated by Fig. 2 can be quantitatively generalized if we ask what determines the *size* of the shift from the mid-point m to g^m. When the distribution of private asset values across traders is imbalanced, then, clearly, the market order flow in one direction will be bigger than in the other. However, these flows alone are insufficient to pin down the average position m of the underlying private values unless the calibration (meaning in this case the functional forms of $\alpha(c)$, $\beta(c)$ and $\gamma(c)$) is fixed. Otherwise, there are many shapes and positions of the value histogram that generate the same MO flows, depending on the parameters of the information distribution among traders. The role of informational dimension will be discussed in Section 4.

Fig. 2 Log mid-price of zero generated by two imbalanced private value distributions



Note: M-point means the mid-point between the high and the low mode of the histogram

3.3 Distribution of limit orders

This subsection presents a number of basic properties of equilibrium limit order histograms in the selected Gaussian setup for both HL and HP markets.

In the sequel, we will posit strict convexity of function k^B and strict concavity of k^S in addition to the earlier assumed strict growth of these two functions. Our leading example will be exponential k^B and k^S, for which this is true. Given these properties of $k^{B,S}$, it is possible to refine further the limit order book equations (22).

The following two technical lemmas, proved in the Appendix, will allow us to identify the set of admissible signals on the market buy order side of the market.

Lemma 3 *Function* φ: $\mathbf{R}\to\mathbf{R}^+$ *introduced in Subsection 3.1 is strictly increasing, with a well-defined strictly increasing inverse* b: $\mathbf{R}^+\to\mathbf{R}$. *We will use the following properties of* φ *and* b.

(a) $\lim_{s\to+\infty}\varphi(s)=+\infty$, $\lim_{s\to-\infty}\varphi(s)=0$;

(b) $\varphi'(s)=s\varphi(s)+1>0$ *for all* s;

(c) $\lim_{s\to+\infty}\varphi'(s) = \lim_{s\to+\infty}\left[s\varphi(s)+1\right] = +\infty$, $\lim_{s\to-\infty}\varphi'(s) = \lim_{s\to-\infty}\left[s\varphi(s)+1\right] = +0$;

(d) $\lim_{r\to+0} b'(r) = \lim_{r\to+0}\frac{1}{1+rb(r)} = +\infty$, $\lim_{r\to+\infty} b'(r) = \lim_{r\to+\infty}\frac{1}{1+rb(r)} = +0$.

Lemma 4 *Under the given notations,* $\tilde{\xi}^B(y^H,v) = \log\left(1-\tilde{w}^B(y^H,v)\right)$ *(cf.* (22)*), and the subjective demand elasticity* $\tilde{w}^B(y^H,v) = \frac{\eta_H}{r_H'(v)}\varphi\left(\frac{y^H - r_H(v)}{\eta_H}\right)$ *perceived by a limit sell submitter has the following properties*:

(a) *For any* $v>0$, $\tilde{w}^B(y^H,v)<1$ *if and only if* $y^H<\theta(v)$, *where*

$$\theta(v) = r_H(v) + \eta_H b\left(\frac{r_H'(v)}{\eta_H}\right);$$

as long as the ratio $\frac{r_H''(v)}{\left(r_H'(v)\right)^2}$ *is bounded on* $\mathbf{R}^+$, *the following asymptotics are valid*:

(b) $\lim_{v\to+0}\theta(v) = -\infty$; $\lim_{v\to+\infty}\theta(v) = +\infty$

(c) $\lim_{v\to+0}\theta'(v) = +\infty$; $\lim_{v\to+\infty}\theta'(v) = 0$.

Note that boundedness property for the ratio $\frac{r_H''(v)}{\left(r_H'(v)\right)^2}$, as required in parts (b) and (c) of Lemma 4, holds in our leading example of exponential k^B and k^S.

The above lemmas describe the conditions under which subjectively optimal limit sell orders are well-defined. When the signal y^H is high, only traders with sufficiently low private asset log-values $\tilde{\xi}^B(y^H,v)$ are willing to place orders at distance v from the inside ask. The mass of such traders goes to zero quickly with increasing absolute value of y^H. As it turns out, for the values of y^H exceeding a certain threshold, the subjective belief of the trader about demand elasticity is so high that there is no internal solution for the optimal limit sell price (every increase in price increases the expected surplus despite the fall in the execution probability). Therefore, it is reasonable to interpret this formal property of $\tilde{\xi}^B(y^H,v)$ as the fact that there are no limit sell orders at distance v from the inside ask by traders with signals y^H unless $y^H < \theta(v)$.

The properties of the limit order expressions are summarized in the following proposition, which is a direct consequence of Lemma 4.

