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The Josefson-Nissenzweig theorem, Grothendieck property, and finitely-supported measures on compact spaces

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THE JOSEFSON–NISSENZWEIG THEOREM, GROTHENDIECK PROPERTY, AND FINITELY-SUPPORTED MEASURES ON COMPACT SPACES

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Abstract. The celebrated Josefson–Nissenzweig theorem implies that for a Banach space $C(K)$ of continuous real-valued functions on an infinite compact space K there exists a sequence of Radon measures $\langle\mu_n\colon\ n\in\omega\rangle$ on K which is weakly* convergent to the zero measure on K and such that $\|\mu_n\|=1$ for every $n\in\omega$. We call such a sequence of measures *a Josefson–Nissenzweig sequence*. In this paper we study the situation when the space K admits a Josefson–Nissenzweig sequence of measures such that its every element has finite support. We prove among the others that K admits such a Josefson–Nissenzweig sequence if and only if $C(K)$ does not have the Grothendieck property restricted to functionals from the space $\ell_1(K)$. We also investigate miscellaneous analytic and topological properties of finitely supported Josefson–Nissenzweig sequences on general Tychonoff spaces.

We prove that various properties of compact spaces guarantee the existence of finitely supported Josefson–Nissenzweig sequences. One such property is, e.g., that a compact space can be represented as the limit of an inverse system of compact spaces based on simple extensions. An immediate consequence of this result is that many classical consistent examples of Efimov spaces, i.e. spaces being counterexamples to the famous Efimov problem, admit such sequences of measures.

Similarly, we show that if K and L are infinite compact spaces, then their product $K\times L$ always admits a finitely supported Josefson–Nissenzweig sequence. As a corollary we obtain a constructive proof that the space $C_p(K\times L)$ contains a complemented copy of the space c_0 endowed with the pointwise topology—this generalizes results of Cembranos and Freniche.

Finally, we provide a direct proof of the Josefson–Nissenzweig theorem for the case of Banach spaces $C(K)$.

1. Introduction

The celebrated Josefson–Nissenzweig theorem, stating that for every infinite-dimensional Banach space there is a sequence $\langle x_n^*\colon\ n\in\omega\rangle$ in the dual space X^* which is weakly* convergent to 0 and such that $\|x_n^*\|=1$ for every $n\in\omega$, is one of the most fundamental results in Banach space theory and has found many remarkable applications. For instance, it was used to obtain such structural results as: (1) for every infinite compact space K and infinite-dimensional Banach space X the space $C(K,X)$ of all continuous functions from K to X is not a Grothendieck space (Khurana [49]); (2) for every infinite compact space K the space $C(K\times K)$ of continuous real-valued functions on $K\times K$ contains a complemented

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copy of the Banach space c_0 (Cembranos [20], Freniche [36]); (3) if X is a Banach space, then X contains a complemented copy of the Banach space ℓ_1 if and only if for every infinite-dimensional Banach space Y there exists a non-compact bounded linear operator $T\colon X \to Y$ (Bator [5]); etc. Several interesting equivalent statements of the theorem have been also found e.g. by Borwein and Fabian [16].

The original proofs of the theorem due to Josefson [45] and Nissenzweig [66] were rather long and complicated. Alternative or simpler proofs were later obtained by Hagler and Johnson [40], Bourgain and Diestel [18], Behrends [9], or Mujica [64]. The conclusion of the theorem was also proved valid separately for several particular classes of Banach spaces, e.g. spaces $C(K)$ of continuous real-valued functions on extremely disconnected compact spaces K (Andô [2], 14 years before the works of Josefson and Nissenzweig!), Banach lattices (Wójtowicz [86]), Banach spaces which are not Asplund but have weakly* sequentially compact dual unit balls (Hájek and Talponen [41]), etc. In Section 3.1 of our paper we provide a simple measure-theoretic proof of the Josefson–Nissenzweig theorem for the class of all Banach spaces $C(K)$ of continuous real-valued functions on compact spaces K.

The Josefson–Nissenzweig theorem does not hold for general topological vector spaces, e.g. it is known that it fails for *Fréchet spaces*, i.e. metrizable and complete locally convex spaces (see Bonet [13], Lindström and Schlumprecht [58], Bonet, Lindström and Valdivia [14]). Banakh, Kąkol and Śliwa [7] studied the validity of the theorem in the class of $C_p(X)$*-spaces*, i.e. spaces of continuous real-valued functions on Tychonoff spaces X endowed with the product topology, and found its remarkable connection with the Separable Quotient Problem for $C_p(X)$-spaces and complementability of the space $(c_0)_p$, the classical Banach space c_0 but equipped with the product topology (see Section 2 for explanation of the notation). Namely, they proved that given a Tychonoff space X, there exists a sequence $\langle \mu_n\colon\ n \in \omega\rangle$ of measures with finite supports on X such that $\lim_{n\to\infty}\int_X f\mathrm{d}\mu_n = 0$ for every $f \in C_p(X)$ and $\|\mu_n\| = 1$ for every $n \in \omega$ if and only if $C_p(X)$ contains a complemented copy of $(c_0)_p$. Note that there is a natural one-to-one correspondence between continuous functionals on the topological vector space $C_p(X)$ and measures on X with finite supports, so their result can be considered as a characterization of those $C_p(X)$-spaces for which the Josefson–Nissenzweig theorem holds. Recall also that by the Riesz representation theorem every continuous functional on a space $C(K)$ of continuous real-valued functions on a compact space K endowed with the supremum norm corresponds similarly to a unique Radon measure on K. We may thus introduce the following notions (cf. [16, page 1122]).

Definition 1.1. A sequence $\langle \mu_n\colon\ n \in \omega\rangle$ of Radon measures on a Tychonoff space X is *a Josefson–Nissenzweig sequence* (or *a JN-sequence*) on X if $\lim_{n\to\infty}\int_X f\mathrm{d}\mu_n = 0$ for every $f \in C(X)$ and $\|\mu_n\| = 1$ for every $n \in \omega$.

Definition 1.2. A sequence $\langle \mu_n\colon\ n \in \omega\rangle$ of Radon measures on a Tychonoff space X is *finitely supported* (resp. *countably supported*) if μ_n has finite (resp. countable) support for every $n \in \omega$.

Definition 1.3. A finitely supported JN-sequence (resp. countably supported JN-sequence) is called in short *an fsJN-sequence* (resp. *a csJN-sequence*).

To show that the Josefson–Nissenzweig theorem does not hold for every space $C_p(X)$, Banakh, Kąkol and Śliwa [7] proved that $\beta\mathbb{N}$, the Čech-Stone compactification of the space $\mathbb{N}$ of natural numbers, does not admit any fsJN-sequences of measures, although it clearly admits *some* JN-sequence by the general Josefson–Nissenzweig theorem. This motivated them to introduce the following property.

Definition 1.4 ([7])**.** Given a Tychonoff space X, we say that $C_p(X)$ has *the Josefson–Nissenzweig property* (*the JNP*, in short) if X admits an fsJN-sequence of measures.

For the sake of precision, we additionally declare the following.

Definition 1.5. A Tychonoff space X has *the finitely supported Josefson–Nissenzweig property* (or *the fsJNP*) if X admits an fsJN-sequence. Similarly, X has *the countably supported Josefson–Nissenzweig property* (or *the csJNP)* if X admits a csJN-sequence.

Thus, given a Tychonoff space X, $C_p(X)$ has the JNP if and only if X has the fsJNP, and it follows that e.g. every metric non-discrete space has the fsJNP and that $\beta\mathbb{N}$ does not have the fsJNP. The question hence arises.

Question 1.6. *Which Tychonoff spaces have finitely supported JN-sequences of measures?*

There are several motivations standing behind Question 1.6, which constitutes the main research problem of this paper. First of all, we would like to continue the line of research presented in [7] and attempt to provide necessary and sufficient conditions for Tychonoff spaces (or, in particular, compact Hausdorff spaces) implying that their C_p-spaces will contain a complemented copy of the space $(c_0)_p$. This would allow us, e.g., to better understand the Separable Quotient Problem for topological vector spaces of the form $C_p(X)$ (see [48] and [6]). Second, because of the utility of the Josefson–Nissenzweig theorem we would like to understand it more thoroughly. In particular, since the original proofs are purely existential, it still seems necessary to understand how the sequence in the theorem can be obtained in a *constructive* way, what properties it may have and to what extent it can be modified. Third, by answering Question 1.6 we could better comprehend the nature and behavior of convergent sequences of Radon measures on compact spaces, objects playing a fundamental role e.g. in probability theory. And the last, we would like to know how *simple* JN-sequences on Tychonoff spaces may be. We approach those issues in the following threefold manner.

In the first main research part of the paper, i.e. Sections 4 and 5, we do assume that a given Tychonoff space has the fsJNP and study what kind of fsJN-sequences it may carry. In particular, starting from a given fsJN-sequence $\langle\mu_n\colon\ n\in\omega\rangle$ of measures on a Tychonoff space X, by manipulating its elements and studying special limit subsets of the union of the supports of μ_n's, we prove that there is another fsJN-sequence $\langle\nu_n\colon\ n\in\omega\rangle$ on X with disjoint supports, that is, $\operatorname{supp}(\nu_n)\cap\operatorname{supp}(\nu_{n'})=\emptyset$ for every $n\neq n'\in\omega$ (Theorem 4.22). Since, by the virtue of the Schur property, fsJN-sequences are never weakly convergent to 0, this result is related to the Dieudonné–Grothendieck characterization of non-weakly compact subsets of the dual Banach space $C(X)^*$ for X compact (see [25, Theorem 14, Chapter VII]).

It is immediate that if a Tychonoff space contains a non-trivial convergent sequence (e.g. if it is metric and non-discrete), then it admits an fsJN-sequence $\langle\mu_n\colon\ n\in\omega\rangle$ such that

$\left|\operatorname{supp}\left(\mu_n\right)\right| = 2$. Example 5.3 shows that the converse does not hold, and in Example 5.1 we study an instance of a compact space with the fsJNP and such that its every fsJN-sequence $\langle\mu_n\colon\ n\in\omega\rangle$ satisfies the conditions $\lim_{n\to\infty}\left|\operatorname{supp}\left(\mu_n\right)\right| = \infty$. These two examples show that the finitely supported Josefson–Nissenzweig property for non-metric compact spaces may be realized in two extremely different ways. However, these two ways are in some sense the only ones. Namely, in Theorem 5.13 we prove that for a given compact space K if there is an fsJN-sequence $\langle\mu_n\colon\ n\in\omega\rangle$ on K for which there exists $M\in\mathbb{N}$ such that $\left|\operatorname{supp}\left(\mu_n\right)\right|\le M$ for every $n\in\omega$, then there is another fsJN-sequence $\langle\nu_n\colon\ n\in\omega\rangle$ on K for which we have $\left|\operatorname{supp}\left(\nu_n\right)\right| = 2$ for every $n\in\omega$. Thus either there is a very simple fsJN-sequence on K, or every fsJN-sequence on K gets more and more complicated. For metric compact spaces, various examples of "complicated" fsJN-sequences are presented in Proposition 4.7.

The second part of the paper is devoted to the study of relations between the finitely supported Josefson–Nissenzweig property of compact spaces and the Grothendieck property of their Banach spaces of continuous functions. Recall that a Banach space X is *a Grothendieck space* or has *the Grothendieck property* if every weakly* convergent sequence of functionals in the dual space X^* of X is also weakly convergent. Similarly, we say that a compact space K has *the Grothendieck property* if $C(K)$ is a Grothendieck space. Grothendieck [39] proved that spaces of the form $\ell_\infty(\Gamma)$ are Grothendieck spaces (or, equivalently, spaces $C(K)$ for K compact and extremely disconnected). Later, many other Banach spaces were recognized to be Grothendieck, e.g. von Neumann algebras (Pfitzner [67]), the space H^∞ of bounded analytic functions on the unit disc (Bourgain [17]), spaces of the form $C(K)$ for K an F-space (Seever [75]; see also Haydon [43], Moltó [63], Schachermayer [72] or Freniche [37]), etc. On the other hand, the space c_0 is not Grothendieck, since a separable Banach space is Grothendieck if and only if it is reflexive. In fact, Cembranos [20] proved that a space $C(K)$ is Grothendieck if and only if it does not contain any complemented copy of c_0. For more information on Grothendieck $C(K)$-spaces we refer the reader to the papers of Haydon [44], Koszmider [52], or Sobota and Zdomskyy [77].

The Josefson–Nissenzweig theorem has found numerous applications in the study of Grothendieck Banach spaces of continuous functions, see e.g. Khurana [49] and Freniche [36]. Also, since the characterization of Grothendieck $C(K)$-spaces due to Cembranos (which we mentioned in the previous paragraph) sounds very similar to the above stated characterization of the Josefson–Nissenzweig property of $C_p(X)$-spaces by Banakh, Kąkol and Śliwa, it seemed natural to seek a connection between the finitely supported Josefson–Nissenzweig property of compact spaces and the Grothendieck property of their spaces of continuous functions. To describe such a relation, in Section 6 we introduce *the ℓ_1-Grothendieck property*. This new property can be described as the restriction of the Grothendieck property of a given space $C(K)$ for K compact to the space of measures on K having countable (equivalently, finite) support, i.e. to the functionals from the subspace $\ell_1(K)$ of the dual $C(K)^*$, see Definition 6.3. Then we prove in Theorem 6.7 that a compact space K has the fsJNP if and only if its space $C(K)$ does not have the ℓ_1-Grothendieck property. It follows immediately that if $C(K)$ is a Grothendieck space, then $C_p(K)$ does not have the JNP—this generalizes the result of [7] stating that $C_p(\beta\mathbb{N})$ does not have the JNP.

The ℓ_1-Grothendieck property follows from the general Grothendieck property, but the converse is not true. Namely, in Section 7 we construct a separable compact space K, in fact a continuous image of $\beta\mathbb{N}$, such that $C(K)$ has the ℓ_1-Grothendieck property but it does not have the Grothendieck property. The construction was suggested to us by G. Plebanek and it generalizes results from his unpublished note [68], where he constructed a compact space L such that $C(L)$ does not have the Grothendieck property but for every separable closed subset $L' \subseteq L$ the space $C(L')$ is Grothendieck.

The third main part of the paper deals with various classes of compact spaces having the finitely supported Josefson–Nissenzweig property. The first major class consists of compact spaces obtained as limits of inverse systems based on so-called simple extensions, see Definition 9.1. Intuitively speaking, such compact spaces may be thought as inverse limits of sequences of compact spaces such that every successor space in a sequence is obtained from its predecessor by splitting only one point into two new points. E.g. every metrizable compact space can be obtained in such a way; see also Koppelberg [50, 51] and Borodulin–Nadzieja [15] for many non-trivial examples coming from the theory of Boolean algebras.

It is a folklore fact that every compact space obtained as the limit of an inverse system based on simple extensions does not have the Grothendieck property. In Corollary 9.11 we generalize this result and prove that every such compact space does not have even the ℓ_1-Grothendieck property, or equivalently, that it admits an fsJN-sequence of measures. As a corollary we obtain that many classical (consistent) examples of Efimov spaces do have the fsJNP, too; consequently, they fail to have the ℓ_1-Grothendieck property (Corollary 9.12). Recall that an infinite compact space K is *an Efimov space* if K neither contains any non-trivial convergent sequences nor any copies of $\beta\mathbb{N}$. The famous Efimov problem asks if there exists an Efimov space (in ZFC). So far many consistent examples of Efimov spaces have been found, see e.g. Fedorchuk [33], Dow [26], Dow and Fremlin [27], Dow and Shelah [29], Sobota [76], but no ZFC example is known. For more information concerning the Efimov problem, we refer the reader to Hart's survey [42]. A weaker form of the problem in terms of the space ℓ_∞ and Grothendieck $C(K)$-spaces was also shortly discussed in Koszmider and Shelah [53, Section 3].

In Section 9.2 we provide a generalization of Corollary 9.11 to a broader class of compact spaces obtained as the limits of inverse systems. The generalization has interesting connections (Corollary 9.22) with the Separable Quotient Problem for $C_p(X)$-spaces in the context of [48]. An interesting tool proved and used in the section is Proposition 9.17 asserting that in particular cases we can obtain an fsJN-sequence on a given compact space by "transporting" it from the standard Cantor space.

Khurana's result, mentioned in the first paragraph, yields that for every infinite compact space K the Banach space $C(K \times K) \cong C(K, C(K))$ is not Grothendieck. This fact, together with the aforementioned theorem of Cembranos, implies that $C(K \times K)$ contains a complemented copy of the space c_0—the result also proved by Freniche [36]. In Section 11 we generalize this theorem by proving that given two infinite compact spaces K and L their product $K \times L$ always admits an fsJN-sequence (Theorem 11.3), thus, in particular, $K \times L$ does not have the ℓ_1-Grothendieck property and the space $C_p(K \times L)$ contains a complemented copy of the space $(c_0)_p$. The result of Cembranos and Freniche follows immediately (see Corollary 6.2). It is worth to mention here that the theorems of Khurana, Cembranos and Freniche are all proved with an aid of the Josefson–Nissenzweig theorem and therefore their original proofs are purely existential. Our proof of Theorem 11.3 is different—we provide a direct and simple definition of the required fsJN-sequence and use basic probability tools to demonstrate its properties.

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2. Preliminaries and notation

The notations and terminology used in the paper are rather standard and follow the books of Kunen [56] (set theory), Engelking [31] (general topology), Frankiewicz and Zbierski [34] (Boolean algebras), Diestel [25] (Banach space theory), Tkachuk [82] (C_p-theory), and Bogachev [11] (measure theory).

In particular, we use the following standard notions and symbols. If X is a set and A its subset, then $A^c = X \setminus A$ and χ_A denotes the characteristic function of A in X. The cardinality of a set X is denoted by $|X|$. ω denotes the first infinite cardinal number and ω_1 denotes the first uncountable cardinal number. If κ is a cardinal, finite or infinite, then by $[X]^\kappa$ we mean the family of all subsets of X of size κ; in particular, $[X]^\omega$ denotes the families of all countable subsets of X. The families of all subsets of X and all finite subsets of X are denoted by $\wp(X)$ and $[X]^{<\omega}$, respectively. The continuum, i.e. the size of the real line $\mathbb{R}$, is denoted either by $\mathfrak{c}$ or 2^ω. We also put simply $\mathbb{R}_+ = [0, \infty)$ and $\omega_+ = \omega \setminus \{0\}$.

Throughout this paper, we assume that all topological spaces we consider are **Tychonoff**, so, e.g., every compact space we deal with is Hausdorff. The weight of a topological space X is denoted by $w(X)$. If X is a space and A its subspace, then $\overline{A}^X$ denotes the closure of A in X. We will often omit the superscript and write simply $\overline{A}$. A° and ∂A denote the interior and the boundary of A in X, respectively. βX denotes the Čech–Stone compactification of X. Given two spaces X and Y, $X \approx Y$ means that they are homeomorphic. The Cantor space will be usually denoted by 2^ω. We also usually identify ω with the discrete space $\mathbb{N}$ of natural numbers.

If $\mathcal{A}$ is a Boolean algebra, then by $St(\mathcal{A})$ we denote its Stone space. Recall that $St(\mathcal{A})$ is a totally disconnected compact space and that the Boolean algebra of clopen subsets of $St(\mathcal{A})$ is isomorphic to $\mathcal{A}$. For every element $A \in \mathcal{A}$ by $[A]_\mathcal{A}$ we denote the corresponding clopen subset of $St(\mathcal{A})$.

If X is a (Tychonoff) space, then by $C_p(X)$ we denote the space of real-valued continuous functions on X endowed with the pointwise topology (i.e. the topology inherited from the product space $\mathbb{R}^X$). If K is a compact space, then $C(K)$ denotes the Banach space of real-valued continuous functions on K endowed with the supremum norm defined as $\|f\|_\infty = \sup\big\{|f(x)|\colon x \in K\big\}$ for every $f \in C(K)$. The symbols ℓ_1, ℓ_∞, c and c_0 denote the usual standard sequence Banach spaces. We also write $(c_0)_p$ for the space $\big\{x \in \mathbb{R}^\omega\colon \lim_{n\to\infty} x(n) = 0\big\}$ but endowed with the product topology inherited from $\mathbb{R}^\omega$.

If we say that μ *is a measure on a topological space* X, then we mean that μ is a signed σ-additive measure defined on the Borel σ-algebra of X and that μ is Radon, i.e. μ is (outer and inner) regular and locally finite. We define the norm $\|\mu\|$ of μ as

$$\|\mu\| = \sup\big\{|\mu(A)| + |\mu(B)|\colon\ A, B \subseteq X \text{ are Borel and disjoint}\big\}$$

If X is compact, then $\|\mu\| < \infty$. $|\mu|$ denotes the variation of μ—it follows that $|\mu|(X) = \|\mu\|$. On the other hand, if we say that μ *is a measure on a Boolean algebra* $\mathcal{A}$, then we assume that it is signed, finitely additive and that the norm $\|\mu\|$ of μ defined similarly as

$$\|\mu\| = \sup\big\{|\mu(A)| + |\mu(B)|\colon\ A, B \in \mathcal{A}, A \wedge B = 0_{\mathcal{A}}\big\}$$

is finite. Note that every measure μ on a Boolean algebra $\mathcal{A}$ (and hence on the Boolean algebra of clopen subsets of $St(\mathcal{A})$) has a unique extension to a measure $\widehat{\mu}$ on $St(\mathcal{A})$ and that $\|\mu\| = \|\widehat{\mu}\|$. We will usually identify μ and $\widehat{\mu}$ and omit $\widehat{\ }$.

A measure μ on a space X is *a probability measure* if $\mu(A) \geq 0$ for every Borel A and $\|\mu\| = 1$. We say that μ *vanishes at points* (or, is *non-atomic*) if $\mu(\{x\}) = 0$ for every $x \in X$.

If μ is a measure on a space X, then by $\operatorname{supp}(\mu)$ we denote *the support* of μ, i.e. the smallest closed subset L of X such that for every open subset $U \subseteq X \setminus L$ we have $|\mu|(U) = 0$. We will say that μ is *finitely (countably) supported* if $\operatorname{supp}(\mu)$ is a finite (countable) set. A sequence $\langle\mu_n\colon\ n \in \omega\rangle$ of measures on X is *finitely (countably) supported* if every μ_n is finitely (countably) supported. The space of all finitely supported measures on X is denoted by $\Delta(X)$. $\ell_1(X)$ denotes on the other hand the space of all countably supported measures on X. Note that the norm $\|\cdot\|$ defined above makes it a Banach space isometrically isomorphic to the space $\ell_1(|X|)$. Obviously, $\Delta(X)$ is a linear subspace of $\ell_1(X)$. If $x \in X$, then by δ_x we mean *the point measure* (or *the Dirac measure*) concentrated at x and defined as $\delta_x(A) = \chi_A(x)$. $\Delta(X)$ may be thus understood as a linear hull of a set $\big\{\delta_x\colon\ x \in X\big\}$ in the space $C_p(C_p(X))$. Also, each element μ of $\Delta(X)$ may be written as:

$$\mu = \sum_{x\in\operatorname{supp}(\mu)} \alpha_x \cdot \delta_x$$

for some non-zero $\alpha_x \in \mathbb{R}$ and every $x \in \operatorname{supp}(\mu)$. Similarly, the variation of μ may be written as $|\mu| = \sum_{x\in\operatorname{supp}(\mu)} \big|\alpha_x\big| \cdot \delta_x$ and thus the norm $\|\mu\|$ is equal to $\sum_{x\in\operatorname{supp}(\mu)} \big|\alpha_x\big|$.

If μ is a measure on a space X, then $L_1(\mu)$ and $L_\infty(\mu)$ denote the spaces of all μ-integrable and μ-essentially bounded functions on X, respectively. Note that if X is compact or μ is finitely supported, then $C(X)$ is a subspace of $L_1(\mu)$. If $f \in L_1(\mu)$, then we write simply $\mu(f) = \int_X f \mathrm{d}\mu$.

If $\langle\mu_n\colon\ n \in \omega\rangle$ is a sequence of measures on a compact space K, then we say that $\langle\mu_n\colon\ n \in \omega\rangle$ is *weakly* convergent* to a measure μ on K if $\lim_{n\to\infty} \mu_n(f) = \mu(f)$ for every

$f \in C(K)$, and that it is *weakly convergent* to μ if $\lim_{n\to\infty} \mu_n(B) = \mu(B)$ for every Borel subset B of K. We also say that $\langle \mu_n\colon\ n \in \omega\rangle$ is *weakly* null* (*weakly null*) if it is weakly* convergent (weakly convergent) to the zero measure 0 on K. Note that, due to the Riesz representation theorem, these notions of weak* and weak convergences coincide with the weak* and weak convergences in the dual space $C(K)^*$ (see also [25, Theorem 11, page 90]). Recall also that $\ell_1(K)$ is a complemented linear subspace of $C(K)^*$ and that it has the Schur property, i.e. every weakly convergent sequence in $\ell_1(K)$ is also norm convergent.

Similarly, if $\langle \mu_n\colon\ n \in \omega\rangle$ is a finitely supported sequence of measures on a space X, then we say that $\langle \mu_n\colon\ n \in \omega\rangle$ is *weakly* convergent* to a measure μ on X if $\lim_{n\to\infty} \mu_n(f) = \mu(f)$ for every $f \in C(X)$, and that $\langle \mu_n\colon\ n \in \omega\rangle$ is *weakly* null* if it is weakly* convergent to the zero measure 0 on X.

Part I. JN-sequences on (non)-compact spaces

3. The Josefson–Nissenzweig theorem for $C(K)$-spaces

Josefson [45] and Nissenzweig [66] proved their theorem for general Banach spaces and both of the proofs are rather long, technical and intricate. However, when we restrict our attention only to the Banach spaces of continuous functions on compact spaces, then it appears that the theorem may be proved in a much easier way. Below we present one of such proofs suggested to the authors by G. Plebanek and relying on measure-theoretic tools (such as the Maharam theorem). Let us note here that another basic proof for the case of $C(K)$-spaces can be also easily extracted from the proof of the general Josefson–Nissenzweig theorem due to Behrends [8, 9], who proved the theorem using famous Rosenthal's ℓ_1-lemma and Banach limits (however, since the space ℓ_1 embeds into $C(K)^*$, we may omit the application of the ℓ_1-lemma and directly go to Case 2 of Behrends' proof presented in [8]). A common point of the two proofs is that both consist of two cases from which the first one concerns sequences of finitely supported measures—a main subject of this paper.

3.1. **A measure-theoretic proof of the Josefson–Nissenzweig theorem for $C(K)$-spaces.** We will prove that every infinite compact space admits a JN-sequence. Let thus K be an infinite compact space. If K is a scattered space, i.e. every subset of K contains an isolated point in the inherited topology, then it is a simple folklore fact that K contains a non-trivial sequence $\langle x_n\colon\ n \in \omega\rangle$ convergent to some point $x \in K$. A sequence $\langle \mu_n\colon\ n \in \omega\rangle$ of measures defined for each $n \in \omega$ by the formula $\mu_n = \frac{1}{2}\big(\delta_{x_n} - \delta_x\big)$ is then a JN-sequence on K.

If K is not scattered, then the proof requires more work. By [74, Theorem 19.7.6], there is a non-atomic probability measure μ on K. It follows from the celebrated Maharam theorem ([61], see also [35]) that there exists a sequence $\langle B_n\colon\ n \in \omega\rangle$ of μ-independent Borel subsets of K such that $\mu\big(B_n\big) = 1/2$ for every $n \in \omega$. (The μ-independence of $\langle B_n\colon\ n \in \omega\rangle$ means here that for every finite sequence $n_1, \ldots, n_k$ of distinct natural numbers and every sequence

$\varepsilon_1, \ldots, \varepsilon_k \in \{-1, 1\}$ we have:

$$\mu\Big(\bigcap_{i=1}^{k} B_{n_i}^{\varepsilon_i}\Big) = \prod_{i=1}^{k} \mu\big(B_{n_i}^{\varepsilon_i}\big) = 1/2^k,$$

where $A^1 = A$ and $A^{-1} = K \setminus A$ for a subset A of K.) For each $n \in \omega$ define the measure μ_n as follows:

$$\mu_n(A) = \mu\big(B_n \cap A\big) - \mu\big(B_n^c \cap A\big),$$

where A is a Borel subset of K; then, $\big\|\mu_n\big\| = 1$. The sequence $\big\langle \mu_n\colon\ n \in \omega\big\rangle$ is a desired JN-sequence on K. Indeed, note that $\mu_n(g) = \int_K g \cdot \big(\chi_{B_n} - \chi_{B_n^c}\big)\mathrm{d}\mu$ for every $n \in \omega$ and $g \in L_1(\mu)$. By the μ-independence of the sequence $\big\langle B_n\colon\ n \in \omega\big\rangle$ and the generalized Riemann–Lebesgue lemma ([80, Page 3]), the bounded sequence $\big\langle \chi_{B_n} - \chi_{B_n^c}\colon\ n \in \omega\big\rangle$ of functions in $L_\infty(\mu)$ has the property that

$$\int_K g \cdot \big(\chi_{B_n} - \chi_{B_n^c}\big)\mathrm{d}\mu = 0$$

for every $g \in L_1(\mu)$, which implies that $\lim_{n\to\infty} \mu_n(g) = 0$ for every $g \in C(K)$, too. The proof of theorem is thus finished.

4. JN-sequences of measures

This section is devoted to the study of basic analytic and topological properties of fsJN-sequences. The first result asserts that in our study of *simple* JN-sequences on compact spaces we can confine our attention to finitely supported JN-sequences only.

Proposition 4.1. *The properties fsJNP and csJNP are equivalent for compact spaces.*

Proof. Let K be a compact space. If K has the fsJNP, then K has trivially also the csJNP, since $\Delta(K) \subseteq \ell_1(K)$. Let us thus assume that K has the csJNP and let $\big\langle \mu_n\colon\ n \in \omega\big\rangle$ be a csJN-sequence. For each $n \in \omega$ let F_n be a finite subset of $\operatorname{supp}\big(\mu_n\big)$ such that $\big\|\mu_n \restriction \big(K \setminus F_n\big)\big\| < 1/n$, so $\big\|\mu_n \restriction F_n\big\| > 1 - 1/n$. For every $n \in \omega$ define the measure ν_n on K as follows:

$$\nu_n = \big(\mu_n \restriction F_n\big)\Big/\big\|\mu_n \restriction F_n\big\|,$$

then, $\nu_n \in \Delta(K)$ and $\big\|\nu_n\big\| = 1$. For every $f \in C(K)$ we have:

$$\big|\nu_n(f)\big| = \big|\big(\mu_n \restriction F_n\big)(f)\big|\Big/\big\|\mu_n \restriction F_n\big\| \le \Big(\big|\mu_n(f)\big| + \big|\big(\mu_n \restriction \big(K \setminus F_n\big)\big)(f)\big|\Big)\Big/\big\|\mu_n \restriction F_n\big\| <$$

$$\Big(\big|\mu_n(f)\big| + \|f\|_\infty/n\Big)\Big/\big(1 - 1/n\big),$$

so $\lim_{n\to\infty} \nu_n(f) = 0$, since $\lim_{n\to\infty} \mu_n(f) = 0$, which implies that $\big\langle \nu_n\colon\ n \in \omega\big\rangle$ is weakly* null. It follows that $\big\langle \nu_n\colon\ n \in \omega\big\rangle$ is an fsJN-sequence on K and hence K has the fsJNP. □

The following lemma shows that measures in an fsJN-sequence have eventually similar absolute values on their negative and positive parts, equal to $\approx \frac{1}{2}$.

Lemma 4.2. *Let $\langle\mu_n\colon\ n\in\omega\rangle$ be an fsJN-sequence on a space X. For every $n\in\omega$ let $P_n=\big\{x\in\operatorname{supp}\big(\mu_n\big)\colon\ \mu_n(x)>0\big\}$ and $N_n=\operatorname{supp}\big(\mu_n\big)\setminus P_n$. Then,*

$$\lim_{n\to\infty}\big\|\mu_n\restriction P_n\big\|=\lim_{n\to\infty}\big\|\mu_n\restriction N_n\big\|=1/2.$$

Proof. Assume there exists a subsequence $\langle\mu_{n_k}\colon\ k\in\omega\rangle$ such that the limit $\alpha=\lim_{k\to\infty}\big\|\mu_{n_k}\restriction P_{n_k}\big\|$ exists and $\alpha\neq 1/2$. Assume first that $\alpha>1/2$. Let $\varepsilon=\big(\alpha-1/2\big)/2$. There exists $K\in\omega$ such that for every $k>K$ we have:

$$\Big|\big\|\mu_{n_k}\restriction P_{n_k}\big\|-\alpha\Big|<\varepsilon$$

and hence

$$1-\big\|\mu_{n_k}\restriction N_{n_k}\big\|=\big\|\mu_{n_k}\restriction P_{n_k}\big\|>1/2+\varepsilon.$$

Then,

$$\big|\mu_{n_k}(X)\big|=\big\|\mu_{n_k}\restriction P_{n_k}\big\|-\big\|\mu_{n_k}\restriction N_{n_k}\big\|>1+2\varepsilon-1=2\varepsilon>0,$$

so $\liminf_{k\to\infty}\big|\mu_{n_k}(X)\big|>0$, a contradiction, since $\langle\mu_n\colon\ n\in\omega\rangle$ is weakly* null.

The proof for $\alpha<1/2$ is similar. Naturally,

$$\lim_{n\to\infty}\big\|\mu_{n_k}\restriction N_{n_k}\big\|=1-\lim_{n\to\infty}\big\|\mu_{n_k}\restriction P_{n_k}\big\|=1/2.$$

□

For a given finitely supported sequence $\langle\mu_n\colon\ n\in\omega\rangle$ of measures on a space X, let us put:

$$S\big(\langle\mu_n\colon\ n\in\omega\rangle\big)=\bigcup_{n\in\omega}\operatorname{supp}\big(\mu_n\big),$$

$$LS\big(\langle\mu_n\colon\ n\in\omega\rangle\big)=\Big\{x\in X\colon\ \limsup_{n\to\infty}\big|\mu_n(\{x\})\big|>0\Big\},$$

$$LI\big(\langle\mu_n\colon\ n\in\omega\rangle\big)=\Big\{x\in X\colon\ \liminf_{n\to\infty}\big|\mu_n(\{x\})\big|>0\Big\},$$

and

$$L\big(\langle\mu_n\colon\ n\in\omega\rangle\big)=\Big\{x\in X\colon\ \lim_{n\to\infty}\mu_n(\{x\})\text{ exists and is not }0\Big\}.$$

We will usually write shorter $S\big(\mu_n\big)$, $LS\big(\mu_n\big)$, $LI\big(\mu_n\big)$ and $L\big(\mu_n\big)$ instead of $S\big(\langle\mu_n\colon\ n\in\omega\rangle\big)$, $LS\big(\langle\mu_n\colon\ n\in\omega\rangle\big)$, $LI\big(\langle\mu_n\colon\ n\in\omega\rangle\big)$ and $L\big(\langle\mu_n\colon\ n\in\omega\rangle\big)$, or even simply S, LS, LI and L if the sequence $\langle\mu_n\colon\ n\in\omega\rangle$ is clear from the context. Of course, always $L\subseteq LI\subseteq LS\subseteq S$, but the reverse inclusions may not hold (cf. Proposition 4.7).

Lemma 4.3. *If $\langle\mu_n\colon\ n\in\omega\rangle$ is an fsJN-sequence on a space X, then S is infinite.*

Proof. If S is finite, then there exists $x_0\in S$ and $\varepsilon>0$ such that $\limsup_{n\to\infty}\big|\mu_n\big(\{x_0\}\big)\big|>\varepsilon$ (if not, then there is $N\in\omega$ such that $\big|\mu_n(\{x\})\big|<1/|S|$ for every $x\in S$ and $n>N$, which implies that $\big\|\mu_n\big\|<1$ for every $n>N$). Let $f\in C(X)$ be such that $f(x_0)=1$ and $f(x)=0$ for every $x\in S\setminus\big\{x_0\big\}$. It follows that $\limsup_{n\to\infty}\big|\mu_n(f)\big|>\varepsilon$, which is a contradiction. □

Note that despite the fact that the set S is a countable subset of X its topology may be very hard to study — see e.g. Levy [57], where it was proved that there exist $2^{\mathfrak{c}}$ many non-homeomorphic countable regular (hence normal) spaces without points of countable character.

Remark 4.4. Let $\langle\mu_n\colon\ n\in\omega\rangle$ be a JN-sequence on a given space X. Then, since S is countable, by induction we can find a subsequence $\langle\mu_{n_k}\colon\ k\in\omega\rangle$ such that $\lim_{k\to\infty}\left|\mu_{n_k}(\{x\})\right|$ exists for every $x\in X$.

Definition 4.5. A sequence $\langle\mu_n\colon\ n\in\omega\rangle$ of finitely supported measures on a space X is *pointwise convergent* if the limit $\lim_{n\to\infty}\mu_n\big(\{x\}\big)$ exists for every $x\in X$.

Note that the definition is equivalent to say that $\lim_{n\to\infty}\mu_n\big(\{x\}\big)=0$ for every $x\in X\setminus L$. It follows that $L\big(\mu_n\big)=LI\big(\mu_n\big)=LS\big(\mu_n\big)\subseteq S\big(\mu_n\big)$ if $\langle\mu_n\colon\ n\in\omega\rangle$ is pointwise convergent. By the previous remark, every fsJN-sequence $\langle\mu_n\colon\ n\in\omega\rangle$ on a space X contains a pointwise convergent fsJN-(sub)sequence $\langle\mu_{n_k}\colon\ k\in\omega\rangle$. Of course, every subsequence of a pointwise convergent sequence of measures is also pointwise convergent.

The proof of the following lemma is left to the reader.

Lemma 4.6. *For every finitely supported sequence $\langle\mu_n\colon\ n\in\omega\rangle$ of measures on a space X and its subsequence $\langle\mu_{n_k}\colon\ k\in\omega\rangle$ it holds:*

(i) $S\big(\langle\mu_{n_k}\colon\ k\in\omega\rangle\big)\subseteq S\big(\langle\mu_n\colon\ n\in\omega\rangle\big)$;
(ii) $LS\big(\langle\mu_{n_k}\colon\ k\in\omega\rangle\big)\subseteq LS\big(\langle\mu_n\colon\ n\in\omega\rangle\big)$;
(iii) $LI\big(\langle\mu_n\colon\ n\in\omega\rangle\big)\subseteq LI\big(\langle\mu_{n_k}\colon\ k\in\omega\rangle\big)$;
(iv) $L\big(\langle\mu_n\colon\ n\in\omega\rangle\big)\subseteq L\big(\langle\mu_{n_k}\colon\ k\in\omega\rangle\big)$.

If $\langle\mu_n\colon\ n\in\omega\rangle$ is pointwise convergent, then

$$L\big(\langle\mu_n\colon\ n\in\omega\rangle\big)=L\big(\langle\mu_{n_k}\colon\ k\in\omega\rangle\big)=LS\big(\langle\mu_{n_k}\colon\ k\in\omega\rangle\big)=LS\big(\langle\mu_n\colon\ n\in\omega\rangle\big).$$

□

The following proposition asserts that the unit square $[0,1]^2$ admits fsJN-sequences satisfying various proper inclusions between sets L, LI, LS and S as well as they have other quantitative properties. It also shows that even in the case of a metric space an fsJN-sequence may be quite intricate.

Proposition 4.7. *Let $\alpha\in(0,1)$. The unit square $[0,1]^2$ admits fsJN-sequences $\langle\mu_n^1\colon\ n\in\omega\rangle$, $\langle\mu_n^2\colon\ n\in\omega\rangle$, $\langle\mu_n^3\colon\ n\in\omega\rangle$ and $\langle\mu_n^4\colon\ n\in\omega\rangle$ such that:*

(1) $\emptyset\neq L\big(\mu_n^1\big)\subsetneq LI\big(\mu_n^1\big)\subsetneq LS\big(\mu_n^1\big)\subsetneq S\big(\mu_n^1\big)$;

(2) *(i)* $LS\big(\mu_n^2\big)=\big([0,1]\cap\mathbb{Q}\big)\times\{0\}$, *so* $LS\big(\mu_n^2\big)$ *is dense-in-itself;*
(ii) $\emptyset=L\big(\mu_n^2\big)=LI\big(\mu_n^2\big)\subsetneq LS\big(\mu_n^2\big)\subsetneq S\big(\mu_n^2\big)$;
(iii) $\mu_n^2(\{x\})\in\{0,1/2\}$ *for every* $x\in LS\big(\mu_n^2\big)$ *and* $n\in\omega$;

(iv) for every $x \in LS\big(\mu_n^2\big)$ we have $\limsup_{n\to\infty} \mu_n^2(\{x\}) = 1/2$, so for every finite $F \subseteq LS\big(\mu_n^2\big)$ it holds:

$$\sum_{x\in F} \limsup_{n\to\infty} \mu_n^2(\{x\}) = |F|/2,$$

and hence:

$$\sum_{x\in LS(\mu_n^2)} \limsup_{n\to\infty} \mu_n^2(\{x\}) = \infty;$$

(3) (i) $L\big(\mu_n^3\big) = \big([0,1]\cap\mathbb{Q}\big)\times\{0\}$, so $L\big(\mu_n^3\big)$ is dense-in-itself;
(ii) $\emptyset \neq L\big(\mu_n^3\big) = LI\big(\mu_n^3\big) = LS\big(\mu_n^3\big) \subsetneq S\big(\mu_n^3\big)$;
(iii)

$$\sum_{x\in L(\mu_n^3)} \lim_{n\to\infty} \mu_n^3(\{x\}) = (1-\alpha)/2 \le 1/2$$

and

$$\lim_{n\to\infty} \big\|\mu_n \restriction L\big\| = (1-\alpha)/2 \le 1/2;$$

(4) (i) $L\big(\mu_n^4\big) = \big\{k/2^{n+1}\colon\ k,n\in\omega,\ 0\le k< 2^{n+1}\big\}\times\{0\}$;
(ii) $\emptyset \neq L\big(\mu_n^4\big) = LI\big(\mu_n^4\big) = LS\big(\mu_n^4\big) = S\big(\mu_n^4\big)$;
(iii) $\big\|\mu_n^4 \restriction L\big\| = 1$ for every $n\in\omega$.

Proof. Put $K=[0,1]^2$ and fix an enumeration $\big\{q_n\colon\ n\in\omega\big\}$ of $[0,1]\cap\mathbb{Q}$.

(1) If $n\in\omega$ is even, then let μ_n^1 be defined as follows:

$$\mu_n^1 = \frac{1}{4}\big(\delta_{(0,0)} - \delta_{(0,1/(n+1))}\big) + \frac{1}{4}\big(\delta_{(1/2,0)} - \delta_{(1/2,1/(n+1))}\big),$$

and if n is odd, then define μ_n^1 as follows:

$$\mu_n^1 = \frac{1}{4}\big(\delta_{(0,0)} - \delta_{(0,1/(n+1))}\big) + \frac{1}{8}\big(\delta_{(1/2,0)} - \delta_{(1/2,1/(n+1))}\big) + \frac{1}{8}\big(\delta_{(1,0)} - \delta_{(1,1/(n+1))}\big).$$

It is immediate that $\big\langle\mu_n^1\colon\ n\in\omega\big\rangle$ is an fsJN-sequence on K and:

$$L\big(\mu_n^1\big) = \big\{(0,0)\big\},$$

$$LI\big(\mu_n^1\big) = \big\{(0,0),\ (1/2,0)\big\},$$

$$LS\big(\mu_n^1\big) = \big\{(0,0),\ (1/2,0),\ (1,0)\big\},$$

$$S\big(\mu_n^1\big) = \big\{(0,0),\ (1/2,0),\ (1,0)\big\} \cup \big\{(x,1/(n+1)\colon\ x\in\{0,1/2,1\},\ n\in\omega\big\},$$

which yields (1).

(2) Let $\big\{P_n\colon\ n\in\omega\big\}$ be a partition of ω into infinite sets. For every $n\in\omega$ and $k\in P_n$ write:

$$\mu_k^2 = \frac{1}{2}\big(\delta_{(q_n,0)} - \delta_{(q_n,1/k)}\big).$$

Then, for each $k \in \omega$ we have $\|\mu_k^2\| = 1$ and it is immediate that for every $n \in \omega$ the sequence $\langle \mu_k^2\colon\ k \in P_n \rangle$ is weakly* null. We will now show that the whole sequence $\langle \mu_k^2\colon\ k \in \omega \rangle$ is weakly* null. Let $f \in C(K)$. We have:

$$\mu_k^2(f) = \frac{1}{2}\big(f(q_n, 0) - f(q_n, 1/k)\big),$$

where $n \in \omega$ and $k \in P_n$. Fix $\varepsilon > 0$. Since K is compact, f is uniformly continuous, so there is $\delta > 0$ such that if for $k, n \in \omega$ we have $1/k < \delta$ and $k \in P_n$, then $\big|f(q_n, 0) - f(q_n, 1/k)\big| < \varepsilon$. So pick $N \in \omega$ such that $1/N < \delta$. For every $k > N$ and $n \in \omega$ such that $k \in P_n$ we have:

$$\big|\mu_k^2(f)\big| = \frac{1}{2}\big|f(q_n, 0) - f(q_n, 1/k)\big| < \varepsilon.$$

Thus, $\langle \mu_k^2\colon\ k \in \omega \rangle$ is weakly* null.

That the conditions (i)–(iv) are satisfied follows directly from the definition of the sequence $\langle \mu_n^2\colon\ n \in \omega \rangle$.