Proposition 1 *For every market state c, the mass of limit sell orders is positive for every price having log-distance v from the inside ask* $v^{ia}(c)$ *in the interval* $(0, v_+)$ *with* $v_+ > k^B\left(c^H\right)$:

$$\tilde{q}^B(c, v^{ia}(c), v) > 0,\ v\in(0,v_+).$$

Therefore, $M^B(c) < \tilde{Q}^B(c, v^{ia}(c), v)$ *for* $v \in (k^B(c^H), v_+)$, *which means that all market buy orders are executed in equilibrium.*

As regards limit buy orders, $\tilde{q}^S(c, v^{ib}(c), v) > 0$ *for all* $v<0$, $c \in \mathbf{R}$, *and* $\lim_{v \to -\infty} \tilde{q}^S(c; v^{ib}(c), v) = 0$. *Consequently, since* $-\infty < k^S(c^L)$, *we have* $M^S(c) < \tilde{Q}^S(c, v^{ib}(c), v)$ *for* $v \in (-\infty, k^S(c^L))$ *and all market sell orders are executed in equilibrium.*

Proposition 1 confirms that in the constructed brokered market model equilibrium trade patterns imply the certain execution of market orders, but the uncertain execution of limit orders for every trader.

Proposition 2 *If the conditions of Lemma 4 are satisfied, then there is no clustering of limit sell orders at the inside ask or bid*:

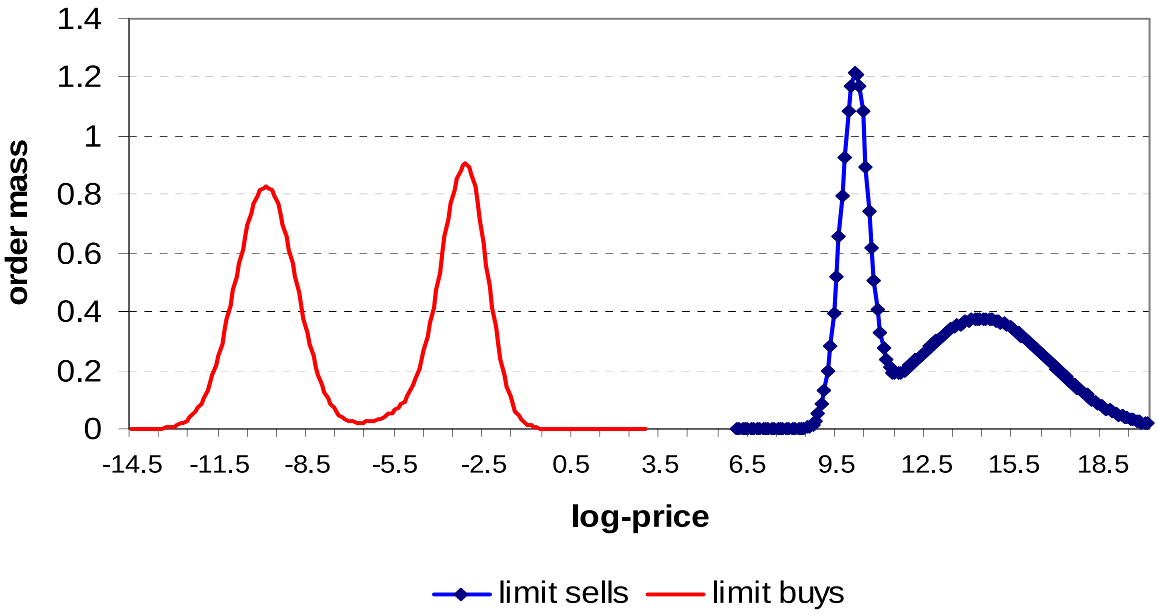
$$\lim_{v \to +0} \tilde{q}^B(c; v^{ia}(c), v) = 0, \ \lim_{v \to -0} \tilde{q}^S(c; v^{ib}(c), v) = 0.$$

Proof: see Appendix.

A typical equilibrium limit order book is shown in Fig. 3.

It is easy to check that the technical conditions required for the validity of the above proposition are satisfied in our leading example of exponential k^B and k^S.

Fig. 3 Equilibrium limit order book



4. Information quality, equilibrium price and trader welfare
4.1 Price impact of the information bias

In the present model, as in most others in microstructure finance, the market price is a result of liquidity-suppliers' (limit order providers here) quotes reaction to the liquidity demanders' (market order submitters) order flow. However, due to the informational dimension of the traders' choices (trader values and signals are unobservable, the market state is observed with a bias at least initially), one cannot determine the price based on the market order flow alone. Typically, what we are able to infer from it is the *direction* of the price change, whilst inferring the *magnitude*, which is the key objective of the microstructure theory, is much less straightforward.