(3) Let us assume additionally that $0 = q_n$ for some $n > 2$. For every $n \in \omega$ define the measure μ_n^3 as follows:

$$\mu_n^3 = (1-\alpha) \cdot \sum_{k=0}^{n} \big(\delta_{(q_k,0)} - \delta_{(q_k,1/(n+1))}\big)/2^{k+2} + \Big(\frac{\alpha}{2} + \frac{1-\alpha}{2^{n+2}}\Big) \cdot \big(\delta_{(0,1-1/(n+1))} - \delta_{(0,1-1/(n+2))}\big).$$

It follows that $\|\mu_n^3\| = 1$. That $\langle \mu_n^3\colon\ n \in \omega \rangle$ is weakly* null follows again from the fact that every $f \in C(K)$ is uniformly continuous—cf. the previous example.

For every $k \in \omega$ and $n \geq k$ we have:

$$(*) \qquad \mu_n^3\big(\{(q_k, 0)\}\big) = (1-\alpha)/2^{k+2},$$

so $(q_k, 0) \in L(\mu_n^3)$. If $x \in K$ is of the form $\big(q_k, 1/(n+1)\big)$ or $\big(0, 1 - 1/n\big)$ for some $k, n \in \omega$, then $\mu_l^3(\{x\}) = 0$ for every $l > n + 2$, so $x \notin L(\mu_n^3)$. Thus, (i) is satisfied. (ii) follows immediately from (i) and the definition of $\langle \mu_n^3\colon\ n \in \omega \rangle$. (iii) follows from $(*)$.

(4) Let $n \in \omega$. Put $P_n = \{0, \ldots, 2^n - 1\}$ and for each $k \in P_n$ write $e_k^n = (2k)/2^{n+1}$ and $o_k^n = (2k+1)/2^{n+1}$. Note that $e_0^n = 0$. Put: $E_n = \{e_k^n\colon k \in P_n\}$, $O_n = \{o_k^n\colon k \in P_n\}$ and $S_n = E_n \cup O_n$. The set S_n will be the support of the measure μ_n^4 we are going to construct.

Note that for every $n \in \omega$ we have $S_n = E_{n+1}$ and $|S_n| = 2|P_n| = 2 \cdot 2^n$, so $|S_{n+1}| = 2|S_n|$. For every $n \in \omega$ let $c_n = 1/2^{n+1}$ and define the auxiliary measure ν_n as follows:

$$\nu_n = \sum_{k \in P_n} \alpha_k^n \cdot \big(\delta_{(e_k^n,0)} - \delta_{(o_k^n,0)}\big),$$

where the coefficients α_k^n's are defined in the following way. For $n = 0$ we simply write $\alpha_0^0 = 1/4$ and for $n > 0$ every $k \in P_n$ we define:

$$\alpha_k^n = \begin{cases} \alpha_{k/2}^{n-1}, & \text{if } e_n^k \in E_{n-1}, \\ c_n/2^n, & \text{otherwise.} \end{cases}$$

Note that if $e_n^k \in E_{n-1}$, then k is even, so the definition is correct. It also holds $\big|\operatorname{supp}\big(\nu_n\big)\big| = 2^{n+1}$.

It follows that $\big\|\nu_n\big\| = 1 - c_n$. Indeed, this is obviously true for $n = 0$, so fix $n \geq 0$ and assume that $\big\|\nu_n\big\| = 1 - c_n$. Since $E_n \subseteq S_n \subseteq S_{n+1}$ and $\big|S_{n+1}\big| = 2\big|S_n\big|$, we have:

$$\big\|\nu_{n+1}\big\| = \big\|\nu_n\big\| + 2 \cdot 2^n \cdot \frac{c_{n+1}}{2^{n+1}} = 1 - c_n + c_{n+1} = 1 - c_{n+1},$$

as required.

We will now show that $\big\langle\nu_n\colon\ n \in \omega\big\rangle$ is weakly* null. Let $f \in C(K)$ and $\varepsilon > 0$. Again, note that f is uniformly continuous, so there is $\delta > 0$ such that for every $n \in \omega$ if $1/2^{n+1} < \delta$, then $\big|f\big(e_k^n, 0\big) - f\big(o_k^n, 0\big)\big| < \varepsilon$. Let thus N be such that $1/2^{n+1} < \delta$ for every $n > N$. We have:

$$\big|\nu_n(f)\big| \leq \sum_{k \in P_n} \alpha_k^n \cdot \big|f\big(e_k^n, 0\big) - f\big(o_k^n, 0\big)\big| < \varepsilon \cdot \sum_{k \in P_n} \alpha_k^n < \varepsilon \cdot \big(1 - c_n\big) < \varepsilon,$$

which yields that $\lim_{n\to\infty} \nu_n(f) = 0$.

Finally, for every $n \in \omega$ let

$$\mu_n^4 = c_n \cdot \delta_{(e_0^n, 0)} + \nu_n,$$

so $\mu_n^4\big(\big\{\big(e_0^n, 0\big)\big\}\big) = c_n + \alpha_0^n$ and hence $\big\|\mu_n^4\big\| = 1$ and $(0,0) \in L\big(\mu_n^4\big)$. Since $\lim_{n\to\infty} c_n = 0$, the sequence $\big\langle\mu_n^4\colon\ n \in \omega\big\rangle$ is weakly* null.

We will now prove (i) and (ii) together. First, notice that $\operatorname{supp}\big(\mu_n^4\big) = S_n \times \{0\}$ for every $n \in \omega$, so

$$S\big(\mu_n^4\big) = \bigcup_{n\in\omega} S_n = \big\{k/2^{n+1}\colon\ k, n \in \omega,\ 0 \leq k < 2^{n+1}\big\} \times \{0\}.$$

Next, if for $x \in (0,1]$ and $n \in \omega$ it holds that $x \in S_n$, then $x \in E_{n+1}$, so $\mu_l^4\big(\{(x,0)\}\big) = \alpha_{n+1}^k$ for some $k \in P_{n+1}$ and every $l > n+1$. It follows that $(x,0) \in L\big(\mu_n^4\big)$. (i) and (ii) are thus proved.

(iii) follows from (ii). □

Let us note here that we presented the constructions of the sequences in Proposition 4.7 in the square $[0,1]^2$ only for simplicity—similar constructions may be carried out also in the unit interval $[0,1]$ or, in fact, any metric compact dense-in-itself space.

The next lemma shows that the value $1/2$ in the property (iii) of $\big\langle\mu_n^3\colon\ n \in \omega\big\rangle$ is not accidental. An intuitive meaning of the lemma is that if for some fixed points of the space X the values of measures of the corresponding singletons grow too much, then they must be nullified by the values on some other points which lie closer and closer to these fixed ones (in the sense of the topology of X), cf. also Lemma 4.10. The property (iv) of $\big\langle\mu_n^2\colon\ n \in \omega\big\rangle$ implies that we cannot relax here limits to inferior limits or superior limits.

Lemma 4.8. *For every fsJN-sequence $\big\langle\mu_n\colon\ n \in \omega\big\rangle$ on a space X it holds:*

$$\sum_{x \in L(\mu_n)} \lim_{n\to\infty} \big|\mu_n(\{x\})\big| \leq 1/2.$$

Proof. Let $\langle\mu_n\colon\ n\in\omega\rangle$ be an fsJN-sequence on a space X. For the sake of contradiction, assume that

$$\sum_{x\in L(\mu_n)}\lim_{n\to\infty}\big|\mu_n(\{x\})\big|>1/2,$$

so there is a finite set $F\subseteq L\big(\mu_n\big)$ such that

$$\sum_{x\in F}\lim_{n\to\infty}\big|\mu_n(\{x\})\big|>1/2.$$

Denote the above sum by α, so $\alpha>1/2$. Let $\varepsilon=\big(\alpha-1/2\big)/2$, so $\alpha=2\varepsilon+1/2$. For every $x\in F$ there is $N_x\in\omega$ such that for every $n>N_x$ we have:

$$\Big|\mu_n(\{x\})-\lim_{k\to\infty}\mu_k(\{x\})\Big|<\varepsilon/|F|.$$

Let $N>\max_{x\in F}N_x$ be such that

$$\operatorname{sgn}\big(\mu_n(\{x\})\big)=\operatorname{sgn}\big(\lim_{k\to\infty}\mu_k(\{x\})\big)$$

for every $x\in F$ and $n>N$. Let $\big\{U_x\colon x\in F\big\}$ be a collection of pairwise disjoint open subsets of X such that $x\in U_x$ for every $x\in F$. There exists a continuous function $f\in C(X)$ such that: $-1\le f\le 1$, $f(x)=\operatorname{sgn}\big(\lim_{k\to\infty}\mu_k(\{x\})\big)$ for every $x\in F$ (so $\|f\|_\infty\le 1$), and $f(y)=0$ for every $y\in X\setminus\bigcup_{x\in F}U_x$. For every $n>N$ it holds:

$$\big(\mu_n\restriction F\big)(f)=\sum_{x\in F}\big|\mu_n(\{x\})\big|,$$

so

$$\begin{gathered}\big|\big(\mu_n\restriction F\big)(f)\big|=\Big|\sum_{x\in F}\big|\mu_n(\{x\})\big|\Big|=\\ \Big|\sum_{x\in F}\big|\mu_n(\{x\})\big|-\sum_{x\in F}\lim_{k\to\infty}\big|\mu_k(\{x\})\big|+\sum_{x\in F}\lim_{k\to\infty}\big|\mu_k(\{x\})\big|\Big|\ge\\ \Big|\sum_{x\in F}\lim_{k\to\infty}\big|\mu_k(\{x\})\big|\Big|-\Big|\sum_{x\in F}\big|\mu_n(\{x\})\big|-\sum_{x\in F}\lim_{k\to\infty}\big|\mu_k(\{x\})\big|\Big|=\\ \alpha-\Big|\sum_{x\in F}\big(\big|\mu_n(\{x\})\big|-\lim_{k\to\infty}\big|\mu_k(\{x\})\big|\big)\Big|\ge\\ \alpha-\sum_{x\in F}\Big|\big|\mu_n(\{x\})\big|-\lim_{k\to\infty}\big|\mu_k(\{x\})\big|\Big|>\\ \alpha-|F|\cdot\varepsilon/|F|=\alpha-\varepsilon=\varepsilon+1/2.\end{gathered}$$

It follows that for every $n>N$ we have:

$$\begin{gathered}\mu_n(f)=\big|\big(\mu_n\restriction F\big)(f)+\big(\mu_n\restriction(X\setminus F)\big)(f)\big|\ge\\ \big|\big(\mu_n\restriction F\big)(f)\big|-\big|\big(\mu_n\restriction(X\setminus F)\big)(f)\big|>\\ \varepsilon+1/2-\|f\|_\infty\cdot\big\|\mu_n\restriction(X\setminus F)\big\|>\varepsilon+1/2-1\cdot(1/2-\varepsilon)=2\varepsilon>0,\end{gathered}$$

so $\limsup_{n\to\infty}\big|\mu_n(f)\big|>2\varepsilon>0$, which is a contradiction. □

Corollary 4.9. *For every fsJN-sequence $\langle\mu_n\colon\ n\in\omega\rangle$ on a space X it holds:*

$$\lim_{x\in L(\mu_n)}\lim_{n\to\infty}\big|\mu_n(\{x\})\big|=0,$$

i.e. for every $\varepsilon>0$ there is a finite subset $F\subseteq L\big(\mu_n\big)$ such that $\lim_{n\to\infty}\big|\mu_n(\{x\})\big|<\varepsilon$ for every $x\in L\big(\mu_n\big)\setminus F$. □

Lemma 4.10. *For every pointwise convergent fsJN-sequence $\langle\mu_n\colon\ n\in\omega\rangle$ on a space X, if $\liminf_{n\to\infty}\big\|\mu_n\restriction L\big(\mu_n\big)\big\|<1$, then the set $S\big(\mu_n\big)\setminus L\big(\mu_n\big)$ is infinite.*

Proof. Let $\langle\mu_{n_k}\colon\ k\in\omega\rangle$ be such a subsequence that $\lim_{k\to\infty}\big\|\mu_{n_k}\big\|=\alpha$, where $\alpha<1$. There is $K\in\omega$ such that for every $k>K$ we have:

$$\Big|\big\|\mu_{n_k}\restriction L\big\|-\alpha\Big|<(1-\alpha)/2,$$

so $\big\|\mu_{n_k}\restriction L\big\|-\alpha/2<1/2$. Since $\langle\mu_{n_k}\colon\ k\in\omega\rangle$ is pointwise convergent, $\lim_{k\to\infty}\mu_{n_k}(\{x\})=0$ for every $x\in S\setminus L$, so if $S\setminus L$ is finite, then there is $K'>K$ such that for every $k>K'$ we have $\big\|\mu_{n_k}\restriction(S\setminus L)\big\|<(1-\alpha)/2$, so $\big\|\mu_{n_k}\restriction(S\setminus L)\big\|+\alpha/2<1/2$, but then for every $k>K'$ we also have:

$$1=\big\|\mu_{n_k}\big\|=\Big(\big\|\mu_{n_k}\restriction(S\setminus L)\big\|+\alpha/2\Big)+\Big(\big\|\mu_{n_k}\restriction L\big\|-\alpha/2\Big)<1/2+1/2=1,$$

a contradiction. □

Note that Proposition 4.7.(4) provides an example of an fsJN-sequence for which the assumption stated in the above lemma does not hold. The following lemma asserts an interesting and useful property of the subspace $S\big(\mu_n\big)$.

Lemma 4.11. *Let $\langle\mu_n\colon\ n\in\omega\rangle$ be an fsJN-sequence on a Tychonoff space X. Then, every function $f\in C(X)$ is bounded on the subspace $\overline{S\big(\mu_n\big)}^X$.*

Proof. Suppose that there is a function $f\in C(X)$, $f\ge 0$, which is unbounded on $S\big(\mu_n\big)$. Passing to a subsequence of $\langle\mu_n\colon\ n\in\omega\rangle$, if necessary, we may assume that there exist a strictly increasing sequence $\langle k_n\in\omega\colon\ n\in\omega\rangle$ and a sequence $\langle x_n\in\operatorname{supp}\big(\mu_n\big)\colon\ n\in\omega\rangle$ such that for every $n\in\omega$ we have

$$f\Big[\bigcup_{l<n}\operatorname{supp}\big(\mu_l\big)\Big]\subset\big(0,k_n-1\big)$$

and $f\big(x_n\big)>k_n$. It follows that $f\big(x_n\big)<k_{n+1}-1$. For every $n\in\omega$ put

$$\epsilon_n=k_{n+1}-1-f\big(x_n\big)$$

and let $\rho_n\colon X\to\big[0,\epsilon_n\big)$ (note that $\epsilon_n>0$) be a continuous function such that:

$$\rho_n\restriction\Big(X\setminus f^{-1}\big[\big(k_n,k_{n+1}-1\big)\big]\Big)\equiv 0,$$

the restriction $\rho_n \restriction \Big(f^{-1}\big(f\big(x_n\big)\big) \cap \operatorname{supp}\big(\mu_n\big)\Big)$ is injective and non-zero, and

$$\rho_n \restriction \Big(f^{-1}\big[\big(k_n, k_{n+1}-1\big)\big] \setminus f^{-1}\big(f\big(x_n\big)\big)\Big) \cap \operatorname{supp}\big(\mu_n\big) \equiv 0.$$

The existence of such a function ρ_n is a direct consequence of the complete regularity of X. By replacing f on $f^{-1}\big[\big(k_n, k_{n+1}-1\big)\big]$ by $f + \rho_n$ for each $n \in \omega$, we may additionally obtain a continuous function h on X such that there is no $x \neq x_n$ in $\operatorname{supp}\big(\mu_n\big)$ for which we have $h(x) = h\big(x_n\big)$ (to see the continuity of h, note that the family $\big\{\rho_n^{-1}\big[\mathbb{R} \setminus \{0\}\big]\colon\ n \in \omega\big\}$ is locally finite).

By induction on $n \in \omega$, construct a continuous function $g_n\colon \big[0, k_{n+1}-1\big] \to \mathbb{R}$ such that

$$\Big|\sum_{x \in \operatorname{supp}(\mu_n)} g_n(h(x)) \cdot \mu_n(x)\Big| > n,$$

and $g_n \restriction \big[0, k_n - 1\big] = g_{n-1}$. Then, the union $g = \bigcup_{n \in \omega} g_n$ is a continuous function $g\colon \mathbb{R}_+ \to \mathbb{R}$ and has the property that

$$\Big|\sum_{x \in \operatorname{supp}(\mu_n)} g(h(x)) \cdot \mu_n(x)\Big| > n$$

for all $n \in \omega$, contradicting the fact that $\big\langle \mu_n\colon\ n \in \omega\big\rangle$ is an fsJN-sequence. □

Corollary 4.12. *If a normal space X admits an fsJN-sequence $\big\langle \mu_n\colon\ n \in \omega\big\rangle$, then the subspace $\overline{S\big(\mu_n\big)}^X$ is pseudocompact.*

Proof. Put $S = \overline{S\big(\mu_n\big)}^X$. Let $f \in C(S)$. By the Tietze extension theorem there is $F \in C(X)$ extending f. By Lemma 4.11, $f = F \restriction S$ is bounded. □

4.1. **Disjointly supported fsJN-sequences.** In this section we will show that if a compact space K has the fsJNP, then K admits an fsJN-sequence with disjoint supports (Theorem 4.22). Let us thus start with the following convenient definition.

Definition 4.13. A finitely supported sequence $\big\langle \mu_n\colon\ n \in \omega\big\rangle$ of measures on a space X is *disjointly supported* if $\operatorname{supp}\big(\mu_n\big) \cap \operatorname{supp}\big(\mu_{n'}\big) = \emptyset$ for every $n \neq n' \in \omega$.

The following two lemmas imply that if a space admits an fsJN-sequence with measures having supports of size 2, then there exists also such a sequence with disjoint supports. In Theorem 4.22 we will generalize this result, however the proof will be much more complicated. For more information on sizes of supports of fsJN-sequences, see Section 5, especially Theorem 5.13.

Lemma 4.14. *Let X be a space. Fix a sequence $\big\langle x_n\colon\ n \in \omega\big\rangle$ in X and a point $x \in X$. For every $n \in \omega$ put $\mu_n = \frac{1}{2}\big(\delta_{x_n} - \delta_x\big)$. Then, $\big\langle \mu_n\colon\ n \in \omega\big\rangle$ is an fsJN-sequence if and only if $x_n \to x$ in X.*

Proof. Easy. □

Lemma 4.15. *Let a space X admit an fsJN-sequence $\langle\mu_n\colon\ n\in\omega\rangle$ defined for every $n\in\omega$ as $\mu_n=\frac{1}{2}\big(\delta_{x_n}-\delta_{y_n}\big)$, where $x_n,y_n\in X$. Then, there exists a disjointly supported fsJN-sequence $\langle\nu_n\colon\ n\in\omega\rangle$ defined for every $n\in\omega$ as $\nu_n=\frac{1}{2}\big(\delta_{u_n}-\delta_{w_n}\big)$, where $u_n,w_n\in X$.*

Proof. If the space X contains a non-trivial convergent sequence $\langle z_n\colon\ n\in\omega\rangle$, then it is easy to see that the measures defined as $\nu_n=\frac{1}{2}\big(\delta_{z_{2n}}-\delta_{z_{2n+1}}\big)$ form an fsJN-sequence satisfying the conclusion of the lemma.

If X does not contain any non-trivial convergent sequences, then, by Lemma 4.14, for every $A\in[\omega]^\omega$ we have $\bigcap_{n\in A}\operatorname{supp}\big(\mu_n\big)=\emptyset$, so there exists a subsequence $\langle\mu_{n_k}\colon\ k\in\omega\rangle$ such that $\operatorname{supp}\big(\mu_{n_k}\big)\cap\operatorname{supp}\big(\mu_{n_l}\big)=\emptyset$ for every $k\neq l\in\omega$. To finish the proof put $\nu_k=\mu_{n_k}$ for every $k\in\omega$. □

Before we present the proof of the main result of this section, Theorem 4.22, we need to prove several auxiliary lemmas, mostly concerning modifying given fsJN-sequences in order to obtain new fsJN-sequences having nicer (or more tamed) properties.

Lemma 4.16. *Fix $\varepsilon>0$. Let $\langle\mu_n\colon\ n\in\omega\rangle$ be a weakly* null sequence of measures on a space X such that $\|\mu_n\|>\varepsilon$ for every $n\in\omega$. Then, $\langle\mu_n/\|\mu_n\|\colon\ n\in\omega\rangle$ is a JN-sequence on X.*

Proof. Let $\nu_n=\mu_n/\|\mu_n\|$ for each $n\in\omega$. Then, $\|\nu_n\|=1$. For every $f\in C(X)$ we have:

$$\big|\nu_n(f)\big|=\big|\mu_n(f)\big|\Big/\big\|\mu_n\big\|<\big|\mu_n(f)\big|/\varepsilon\longrightarrow 0$$

as $n\to\infty$, so $\langle\nu_n\colon\ n\in\omega\rangle$ is weakly* null. It follows that it is a JN-sequence. □

Lemma 4.17. *Let $\langle\mu_n\colon\ n\in\omega\rangle$ and $\langle\nu_n\colon\ n\in\omega\rangle$ be two finitely supported sequences of measures on a space X such that $\lim_{n\to\infty}\|\mu_n-\nu_n\|=0$. Assume that $\langle\mu_n\colon\ n\in\omega\rangle$ is an fsJN-sequence on X, $\|\nu_n\|=1$ for every $n\in\omega$, and that every function $f\in C(X)$ is bounded on $S\big(\nu_n\big)$. Then, $\langle\nu_n\colon\ n\in\omega\rangle$ is also an fsJN-sequence on X.*

Proof. It is only necessary to prove that $\lim_{n\to\infty}\nu_n(f)=0$ for every $f\in C(X)$. Let thus $f\in C(X)$ and put $\alpha=\sup\big\{|f(x)|\colon\ x\in S\big(\mu_n\big)\big\}$ and $\beta=\sup\big\{|f(x)|\colon\ x\in S\big(\nu_n\big)\big\}$. By Lemma 4.11 the function f is bounded on $S\big(\mu_n\big)$, so $\alpha<\infty$. Similarly, $\beta<\infty$ by the assumption. We then have:

$$\big|\nu_n(f)\big|\le\big|\mu_n(f)-\nu_n(f)\big|+\big|\mu_n(f)\big|\le(\alpha+\beta)\big\|\mu_n-\nu_n\big\|+\big|\mu_n(f)\big|,$$

so $\lim_{n\to\infty}\nu_n(f)=0$. It follows that $\langle\nu_n\colon\ n\in\omega\rangle$ is an fsJN-sequence on X. □

Lemma 4.18. *Let $\langle\mu_n\colon\ n\in\omega\rangle$ and $\langle\nu_n\colon\ n\in\omega\rangle$ be two sequences of measures on a space X such that $\lim_{n\to\infty}\|\mu_n-\nu_n\|=0$. If $\langle\mu_n\colon\ n\in\omega\rangle$ is pointwise convergent, then $\langle\nu_n\colon\ n\in\omega\rangle$ is also pointwise convergent and $L\big(\mu_n\big)=L\big(\nu_n\big)$.*

Proof. For every $x\in X$ we have $\big|\mu_n(\{x\})-\nu_n(\{x\})\big|\le\big\|\mu_n-\nu_n\big\|$, so if $\lim_{n\to\infty}\mu_n(\{x\})$ exists, then $\lim_{n\to\infty}\nu_n(\{x\})$ must exist, too. Similarly, $\lim_{n\to\infty}\mu_n(\{x\})\neq 0$ if and only if $\lim_{n\to\infty}\nu_n(\{x\})\neq 0$, so $L\big(\mu_n\big)=L\big(\nu_n\big)$. □

Lemma 4.19. *For every fsJN-sequence $\langle\mu_n\colon\ n\in\omega\rangle$ on a space X there exists a pointwise convergent fsJN-sequence $\langle\nu_k\colon\ k\in\omega\rangle$ on X and an increasing sequence $\langle n_k\colon\ k\in\omega\rangle$ of indices such that:*

(1) $\operatorname{supp}\big(\nu_k\big)\subseteq\operatorname{supp}\big(\mu_{n_k}\big)$ *for every* $k\in\omega$*,*

(2) for every $k\in\omega$ *there exists* $\alpha_k\in\big(1,1+1/k\big)$ *such that*

$$\nu_k=\alpha_k\cdot\Big(\mu_{n_k}\restriction\operatorname{supp}\big(\nu_k\big)\Big)\quad\text{and}\quad\lim_{k\to\infty}\big\|\nu_k-\mu_{n_k}\big\|=0,$$

(3) $L\big(\langle\nu_k\colon\ k\in\omega\rangle\big)=LS\big(\langle\nu_k\colon\ k\in\omega\rangle\big)=LS\big(\langle\mu_{n_k}\colon\ k\in\omega\rangle\big)$*,*

(4) $\operatorname{supp}\big(\nu_l\big)\cap\operatorname{supp}\big(\nu_{l'}\big)\subseteq L\big(\langle\nu_k\colon\ k\in\omega\rangle\big)$ *for every* $l\neq l'\in\omega$*.*

Proof. By Remark 4.4, without loss of generality we may assume that $\langle\mu_n\colon\ n\in\omega\rangle$ is pointwise convergent. Put $Y=LS\big(\langle\mu_n\colon\ n\in\omega\rangle\big)$.

Let $n_0=0$. There exists $n_1>n_0$ such that for every $n\geq n_1$ we have:

$$\big\|\mu_n\restriction\big(\operatorname{supp}\big(\mu_{n_0}\big)\setminus Y\big)\big\|<1/2.$$

Again, there exists $n_2>n_1$ such that for every $n\geq n_2$ we have:

$$\big\|\mu_n\restriction\Big(\big(\operatorname{supp}\big(\mu_{n_0}\big)\cup\operatorname{supp}\big(\mu_{n_1}\big)\big)\setminus Y\Big)\big\|<1/3.$$

Continuing in this manner, we will get a subsequence $\langle\mu_{n_k}\colon\ k\in\omega\rangle$ such that for every $k\in\omega$ it holds:

$$(*)\qquad\big\|\mu_{n_k}\restriction\big(\bigcup_{i=0}^{k-1}\operatorname{supp}\big(\mu_{n_i}\big)\setminus Y\big)\big\|<1/(k+1).$$

For every $k\in\omega$ define the measures ν_k' and ν_k as follows:

$$\nu_k'=\mu_{n_k}\restriction\operatorname{supp}\big(\mu_{n_k}\big)\setminus\big(\bigcup_{i=0}^{k-1}\operatorname{supp}\big(\mu_{n_i}\big)\setminus Y\big),$$

and

$$\nu_k=\alpha_k\cdot\nu_k',$$

where $\alpha_k=\big\|\nu_k'\big\|^{-1}$, so $\big\|\nu_k\big\|=1$. Note that:

$$\nu_k=\alpha_k\cdot\Big(\mu_{n_k}\restriction\operatorname{supp}\big(\nu_k\big)\Big).$$

(1) follows immediately. We have $k/(k+1)<\big\|\nu_k'\big\|<1$ for every $k\in\omega$, so by $(*)$ it holds:

$$\begin{aligned}\big\|\nu_k-\mu_{n_k}\big\|=\big\|\nu_k-\big(\mu_{n_k}\restriction\operatorname{supp}\big(\nu_k\big)\big)-\big(\mu_{n_k}\restriction\operatorname{supp}\big(\nu_k\big)^c\big)\big\|\leq\\ \big\|\nu_k-\big(\mu_{n_k}\restriction\operatorname{supp}\big(\nu_k\big)\big)\big\|+\big\|\mu_{n_k}\restriction\operatorname{supp}\big(\nu_k\big)^c\big\|=\\ (\alpha_k-1)\big\|\mu_{n_k}\restriction\operatorname{supp}\big(\nu_k\big)\big\|+\big\|\mu_{n_k}\restriction\operatorname{supp}\big(\nu_k\big)^c\big\|<\\ \big\|\mu_{n_k}\restriction\operatorname{supp}\big(\nu_k\big)\big\|/k+1/(k+1)\leq1/k+1/(k+1),\end{aligned}$$

which converges to 0 as $k \to \infty$, hence (2) holds as well. Since $S(\nu_k) \subseteq S(\mu_{n_k})$, by Lemmas 4.11 and 4.17, $\langle \nu_k\colon\ n \in \omega \rangle$ is an fsJN-sequence on X. By Lemma 4.18, $\langle \nu_k\colon\ n \in \omega \rangle$ is pointwise convergent.

We now show that (3) holds. Let $x \in LS(\nu_k)$, so $\limsup_{k\to\infty} |\nu_k(\{x\})| > 0$. We have:

$$\limsup_{k\to\infty} |\mu_{n_k}(\{x\})| = \limsup_{k\to\infty} \alpha_k^{-1} |\nu_k(\{x\})| \geq \limsup_{k\to\infty} \frac{k}{k+1} |\nu_k(\{x\})| > 0,$$

which proves that $x \in LS(\mu_{n_k})$. Conversely, for every $x \in LS(\mu_{n_k})$ (so $\limsup_{k\to\infty} |\mu_{n_k}(\{x\})| > 0$) we have:

$$\limsup_{k\to\infty} |\nu_k(\{x\})| = \limsup_{k\to\infty} \alpha_k |\mu_{n_k}(\{x\})| \geq \limsup_{k\to\infty} |\mu_{n_k}(\{x\})| > 0,$$

which yields that $x \in LS(\nu_k)$. Since $\langle \nu_k\colon\ k \in \omega \rangle$ is pointwise convergent, $L(\nu_k) = LS(\nu_k)$ and (3) is satisfied.

It is left to prove (4). Note that immediately by the definition of ν_k's we have $\operatorname{supp}(\nu_l) \cap \operatorname{supp}(\nu_{l'}) \subseteq Y$ for every $l \neq l' \in \omega$. Since $\langle \mu_n\colon\ n \in \omega \rangle$ is pointwise convergent, Lemma 4.6 implies that $LS(\mu_{n_k}) = Y$. By (3), it follows that $\operatorname{supp}(\nu_l) \cap \operatorname{supp}(\nu_{l'}) \subseteq LS(\nu_k) = L(\nu_k)$ for every $l \neq l'$, so (4) holds. □

If the set $LS(\mu_n)$ for a given fsJN-sequence $\langle \mu_n\colon\ n \in \omega \rangle$ on a space X has an isolated point x, then it is easy to construct an fsJN-sequence $\langle \nu_n\colon\ n \in \omega \rangle$ on X with $LS(\nu_n) = \{x\}$—intuitively speaking, such an fsJN-sequence is "concentrated" in a sense around the point x.

Proposition 4.20. *Let $\langle \mu_n\colon\ n \in \omega \rangle$ be an fsJN-sequence on a space X such that the set $LS(\mu_n)$ has an isolated point x in the relative topology. Then, there exists an increasing sequence $\langle n_k\colon\ k \in \omega \rangle$ and an fsJN-sequence $\langle \nu_k\colon\ k \in \omega \rangle$ such that $LS(\nu_k) = \{x\}$ and $\operatorname{supp}(\nu_k) \subseteq \operatorname{supp}(\mu_{n_k})$ for every $k \in \omega$.*

Proof. Let U be an open set in X such that $U \cap LS(\mu_n) = \{x\}$ and let:

$$\alpha = \limsup_{n\to\infty} |\mu_n(\{x\})|.$$

There exists an increasing sequence $\langle n_k\colon\ k \in \omega \rangle$ such that $\lim_{k\to\infty} |\mu_{n_k}(\{x\})| = \alpha$ and $|\mu_{n_k}(\{x\})| > \alpha/2$ for every $k \in \omega$. There exists a function $g \in C(X)$ such that $0 \leq g \leq 1$, $g(x) = 1$ and $g \restriction U^c \equiv 0$. For every $k \in \omega$ we define the measure ν_k as follows:

$$\nu_k = g\mathrm{d}\mu_{n_k} \ / \ \|g\mathrm{d}\mu_{n_k}\|.$$

Note that $\|g\mathrm{d}\mu_{n_k}\| \neq 0$, since

$$1 \geq \|g\mathrm{d}\mu_{n_k}\| \geq |g(x)| \cdot |\mu_{n_k}(\{x\})| = |\mu_{n_k}(\{x\})| > \alpha/2 > 0,$$

so ν_k is well-defined and $\|\nu_k\| = 1$. Obviously, $\operatorname{supp}(\nu_k) \subseteq \operatorname{supp}(\mu_{n_k}) \cap U$ for every $k \in \omega$. Since

$$|\nu_k(f)| < |\mu_{n_k}(f \cdot g)| \cdot 2/\alpha$$

for every $f \in C(X)$ and $k \in \omega$, the sequence $\langle \nu_k \colon\ k \in \omega \rangle$ is weakly* convergent to 0 and hence it is an fsJN-sequence. Finally, $\big|\nu_k(\{x\})\big| > \alpha/2$ for every $k \in \omega$, so $\{x\} \subseteq LS\big(\nu_k\big)$. To prove the converse inclusion, let $y \in U \setminus \{x\}$. If $y \notin \operatorname{supp}\big(\mu_{n_k}\big)$ for some $k \in \omega$, then $\nu_k(\{y\}) = 0$. If $y \in \operatorname{supp}\big(\mu_{n_k}\big)$ for some $k \in \omega$, then

$$\big\|\nu_k(\{y\})\big| = \big|g(y) \cdot \mu_{n_k}(\{y\})\big| / \big\|g \mathrm{d}\mu_{n_k}\big\| < \big|g(y) \cdot \mu_{n_k}(\{y\})\big| \cdot 2/\alpha,$$

which converges to 0 as $k \to \infty$, since $y \notin LS\big(\mu_{n_k}\big)$ (by Lemma 4.6). It follows that $LS\big(\nu_k\big) = \{x\}$. □

Lemma 4.21. *Assume that a space X admits an fsJN-sequence $\langle \mu_n \colon\ n \in \omega \rangle$ such that $\lim_{m\to\infty} \big\|\mu_m \restriction L\big(\mu_n\big)\big\| = 1$. Then, $\langle \mu_n \colon\ n \in \omega \rangle$ is pointwise convergent and there exist a pointwise convergent fsJN-sequence $\langle \nu_k \colon\ k \in \omega \rangle$ on X and an increasing sequence $\langle n_k \colon\ k \in \omega \rangle$ of indices such that:*

(1) $\operatorname{supp}\big(\nu_k\big) \subseteq \operatorname{supp}\big(\mu_{n_k}\big) \cap L\big(\mu_n\big)$ *for every $k \in \omega$,*

(2) for every $k \in \omega$ there exists $\alpha_k \in \big[1, 1 + 1/k\big)$ such that

$$\nu_k = \alpha_k \cdot \Big(\mu_{n_k} \restriction \operatorname{supp}\big(\nu_k\big)\Big) \quad \textit{and} \quad \lim_{k\to\infty} \big\|\nu_k - \mu_{n_k}\big\| = 0,$$

(3) $L\big(\langle \nu_k \colon\ k \in \omega \rangle\big) = L\big(\langle \mu_n \colon\ n \in \omega \rangle\big)$.

In particular, $\operatorname{supp}\big(\nu_l\big) \subseteq L\big(\nu_k\big)$ *for every $l \in \omega$.*

Proof. Let $L = L\big(\mu_n\big)$. For every $k \in \omega$ let $n_k \in \omega$ be such that $\big\|\mu_m \restriction L^c\big\| < 1/(k+1)$ for every $m \geq n_k$ and put:

$$\nu_k = \Big(\mu_{n_k} \restriction L\Big) / \big\|\mu_{n_k} \restriction L\big\|.$$

(1) follows immediately. For every $k \in \omega$ we have $\big\|\nu_k\big\| = 1$ and $k/(k+1) < \big\|\mu_{n_k} \restriction L\big\| \leq 1$, so $\nu_k = \alpha_k \cdot \Big(\mu_{n_k} \restriction \operatorname{supp}\big(\nu_k\big)\Big)$ for some $\alpha_k \in \big[1, 1 + 1/k\big)$. It holds that:

$$\big\|\nu_k - \mu_{n_k}\big\| = \big\|\big(\nu_k \restriction L\big) - \big(\mu_{n_k} \restriction L\big)\big\| + \big\|\mu_{n_k} \restriction L^c\big\| = \big(1 - \big\|\mu_{n_k} \restriction L\big\|\big) + \big\|\mu_{n_k} \restriction L^c\big\| =$$
$$= 2\big\|\mu_{n_k} \restriction L^c\big\| < 2/(k+1),$$

which converges to 0 as $k \to \infty$. Thus, (2) is also proved.

To see that $\langle \mu_n \colon\ n \in \omega \rangle$ is pointwise convergent, note that for every $x \in L$ the limit $\lim_{n\to\infty} \mu_n(\{x\})$ exists by the definition and for every $x \in X \setminus L$ we have:

$$\big|\mu_n(\{x\})\big| = 1 - \big\|\mu_n \restriction X \setminus \{x\}\big\| \leq 1 - \big\|\mu_n \restriction L\big\|,$$

which converges to 0 as $n \to \infty$. The sequence $\langle \nu_k \colon\ k \in \omega \rangle$ is then pointwise convergent by Lemma 4.18. Similarly as in the proof of Lemma 4.19, we conclude that $\langle \nu_k \colon\ k \in \omega \rangle$ is an fsJN-sequence on X.

Since $\langle \mu_n \colon\ n \in \omega \rangle$ is pointwise convergent, $L\big(\mu_n\big) = L\big(\mu_{n_k}\big)$. By Lemma 4.18, $L\big(\nu_k\big) = L\big(\mu_{n_k}\big)$, which proves (3). □

We are now in the position to prove the main theorem.

Theorem 4.22. *Assume that a space X has the fsJNP. Then, X admits a disjointly supported fsJN-sequence.*

Proof. Let $\langle \mu_n\colon\ n \in \omega\rangle$ be an fsJN-sequence on X. Apply Lemma 4.19 to $\langle \mu_n\colon\ n \in \omega\rangle$ to obtain a pointwise convergent fsJN-sequence $\langle \nu_n\colon\ n \in \omega\rangle$ such that

$$(\#) \qquad \operatorname{supp}\big(\nu_l\big) \cap \operatorname{supp}\big(\nu_{l'}\big) \subseteq L\big(\langle \nu_n\colon\ n \in \omega\rangle\big)$$

for every $l \neq l' \in \omega$. Let $L = L\big(\nu_n\big)$. By going to a further subsequence, we may assume that the limit $\lim_{n\to\infty} \big\|\nu_n \restriction L\big\|$ exists and is equal to some $\alpha \in \mathbb{R}$. If $L = \emptyset$, then we are done by the properties of $\langle \nu_n\colon\ n \in \omega\rangle$. Let us thus assume that $|L| > 0$ and enumerate $L = \big\{q_k\colon k < |L|\big\}$ (note that $|L|$ may be finite or ω). By going again to a subsequence, we may assume that for every $k < |L|$ and $l \geq k$ we have:

$$(\dagger) \qquad \Big|\nu_l\big(\{q_k\}\big) - \lim_{n\to\infty} \nu_n\big(\{q_k\}\big)\Big| < \frac{1}{l+1} \cdot \big|\lim_{n\to\infty} \nu_n\big(\{q_k\}\big)\big|.$$

It follows that $\operatorname{supp}\big(\nu_k\big) \cap L \subseteq \operatorname{supp}\big(\nu_l\big) \cap L$ for every pair $k, l \in \omega$ such that $k \leq l$. We now need to consider three cases depending on the value of α.

(1) If $\alpha = 0$, then $L = \emptyset$ and we are done.

(2) If $\alpha \in (0,1)$, then there is $N \in \omega$ such that for every $n > N$ we have

$$\big\|\nu_n \restriction L\big\| < \alpha + \frac{1-\alpha}{2} = \frac{1+\alpha}{2},$$

so

$$\big\|\nu_n \restriction L^c\big\| = 1 - \big\|\nu_n \restriction L\big\| > 1 - \frac{1+\alpha}{2} = \frac{1-\alpha}{2} > 0.$$

Since for every $n \in \omega$ we have $\operatorname{supp}\big(\nu_{2n}\big) \cap \operatorname{supp}\big(\nu_{2n+1}\big) \subseteq L$, for every $n > N$ it holds:

$$(*) \qquad \big\|\nu_{2n} - \nu_{2n+1}\big\| \geq \big\|\nu_{2n} \restriction L^c\big\| + \big\|\nu_{2n+1} \restriction L^c\big\| > 2 \cdot \frac{1-\alpha}{2} = 1 - \alpha > 0.$$

Let us remove the first N elements from the sequence $\langle \nu_n\colon\ n \in \omega\rangle$, so we may assume that $(*)$ holds actually for every $n \in \omega$. Let $\beta = 1 - \alpha$.

(3) If $\alpha = 1$, then we can substitute the current fsJN-sequence $\langle \nu_n\colon\ n \in \omega\rangle$ by the pointwise convergent fsJN-sequence $\langle \nu_n\colon\ n \in \omega\rangle$ from Lemma 4.21, i.e. $\operatorname{supp}\big(\nu_l\big) \subseteq L\big(\nu_n\big)$ for every $l \in \omega$. By going again to a subsequence we may assume that for every $k \in \omega$ and $x \in \operatorname{supp}\big(\nu_{2k}\big)$ we have:

$$(\dagger\dagger) \qquad \big|\nu_{2k+1}(\{x\}) - \lim_{n\to\infty} \nu_n(\{x\})\big| < \frac{1}{4 \cdot \big|\operatorname{supp}\big(\nu_{2k}\big)\big|}.$$

It follows by $(\dagger\dagger)$ and Lemma 4.8 that for every $n \in \omega$ it holds:

$$\big\|\nu_{2k+1} \restriction \operatorname{supp}\big(\nu_{2k}\big)\big\| = \sum_{x \in \operatorname{supp}(\nu_{2k})} \big|\nu_{2k+1}(\{x\})\big| \leq \sum_{x \in \operatorname{supp}(\nu_{2k})} \Big(\frac{1}{4 \cdot \big|\operatorname{supp}\big(\nu_{2k}\big)\big|} + \big|\lim_{n\to\infty} \nu_n(\{x\})\big|\Big) =$$

$$\sum_{x\in\operatorname{supp}(\nu_{2k})}\Big(\frac{1}{4\cdot\big|\operatorname{supp}\big(\nu_{2k}\big)\big|}+\lim_{n\to\infty}\big|\nu_n(\{x\})\big|\Big)\le\big|\operatorname{supp}\big(\nu_{2k}\big)\big|\cdot\frac{1}{4\cdot\big|\operatorname{supp}\big(\nu_{2k}\big)\big|}+1/2=3/4,$$

so $\big\|\nu_{2k+1}\restriction\operatorname{supp}\big(\nu_{2k}\big)^c\big\|\ge 1/4$ and hence:

$$(**)\qquad \big\|\nu_{2k}-\nu_{2k+1}\big\|\ge\big\|\nu_{2k+1}\restriction\operatorname{supp}\big(\nu_{2k}\big)^c\big\|\ge 1/4>0.$$

Let $\beta=1/4$.

Note that in both cases $(*)$ and $(**)$ the inequality $\big\|\nu_{2n}-\nu_{2n+1}\big\|\ge\beta>0$ holds for every $n\in\omega$. For every $n\in\omega$ we define the measure θ_n as follows:

$$\theta_n=\big(\nu_{2n}-\nu_{2n+1}\big)\ /\ \big\|\nu_{2n}-\nu_{2n+1}\big\|.$$

For every $f\in C(X)$ and $n\in\omega$ we have:

$$\big|\theta_n(f)\big|\le\beta^{-1}\big(\big|\nu_{2n}(f)\big|+\big|\nu_{2n+1}(f)\big|\big),$$

so $\langle\theta_n\colon\ n\in\omega\rangle$ is an fsJN-sequence. We also claim that $LS\big(\theta_n\big)=\emptyset$. Let thus $x\in X$. If $x\notin L$, then immediately by (#) and the definition of θ_n's we have that $\lim_{n\to\infty}\theta_n(\{x\})=0$, so $x\notin LS\big(\theta_n\big)$. On the other hand, if $x\in L$, then $x=q_k$ for some $k\in\omega$, but then by (†) for every $l\ge k$ we have:

$$\big|\theta_l\big(\big\{q_k\big\}\big)\big|=\big|\nu_{2l}\big(\big\{q_k\big\}\big)-\nu_{2l+1}\big(\big\{q_k\big\}\big)\big|\ /\ \big\|\nu_{2l}-\nu_{2l+1}\big\|\le$$
$$\beta^{-1}\cdot\Big|\ \Big(\nu_{2l}\big(\big\{q_k\big\}\big)-\lim_{n\to\infty}\nu_n\big(\big\{q_k\big\}\big)\Big)-\Big(\nu_{2l+1}\big(\big\{q_k\big\}\big)-\lim_{n\to\infty}\nu_n\big(\big\{q_k\big\}\big)\Big)\ \Big|\le$$
$$\beta^{-1}\cdot\Big(\ \Big|\nu_{2l}\big(\big\{q_k\big\}\big)-\lim_{n\to\infty}\nu_n\big(\big\{q_k\big\}\big)\Big|+\Big|\nu_{2l+1}\big(\big\{q_k\big\}\big)-\lim_{n\to\infty}\nu_n\big(\big\{q_k\big\}\big)\Big|\ \Big)<$$
$$\frac{2\beta^{-1}}{2l+1}\cdot\big|\lim_{n\to\infty}\nu_n\big(\big\{q_k\big\}\big)\big|,$$

which goes to 0 as $l\to\infty$, so again $\lim_{l\to\infty}\theta_l(\{x\})=0$ and hence $x\notin LS\big(\theta_n\big)$. This proves that $LS\big(\theta_n\big)=\emptyset$ indeed.