In my model, the mapping from the two-dimensional market state vector into the inside quotes is characterized indirectly by two market clearing conditions on the buy and the sell side. But, the dimension of the underlying state space in a broader sense is certainly higher than two. At the least, there is the "bull" mode weight α, the distance between modes β, and the shift from the between-modes point m to the mid-quote g^m, denoted by γ. So, there is at least one dimension along which the unobservable market state transmission into price remains non-unique until one selects a calibration. As was explained in Subsection 3.2, for any observed mid-quote g^m and half-spread h, and under a fixed distance β between the modes, (23) defines two possible maps from the (narrowly understood) market state c into the pair (α,γ). In addition, there is the common initial prior c_p=(c_p^H, c_p^L) (more exactly, the mean of the prior distribution of the market state common to all traders) which co-determines the solution to the equilibrium equations (23). This means that the so called "price impact of trades" varies depending on the trader private value and information distribution z. In itself, this statement is trivial, given that the solution should normally depend on parameters entering the equations. What is less trivial is the quantitative relationship that we have therewith established between the equilibrium price and a factor that is present in investors' minds prior to and irrespective of, subsequent trades. Specifically, the post-trade prices are different for investor populations with the same private value distribution but with differently distributed common prior beliefs.

The exact place where the initial bias c_p enters the solution for equilibrium inside prices is the following. (We only discuss the HP-case for simplicity. The LP-case involves a messier notation with no possibility of numerical experiments at the present stage.) Using (20), one can write for the exercised limit order quantities:

$$L^B(c,g) = \int_{-\infty}^{\theta(k^B(c^H))} \Psi^-\left(c, g+v+\tilde{\xi}^B(y^H, k^B(c^H))\right)\bar{f}^H(\bar{c}^H - y^H)dy^H\,, \tag{24a}$$

$$L^S(c,g) = \int_{-\infty}^{+\infty} \Psi^+\left(c, g+v+\tilde{\xi}^S(y^L, k^S(c^L))\right)\bar{f}^L(\bar{c}^L - y^L)dy^L\,. \tag{24b}$$

Note that the only place where the prior beliefs appear are the expressions for the informational density $\bar{f}^{H,L}$, whereas all the other terms on the right hand side of (24) depend only on the true market state c. Namely, the terms $\Psi^{\pm}$ give the counts of traders with a private value in the right interval, the information parameter fixed. These are quantities derivative of the true market state. On the contrary, the counts of traders with the given *information* are bias-dependent (cf. (18hist)).

On the market order side, in the expressions

$$M^B(c,g) = \Psi^+\left(c, g + \int_0^{k^B(c^H)} \frac{\tilde{q}^B(c,g,v)dv}{L^B(c,g)}\right),\ M^S(c,g) = \Psi^-\left(c, g + \int_{k^S(c^L)}^0 \frac{\tilde{q}^S(c,g,v)dv}{L^S(c,g)}\right),$$

the bias value appears only in $\tilde{q}^{B,S}$ and $L^{B,S}$ inside the integrals. Here, the MO-submitters know both c and c_p and can reconstruct (24) exactly. Altogether, (23) is sufficient to calculate the partial derivatives of inside prices g^a, g^b, with respect to $c_p^{H,L}$. The following table shows the values of $\frac{\partial g^a}{\partial c_p^H}$, $\frac{\partial g^b}{\partial c_p^L}$ calculated after solving for $(\alpha(c),\gamma(c))$ at the point $c^{H,L} = c_p^{H,L}$ for a number of values of c, with g^m=0, h and β fixed. Of the two possible solutions $(\alpha(c),\gamma(c))$, we have selected one for which α>0.5, γ>0 for c^H≥0, c^L≤0. This roughly corresponds to selecting the equilibrium in which a wider range of executed limit buys (c^H>0) and/or a narrower range of executed limit sells (c^L>0) is consistent with the bullish investors outnumbering the bearish ones and the mid-price exceeding the point between the modes of the private value histogram.

Table 1 Impact of prior beliefs on inside prices

c^H / c^L	**-1**	**0**	**1**
-1	0.094 -0.0207	0.0953 -0.6761	0.101 -0.1184
0	0.0935 -0.0193	0.0949 -0.1855	0.0967 -0.1974
1	0.0923 -0.018	0.0944 -0.0176	0.0963 -0.0173

Note: In every cell, the upper line gives the value of $\frac{\partial g^a}{\partial c_p^H}$ and the lower one – the value of $\frac{\partial g^b}{\partial c_p^L}$. The following parameters have been used: h=0.1, β=1.4, $\sigma_H=\sigma_L$=0.1, $\eta_H=\eta_L$=0.5, $\lambda_H=\lambda_L$=17, $\kappa_H=\kappa_L$=0.5.

The above findings may be important for the progress in the as yet unresolved debate between the "flow-centrist" and "fundamentalists" among asset price theorists. Our model demonstrates that trades indeed drive prices, but they only do so subject to a given attitude (private value + information) profile of the investor population. Most probably, the longer the horizon, the more important becomes the common element in beliefs along the informational coordinate. This common element can be linked to "fundamentals", even if not identical with them, since the difference between c and c_p can just as well be caused by a groundless sentiment. So, in a longer run than just one period of intensive trade, price evolution may to a large extent be the attitude evolution, even though traded order flow is always instrumental in channeling the attitude into the price.