Finally, by Lemma 4.19, there exists an fsJN-sequence $\langle\rho_n\colon\ n\in\omega\rangle$ on X such that for every $m\neq m'$ we have:

$$\operatorname{supp}\big(\rho_m\big)\cap\operatorname{supp}\big(\rho_{m'}\big)\subseteq LS\big(\langle\rho_n\colon\ n\in\omega\rangle\big)\subseteq LS\big(\langle\theta_n\colon\ n\in\omega\rangle\big)=\emptyset,$$

and the proof is finished. □

In Lemma 4.15 we stated that if the supports of a given fsJN-sequence on a space X have cardinality 2, then there is a disjointly supported fsJN-sequence on X having the same property. It is an easy topological fact that if $\langle F_n\colon\ n\in\omega\rangle$ is a countable collection of pairwise disjoint finite subsets of a space X for which there exists $M\in\omega$ such that $\big|F_n\big|\le M$, then there exist a subsequence $\langle F_{n_k}\colon\ k\in\omega\rangle$ and a sequence $\langle U_k\colon\ k\in\omega\rangle$ of pairwise disjoint open subsets of X such that $F_{n_k}\subseteq U_k$ for every $k\in\omega$. It follows that the union $\bigcup_{k\in\omega}F_{n_k}$ is a discrete subspace of X. Consequently, if there is a disjointly supported fsJN-sequence $\langle\mu_n\colon\ n\in\omega\rangle$ on X such that $\big|\operatorname{supp}\big(\mu_n\big)\big|=2$ for every $n\in\omega$, then there is a subsequence $\langle\mu_n\colon\ k\in\omega\rangle$ such that the union $\bigcup_{k\in\omega}\operatorname{supp}\big(\mu_{n_k}\big)$ is discrete (cf. Theorem 5.13). We do not know whether the same property holds for every fsJN-sequence.

Question 4.23. *Does every space X with the fsJNP admit an fsJN-sequence $\langle \mu_n\colon\ n \in \omega\rangle$ such that the union $\bigcup_n \operatorname{supp}\big(\mu_n\big)$ is a discrete subspace of X?*

The positive answer to Question 4.23 would bring much simplification to the study of the finitely supported Josefson–Nissenzweig property.

5. Sizes of supports in fsJN-sequences

In this section we will study possible cardinalities of supports of measures from fsJN-sequences. We have two cases here: either (1) a space X admits an fsJN-sequence $\langle \mu_n\colon\ n \in \omega\rangle$ for which there exists $M \in \omega$ such that $\big|\operatorname{supp}\big(\mu_n\big)\big| \le M$ for every $n \in \omega$, or (2) every fsJN-sequence $\langle \mu_n\colon\ n \in \omega\rangle$ on X has the property that $\lim_{n\in\omega}\big|\operatorname{supp}\big(\mu_n\big)\big| = \infty$. As an example of the former case we may name any space X having a non-trivial convergent sequence. An appropriate example for the latter case is more difficult to find—however, it appears that the space K considered in Banakh, Kąkol and Śliwa [7, Section 4] (*Plebanek's example*) has the required property. In Subsection 5.1 we prove this statement as well as we present another example (due to Schachermayer) which is in many places very similar to Plebanek's one but satisfies the case (1).

In Subsection 5.2 we will provide several general statements concerning cardinalities of supports. In particular, we prove in Theorem 5.13 that if a compact space K satisfies the case (1), then there exists an fsJN-sequence $\langle \mu_n\colon\ n \in \omega\rangle$ such that $\big|\operatorname{supp}\big(\mu_n\big)\big| = 2$ for every $n \in \omega$.

5.1. **Two examples.**

Example 5.1. In Banakh, Kąkol and Śliwa [7, Section 4], the authors provided the following example of a Boolean algebra $\mathcal{D}$ due to Plebanek:

$$\mathcal{D} = \Big\{A \in \wp(\omega)\colon\ \lim_{n\to\infty}\frac{|A \cap \{0,\dots,n-1\}|}{n} \in \{0,1\}\Big\}.$$

Let $\mathcal{D}$ be called *the density Boolean algebra*. Since for each $n \in \omega$ the set $\{n\}$ belongs to $\mathcal{D}$ and is an atom therein, we may consider $St(\mathcal{D})$ as a compactification of ω. Let us additionally define the ideal $\mathcal{Z}$ and the ultrafilter p in $\mathcal{D}$ as follows:

$$\mathcal{Z} = \Big\{A \in \wp(\omega)\colon\ \lim_{n\to\infty}\frac{|A \cap \{0,\dots,n-1\}|}{n} = 0\Big\}$$

and

$$p = \mathcal{D} \setminus \mathcal{Z}.$$

Proposition 5.2. *The density Boolean algebra $\mathcal{D}$ has the following properties:*

1. *$St(\mathcal{D})$ does not have any non-trivial convergent sequences;*
2. *if $X \subseteq St(\mathcal{D})$ is infinite, then there exists an infinite subset $Y \subseteq X$ such that $\overline{Y}^{St(\mathcal{D})}$ is homeomorphic to $\beta\omega$;*
3. *$St(\mathcal{D})$ has the fsJNP;*
4. *every fsJN-sequence $\langle \mu\colon\ n \in \omega\rangle$ on $St(\mathcal{D})$ has the property that $\lim_{n\to\infty}\big|\operatorname{supp}\big(\mu_n\big)\big| = \infty$.*

Proof. For (1)–(3), see [7, Section 4, Fact 1–3, page 3026]. We now prove (4), so for the sake of contradiction let us assume that there exists an fsJN-sequence $\langle \mu_n \colon\ n \in \omega \rangle$ on $St(\mathcal{D})$ and an integer $M > 1$ such that $\left| \operatorname{supp}\left(\mu_n\right) \right| = M$ for every $n \in \omega$. By Theorem 5.13, we may assume that $\mu_n = \frac{1}{2}\big(\delta_{x_n} - \delta_{y_n}\big)$. By Lemma 4.15, we may also assume that $\big\{x_n, y_n\big\} \cap \big\{x_{n'}, y_{n'}\big\} = \emptyset$ for every $n \neq n' \in \omega$ and that $p \notin \big\{x_n, y_n\big\}$ for every $n \in \omega$. We need to consider several cases:

(i) There is $Q \in [\omega]^\omega$ such that $\big\{x_n, y_n\big\} \subseteq \omega$ for every $n \in Q$. We then go to a subsequence $\langle n_k \in Q \colon\ k \in \omega \rangle$ such that $A = \bigcup_{k\in\omega} \big\{x_{n_k}, y_{n_k}\big\} \in \mathcal{Z}$. Since $[A]_\mathcal{D}$ is homeomorphic to $\beta\omega$, it follows that $\langle \mu_{n_k} \restriction [A]_\mathcal{D} \colon\ k \in \omega \rangle$ gives rise to an fsJN-sequence in $\beta\omega$, which is impossible.

(ii) There is $Q \in [\omega]^\omega$ such that $\big\{x_n, y_n\big\} \cap \omega = \emptyset$ for every $n \in Q$. We find $A_n \in \mathcal{Z}$ such that $\big\{x_n, y_n\big\} \subseteq \big[A_n\big]_\mathcal{D}$ for every $n \in Q$. By [7, Section 4, Fact 1, page 3026], there is infinite $B \in \mathcal{Z}$ such that $A_n \setminus B$ is finite for every $n \in Q$. Since $\big\{x_n, y_n\big\} \cap \omega = \emptyset$ for every $n \in Q$, it follows that $A_n \setminus B \notin x_n$ and $A_n \setminus B \notin y_n$, and hence $\big\{x_n, y_n\big\} \subseteq [B]_\mathcal{D}$. Again, since $[B]_\mathcal{D}$ is homeomorphic to $\beta\omega$, we obtain an fsJN-sequence on $\beta\omega$, which is a contradiction.

(iii) There is $Q \in [\omega]^\omega$ such that $\left|\big\{x_n, y_n\big\} \cap \omega\right| = 1$ for every $n \in Q$. Without loss of generality, we may assume that $x_n \in \omega$ for every $n \in Q$. First, let us find $R \in [Q]^\omega$ such that $\big\{x_n \colon\ n \in R\big\} \in \mathcal{Z}$. Then, similarly as in (ii), let us find $B \in \mathcal{Z}$ such that $\big\{y_n \colon\ n \in R\big\} \subseteq [B]_\mathcal{D}$. Since $\mathcal{Z}$ is an ideal, $C = \big\{x_n \colon\ n \in R\big\} \cup B \in \mathcal{Z}$. It follows that $[C]_\mathcal{D}$ is homeomorphic to $\beta\omega$ and $\langle \mu_n \restriction [C]_\mathcal{D} \colon\ n \in R \rangle$ is an fsJN-sequence on $[C]_\mathcal{D}$, a contradiction.

□

Example 5.3. In [72, Example 4.10] Schachermayer provided a simple example of a Boolean algebra $\mathcal{S}$ without the Grothendieck property (see Section 6) and such that $St(\mathcal{S})$ does not have any non-trivial convergent sequences. The algebra $\mathcal{S}$ is defined as a subalgebra of $\wp(\omega)$ in the following way:

$$\mathcal{S} = \big\{A \in \wp(\omega) \colon\ \big(\forall^\infty k \in \omega\big)\ \big(2k \in A \equiv 2k+1 \in A\big)\big\}.$$

Since $\{n\} \in \mathcal{S}$ for every $n \in \omega$, we may—as previously—identify isolated points of the Stone space $St(\mathcal{S})$ with the elements of ω, so $St(\mathcal{S})$ is a compactification of ω. $St(\mathcal{S})$ has the fsJNP—indeed, we can define an fsJN-sequence as follows:

$$\mu_n = \frac{1}{2}\big(\delta_{2n} - \delta_{2n+1}\big),\ n \in \omega.$$

Note that for every $A \in \mathcal{S}$ we have $\mu_n(A) = 0$ for sufficiently large $n \in \omega$.

The next proposition is similar to 5.2.

Proposition 5.4. *Schachermayer's Boolean algebra $\mathcal{S}$ has the following properties:*

(1) $St(\mathcal{S})$ does not have any non-trivial convergent sequences;

(2) if $X \subseteq St(\mathcal{S})$ is infinite, then there exists an infinite subset $Y \subseteq X$ such that $\overline{Y}^{St(\mathcal{D})}$ is homeomorphic to $\beta\omega$;

(3) there exists an fsJN-sequence $\langle \mu_n\colon\ n \in \omega\rangle$ *such that* $\big|\operatorname{supp}\big(\mu_n\big)\big| = 2$ *for every* $n \in \omega$.

Proof. For (1) and (3), see Example 5.3. We now show (2), so let $X \subseteq St(\mathcal{S})$ be infinite. We have two cases:

(i) $X \cap \omega$ is infinite, so without loss of generality we may assume that $X \subseteq \omega$. Enumerate X as a strictly increasing sequence $\langle a_n \in X\colon\ n \in \omega\rangle$ and go to a subsequence $\langle a_{n_k}\colon\ k \in \omega\rangle$ such that $a_{n_{k+1}} - a_{n_k} > 1$ for every $k \in \omega$. For every $k \in \omega$ let A_k be a subset of ω of size 2 such that:
- $a_{n_k} \in A_k$, and
- for every $l \in \omega$ we have: $2l \in A_k$ if and only if $2l + 1 \in A_k$.

Then, $A_k \in \mathcal{S}$. Let $U, V \in [\omega]^\omega$ be two disjoint sets such that $U \cup V = \omega$. It follows that $\bigcup_{k\in U} A_k \in \mathcal{S}$, $\bigcup_{k\in V} A_k \in \mathcal{S}$, and $\bigcup_{k\in U} A_k \cap \bigcup_{k\in V} A_k = \emptyset$, and hence finally

$$\overline{\big\{a_{n_k}\colon\ k \in U\big\}}^{St(\mathcal{S})} \subseteq \Big[\bigcup_{k\in U} A_k\Big]_{\mathcal{S}} \quad\text{and}\quad \overline{\big\{a_{n_k}\colon\ k \in V\big\}}^{St(\mathcal{S})} \subseteq \Big[\bigcup_{k\in V} A_k\Big]_{\mathcal{S}},$$

so

$$\overline{\big\{a_{n_k}\colon\ k \in U\big\}}^{St(\mathcal{S})} \cap \overline{\big\{a_{n_k}\colon\ k \in V\big\}}^{St(\mathcal{S})} = \emptyset.$$

Since U and V were arbitrary, we get that $\overline{\big\{a_{n_k}\colon\ k \in \omega\big\}}^{St(\mathcal{S})}$ is homeomorphic to $\beta\omega$.

(ii) $X \cap \omega$ is finite, so without loss of generality we may assume that $X \cap \omega = \emptyset$. Let $Y = \big\{x_n \in X\colon\ n \in \omega\big\}$ be a discrete subset of X and find a sequence $\langle A_n\colon\ n \in \omega\rangle$ of pairwise disjoint elements of $\mathcal{S}$ such that $x_n \in \big[A_n\big]_{\mathcal{S}}$ for every $n \in \omega$. For each $n \in \omega$ let $B_n \in \mathcal{S}$ be such that:
- $B_n \leq A_n$,
- $A_n \setminus B_n$ is finite,
- for every $l \in \omega$ we have: $2l \in B_n$ if and only if $2l + 1 \in B_n$.

Since for each $n \in \omega$ we have $x_n \notin \omega$, $x_n \in B_n$. Now, notice that for every $W \in [\omega]^\omega$ we have $\bigcup_{n\in W} B_n \in \mathcal{S}$ and proceed as in the previous case to prove that $\overline{\big\{x_n\colon\ n \in \omega\big\}}^{St(\mathcal{S})}$ is homeomorphic to $\beta\omega$.

□

Remark 5.5. Let us note that we can provide a completely different proof of Proposition 5.4.(2) in the case when $X = [A]_{\mathcal{S}}$ for some infinite $A \in \mathcal{S}$. Indeed, let $B \subseteq A$ be an infinite subset such that for every $k \in \omega$ we have: $2k \in B \equiv 2k + 1 \in B$. Put:

$$\mathcal{B} = \big\{C \in \mathcal{A}\colon\ C \leq B\big\};$$

then, $\mathcal{B}$ is a Boolean algebra with obvious operations and the unit element B. $\mathcal{B}$ is also isomorphic with $\mathcal{S}$. By Koszmider and Shelah [53, Proposition 2.5], $\mathcal{B}$ has the so-called Weak Subsequential Separation Property, and hence, by [53, Theorem 1.4], it contains an independent family $\mathcal{F}$ of size $\mathfrak{c}$. Let $\mathcal{C}$ be the subalgebra of $\mathcal{B}$ generated by $\mathcal{F}$. Since $|\mathcal{F}| = \mathfrak{c}$, there is a homomorphism φ from $\mathcal{C}$ onto $\wp(\omega)$. Since $\wp(\omega)$ is complete, by the Sikorski extension theorem, there is an extension Φ of φ onto $\mathcal{B}$. By the Stone duality, it follows that $\beta\omega$ is a subspace of $St(\mathcal{B}) \approx [B]_{\mathcal{S}} \subseteq [A]_{\mathcal{S}}$.

5.2. **Study of sizes of supports.** We will now restrict our study to those compact space which admits fsJN-sequences with bounded sizes of supports, i.e. such fsJN-sequences $\langle\mu_n\colon\ n \in \omega\rangle$ that there exists $M \in \omega$ such that $\left|\operatorname{supp}\left(\mu_n\right)\right| \le M$ for every $n \in \omega$.

Proposition 5.6. *Let $\langle\mu_n\colon\ n \in \omega\rangle$ be an fsJN-sequence on a compact space K. Then, there is $N \in \omega$ such that for every $n > N$ the support $\operatorname{supp}\left(\mu_n\right)$ is not a singleton.*

Proof. Assume for the sake of contradiction that there exists a subsequence $\langle\mu_{n_k}\colon\ k \in \omega\rangle$ such that $\left|\operatorname{supp}\left(\mu_{n_k}\right)\right| = 1$ for every $k \in \omega$. Then, for each $k \in \omega$ there exist $x_k \in K$ and $\alpha_k \in \{-1, 1\}$ such that $\mu_{n_k} = \alpha_k \delta_{x_k}$. It follows that $\left|\mu_{n_k}(K)\right| = 1$ for every $k \in \omega$, which contradicts the fact that $\left|\mu_{n_k}(K)\right| \to 0$ as $k \to \infty$. □

Proposition 5.7. *Let a compact space K admit an fsJN-sequence $\langle\mu_n\colon\ n \in \omega\rangle$ such that there exists $M \ge 2$, $M \in \omega$, for which we have $\left|\operatorname{supp}\left(\mu_n\right)\right| = M$ for every $n \in \omega$. For each $n \in \omega$ write $\operatorname{supp}\left(\mu_n\right) = \left\{x_1^n, \ldots, x_M^n\right\}$. Then, there exist $\alpha_1, \ldots, \alpha_M \in \mathbb{R}$ and an increasing sequence $\langle n_k\colon\ k \in \omega\rangle$ such that the measures $\nu_k = \sum_{i=1}^M \alpha_i \delta_{x_i^{n_k}}$, $k \in \omega$, form an fsJN-sequence such that $\left\|\nu_k - \mu_{n_k}\right\| \to 0$ as $k \to \infty$.*

Proof. For each $n \in \omega$ let $\mu_n = \sum_{i=1}^M \alpha_i^n \delta_{x_i^n}$. Since $\alpha_1^n \in [-1, 1]$ for each $n \in \omega$, there is $A_1 \in [\omega]^\omega$ and $\alpha_1 \in \mathbb{R}$ such that

$$\lim_{\substack{n\to\infty\\ n\in A_1}} \alpha_1^n = \alpha_1.$$

Similarly, since $\alpha_2^n \in [-1, 1]$ for each $n \in A_1$, there is $A_2 \in \left[A_1\right]^\omega$ and α_2 such that

$$\lim_{\substack{n\to\infty\\ n\in A_2}} \alpha_2^n = \alpha_2.$$

We continue in this way, until we get a sequence $A_1 \supset \ldots \supset A_M$ of infinite subsets of ω and a sequence of real numbers $\alpha_1, \ldots, \alpha_M \in \mathbb{R}$ such that

$$\lim_{\substack{n\to\infty\\ n\in A_i}} \alpha_M^n = \alpha_i$$

for each $i = 1, \ldots, M$. We claim that $\sum_{i=1}^M \left|\alpha_i\right| = 1$. To see this, assume that $\sum_{i=1}^M \left|\alpha_i\right| = \alpha \neq 1$. Assume first that $\alpha < 1$. Let $\varepsilon = (1 - \alpha)/M$. There is $N \in \omega$ such that for every $n \in A_M \setminus \{0, \ldots, N\}$ and $i = 1, \ldots, M$ we have: $\left|\alpha_i^n - \alpha_i\right| < \varepsilon$. But then for those n it holds:

$$1 = \sum_{i=1}^M \left|\alpha_i^n\right| \le \sum_{i=1}^M \left(\left|\alpha_i^n - \alpha_i\right| + \left|\alpha_i\right|\right) = \sum_{i=1}^M \left|\alpha_i^n - \alpha_i\right| + \sum_{i=1}^M \left|\alpha_i\right| < M \cdot \varepsilon + \alpha = 1,$$

a contradiction. Similarly, if $\alpha > 1$, then let $\varepsilon = (\alpha - 1)/M$, and again let $N \in \omega$ be such such that for every $n \in A_M \setminus \{0, \ldots, N\}$ and $i = 1, \ldots, M$ we have: $\left|\alpha_i^n - \alpha_i\right| < \varepsilon$. But then for every such n we have:

$$\alpha - 1 = M \cdot \varepsilon > \sum_{i=1}^{M} \left|\alpha_i^n - \alpha_i\right| \geq \sum_{i=1}^{M} \Big(\left|\alpha_i\right| - \left|\alpha_i^n\right|\Big) = \sum_{i=1}^{M} \left|\alpha_i\right| - \sum_{i=1}^{M} \left|\alpha_i^n\right| = \alpha - 1,$$

again a contradiction, so $\alpha = 1$.

Enumerate $A_M = \langle n_k\colon\ k \in \omega\rangle$ and define for every $k \in \omega$ the measure ν_k as follows:

$$\nu_k = \sum_{i=1}^{M} \alpha_i \delta_{x_i^{n_k}};$$

then, $\left\|\nu_k\right\| = 1$, by the previous argument. To finish the proof notice that

$$\left\|\nu_k - \mu_{n_k}\right\| = \sum_{i=1}^{M} \left|\alpha_i - \alpha_i^{n_k}\right| \longrightarrow 0$$

as $k \to \infty$ and appeal to Lemma 4.17 to conclude that $\langle \nu_k\colon\ k \in \omega\rangle$ is an fsJN-sequence on K. □

By Lemma 4.2, we immediately get the following corollary.

Corollary 5.8. *If $\langle \mu_n\colon\ n \in \omega\rangle$ is an fsJN-sequence on a compact space K and there exist numbers $\alpha_1, \ldots, \alpha_M \in \mathbb{R}$ such that every μ_n can be written in the form $\mu_n = \sum_{i=1}^{M} \alpha_i \delta_{x_i^n}$, for some $x_1^n, \ldots, x_M^n \in K$, then:*

$$\sum \big\{\alpha_i\colon\ \alpha_i > 0, 1 \leq i \leq M\big\} = 1/2 = -\sum \big\{\alpha_i\colon\ \alpha_i < 0, 1 \leq i \leq M\big\}.$$

□

Note that Proposition 5.7 does not say that $\alpha_i \neq 0$ for all $i = 1, \ldots, M$, but of course we may remove from the definition of ν_k all such points $x_i^{n_k}$ for which we have $\alpha_i = 0$ and obtain a sequence $\langle \nu_n\colon\ n \in \omega\rangle$ such that $\left|\operatorname{supp}\left(\nu_n\right)\right| < M$ for every $n \in \omega$. The next lemma is thus a variant of Proposition 5.7 (with an alternative proof).