The current understanding of the model suggests that the relative importance of prior information about, as opposed to the actual momentary state of, private value profile for a persistent price change is yet to be explored. Formally, at the beginning of Subsection 3.1, I have defined heterogeneous trader information as an outcome of Bayesian updating of a common *biased prior belief* by heterogeneous unbiased signals. This definition might evoke an impression of the prior beliefs as a deviation from the true market state that should be responsible for the fundamental value. However, nothing precludes us from making an inverse interpretation. Hypothetically, there may be more fundamental truth to the common prior knowledge c_p of the market state than to the specific realization c of this state at a point in time. Seen from this perspective, the only fundamental input into the price is fed through the initial belief c_p, whereas all other determinants of equilibrium trades and quotes, summarized by c, are transient. As usual, the reality is most likely to lie between the extremes, so that we may never be sure as to whether the observed order flow takes the price towards or away from, the unobserved fundamental value.

4.2 Information and trader surplus

Beside the bias c_p-c, the aggregate quality of information in the present model is characterized by the values of standard deviation parameters η_H, η_L (the lower they are, the better the trader population as a whole is informed about its own value-and-signal histogram). However, the model does not possess a meaningful "full information" limit when η_H, $\eta_L \to +0$. Indeed, at such a limit, if it existed, traders would know the last executable limit prices i^b+k^S, i^a+k^B exactly, which fact would induce them to place their orders exactly there. Expected market order execution prices would then converge to the same values for all traders, b to i^b+k^S and a to i^a+k^B. Inside quotes would loose their meaning, and the only log-price $\rho= i^b+k^S=i^a+k^B$ that would satisfy the degenerate market clearing condition (25) would be i^m for which $\Psi^-(c_0,i^m)=\Psi^+(c_0,i^m)$. This outcome would not be an equilibrium since for a subset of traders with a positive mass, undercutting or overbidding would be a best response to i^m-quoting by others. This is just a special case of the well-known indeterminacy result in a private value auction with full information.

On the other hand, the present model allows for the study of welfare as a function of the private information value for an individual trader. Since traders have been assumed to be risk-neutral, the surpluses they derive from each of the four types of orders can be analyzed separately.

Limit order submitter surplus To handle this case, we shall need to invert the functions $\left(y^H,v\right)\mapsto \Xi^B\left(y^H,v\right)=v+\tilde{\xi}^B\left(y^H,v\right)$ and $\left(y^L,v\right)\mapsto \Xi^S\left(y^L,v\right)=v+\tilde{\xi}^S\left(y^L,v\right)$ with respect to the second argument (recall that they are both strictly increasing in v). So, let us define functions $V^{a,b}$ such that

$$\Xi^B\left(y^H,V^a(y^H,x-i^a(c))\right)\equiv x-i^a(c),\ \Xi^S\left(y^L,V^b(y^L,x-i^b(c))\right)\equiv x-i^b(c) \qquad (25)$$

for all y and c. $V^a(x$-$i^a(c))$ is the relative quote placed by a limit sell submitter with private value x and information y, and a similar interpretation is valid for V^b.

Clearly, the ex ante expected surplus from the limit order placed by such a trader, is equal to

$$B(y,p)\left(p-e^x\right)=\tilde{B}(y,V^a(x-i^a(c),y))\left(e^{i^a(c)+V^a(x-i^a(c),y)}-e^x\right)$$

$$=N\left(\frac{y^H-r_H\left(V^a(x-i^a(c),y^H)\right)}{\eta_H}\right)\left(e^{i^a(c)+V^a(x-i^a(c),y^H)}-e^x\right).$$

Then, by the Envelope Theorem, the derivative of this surplus w.r.t. information y^H equals

$$\frac{1}{\eta_H} n\left(\frac{y^H - r_H\left(V^a(x - i^a(c), y^H)\right)}{\eta_H}\right)\left(e^{i^a(c)+V^a(x-i^a(c),y^H)} - e^x\right),$$

which is positive everywhere. That is, the ex ante surplus from a limit sell grows with the trader's optimism regarding the executable limit sell range. However, for large signal values the increments in expected surplus become very small.

For the same reason, the limit buy submitter's ex ante surplus, equal to

$$S(y,p)\left(e^x - p\right) = \tilde{S}(y, V^b(x - i^b(c), y))\left(e^x - e^{i^b(c)+V^b(x-i^b(c),y)}\right)$$

$$= N\left(\frac{r_L\left(V^b(x - i^b(c), y^L)\right) - y^L}{\eta_L}\right)\left(e^x - e^{i^b(c)+V^b(x-i^b(c),y^L)}\right),$$

is a strictly decreasing function of information y^L, meaning that this surplus grows with the trader's pessimism regarding the range of executable limit buys. The surplus growth becomes negligible for large negative values of the signal.