Lemma 5.9. *Let a compact space K admit an fsJN-sequence $\langle \mu_n\colon\ n \in \omega\rangle$ such that there exists $M > 2$, $M \in \omega$, for which we have $\left|\operatorname{supp}\left(\mu_n\right)\right| = M$ for every $n \in \omega$. If there exists a sequence $\langle x_n\colon\ n \in \omega\rangle$ such that $x_n \in \operatorname{supp}\left(\mu_n\right)$ for every $n \in \omega$ and $\lim_{n\to\infty} \mu_n\big(\{x_n\}\big) = 0$, then K admits an fsJN-sequence $\langle \nu_n\colon\ n \in \omega\rangle$ such that $\left|\operatorname{supp}\left(\nu_n\right)\right| = M - 1$ for every $n \in \omega$.*

Proof. Let $\langle n_k\colon\ k \in \omega\rangle$ be such an increasing sequence that $\left|\mu_{n_k}\big(\{x_{n_k}\}\big)\right| < 1/(k+1)$, so $\left\|\mu_{n_k} \restriction \big(K \setminus \{x_{n_k}\}\big)\right\| > 1 - 1/(k+1)$. For every $k \in \omega$ put:

$$\nu_k = \Big(\mu_{n_k} \restriction \big(K \setminus \{x_{n_k}\}\big)\Big) \Big/ \left\|\mu_{n_k} \restriction \big(K \setminus \{x_{n_k}\}\big)\right\|.$$

Obviously, $\big|\operatorname{supp}\big(\nu_k\big)\big| = M-1$ and $\big\|\nu_k\big\| = 1$ for every $k\in\omega$. We need to show that $\big\langle\nu_k\colon\ k\in\omega\big\rangle$ is weakly* null. Let $f\in C(K)$. For every $k\in\omega$ we have:

$$\big|\nu_k(f)\big| = \Big|\int_{K\setminus\{x_{n_k}\}} f\mathrm{d}\mu_{n_k}\Big|\Big/\big\|\mu_{n_k}\restriction\big(K\setminus\{x_{n_k}\}\big)\big\| \le$$

$$\Big(\big|\mu_{n_k}(f)\big| + \big|f\big(x_{n_k}\big)\cdot\mu_{n_k}\big(\{x_{n_k}\}\big)\big|\Big)\Big/\big\|\mu_{n_k}\restriction\big(K\setminus\{x_{n_k}\}\big)\big\| <$$

$$\Big(\big|\mu_{n_k}(f)\big| + \|f\|_\infty\cdot 1/(k+1)\Big)\Big/\big(1-1/(k+1)\big),$$

so, since $\lim_{k\to\infty}\mu_{n_k}(f)=0$, $\lim_{k\to\infty}\nu_k(f)=0$, too. This proves that $\big\langle\nu_k\colon\ k\in\omega\big\rangle$ is weakly* null. □

Note that a compact space K admits an fsJN-sequence $\big\langle\mu_n\colon\ n\in\omega\big\rangle$ of the form $\mu_n=\frac{1}{2}\big(\delta_{x_n}-\delta_{y_n}\big)$, where $x_n,y_n\in K$, if and only if there exist two disjoint sequences $\big\langle x_n\in K\colon\ n\in\omega\big\rangle$ and $\big\langle y_n\in K\colon\ n\in\omega\big\rangle$ such that for every $f\in C(K)$ and $\varepsilon>0$ there exists $N\in\omega$ such that for every $n>N$ we have $\big|f\big(x_n\big)-f\big(y_n\big)\big|<\varepsilon$. If K is totally disconnected, this observation boils down to the following one: K admits an fsJN-sequence $\big\langle\mu_n\colon\ n\in\omega\big\rangle$ of the form $\mu_n=\frac{1}{2}\big(\delta_{x_n}-\delta_{y_n}\big)$, where $x_n,y_n\in K$, if and only if there exist two disjoint sequences $\big\langle x_n\in K\colon\ n\in\omega\big\rangle$ and $\big\langle y_n\in K\colon\ n\in\omega\big\rangle$ such that for every clopen set U there is $N\in\omega$ such that for every $n>N$ either $x_n,y_n\in U$ or $x_n,y_n\in U^c$. These two observations, after appropriate generalizations, are crucial for proving Theorem 5.13—see the next two lemmas, where $\mathcal{K}(K)$ denotes the space of all non-empty closed subsets of a compact space K endowed with the Vietoris topology.

Lemma 5.10. *Let a compact space K have the fsJNP and assume that $M\in\omega$ is the minimal natural number for which there exists an fsJN-sequence $\big\langle\mu_n\colon\ n\in\omega\big\rangle$ such that $\big|\operatorname{supp}\big(\mu_n\big)\big|=M$ for every $n\in\omega$. For every $n\in\omega$ put $F_n=\operatorname{supp}\big(\mu_n\big)$. Then, the set $\mathcal{F}=\big\{F_n\colon\ n\in\omega\big\}$ has the following two properties as a subset of the space $\mathcal{K}(K)$:*

(1) every limit point of $\mathcal{F}$ is a singleton;
(2) $\mathcal{F}$ is not closed.

Proof. (1) By Lemma 5.9 and the minimality of M, we may assume that there exists $\varepsilon>0$ such that for every $n\in\omega$ and $x\in F_n$ we have $\big|\mu_n(\{x\})\big|>\varepsilon$. By Proposition 5.6, $M>1$. Let $F\in\mathcal{K}(K)$ be a limit point of $\mathcal{F}$. We claim that $|F|=1$. To see this, let us suppose that $|F|>1$, so there exist distinct $x_0,x_1\in F$. Let U_0, and U_1 be two open subsets of K such that $x_0\in U_0$, $x_1\in U_1$ and $\overline{U_0}\cap\overline{U_1}=\emptyset$. Put:

$$I=\big\{n\in\omega\colon\ F_n\cap U_0\neq\emptyset,\ F_n\cap U_1\neq\emptyset\big\}.$$

Since F is a limit point of $\big\{F_n\colon\ n\in\omega\big\}$, I is infinite. Let $g\in C(K)$ be a continuous function such that $0\le g\le 1$, $g\restriction\overline{U_0}\equiv 1$ and $g\restriction\overline{U_1}\equiv 0$. For every $n\in I$ define the measure θ_n as follows:

$$\theta_n = g\mathrm{d}\mu_n\Big/\big\|g\mathrm{d}\mu_n\big\|.$$

Then, $\langle\theta_n\colon\ n \in I\rangle$ is an fsJN-sequence. Indeed, for each $n \in I$ we have $\|\theta_n\| = 1$ and since $F_n \cap U_0 = \operatorname{supp}(\mu_n) \cap U_0 \neq \emptyset$, it follows that

$$\|g\mathrm{d}\mu_n\| \geq \|(g\mathrm{d}\mu_n) \restriction U_0\| = \|\mu_n \restriction U_0\| > \varepsilon,$$

so if $f \in C(K)$, then for every $n \in I$ we have $\theta_n(f) = \mu_n(f \cdot g)/\|g\mathrm{d}\mu_n\|$ and

$$|\theta_n(f)| = |\mu_n(f \cdot g)| \Big/ \|g\mathrm{d}\mu_n\| < |\mu_n(f \cdot g)|/\varepsilon.$$

Since

$$\lim_{\substack{n\to\infty\\ n\in I}} \mu_n(f \cdot g) = 0,$$

it follows that

$$\lim_{\substack{n\to\infty\\ n\in I}} \theta_n(f) = 0.$$

This proves that $\langle\theta_n\colon\ n \in I\rangle$ is weakly* null and hence an fsJN-sequence. Since $g \restriction \overline{U_1} \equiv 0$ and for each $n \in I$ it holds that $\operatorname{supp}(\theta_n) \subseteq \operatorname{supp}(\mu_n)$, it follows that $\operatorname{supp}(\theta_n) \subsetneq \operatorname{supp}(\mu_n)$, so $|\operatorname{supp}(\theta_n)| < M$, which is a contradiction with the assumption that M is minimal. This proves that F is a singleton.

(2) By (1), each limit point of $\mathcal{F}$ is a singleton, so since, by Proposition 5.6, none of the elements of $\mathcal{F}$ is a singleton, $\mathcal{F}$ cannot be closed. □

Lemma 5.11. *Assume that a compact space K admits an fsJN-sequence $\langle\mu_n\colon\ n \in \omega\rangle$ such that there exists $M \in \omega$ for which we have $|\operatorname{supp}(\mu_n)| = M$ for every $n \in \omega$. Then, there exists an fsJN-sequence $\langle\nu_n\colon\ n \in \omega\rangle$ such that $\nu_n = \frac{1}{2}(\delta_{x_n} - \delta_{y_n})$ for every $n \in \omega$, where $x_n, y_n \in K$.*

Proof. Let M be minimal such that there exists an fsJN-sequence $\langle\mu_n\colon\ n \in \omega\rangle$ on K for which $|\operatorname{supp}(\mu_n)| = M$ for every $n \in \omega$. By Proposition 5.6, $M > 1$. We shall show that $M = 2$.

By Lemma 5.9 and the minimality of M, we may assume that there is $\varepsilon > 0$ such that for every $n \in \omega$ and $x \in \operatorname{supp}(\mu_n)$ it holds $|\mu_n(\{x\})| > \varepsilon$. For every $n \in \omega$ put $F_n = \operatorname{supp}(\mu_n)$; then, $|F_n| = M$. Let $\mathcal{F} = \{F_n\colon\ n \in \omega\}$; by Lemma 5.10 every limit point of $\mathcal{F}$ in the Vietoris topology of $\mathcal{K}(K)$ is a singleton.

For every $n \in \omega$ choose $x_n \neq y_n \in F_n$ and define the measure ν_n as $\nu_n = \frac{1}{2}(\delta_{x_n} - \delta_{y_n})$. We claim that the sequence $\langle\nu_n\colon\ n \in \omega\rangle$ is weakly* null and hence an fsJN-sequence. To see this, assume that there exists $f \in C(K)$ and $\eta > 0$ such that the set

$$J = \left\{n \in \omega\colon\ \frac{1}{2}\left|f(x_n) - f(y_n)\right| > \eta\right\}$$

is infinite. Let $z \in K$ be such that $\{z\}$ is a limit point of the set $\{F_n\colon\ n \in J\}$ in $\mathcal{K}(K)$. Let U be a neighborhood of z such that for every $x, y \in U$ we have $|f(x) - f(y)| < 2\eta$. Since $\{z\}$ is a limit point of $\{F_n\colon n \in J\}$, there is $n \in J$ such that $F_n \subseteq U$, and hence $x_n, y_n \in U$, which is a contradiction, as $|f(x_n) - f(y_n)| > 2\eta$. □

Remark 5.12. Let us note that if K is totally disconnected, then we can prove Lemma 5.11 without appealing to Lemma 5.10. Indeed, let $\langle\mu_n\colon n\in\omega\rangle$ and M be as in Lemma 5.11. By Lemma 5.7, we may assume that there exist non-zero $\alpha_1,\ldots,\alpha_M\in[-1,1]$ such that for every $n\in\omega$ the measure μ_n is of the form $\mu_n=\sum_{i=1}^M\alpha_i\delta_{x_i^n}$ for some $x_1^n,\ldots,x_M^n\in K$. Note that for every clopen set $U\subseteq K$ the sequences $\langle\mu_n\restriction U\colon n\in\omega\rangle$ and $\langle\mu_n\restriction U^c\colon n\in\omega\rangle$ are weakly* null, so it follows that for sufficiently large $n\in\omega$ either $x_1^n,\ldots,x_M^n\in U$, or $x_1^n,\ldots,x_M^n\in U^c$—otherwise, we would get a contradiction with the minimality of M. Now, the formula $\nu_n=\frac{1}{2}\big(\delta_{x_1^n}-\delta_{x_2^n}\big)$ defines a fsJN-sequence on K, with the property that $\big|\operatorname{supp}\big(\nu_n\big)\big|=2$ for every $n\in\omega$. Since M is minimal, it follows that $M=2$.

We now immediately obtain the main theorem of this section.

Theorem 5.13. *Let K be a compact space with the fsJNP. Then, either there is an fsJN-sequence $\langle\mu_n\colon n\in\omega\rangle$ of measures on K such that for every $n\in\omega$ the measure μ_n is of the form $\mu_n=\frac{1}{2}\big(\delta_{x_n}-\delta_{y_n}\big)$, where $x_n,y_n\in K$, or every fsJN-sequence $\langle\mu_n\colon n\in\omega\rangle$ satisfies the equality $\lim_{n\to\infty}\big|\operatorname{supp}\big(\mu_n\big)\big|=\infty$.*

Proof. Assume that there exists an fsJN-sequence $\langle\mu_n\colon n\in\omega\rangle$ and an integer $M>1$ such that $\big|\operatorname{supp}\big(\mu_n\big)\big|=M$ for every $n\in\omega$, and apply Lemma 5.11. □

An immediate corollary to Theorem 5.13 is the following general form of the Josefson–Nissenzweig theorem for $C(K)$-spaces.

Corollary 5.14. *Let K be an infinite compact space. Then, either there is a JN-sequence $\langle\mu_n\colon n\in\omega\rangle$ on K such that $\big|\operatorname{supp}\big(\mu_n\big)\big|=2$ for every $n\in\omega$, or $\lim_{n\to\infty}\big|\operatorname{supp}\big(\mu_n\big)\big|=\infty$ for every JN-sequences on K.* □

Note that if a compact space K admits an fsJN-sequence $\langle\mu_n\colon n\in\omega\rangle$ such that $\big|\operatorname{supp}\big(\mu_n\big)\big|=2$ for every $n\in\omega$, then by Lemma 4.15 there exists a disjointly supported fsJN-sequence on K having the same property.

Part II. The Grothendieck property of $C(K)$-spaces

6. The ℓ_1-Grothendieck property and the fsJNP

Recall that a compact space K has *the Grothendieck property* if and only if every weakly* convergent sequence of measures on K is weakly convergent, or, in other words, the Banach space $C(K)$ is a Grothendieck space. We also say that a Boolean algebra $\mathcal{A}$ has *the Grothendieck property* if its Stone space $St(\mathcal{A})$ has the Grothendieck property.

It appears that the property is closely related to the finitely supported Josefson–Nissenzweig property of compact spaces. Namely, its variant, the ℓ_1-Grothendieck property, is equivalent to the negation of the fsJNP—in this section we shall show different approaches to this fact, starting with the issue of complementability of the Banach space c_0 in $C(K)$. Recall that Schachermayer [72, Proposition 5.3] and Cembranos [20, Corollary 2] proved that a compact space K has the Grothendieck property if and only if $C(K)$ does not contain any complemented copy of c_0.

Proposition 6.1. *Let K be a compact space such that in $C_p(K)$ there is a complemented closed subspace E isomorphic to $(c_0)_p$. Then, E with the norm topology of $C(K)$ is complemented in $C(K)$ and isomorphic to the Banach space c_0.*

Proof. Let F be a closed subspace of $C_p(K)$ such that $C_p(K) = E \oplus F$. Then, since the norm topology of $C(K)$ is finer than the product topology of $C_p(K)$, the spaces $(E, \|\cdot\|)$ and $(F, \|\cdot\|)$ (i.e. endowed with the inherited norm topology of $C(K)$) are still closed in $C(K)$ and $C(K) = E \oplus F$. It is enough now to show that $(E, \|\cdot\|)$ is isomorphic to the Banach space c_0. Since (E, τ_p) (i.e. with the inherited product topology of $C_p(K)$) is isomorphic to $(c_0)_p$, there is a topology τ on E stronger than τ_p and such that (E, τ) is isomorphic to c_0. The identity operator $T\colon (E, \|\cdot\|) \to (E, \tau)$ has the closed graph, so it is continuous, and hence τ is a Banach space topology on E smaller than the norm topology of E. On the other hand, the identity operator $S\colon (E, \tau) \to (E, \|\cdot\|)$ has the closed graph, too, so it is also continuous, and hence the topology τ on E is greater than the norm topology of E. It follows that the both topologies are equal, and hence $(E, \|\cdot\|)$ is isomorphic to the Banach space c_0. □

Corollary 6.2. *Let K be an infinite compact space. If $C_p(K)$ contains a complemented copy of $(c_0)_p$, then $C(K)$ contains a complemented copy of c_0.* □

By the result of Banakh, Kąkol and Śliwa [7, Theorem 1], mentioned in Introduction, stating that the fsJNP of a space X is equivalent to the complementability of $(c_0)_p$ in $C_p(X)$, it follows that the Grothendieck property of a compact space K implies the lack of the fsJNP of K. Below, we provide an alternative proof of this fact (see Corollary 6.6) and strengthen it in Theorem 6.7.

Definition 6.3. A compact space K has *the ℓ_1-Grothendieck property* (resp. *the Δ-Grothendieck property*) if and only if every weakly* convergent sequence of measures $\langle \mu_n \in \ell_1(K)\colon\ n \in \omega\rangle$ (resp. $\langle \mu_n \in \Delta(K)\colon\ n \in \omega\rangle$) is weakly convergent.

A Boolean algebra $\mathcal{A}$ has *the ℓ_1-Grothendieck property* (resp. *the Δ-Grothendieck property*) if its Stone space $St(\mathcal{A})$ has the property.

Proposition 6.4. *The Δ-Grothendieck property and ℓ_1-Grothendieck property are equivalent.*

Proof. Let K be a compact space. As $\Delta(K) \subseteq \ell_1(K)$, the ℓ_1-Grothendieck property implies immediately the Δ-property. Assume now that K has the $\Delta(K)$-Grothendieck property and let $\langle \mu_n \in \ell_1(K)\colon\ n \in \omega\rangle$ be weakly* convergent. For each $n \in \omega$ find a finite set $F_n \subseteq K$ such that $\|\mu_n \restriction (K \setminus F_n)\| < 1/n$. For every $x^{**} \in C(K)^{**}$ we have:

$$|x^{**}(\mu_n)| \le |x^{**}(\mu_n \restriction F_n)| + |x^{**}(\mu_n \restriction (K \setminus F_n))| \le |x^{**}(\mu_n \restriction F_n)| + \|x^{**}\| \cdot 1/n,$$

so $\lim_{n\to\infty} |x^{**}(\mu_n)| = 0$, since $\lim_{n\to\infty} |x^{**}(\mu_n \restriction F_n)| = 0$. This proves that K has the ℓ_1-Grothendieck property. □

Proposition 6.5. *Let K be a compact space with the Grothendieck property. Assume that $\langle \mu_n\colon\ n \in \omega\rangle$ is a JN-sequence on K. For each $n \in \omega$ write $\mu_n = \nu_n + \theta_n$, where $\nu_n \in \ell_1(K)$ and θ_n is non-atomic. Then, $\|\nu_n\| \to 0$, or equivalently $\|\theta_n\| \to 1$, as $n \to \infty$.*

Proof. By the Grothendieck property, $\langle\mu_n\colon\ n\in\omega\rangle$ is weakly null, so $\lim_{n\to\infty}\mu_n(B)=0$ for every Borel set $B\subseteq K$. Let $L=\bigcup_{n\in\omega}\operatorname{supp}(\nu_n)$. As L is countable, $\lim_{n\to\infty}\nu_n(B)=\lim_{n\to\infty}\mu_n(B)=0$ for every $B\in\wp(L)$, so $\langle\nu_n\colon\ n\in\omega\rangle$ is weakly null. It follows that the sequence $\langle\nu_n\colon\ n\in\omega\rangle$ is also weakly null as a sequence of elements of the space $\ell_1(L)$. Since the space $\ell_1(L)$ has the Schur property, it follows that $\lim_{n\to\infty}\left\|\nu_n\right\|_{C(K)^*}=\lim_{n\to\infty}\left\|\nu_n\right\|_{\ell_1(L)}=0$. □

Since in every fsJN-sequence every measure has norm 1 and belongs to $\ell_1(K)$, we immediately get the following corollary.

Corollary 6.6. *If a compact space K has the Grothendieck property, then it does not have the fsJNP.* □

Corollary 6.6 can be generalized to the following characterization of the finitely supported Josefson–Nissenzweig property.

Theorem 6.7. *Let K be a compact space. Then, the following are equivalent:*

(1) K has the ℓ_1-Grothendieck property,
(2) K has the Δ-Grothendieck property,
(3) K does not have the fsJNP,
(4) K does not have the csJNP.

Proof. The equivalences (1)⇔(2) and (3)⇔(4) were proved in Propositions 6.4 and 4.1, respectively. We now show the equivalence (2)⇔(3). To show the implication (2)⇒(3) we may easily adopt the proof of Proposition 6.5, but we proceed differently. Namely, for the sake of contradiction, let us assume that K has the fsJNP. By Theorem 4.22, there exists a disjointly supported fsJN-sequence $\langle\mu_n\colon\ n\in\omega\rangle$. Let for each $n\in\omega$ the set P_n be as in Lemma 4.2. Put $P=\bigcup_{n\in\omega}P_n$. Since K has the Δ-Grothendieck property, the sequence $\langle\mu_n\colon\ n\in\omega\rangle$ is weakly null and hence it is convergent to 0 on every Borel subset of K, in particular on P. But this contradicts Lemma 4.2.

To show (3)⇒(2), let us assume that there exists weakly* null sequence $\langle\mu_n\in\Delta(K)\colon\ n\in\omega\rangle$ which is not weakly null. Since the weak topology is weaker than the norm topology, it follows that there exists a subsequence $\langle\mu_{n_k}\colon\ k\in\omega\rangle$ and $\varepsilon>0$ such that $\left\|\mu_{n_k}\right\|>\varepsilon$ for every $k\in\omega$. But then, by Lemma 4.16, the sequence $\langle\mu_{n_k}/\left\|\mu_{n_k}\right\|\colon\ k\in\omega\rangle$ is an fsJN-sequence on K, a contradiction. □

7. The ℓ_1-Grothendieck property vs. the Grothendieck property

In his unpublished note [68] Plebanek constructed in ZFC a compact space K such that its every separable closed subspace L has the Grothendieck property, but K itself does not have the property (cf. Bielas [10]). It follows that K is not separable, but it has the ℓ_1-Grothendieck property.

Following the ideas of [68] and Plebanek's suggestions provided in the private communication, we will construct in this section a separable compact space—in fact, a continuous image of $\beta\omega$—without the Grothendieck property, but with the ℓ_1-Grothendieck property.

Lemma 7.1. *Let K be a totally disconnected compact space and μ a probability measure on K. Let $\langle A_n\colon\ n\in\omega\rangle$ be a sequence of clopen mutually disjoint subsets of K such that $\mu\big(A_n\big)>0$ for every $n\in\omega$. Define the set F as follows: $x\in F$ if and only if for every clopen neighborhood U of x the following inequality is satisfied:*

$$\limsup_{n\to\infty}\frac{\mu\big(A_n\cap U\big)}{\mu\big(A_n\big)}>0.$$

Then, F is closed and non-empty, and the quotient space K/F does not have the Grothendieck property.

Proof. We first show that $F\neq\emptyset$. Assume for the sake of contradiction that for every $x\in K$ there exists its clopen neighborhood U_x such that $\lim_n\mu\big(A_n\cap U_x\big)/\mu\big(A_n\big)=0$. By compactness of K, there exists a finite cover $U_{x_1},\ldots,U_{x_k}$ of K. We then have:

$$1=\lim_{n\to\infty}\frac{\mu\big(A_n\cap K\big)}{\mu\big(A_n\big)}\leq\sum_{i=1}^{k}\lim_{n\to\infty}\frac{\mu\big(A_n\cap U_{x_i}\big)}{\mu\big(A_n\big)}=0,$$

a contradiction.