Turning to the limit sell submitter's ex post surplus, equal to $e^{i^a(c)+V^a(x-i^a(c),y^H)} - e^x$ if the order is executed, it grows/decreases with y^H if and only if the optimal quote function V^a does. By the Implicit Function Theorem, the partial derivative of V^a w.r.t. y^H equals

$$\frac{\partial V^a}{\partial y^H} = -\frac{\tilde{\xi}^B_{y^H}}{1+\tilde{\xi}^B_v} = \frac{1}{1-\tilde{w}^B}\frac{1}{1+\tilde{\xi}^B_v}\frac{1}{r_H{}'(V^a)}\varphi'\left(\frac{y^H - r_H(V^a)}{\eta_H}\right);$$

this expression is always positive. So, also the ex post surplus of a limit sell submitter increases with the signal value. However, this surplus falls to zero the moment the optimal quote reaches the last executed quote $k^B(c^H)$. It can be checked that the corresponding critical information value is equal to

$$\hat{y}^H = c^H + \eta_H b\left(\frac{1 - e^{x - i^a(c) - k^B(c^H)}}{\eta_H k^{B\prime}(c^H)}\right). \tag{26}$$

We conclude that the highest welfare from a limit sell is not derived by the traders with the most precise information but by the ones with the right relation of the information bias to their own asset valuation, as defined by (26).

Note that (26) defines a cut-off information value that is within the range of admissible values with the upper bound $\theta(k^B(c^H))$ valid for the traders who submit limit sells at the maximum distance $k^B(c^H)$ from the best ask. It is easy to see by comparing the right hand side of (26) with the definition of function θ given in Lemma 4.

Analogous arguments lead to the following expression for the minimal value of information y^L for which the limit buy is executed, at the same time being the information of the traders with the highest ex post surplus derived from the limit buy:

$$\hat{y}^L = c^L - \eta_L b \left(\frac{1 - e^{x - i^B(c) - k^S(c^L)}}{\eta_L k^{S'}(c^L)} \right). \tag{27}$$

Again, the optimal bias of the limit buyer's information is not zero but is a non-trivial function of the private asset value, x, as given by (27).

Market order submitter surplus Evidently, the ex ante market buy submitter surplus with private value x and information y is given by e^x-$e^{a(y)}$. Only traders for which this value is positive place market buys. Ex post (and, therefore, always in the HP-case), the realized surplus depends on the order matching by the broker (recall that a market buy has to pay $e^{i^a(c)+v}$ with probability $\dfrac{\tilde{q}^B(c, i^a(c), v)dv}{\tilde{Q}^B(c, i^a(c), k^B(c^H))}$ for $0<v<k^B(c^H)$). If one takes expectations with respect to this order matching uncertainty, one gets a measure of realized surplus which depends on the signal in a piecewise-constant manner. Specifically, there is a nontrivial set of signal values for which the expected execution price becomes underestimated and the trader's realized surplus - negative.

Returning to the ex ante surplus (which is relevant only in the HL-case), its dependence on information is given by $\dfrac{\partial a}{\partial y^{H,L}}(y) = \int \hat{a}(\gamma, i^a(\gamma)) \dfrac{\gamma^{H,L} - y^{H,L}}{\eta^2_{H,L}} f(\gamma - y) d\gamma$. This expression can take both signs depending on the value of y and the parameters of the model. Depending on the sign of the log-price realizations in the square brackets inside the integral, the expected execution price as a function of y may have both local maxima and minima. Qualitatively, in precise information cases (i.e. f concentrated around the true value of the market state), one would expect this extreme value to be close to the market state true value as well.

Similar considerations are valid for the market sell submitter surplus.

5. Conclusion

In this paper, I model price formation in a fast order-driven market. Trading is considered at a point in time. There are many participants who may submit both limit and market orders for a small standard quantity of the asset. Every trader is uncertain about other participants' preferences. Terms of trade facing every potential order are partially unobservable. One possibility is that the time window of opportunity to trade is so short as if all market and limit orders were being submitted at once. This also means that market order submitters do not observe - or observe for insufficient time to react - the book against which their orders will be matched. The less severe case of terms of trade unobservability obtains when the book is observed and fully analyzed, but due to a large number of market orders arriving nearly simultaneously, no trader, for uncertain priority reasons, can be sure about the exact execution price. For both variants, I have derived the equations for equilibrium inside bid and ask price at which all market orders are executed whereas a part of limit orders are not.

The model is best suited to analyze brokered markets with high frequency of order arrival and unclear fundamental characteristics of the traded asset (gilt bonds, FX). It is constructed in such a way that does not preclude the existence of market makers servicing the same set of investors. It can be used to obtain an instantaneous capture of the market value distribution with two modes, high (bullish traders, potential buyers) and low (bearish traders, potential sellers) at any moment when both inside quotes of a broker are available. Other data from the same market and time, such as the state of the book and order flows on both sides of the market, would allow the calibration of the model with respect to the involved trader set size and other parameters.

The model is helpful in analyzing the role of market order flow in price setting in an order-driven market. More exactly, we ask whether the two-dimensional vector of market buy and market sell order quantities is a sufficient statistic for the asset price. The answer is no, unless one artificially restricts the distribution of private asset values and information endowments. Otherwise, there exist parameters describing the trader population (including one that we may loosely identify with the popular notion of "market sentiment") which influence the price, making it indeterminate for any given market order flow value. Formally, we have attached the meaning of market sentiment to the initial bias in the investor private information. Therefore, even though trades do transmit preferences into prices, they alone are not enough to pin down the price uniquely. As a result, one cannot claim that price and trades carry the same information. Selecting a particular calibration of the model so as to approximate an actually observed order book, one can remove the indeterminacies, including the one linked to the "sentiment". Accordingly, whether the observed price movements reflect fundamental or transient factors appears to be an empirical question.

Appendix: Proofs

Proof of Lemma 2: Below, I will prove that the g-derivative of function J^S defined as

$$J^S(c,g) = \int_{-\infty}^{+\infty} \Psi^+\left(c, g + k^S(c^L) + \tilde{\xi}^S(y^L, k^S(c^L))\right)\bar{f}^L(\bar{c}^L - y^L)dy^L$$
$$- \Psi^-\left(c, g + k^S(c^L) + \int_{k^S(c^L)}^{0} \frac{\tilde{Q}^S(c,g,v)dv}{L^S(c,g)}\right),$$

Is negative, the J^B-case being fully analogous. Recall that $\Psi^{\pm}_{\rho}(c,\rho) = \mp\Phi(c,\rho)$. Therefore, to prove the lemma it is sufficient to demonstrate that the partial g-derivative of

$$L^S(c,g) = \int_{-\infty}^{+\infty} \Psi^+\left(c, g + k^S(c^L) + \tilde{\xi}^S(y^L, k^S(c^L))\right)\bar{f}^L(\bar{c}^L - y^L)dy^L$$

is negative, whereas the partial g-derivative of

$$\hat{b}(c,g) = g + k^S(c^L) + \int_{k^S(c^L)}^{0} \frac{\tilde{Q}^S(\gamma,g,v)dv}{\tilde{Q}^S\left(\gamma,g,k^S(c^L)\right)}$$

is positive everywhere.

Observe that, according to (22b), for every $v^b<0$

$$\tilde{Q}^S(c,g,v^b)=\int_{v^b}^{0}\tilde{q}^S(c,g,v)dv=\int_{v^b}^{0}\int z\left(g+v+\tilde{\xi}^S(y^L,v),y^L,c\right)\left[1+\tilde{\xi}_v^S(y^L,v)\right]dy^L dv$$

and

$$\tilde{Q}_g^S(c,g,v)=-\psi^S(c,g,v)=-\int_{-\infty}^{+\infty}z\left(g+v+\tilde{\xi}^S(y^L,v),y^L,c\right)dy^L .$$

Suppressing the argument of k^S for the sake of brevity, and with the help of some standard algebra, we obtain the following expression for $\hat{b}_g(c,g)$:

$$\hat{b}_g(c',g)=\int_{k^S}^{0}\frac{\tilde{Q}^S(c,g,v)dv}{\tilde{Q}^S(c,g,k^S)^2}\psi^S(c,g,k^S)+\int_{k^S}^{0}\int\frac{z\left(g+v+\tilde{\xi}^S(y^L,v),y^L,c\right)\tilde{\xi}_v^S(y^L,v)dy^L dv}{\tilde{Q}^S(c,g,k^S)} .$$

The first term in the last equality is clearly positive. To make sure about the positive sign of the second term, one should recall (see Assumption 3 in Section 3) that $\tilde{\xi}_v^S>0$ everywhere, completing the proof •

Proof of Lemma 3: The identity $\varphi'(s)=s\varphi(s)+1$ is checked directly.

(a) The statement about the limit value of φ at plus-infinity is trivial. To check the statement about the limit value of this function at the minus-infinity, use L'Hospital's rule:

$$\lim_{s\to=\infty}\frac{N(s)}{n(s)}=\lim_{s\to=\infty}\frac{n(s)}{-sn(s)}=0 .$$

(b) It is necessary and sufficient to prove that the expression $D(s)=n(s)-sN(-s)$ is positive for all $s\geq0$. Obviously, $D(0)=\left(2\pi\right)^{-\frac{1}{2}}>0$, $D(+\infty)=0$ and, by direct inspection, one easily finds out that $D'(s)=-N(-s)<0$ for all $s\geq0$. That is, D is indeed strictly positive on the whole positive half-axis.

(c) The statement $\lim_{s\to+\infty}\varphi'(s)=\lim_{s\to+\infty}\left[s\varphi(s)+1\right]=+\infty$ is trivial. To prove the other one, we apply L'Hospital's rule as follows:

$$\lim_{s\to-\infty}\frac{sN(s)+n(s)}{n(s)}=\lim_{s\to-\infty}\frac{N(s)+sn(s)-sn(s)}{-sn(s)}=\lim_{s\to-\infty}\frac{n(s)}{-n(s)+s^2n(s)}=\lim_{s\to-\infty}\frac{1}{s^2-1}=0 .$$

From (b), we already know that $s\varphi(s)+1$ is strictly positive everywhere, which completes the proof.