Let us now prove that K/F does not have the Grothendieck property. Let $\varphi\colon K\to K/F$ be the quotient map. Denote $p=\varphi[F]$. For every $n\in\omega$ define a measure μ_n on K/F as follows:

$$\mu_n(A)=\frac{\mu\big(A_n\cap\varphi^{-1}[A]\big)}{\mu\big(A_n\big)},$$

where A is a clopen subset of K/F. Then, μ_n converges weakly* to δ_p on K/F. Indeed, if A is a clopen in K/F not containing p, then $\varphi^{-1}[A]\cap F=\emptyset$ and hence, by compactness of $\varphi^{-1}[A]$, we have $\limsup_n\mu\big(A_n\cap\varphi^{-1}[A]\big)/\mu\big(A_n\big)=0$, and so $\lim_n\mu_n(A)=0$. On the other hand, if $p\in A$, then:

$$\lim_{n\to\infty}\mu_n(A)=\lim_{n\to\infty}\Big(\mu_n(K/F)-\mu_n(A^c)\Big)=1-\lim_{n\to\infty}\mu_n(A^c)=1-0=1.$$

Had K/F the Grothendieck property, μ_n would converge weakly to δ_p and hence $\mu_n(\{p\})$ would converge to 1, which is not the case, since $A_n\cap F=\emptyset$ for every $n\in\omega$ as elements of $\langle A_n\colon\ n\in\omega\rangle$ are mutually disjoint. □

Lemma 7.2. *Let K be an extremely disconnected compact space. Let μ, $\langle A_n\colon\ n\in\omega\rangle$ and F be like in Lemma 7.1. Let $\mathcal{Z}$ denote the family of all clopen subsets C of K such that $\lim_{n\to\infty}\frac{\mu\left(A_n\cap C\right)}{\mu\left(A_n\right)}=0$, i.e.,*

$$\mathcal{Z}=\big\{C\subset K\ :\ K\text{ is clopen and }A\cap F=\emptyset\big\}.$$

Then $\mathcal{Z}$ has the following pseudo-intersection-like property: for every sequence $\langle C_n\colon\ n\in\omega\rangle$ of elements in $\mathcal{Z}$ there exists $C\in\mathcal{Z}$ such that

$$\forall n\in\omega\ \exists m\in\omega\ (C_n\setminus C\subset\bigcup_{j\leq m}A_j).$$

Proof. We shall show only the latter property, since the remaining two are obvious. The proof is now similar to the standard one showing that the density ideal on ω is a P-ideal. Namely, inductively find a strictly increasing sequence $\langle n_k\colon\ k \in \omega\rangle$ of indices such that

$$\frac{\mu\Big(A_n \cap \bigcup_{i=0}^k C_i\Big)}{\mu\big(A_n\big)} < \frac{1}{k+1}$$

for every $n > n_k$ and $k \in \omega$. Put:

$$C = \overline{\bigcup_{k\in\omega}\Big(C_k \setminus \bigcup_{j=0}^{n_k} A_j\Big)}.$$

Since K is extremely disconnected, C is a clopen set. It follows easily that for every $k \in \omega$ we have:

$$C_k \setminus C \subseteq \bigcup_{j=0}^{n_k} A_j.$$

We shall now show that $C \in \mathcal{Z}$. Fix $n \in \omega$, $n > n_0$, and let $k \in \omega$ be such that $n_k < n \leq n_{k+1}$. We have:

$$\frac{\mu\big(A_n \cap C\big)}{\mu\big(A_n\big)} \leq \frac{\mu\Big(A_n \cap \bigcup_{i=0}^k C_i\Big)}{\mu\big(A_n\big)} + \frac{\mu\big(A_n \cap D\big)}{\mu\big(A_n\big)} < \frac{1}{k+1} + \frac{\mu\big(A_n \cap D\big)}{\mu\big(A_n\big)},$$

where:

$$D = C \setminus \Big(\bigcup_{i\in\omega}\big(C_i \setminus \bigcup_{j=0}^{n_i} A_j\big)\Big),$$

so it is just the set added to $\bigcup_{i\in\omega}\big(C_i \setminus \bigcup_{j=0}^{n_i} A_j\big)$ after taking the closure. Since $k \to \infty$ if $n \to \infty$, it is enough to show that $A_n \cap D = \emptyset$, whence $\mu\big(A_n \cap D\big)/\mu\big(A_n\big) = 0$. Assume to the contrary that there exists $x \in A_n \cap D$. Since $x \in D$, every neighborhood of x intersects $C_k \setminus \bigcup_{j=0}^{n_k} A_j$ for infinitely many k. In particular, A_n must intersect $C_k \setminus \bigcup_{j=0}^{n_k} A_j$ for some k with $n_k > n$, which is impossible. □

Theorem 7.3. *For every extremely disconnected compact space K there exists a compact space L and a continuous surjection $\varphi\colon K \to L$ such that L does not have the Grothendieck property but it has the ℓ_1-Grothendieck property.*

Proof. Let μ, $\langle A_n\colon\ n \in \omega\rangle$, F and $\mathcal{Z}$ be like in Lemmas 7.1 and 7.2. Note that since K is not scattered we may assume that μ vanishes on points (see [74, Theorem 19.7.6]). Put $L = K/F$ and let φ be the quotient map. It follows from Lemma 7.1 that L does not have Grothendieck property. For the sake of contradiction assume that L does not have the ℓ_1-Grothendieck property either, so there is a disjointly supported JN-sequence $\langle \mu_n\colon\ n \in \omega\rangle$ on L. We may assume that $\varphi[F] \cap \operatorname{supp}\big(\mu_n\big) = \emptyset$ for every $n \in \omega$, and hence for every $n \in \omega$ we can find $C_n \in \mathcal{Z}$ such that $\operatorname{supp}\big(\mu_n\big) \subseteq \varphi\big[C_n\big]$. Let $C \in \mathcal{Z}$ be like in Lemma 7.2 for the sequence $\langle C_n\colon\ n \in \omega\rangle$, i.e., $C_n \setminus C \subset \bigcup_{j=0}^{m_n} A_j$ for some increasing number sequence

$\langle m_n\colon\ n\in\omega\rangle$. By Proposition 6.5, the Grothendieck property of K (and hence of $\varphi[C]$) yields

$$\lim_{n\to\infty}\big\|\mu_n\restriction\varphi[C]\big\|=0,$$

which together with $\|\mu_n\restriction\varphi[C_n]\|=1$ and $C_n\setminus C\subset\bigcup_{j=0}^{m_n}A_j$ gives

$$\lim_{n\to\infty}\big\|\mu_n\restriction\bigcup_{j=0}^{m_n}\varphi\big[A_j\big]\big\|=1.$$

On the other hand, since for every $Q\in[\omega]^{<\omega}$ we have:

$$\lim_{n\to\infty}\big\|\mu_n\restriction\bigcup_{j\in Q}\varphi\big[A_j\big]\big\|=0,$$

it follows that there exists a subsequence $\langle\mu_{n_k}\colon\ k\in\omega\rangle$ and a sequence $\langle Q_k\in[\omega]^{<\omega}\colon\ k\in\omega\rangle$ of pairwise disjoint sets such that for every $k\in\omega$ we have:

$$\big\|\mu_{n_k}\restriction\bigcup_{j\in Q_k}\varphi\big[A_j\big]\big\|>1/2.$$

Let B_k be a clopen subset of $\bigcup_{j\in Q_k}A_j$ containing $\operatorname{supp}\mu_{n_k}\cap\bigcup_{j\in Q_k}A_j$ and such that $\frac{\mu(B_k\cap A_j)}{\mu(A_j)}<1/2^k$ for all $j\in Q_k$ (this is the only place where we use that μ vanishes on points). Set $D=\overline{\bigcup_{k\in\omega}B_k}$ and note that D is a clopen subset of K such that $D\cap A_j=B_k\cap A_j$ for all $k\in\omega$ and $j\in Q_k$, and $D\cap A_j=\emptyset$ for $j\in\omega\setminus\bigcup_{k\in\omega}Q_k$. Thus

$$\lim_{j\to\infty}\frac{\mu(A_j\cap D)}{\mu(A_j)}=0,$$

which means $D\in\mathcal{Z}$, i.e., $D\cap F=\emptyset$. It follows from the above that

$$\big\|\mu_{n_k}\restriction\varphi[D]\big\|=\big\|\mu_{n_k}\restriction\bigcup_{j\in Q_k}\varphi\big[A_j\big]\big\|>1/2$$

for every $k\in\omega$, which is a contradiction, since K (and hence $\varphi[D]$) has the Grothendieck property. □

Considering $K=\beta\omega$ we obtain the following important corollary.

Corollary 7.4. *There exists a separable compact space L such that it does not have the Grothendieck property but it has the ℓ_1-Grothendieck property.* □

By the Stone duality we also obtain the following corollary saying that every complete Boolean algebra may be slimmed down in such a way that it loses the Grothendieck property but preserves the ℓ_1-Grothendieck property.

Corollary 7.5. *For every σ-complete Boolean algebra $\mathcal{A}$ there exists a subalgebra $\mathcal{B}\subseteq\mathcal{A}$ such that $\mathcal{B}$ does not have the Grothendieck property but it has the ℓ_1-Grothendieck property.*
□

Analyzing the proof of Theorem 7.3, it seems that the theorem might also hold for those totally disconnected compact spaces which are the Stone spaces of Boolean algebras with e.g. Haydon's Subsequential Completeness Property (see Haydon [43]). We do not know however how far the σ-completeness of $\mathcal{A}$ can be weakened in Corollary 7.5, which motivates the following question.

Question 7.6. *Let $\mathcal{A}$ be a Boolean algebra with the Grothendieck property. Does there exist a Boolean subalgebra $\mathcal{B}$ of $\mathcal{A}$ which fails to have the Grothendieck property but which nonetheless has the ℓ_1-Grothendieck property?*

8. The Nikodym property

A property closely related to the Grothendieck property is the Nikodym property defined for Boolean algebras or, equivalently, for totally disconnected compact spaces as follows.

Definition 8.1. A Boolean algebra $\mathcal{A}$ has *the Nikodym property* if every pointwise convergent sequence of measures on $\mathcal{A}$ is also weakly* convergent.

Equivalently, a Boolean algebra $\mathcal{A}$ has the Nikodym property if every pointwise bounded sequence of measures on $\mathcal{A}$ is also uniformly bounded, see Sobota and Zdomskyy [77, Proposition 2.4]. (Recall that a sequence $\langle \mu_n\colon\ n \in \omega \rangle$ of measures on a Boolean algebra $\mathcal{A}$ is *pointwise convergent* to a measure μ on $\mathcal{A}$ if $\lim_{n\to\infty} \mu_n(A) = \mu(A)$ for every $A \in \mathcal{A}$, *pointwise bounded* if $\sup_{n\in\omega} \left|\mu_n(A)\right| < \infty$ for every $A \in \mathcal{A}$, and *uniformly bounded* if $\sup_{n\in\omega} \left\|\mu_n\right\| < \infty$.)

Nikodym [65] (see also Dieudonné [24], Darst [21] or Rosenthal [71]) proved that every σ-complete Boolean algebra has the Nikodym property. Later on, many weakenings of σ-completeness of Boolean algebras also implying the property have been found, see e.g. Seever [75], Moltó [63], Freniche [37], Aizpuru [1] etc. The property has been also studied in the context of topological vector spaces, see e.g. Valdivia [84, 85] or Kąkol and López-Pellicer [46].

It seems hard to distinguish the Nikodym property from the Grothendieck property. So far, only a few examples of Boolean algebras having only one of the properties have been found. Schachermayer [72, Propositions 3.2 and 3.3] proved that the Jordan algebra $\mathcal{J}$ of Jordan measurable subsets of the interval $[0, 1]$ (i.e. such subsets A of $[0, 1]$ that have boundary of Lebesgue measure 0: $\lambda(\partial A) = 0$) has the Nikodym property, but lacks the Grothendieck property (see also Graves and Wheeler [38] for generalizations). Recently, an example of a minimally generated Boolean algebra with similar properties has been also obtained under the set-theoretic assumption of the Diamond Principle $\diamondsuit$ by Sobota and Zdomskyy [78]. On the other hand, Talagrand [81], assuming the Continuum Hypothesis (CH, in short), constructed a Boolean algebra with the Grothendieck property, but without the Nikodym property (no ZFC example is known). It seems thus natural to ask about the relation of the Nikodym property and the fsJNP, however it appears that both properties are independent of each other. Indeed, we have the following examples:

(1) σ-complete Boolean algebras have the Nikodym property, but they lack the fsJNP (since they have the Grothendieck property);

(2) Talagrand's example is (consistently) an example of a Boolean algebra without the fsJNP and without the Nikodym property;
(3) Schachermayer proved that $\mathcal{J}$ has the Nikodym property, but lacks the Grothendieck property—in fact, in [72, Proposition 3.2] he proves that $\mathcal{J}$ does not have the ℓ_1-Grothendieck property, so $\mathcal{J}$ has the fsJNP;
(4) Schachermayer's algebra $\mathcal{S}$ (see Example 5.3) has the fsJNP, but it does not have the Nikodym property, as well as its Stone space $St(\mathcal{S})$ does not contain non-trivial convergent sequences;
(5) Boolean algebras whose Stone spaces have non-trivial convergent sequences are examples of Boolean algebras with the fsJNP, but without the Nikodym property.

The following table summarizes the above points.

Example	**fsJNP**	**the Nikodym property**	**conv. sequences**
σ-complete Boolean algebras	no	yes	no
Talagrand's example (under CH)	no	no	no
Jordan algebra $\mathcal{J}$	yes	yes	no
Schachermayer's example $\mathcal{S}$	yes	no	no
$St(\mathcal{A})$ contains convergent sequences	yes	no	yes

However, as the next two propositions show, in some cases the lack of the Nikodym property implies the fsJNP and *vice versa*. The proofs are straightforward (in the second statement of Proposition 8.3 and Corollary 8.4 we need to appeal to Theorem 5.13).

Proposition 8.2. *Let $\mathcal{A}$ be a Boolean algebra such that there exists a sequence $\langle \mu_n\colon\ n \in \omega\rangle$ of finitely supported measures on $St(\mathcal{A})$ which is pointwise convergent on $\mathcal{A}$ but not uniformly bounded. Then, the sequence $\langle \mu_n/\|\mu_n\|\colon\ n \in \omega\rangle$ is an fsJN-sequence on $St(\mathcal{A})$, so $St(\mathcal{A})$ has the fsJNP.*

In other words, if there is a sequence of finitely supported measures on a Boolean algebra $\mathcal{A}$ witnessing the lack of the Nikodym property, then $\mathcal{A}$ does not have the ℓ_1-Grothendieck property, too. □

Proposition 8.3. *Let $\mathcal{A}$ be a Boolean algebra such that $St(\mathcal{A})$ admits an fsJN-sequence $\langle \mu_n\colon\ n \in \omega\rangle$ of the form $\mu_n = \frac{1}{2}(\delta_{x_n} - \delta_{y_n})$. Then, the sequence $\langle n\cdot\mu_n\colon\ n \in \omega\rangle$ is pointwise convergent but not uniformly bounded.*

In other words, if there is an fsJN-sequence on $St(\mathcal{A})$ with bounded sizes of supports, then $\mathcal{A}$ has neither the Grothendieck property, nor the Nikodym property. □

Corollary 8.4. *If a Boolean algebra $\mathcal{A}$ has the Nikodym property but not the ℓ_1-Grothendieck property, then for every fsJN-sequence $\langle \mu_n\colon\ n \in \omega\rangle$ on $St(\mathcal{A})$ we have $\lim_{n\to\infty}\left|\operatorname{supp}(\mu_n)\right| = \infty$.* □

In Section 5.1, we mentioned that the density Boolean algebra $\mathcal{D}$ and Schachermayer's algebra $\mathcal{S}$ have the following properties:

- their Stone spaces do not have any non-trivial convergent sequences;

- every infinite subset of their Stone spaces contains a subset Y such that $\overline{Y}$ is homeomorphic to $\beta\omega$;
- they have the fsJNP;
- every fsJN-sequence on $St(\mathcal{D})$ has supports with cardinalities convergent to ∞, while $St(\mathcal{S})$ has an fsJN-sequence with supports of size 2.

As said above, the Jordan algebra $\mathcal{J}$ has the Nikodym property, so its Stone space lacks any non-trivial convergent sequences, and it has the fsJNP. Schachermayer [72, Proposition 3.11] provided also a proof that if X is an infinite subset of $St(\mathcal{J})$, then there exists a subset Y of X such that $\overline{Y}^{St(\mathcal{J})}$ is homeomorphic to $\beta\omega$. Regarding sizes of supports of fsJN-sequences on $St(\mathcal{J})$, by Corollary 8.4, $\mathcal{J}$ must necessarily have the same property as $\mathcal{D}$.

Proposition 8.5. *Let $\langle \mu_n\colon\ n \in \omega\rangle$ be an fsJN-sequence on $St(\mathcal{J})$. Then, $\lim_{n\to\infty}\big|\operatorname{supp}(\mu_n)\big| = \infty$.* □

Part III. Examples of classes of spaces with the fsJNP

9. Systems of simple extensions and the fsJNP

In this section we will show, combining several already known results, that the limit of every inverse system of simple extensions of compact spaces has the fsJNP (Theorem 9.10). This yields a corollary that many consistent examples of Efimov spaces from the literature (e.g. [32], [28], [29]), constructed under such axioms as the Continuum Hypothesis or Martin's axiom, have the fsJNP as well. In Subsection 9.2 we will generalize this result.

9.1. **Systems of simple extensions.** We start this subsection with recalling what an inverse system of simple extensions is. For general information on limits of inverse systems, see Engelking [31, Chapters 2.5 and 3.2].

Definition 9.1. An inverse system $\langle K_\alpha, \pi^\beta_\alpha\colon \alpha \le \beta \le \delta\rangle$ of totally disconnected compact spaces is *a system of simple extensions* if

- it is *continuous*, i.e. for every limit ordinal $\gamma \le \delta$ the space K_γ is the limit of the inverse system $\langle K_\alpha, \pi^\beta_\alpha\colon \alpha \le \beta \le \gamma\rangle$,
- $K_0 = 2^\omega$ and each K_α is *perfect*, i.e. has no isolated points,
- for every $\alpha < \delta$ the space $K_{\alpha+1}$ is *a simple extension of* K_α, i.e. there is $x_\alpha \in K_\alpha$ such that $\Big|\big(\pi^{\alpha+1}_\alpha\big)^{-1}(x_\alpha)\Big| = 2$ and for every $y \in K_\alpha \setminus \{x_\alpha\}$ it holds $\Big|\big(\pi^{\alpha+1}_\alpha\big)^{-1}(y)\Big| = 1$.

To prove the main result of this section, Theorem 9.10, we also need to provide several definitions concerning complexity of probability measures on compact spaces.

Definition 9.2. *The Maharam type* of a probability measure μ on a compact space K is the minimal cardinality of a family $\mathcal{C}$ of Borel subsets of K such that for every Borel subset B of K and $\varepsilon > 0$ there exists $C \in \mathcal{C}$ such that $\mu(B \triangle C) < \varepsilon$.

Equivalently, the Maharam type of a probability measure μ is the density of the Banach space $L_1(\mu)$ of all μ-integrable functions. For more information on the topic, see Maharam [61], Fremlin [35], or Plebanek and Sobota [69].

A notion closely related to the countable Maharam type is the uniform regularity, introduced by Babiker [3] and later studied by Pol [70] and Mercourakis [62]; see also Krupski and Plebanek [54].

Definition 9.3. A probability measure μ on a compact space K is *uniformly regular* if there exists a countable family $\mathcal{C}$ of zero subsets of K such that for every open subset U of K and every $\varepsilon > 0$ there exists $F \in \mathcal{C}$ such that $F \subseteq U$ and $\mu(U \setminus F) < \varepsilon$.

Uniformly regular measures are also called *strongly countably determined* (cf. Pol [70]). Note that every uniformly regular probability measure μ has necessarily separable support and countable Maharam type.

It is an easy fact that every zero set in a normal space is a closed $\mathbb{G}_\delta$-set, thus the definition of uniformly regular measures may be stated in terms of closed $\mathbb{G}_\delta$-sets. Recall that a subset Y of a space K is *a* $\mathbb{G}_\delta$*-subset* if there exists a countable collection $\mathcal{U}$ of open subsets of K such that $Y = \bigcap \mathcal{U}$. An element $x \in X$ is called *a* $\mathbb{G}_\delta$*-point* if $\{x\}$ is a $\mathbb{G}_\delta$-subset of X.

Proposition 9.4. *Let μ be a uniformly regular measure on a compact space K and $x \in K$ be such that $\mu(\{x\}) > 0$. Then, x is a $\mathbb{G}_\delta$-point.*

Proof. Let $\mathcal{C}$ be a countable collection of zero sets (closed $\mathbb{G}_\delta$'s) witnessing that μ is uniformly regular. Put $\mathcal{C}' = \big\{F \in \mathcal{C}\colon\ x \in F\big\}$. It follows that $\{x\} = \bigcap \mathcal{C}'$. To see this, assume that there is $y \in \bigcap \mathcal{C}'$ such that $x \neq y$. Put $\varepsilon = \mu(\{x\})$; so $\varepsilon > 0$. Using the regularity of μ, it is easy to see that there is an open neighborhood U of x not containing y and such that $\mu(U \setminus \{x\}) < \varepsilon/3$. Note that $\varepsilon \leq \mu(U) < 4\varepsilon/3$. However, there is no $F \in \mathcal{C}$ such that $F \subseteq U$ and $\mu(U \setminus F) < \varepsilon/3$, since otherwise $x \in F$ and hence $y \in F \in \mathcal{C}'$ and $y \in U$, which is a contradiction. Since the intersection of a countable collection of $\mathbb{G}_\delta$-sets is $\mathbb{G}_\delta$, x is a $\mathbb{G}_\delta$-point. □

Remark 9.5. Note that from Proposition 9.4 it immediately follows that if a uniformly regular measure on a compact space K has an atom (i.e. it does not vanish on points), then K contains a non-trivial convergent sequence.

Definition 9.6. A probability measure μ on a compact space K *admits a uniformly distributed sequence* $\big\langle x_n \in K\colon\ n \in \omega\big\rangle$ if $\frac{1}{n}\sum_{i=0}^{n-1} \delta_{x_i}$ converges weakly* to μ. We then say that $\big\langle x_n\colon\ n \in \omega\big\rangle$ is *μ-uniformly distributed.*

Uniformly distributed sequences constitute a useful tool for investigating various properties of probability measures as they allow to treat those measure in a way similar to the classical Jordan measure on the real line, see e.g. the monograph of Kuipers and Niederreiter [55], Losert [59, 60], or Mercourakis [62].

Recall that a sequence $\big\langle x_n\colon\ n \in \omega\big\rangle$ in a space X is *injective* if $x_n \neq x_{n'}$ for every $n \neq n' \in \omega$. The following proposition will be crucial for the proof of the main theorem of this section.