(d) Statements about b follow directly from the properties of the inverse function derivative •

Proof of Lemma 4:
(a) Condition on y^H can be checked directly.
(b) We write θ as

$$\theta(v)=r_H(v)\left[1+\eta_H\frac{b\left(r_H{}'(v)/\eta_H\right)}{r_H(v)}\right]. \qquad \text{(A1)}$$

As $v \to +0$, $r_H'(v) \to +\infty$, so that both the numerator and the denominator of the ratio $\dfrac{b\left(r_H'(v)/\eta_H\right)}{r_H(v)}$ converge to infinity in absolute value. As $v \to +\infty$, $r_H'(v) \to +0$, and both the numerator and the denominator of the said ratio converge to infinity in absolute value again. By L'Hospital's rule, this ratio must have the same limit as

$$\frac{b'\left(\frac{r_H'(v)}{\eta_H}\right) r_H''(v)}{r_H'(v)} = \frac{r_H''(v)}{\left(r_H'(v)\right)^2} \frac{\frac{r_H'(v)}{\eta_H}}{1+\frac{r_H'(v)}{\eta_H} b\left(\frac{r_H'(v)}{\eta_H}\right)} = \frac{r_H''(v)}{\left(r_H'(v)\right)^2} \frac{1}{\frac{\eta_H}{r_H'(v)} + b\left(\frac{r_H'(v)}{\eta_H}\right)} \quad \text{(A2)}$$

(the first equality in (A2) follows from

$$\frac{db\left(r_H'(v)/\eta_H\right)}{dv} = \frac{r_H''(v)/\eta_H}{1 + r_H'(v)/\eta_H b_{\pm}\left(r_H'(v)/\eta_H\right)}).$$

One can easily check that the expression $\dfrac{\eta_H}{r_H'(v)} + b\left(\dfrac{r_H'(v)}{\eta_H}\right)$ is strictly positive (follows from part (d) of Lemma 3) and converges to infinity in absolute value both for $v \to +0$ and $v \to +\infty$. Due to the assumption of boundedness of the ratio $\dfrac{r_H''(v)}{\left(r_H'(v)\right)^2}$, we conclude that the expression in square brackets in (A1) converges to unity. Accordingly, $\theta(v)$ has the same limit behavior as $r_H(v)$, as claimed.

(c) Writing the derivative of θ as

$$\theta'(v) = r_H'(v) \left\{ 1 + \frac{r_H''(v)}{\left(r_H'(v)\right)^2} \frac{1}{\frac{1}{r_H'(v)} + \frac{1}{\eta_H} b\left(\frac{r_H'(v)}{\eta_H}\right)} \right\},$$

we use the same arguments as in (b) to conclude that θ' has the same limit behavior as r_H' •

Proof of Proposition 2: The mass of limit sells placed at v is given by equation (14a), which we adapt to the specific functional forms of Section 3 as follows:

$$\tilde{q}^B\left(c, i^a(c), v\right) = \frac{1}{\eta_H} \int_{-\infty}^{\theta(v)} \Phi\left(c, i^a(c) + v + \tilde{\xi}^B(y^H, v)\right) n\left(\frac{c^H - y^H}{\eta_H}\right) \left[1 + \tilde{\xi}_v^B(y^H, v)\right] dy^H . \quad \text{(A3)}$$

Since $\lim_{v\to+0}\theta(v) = -\infty$ (Lemma 4(b)), it is sufficient to prove that the integrand in (A3) is *v*-uniformly bounded when y^H is close to $\theta(v)$ and *v* is close to the origin. In fact, we are able to prove even more than that.

Indeed, direct calculation with the help of Lemma 3(b) leads to the following expression for $1+\tilde{\xi}_v^B$:

$$1+\tilde{\xi}_v^B(y^H,v) = \frac{2+\left(\frac{\eta_H r_H''(v)}{\left(r_H'(v)\right)^2}+\frac{y^H-r_H(v)}{\eta_H}-\frac{\eta_H}{r_H'(v)}\right)\varphi\left(\frac{y^H-r_H(v)}{\eta_H}\right)}{1-\frac{\eta_H\varphi\left(\frac{y^H-r_H(v)}{\eta_H}\right)}{r_H'(v)}}. \tag{A4}$$

By definition of $\theta(v)$ (Lemma 4), when $y^H \to \theta(v)-0$, $\frac{y^H-r_H(v)}{\eta_H}\to b\left(\frac{r_H'(v)}{\eta_H}\right)-0$ and $\varphi\left(\frac{y^H-r_H(v)}{\eta_H}\right)\to\frac{r_H'(v)}{\eta_H}-0$. Therefore, for any $v>0$, the numerator in (A4) converges to

$$1+\frac{r_H''(v)}{r_H'(v)}+\frac{r_H'(v)}{\eta_H}b\left(\frac{r_H'(v)}{\eta_H}\right)$$

as $y^H \to \theta(v)-0$. This expression, although it grows to plus infinity as $v\to+0$, does so at a lower speed than $n\left(\frac{c^H-\theta(v)}{\eta_H}\right)$, which means that the numerator of (A4) times $n\left(\frac{c^H-y^H}{\eta_H}\right)$ converges to zero uniformly in $y^H<\theta(v)$ as $v\to+0$. The denominator of (A4) is equal to $e^{\tilde{\xi}^B(y^H,v)}$ and converges to zero when $y^H \to \theta(v)-0$. However, the expression $\Phi\left(c,i^a(c)+v+\tilde{\xi}^B(y^H,v)\right)$ appearing in (A3) falls to zero uniformly in *v* at an even higher speed, as $y^H \to \theta(v)-0$. Therefore, (A3) as a whole vanishes with $v\to+0$, proving the first part of the proposition.

To prove the statement concerning the market sell side, we adapt the expression (14b) to the specific functional forms of Section 3 as

$$\tilde{q}^S\left(c;i^b(c),v\right)=\frac{1}{\eta_L}\int_{-\infty}^{\infty}\Phi\left(c,i^b(c)+v+\tilde{\xi}^S(y^L,v)\right)n\left(\frac{c^L-y^L}{\eta_L}\right)\left[1+\tilde{\xi}_v^S(y^L,v)\right]dy^L. \tag{A5}$$

Next, analogously to (A4), we derive the following expression for $1+\tilde{\xi}_v^S$:

$$1+\tilde{\xi}_v^S(y^L,v)=\frac{2+\left(\frac{\eta_L}{r_L^{'}(v)}-\frac{\eta_L r_L^{''}(v)}{\left(r_L^{'}(v)\right)^2}+\frac{r_L(v)-y^L}{\eta_L}\right)\varphi\left(\frac{r_L(v)-y^L}{\eta_L}\right)}{1+\frac{\eta_L\varphi\left(\frac{r_L(v)-y^L}{\eta_L}\right)}{r_L^{'}(v)}}. \tag{A6}$$

As $v\to -0$, both $r_L(v)$ and $r_L^{'}(v)$ converge to plus infinity. It suffices to observe three facts. First, for big y^L (close to plus infinity), both (A6) and $\Phi\left(i^b(c)+v+\tilde{\xi}^S(y^S,v),c\right)$ are uniformly bounded in v, whereas the density $n\left(\frac{c^L-y^L}{\eta_L}\right)$ converges to zero. Second, for small y^L (close to minus infinity), $\tilde{\xi}^S(y^L,v)$ grows to plus infinity and $\Phi\left(i^b(c)+v+\tilde{\xi}^S(y^S,v),c\right)$ falls to zero faster than the numerator in (A6). Third, for intermediate y^L, $\tilde{\xi}^S(y^L,v)$ also grows to plus infinity uniformly in y^L, whereas (A6) and the whole integrand in (A5) vanishes as $v\to -0$. This completes the proof •

References

1. Biais, B., P. Hillion, and C. Spatt (1995) An empirical analysis of the limit order book and the order flow in the Paris Bourse. *Journal of Finance* L, No. 5, 1655-1689.
2. Daníelsson, J., and R. Payne (2002) Real trading patterns and prices in spot foreign exchange markets. *Journal of International Money and Finance* 21, 203-222.
3. Derviz, A. (2003) *FOREX Microstructure, Invisible Price Determinants, and the Central Bank's Understanding of Exchange Rate Formation*. Czech national Bank, Research Department WP No. 6 (July).
4. Evans, M., and R. Lyons (2002) Order flow and exchange rate dynamics. *Journal of Political Economy* 110, 170-180.
5. Foucault, T. (1999) Order flow composition and trading costs in a dynamic limit order market. *Journal of Financial Markets* 2, 99-134.
6. Froot, K. A., and T. Ramadorai (2005) Currency Returns, Intrinsic Value, and Institutional Investor Flows. Journal of Finance 60, no. 3 (June), 1535-1566.
7. Glosten, L. (1989) Insider trading, liquidity, and the role of the monopolist specialist. *Journal of Business* 62 (2), 211-235.
8. Glosten, L. (1994) Is the electronic open limit order book inevitable? *Journal of Finance* XLIX, No. 4, 1127-1161.
9. Goettler, R., C. Parlour and U. Rajan (2003) *Equilibrium in a Dynamic Limit Order Market*. Mimeo, Carnegie Mellon University (forthcoming in the *Journal of Finance*).
10. Handa, P., and R. Schwartz (1996) Limit order trading. *Journal of Finance* LI, No. 5, 1835-1861.
11. Handa, P., R. Schwartz, and A. Tiwari (2003) Quote setting and price formation in an order driven market. *Journal of Financial Markets* 6, 461-489.
12. Kyle, A. (1985) Continuous auctions and insider trading. *Econometrica* 53, 1315-1335.
13. Parlour, C. (1998) Price dynamics in limit order markets. *Review of Financial Studies* 11, 789-816.

14. Rime, D. (2003) *New Electronic Trading Systems in Foreign Exchange Markets*. Mimeo, Norges Bank and Stockholm Inst. For Financial Research (January).