Proposition 9.7. *If μ is a non-atomic uniformly regular measure on a compact space K, then μ admits a uniformly distributed injective sequence.*

Proof. Let μ^∞ be the product measure on the countable product space K^ω induced by μ. [62, Corollary 2.8] implies that $\mu^\infty(S) = 1$, where S denotes the subspace of K^ω consisting of all μ-uniformly distributed sequences in K. Since by the non-atomicity of μ the subspace T of K^ω consisting of all non-injective sequences in K satisfies $\mu(T) = 0$, it follows that there exists a μ-uniformly distributed sequence in S which is injective. □

Proposition 9.8. *Let μ be a probability measure on a compact space K and $\langle x_n\colon\ n \in \omega\rangle$ be a μ-uniformly distributed injective sequence. Then, K has the fsJNP.*

Proof. Let $\omega = \bigcup_{n\in\omega} P_n$ be a partition of ω into finite sets such that

$$\max P_n < \min P_{n+1} \quad \text{and} \quad \big|P_n\big| / \max P_n \geq 1/2$$

for every $n \in \omega$ (see the classical proof that $\sum_{n\in\omega} 1/n = \infty$). For every $n \in \omega$ let us write:

$$\nu_n = \frac{1}{\max P_{n+1}} \sum_{k\leq \max P_{n+1}} \delta_{x_k} - \frac{1}{\max P_n} \sum_{k\leq \max P_n} \delta_{x_k}.$$

Then, by the injectivity of $\langle x_n\colon\ n \in \omega\rangle$,

$$\big\|\nu_n\big\| \geq \frac{1}{\max P_{n+1}} \cdot \big|P_{n+1}\big| \geq 1/2.$$

Since either sum in the definition of ν_n's is weakly* convergent to μ, $\langle \nu_n\colon\ n \in \omega\rangle$ converges weakly* to 0. Normalizing $\mu_n = \nu_n / \big\|\nu_n\big\|$, $\langle \mu_n\colon\ n \in \omega\rangle$ is an fsJN-sequence on K. □

The following theorem was proved by Borodulin-Nadzieja [15] in the language of minimally generated Boolean algebras, the dual notion to the limits of inverse systems of simple extensions, cf. Koppelberg [51].

Theorem 9.9 (Borodulin-Nadzieja)**.** *The following assertions hold for every totally disconnected compact space K:*

(1) [15, Theorem 4.6] *K carries either a uniformly regular measure or a measure of uncountable Maharam type;*
(2) [15, Theorem 4.9] *If K is the limit of an inverse system of simple extensions, then every measure on K has countable Maharam type. In particular, there exists a uniformly regular measure on K.* □

Let us recall here that Džamonja and Plebanek [30, Lemma 4.1] proved that if an inverse system of simple extensions has length at most ω_1, then every measure on its limit is uniformly regular.

We are in the position to prove the main theorem of this section.

Theorem 9.10. *If K is the limit of an inverse system of simple extensions, then K has the fsJNP.*

Proof. By Theorem 9.9.(2) there exists a uniformly regular measure μ on K. If μ has an atom, then K contains a non-trivial convergent sequence by Proposition 9.4 and Remark 9.5, and hence K has trivially the fsJNP. If on the other hand μ is non-atomic, then Proposition

9.7 implies that μ admits a uniformly distributed injective sequence. Now, Proposition 9.8 yields an fsJN-sequence on K. □

The following corollary generalizes the well-known fact that no minimally generated Boolean algebra has the Grothendieck property.

Corollary 9.11. *If $\mathcal{A}$ is a minimally generated Boolean algebra, then $\mathcal{A}$ does not have the ℓ_1-Grothendieck property.* □

As a corollary to Theorem 9.10 we obtain also that many Efimov spaces constructed in the literature have the fsJNP.

Corollary 9.12. *If K is an Efimov space obtained as the limit of an inverse system of simple extensions, then K has the fsJNP. In particular, the examples of Efimov spaces by Fedorchuk (under $\diamondsuit$; see* [32]*), Dow and Pichardo-Mendoza (under CH; see* [28]*), or Dow and Shelah (under Martin's axiom; see* [29]*) have the fsJNP.* □

Let us note that consistently there exist Efimov spaces with the Grothendieck property and hence without the fsJNP, see e.g. Talagrand [79], Brech [19], or Sobota and Zdomskyy [77].

In the next section we will prove that some other classes of Efimov spaces do have the fsJNP, too.

9.2. **τ-simple extensions.** The aim of this section is to generalize Theorem 9.10 to a broader class of inverse systems of compact spaces (however of length at most $\mathfrak{c}$). We start with the following simple observations.

Lemma 9.13. *Let $\left\langle K_\alpha, \pi_\alpha^\beta\colon \alpha \le \beta \le \delta\right\rangle$ be an inverse system of simple extensions. For every $\alpha < \delta$ and every subset $X \subseteq K_{\alpha+1}$ we have $\partial\pi_\alpha^{\alpha+1}[X] \setminus \pi_\alpha^{\alpha+1}[\partial X] \subseteq \left\{x_\alpha\right\}$.*

Proof. Fix $\alpha < \delta$ and a subset $X \subseteq K_{\alpha+1}$. For the sake of contradiction, assume there is $x \in \partial\pi_\alpha^{\alpha+1}[X]\setminus\pi_\alpha^{\alpha+1}[\partial X]$ such that $x \ne x_\alpha$. Let V be a clopen subset of K_α such that $x_\alpha \in V$ but $x \notin V$. Put $X' = X \setminus \left(\pi_\alpha^{\alpha+1}\right)^{-1}[V]$. Since $\left(\pi_\alpha^{\alpha+1}\right)^{-1}[V]$ is closed, $x \in \partial\pi_\alpha^{\alpha+1}[X'] \setminus \pi_\alpha^{\alpha+1}[\partial X']$. But, as $K_{\alpha+1} \setminus \pi_\alpha^{\alpha+1}[V]$ is homeomorphic to $K_\alpha \setminus V$, we have $\partial\pi_\alpha^{\alpha+1}[X'] = \pi_\alpha^{\alpha+1}[\partial X']$, a contradiction. □

Lemma 9.14. *Let $\left\langle K_\alpha, \pi_\alpha^\beta\colon \alpha \le \beta \le \delta\right\rangle$ be an inverse system of simple extensions. Then, π_α^β is irreducible for any $\alpha < \beta \le \delta$.*

Proof. By the continuity of this inverse system, it is enough to prove that $\pi_\alpha^{\alpha+1}$ is irreducible for every $\alpha < \delta$. But this is fairly simple, and is actually proved in the first paragraph of the proof of Proposition 9.21—one just needs to consider the case when $\left|G_\alpha\right| = 1$. □

Let us also note that if $\left\langle K_\alpha, \pi_\alpha^\beta\colon \alpha \le \beta \le \delta\right\rangle$ is an inverse system of simple extensions, then $w\left(K_\alpha\right) = w\left(K_{\alpha+1}\right)$ for every $\alpha < \delta$. Motivated by these three observations, we introduce the following generalization of systems of simple extensions.

Definition 9.15. Let $\tau \le \mathfrak{c}$ be a cardinal number. An inverse system $\left\langle K_\alpha, \pi_\alpha^\beta\colon \alpha \le \beta \le \delta\right\rangle$ of totally disconnected compact spaces is *a system of τ-simple extensions* if

- it is continuous,
- $K_0 = 2^\omega$ and each K_α is perfect, i.e. has no isolated points,
- for every $\alpha < \delta$ the space $K_{\alpha+1}$ is *a τ-simple extension of* K_α, i.e. $\left|\partial\pi_\alpha^{\alpha+1}[U] \setminus \pi_\alpha^{\alpha+1}[\partial U]\right| \leq \tau$ for every closed subset $U \subseteq K_{\alpha+1}$,
- the map π_α^β is irreducible for every $\alpha < \beta \leq \delta$,
- $w\big(K_\alpha\big) = w\big(K_{\alpha+1}\big)$ for every $\alpha < \delta$.

Theorem 9.19 states that systems of τ-simple extensions of length at most $\mathfrak{c}$ have the fsJNP. In order to show this, we need first to prove several technical results.

Lemma 9.16. *Let K and L be two compact spaces and $f\colon K \to L$ a continuous surjection. Assume that for a clopen subset $U \subseteq K$ the interior $\big(f[U] \cap f[K \setminus U]\big)^\circ = \emptyset$. Then, $f[U] \cap f[K \setminus U] = \partial f[U] \cup \partial f[K \setminus U]$.*

Proof. We have:

$$\partial f[U] = \overline{f[U]} \cap \overline{L \setminus f[U]} = f[U] \cap \overline{L \setminus f[U]} \subseteq f[U] \cap \overline{f[K \setminus U]} = f[U] \cap f[K \setminus U],$$

where the only inclusion follows from the surjectivity of f and the last equality from the closedness of f. We show similarly that $\partial f[K \setminus U] \subseteq f[U] \cap f[K \setminus U]$, whence we get:

$$f[U] \cap f[K \setminus U] \subseteq \partial f[U] \cup \partial f[K \setminus U].$$

Let now $x \in f[U] \cap f[K \setminus U]$. By the assumption, for every open neighborhood V of x we have $V \not\subseteq f[U] \cap f[K \setminus U]$, so either $V \setminus f[U] \neq \emptyset$ or $V \setminus f[K \setminus U] \neq \emptyset$. If for every V we have $V \setminus f[U] \neq \emptyset$, then $x \in \partial f[U]$. So let us assume that there exists an open neighborhood V of x such that $V \setminus f[U] = \emptyset$, equivalently $V \subseteq \big(f[U]\big)^\circ$. It follows that $x \in \partial f[K \setminus U]$, since otherwise there is an open neighborhood W of x such that $W \setminus f[K \setminus U] = \emptyset$, so $W \subseteq \big(f[K \setminus U]\big)^\circ$, and hence:

$$x \in V \cap W \subseteq f[U]^\circ \cap f[K \setminus U]^\circ = \big(f[U] \cap f[K \setminus U]\big)^\circ,$$

a contradiction. We get thus:

$$\partial f[U] \cup \partial f[K \setminus U] \subseteq f[U] \cap f[K \setminus U].$$

□

Proposition 9.17, shows how fsJN-sequences may be recovered from the Cantor space via continuous surjections. Note that if for every $n \in \omega$, $i \in 2$ and $s \in 2^n$ we put $x_s^i = s^\frown(i)$, where (i) denotes the constant sequence of length ω all of whose members equal i, then the measures defined as

$$\mu_n = \frac{1}{2^{n+1}} \sum_{s \in 2^n} \big(\delta_{x_s^1} - \delta_{x_s^0}\big)$$

form an fsJN-sequence on the Cantor space 2^ω. Recall that λ denotes the standard product measure on 2^ω.

Proposition 9.17. *Let Y be a totally disconnected compact space and $f\colon Y\to 2^\omega$ a continuous surjection such that $\lambda\big(f[U]\cap f[Y\setminus U]\big)=0$ for every clopen $U\subseteq Y$. For every $n\in\omega$, $i\in 2$ and $s\in 2^n$ fix $y_s^i\in f^{-1}\big(x_s^i\big)$ and define the measure on Y as follows:*

$$\nu_n=\frac{1}{2^{n+1}}\sum_{s\in 2^n}\big(\delta_{y_s^1}-\delta_{y_s^0}\big).$$

Then, the sequence $\big\langle\nu_n\colon\ n\in\omega\big\rangle$ is an fsJN-sequence on Y.

Proof. We need only to show that $\big\langle\nu_n(U)\colon\ n\in\omega\big\rangle$ converges to 0 for every clopen $U\subseteq Y$. Fix $\varepsilon>0$. Since $\lambda\big(f[U]\cap f[Y\setminus U]\big)=0$, it follows that $f[U]\cap f[Y\setminus U]$ has empty interior in 2^ω. By Lemma 9.16, there is a clopen set $B\subseteq 2^\omega$ such that $\lambda(B)<\varepsilon$ and

$$\partial f[U]\cup\partial f[Y\setminus U]=f[U]\cap f[Y\setminus U]\subseteq B.$$

It follows that $f[U]\setminus B$ and $f[Y\setminus U]\setminus B$ are also clopen sets. Since

$$f^{-1}\big[f[U]\setminus B\big]\subseteq U$$

and

$$f^{-1}\big[f[Y\setminus U]\setminus B\big]\subseteq Y\setminus U,$$

we have that $y_s^i\in U$ if $x_s^i\in f[U]\setminus B$, and $y_s^i\in Y\setminus U$ if $x_s^i\in f[Y\setminus U]\setminus B$. Let $n_0\in\omega$ be such that $x_s^0\in f[U]\setminus B$ if and only if $x_s^1\in f[U]\setminus B$, for all $n\ge n_0$ and $s\in 2^n$. Then,

$$\big|\nu_n(U)\big|=\Big|\frac{1}{2^{n+1}}\sum_{s\in 2^n}\big(\delta_{y_s^1}(U)-\delta_{y_s^0}(U)\big)\Big|=$$

$$\frac{1}{2^{n+1}}\Big|\sum_{s\in 2^n}\big(\delta_{y_s^1}(U)-\delta_{y_s^0}(U)\big)-\sum_{s\in 2^n}\big(\delta_{x_s^1}(f[U]\setminus B)-\delta_{x_s^0}(f[U]\setminus B)\big)\Big|=$$

$$\frac{1}{2^{n+1}}\Big|\sum_{s\in 2^n}\big(\delta_{y_s^1}(U)-\delta_{x_s^1}(f[U]\setminus B)\big)-\sum_{s\in 2^n}\big(\delta_{y_s^0}(U)-\delta_{x_s^0}(f[U]\setminus B)\big)\Big|\le$$

$$\frac{1}{2^{n+1}}\Big(\big|\sum_{f(y_s^1)\in B}\delta_{y_s^1}(U)\big|+\big|\sum_{f(y_s^0)\in B}\delta_{y_s^0}(U)\big|\Big)=\frac{1}{2^{n+1}}\Big(\big|\sum_{x_s^1\in B}\delta_{y_s^1}(U)\big|+\big|\sum_{x_s^0\in B}\delta_{y_s^0}(U)\big|\Big)\le$$

$$\frac{1}{2^{n+1}}\big(|\{s\in 2^n:x_s^1\in B\}|+|\{s\in 2^n:x_s^0\in B\}|\big)$$

for all $n\ge n_0$. Let $n_1\in\omega$ be such that for every $n\ge n_1$ there exists $S_n\subseteq 2^n$ for which $B=\bigcup_{s\in S_n}[s]$. Then $\lambda(B)=\big|S_n\big|/2^n<\varepsilon$ for all $n\ge n_1$. Then, for all $n\ge\max\{n_0,n_1\}$ we have:

$$\big|\nu_n(U)\big|\le\frac{1}{2^{n+1}}\Big(\big|\big\{s\in 2^n\colon\ x_s^1\in B\big\}\big|+\big|\big\{s\in 2^n\colon\ x_s^0\in B\big\}\big|\Big)\le\frac{1}{2^{n+1}}\cdot 2\cdot\big|S_n\big|=\frac{\big|S_n\big|}{2^n}<\varepsilon,$$

which completes the proof. □

Lemma 9.18. *Fix three cardinal numbers $\delta,\kappa,\tau<\mathfrak{c}$, where κ is infinite. Assume that $\big\langle K_\alpha,\pi_\alpha^\beta\colon\alpha\le\beta\le\delta\big\rangle$ is an inverse system of τ-simple extensions. Then, for any $\alpha<\beta\le\delta$ and every closed set $U\subseteq X_\beta$ such that $|\partial U|\le\kappa$, we have $\big|\partial\pi_\alpha^\beta[U]\big|\le|\beta|\cdot\tau\cdot\kappa$.*

Proof. Let us first observe that the case $\beta = \alpha + 1$, where $\alpha < \delta$, follows immediately from Definition 9.15, thus we need only to prove the case where $\alpha + 1 < \beta$. Fix $\alpha < \delta$. The proof is by induction on $\beta > \alpha$. Let us thus fix also $\beta \leq \delta$ and assume that the thesis holds for every $\alpha < \xi < \beta$. We have two cases:

(1) $\beta = \xi + 1$ for some $\alpha < \xi < \delta$. Then, by the beginning remark, $\left|\partial\pi^\beta_\xi[U]\right| \leq |\beta| \cdot \tau \cdot \kappa$. By the inductive assumption used for an ordinal number ξ, a closed set $\pi^\beta_\xi[U]$, and the cardinal $|\beta| \cdot \tau \cdot \kappa$, we conclude that:

$$\left|\partial\pi^\beta_\alpha[U]\right| = \left|\partial\pi^\xi_\alpha\big[\pi^\beta_\xi[U]\big]\right| \leq |\xi| \cdot \tau \cdot |\beta| \cdot \tau \cdot \kappa = |\beta| \cdot \tau \cdot \kappa.$$

(2) β is limit. First note that $w\big(K_\beta\big) \leq |\beta| + \omega \leq |\beta| \cdot \kappa$, because the inverse system is based on τ-simple extensions. It follows that $\partial U = \bigcap_{\iota < |\beta|\cdot\kappa} A_\iota$ for some family $\big\{A_\iota\colon\ \iota < |\beta| \cdot \kappa\big\}$ of clopen subsets of K_β. Then,

$$\pi^\beta_\alpha[U \setminus \partial U] = \bigcup_{\iota<|\beta|\cdot\kappa} \pi^\beta_\alpha\big[U \setminus A_\iota\big].$$

We now claim that

$$(*) \qquad \partial\pi^\beta_\alpha[U] \subseteq \bigcup_{\iota<|\beta|\cdot\kappa} \partial\pi^\beta_\alpha\big[U \setminus A_\iota\big] \cup \pi^\beta_\alpha[\partial U].$$

To see this, fix $x \in \partial\pi^\beta_\alpha[U] \setminus \pi^\beta_\alpha[\partial U]$ and note that $\big(\pi^\beta_\alpha\big)^{-1}(x) \cap U \subseteq U^\circ$, and hence there exists $\iota < |\beta|\cdot\kappa$ such that $\big(\pi^\beta_\alpha\big)^{-1}(x) \cap U \subseteq U \setminus A_\iota$. It follows that $x \in \pi^\beta_\alpha\big[U \setminus A_\iota\big]$, and hence $x \in \partial\pi^\beta_\alpha\big[U \setminus A_\iota\big]$, because otherwise $x \in \big(\pi^\beta_\alpha\big[U \setminus A_\iota\big]\big)^\circ \subseteq \big(\pi^\beta_\alpha[U]\big)^\circ$, thus contradicting $x \in \partial\pi^\beta_\alpha[U]$.

Since for every $\iota < |\beta| \cdot \kappa$ the set $U \setminus A_\iota$ is clopen in K_β, for every $\iota < |\beta| \cdot \kappa$ there are $\xi_\iota \in \beta \setminus \alpha$ and clopen $B_\iota \subseteq K_{\xi_\iota}$ such that $U \setminus A_\iota = \big(\pi^\beta_{\xi_\iota}\big)^{-1}\big[B_\iota\big]$, and hence $\pi^\beta_\alpha\big[U \setminus A_\iota\big] = \pi^{\xi_\iota}_\alpha\big[B_\iota\big]$ for all $\iota < |\beta| \cdot \kappa$. It follows from our inductive assumption that

$$\partial\pi^\beta_\alpha\big[U \setminus A_\iota\big] = \partial\pi^{\xi_\iota}_\alpha\big[B_\iota\big] \leq \big|\xi_i\big| \cdot \tau \cdot \kappa \leq |\beta| \cdot \tau \cdot \kappa,$$

and hence we conclude from $(*)$ that

$$\left|\partial\pi^\beta_\alpha[U]\right| \leq |\beta| \cdot \kappa \cdot |\beta| \cdot \tau \cdot \kappa + \kappa \leq |\beta| \cdot \tau \cdot \kappa,$$

which completes our proof.

□

We are in the position to prove the main theorem of this section.

Theorem 9.19. *Let $\tau < \mathfrak{c}$ be a cardinal number. Assume that $\big\langle K_\alpha, \pi^\beta_\alpha\colon\ \alpha \leq \beta \leq \delta\big\rangle$ is an inverse system of τ-simple extensions with $\delta \leq \mathfrak{c}$. Then, K_δ has the fsJNP.*

Proof. Since π^β_α are irreducible for any $\alpha < \beta \leq \delta$, for every clopen $U \subseteq K_\beta$ we have $\big(\pi^\beta_\alpha[U] \cap \pi^\beta_\alpha\big[K_\beta \setminus U\big]\big)^\circ = \emptyset$ and hence, by Lemma 9.16,

$$\pi^\beta_\alpha[U] \cap \pi^\beta_\alpha\big[K_\beta \setminus U\big] = \partial\pi^\beta_\alpha[U] \cup \partial\pi^\beta_\alpha\big[K_\beta \setminus U\big].$$

Let $U \subseteq K_\delta$ be clopen. If $\delta < \mathfrak{c}$, then it follows from the above equality and Lemma 9.18 (with $\kappa = \omega$—note that $|\partial U| = 0$) that

$$\pi_0^\delta[U] \cap \pi_0^\delta\big[K_\delta \setminus U\big] \leq |\delta| \cdot \tau \cdot \omega < \mathfrak{c}.$$

If $\delta = \mathfrak{c}$, then $U = \big(\pi_\beta^\delta\big)^{-1}[W]$ for some $\beta < \delta$ and clopen $W \subseteq K_\beta$, and hence, again by Lemma 9.18,

$$\big|\pi_0^\delta[U] \cap \pi_0^\delta\big[K_\delta \setminus U\big]\big| = \big|\pi_0^\beta[W] \cap \pi_0^\beta\big[K_\delta \setminus W\big]\big| \leq |\beta| \cdot \tau \cdot \omega < \mathfrak{c}.$$

Thus, $\big|\pi_0^\delta[U] \cap \pi_0^\delta\big[K_\delta \setminus U\big]\big| < \mathfrak{c}$ in any case. Since $\pi_0^\delta[U] \cap \pi_0^\delta[X_\delta \setminus U]$ is a closed subset of 2^ω of size $< \mathfrak{c}$, we conclude that it is countable, and hence it must have Lebesgue measure 0. It remains to apply Proposition 9.17 for $Y = K_\delta$. □

Rephrasing the theorem, we get that the limits of inverse systems of τ-simple extensions of length at most $\mathfrak{c}$ do not have the ℓ_1-Grothendieck property, which generalizes Corollary 9.11.

The assumption in Theorem 9.19 that the ordinal number δ is not greater than $\mathfrak{c}$ seems to be essential as it allows us to appeal to Proposition 9.17 in order to "transport" the fsJN-sequence $\langle\mu_n\colon\ n \in \omega\rangle$ from the Cantor space onto K_δ. We do not know whether the conclusion of the theorem holds true without this assumption.

Question 9.20. *Let $\tau < \mathfrak{c}$ be a cardinal number. Assume that $\big\langle K_\alpha, \pi_\alpha^\beta\colon \alpha \leq \beta \leq \delta\big\rangle$ is an inverse system of τ-simple extensions with $\delta > \mathfrak{c}$. Does K_δ necessarily have the fsJNP?*

As the application of Theorem 9.19, we will show that some special Efimov spaces have the fsJNP, too. Namely, in [22, 23], under $\diamondsuit$, de la Vega introduced continuous inverse systems $\big\langle K_\alpha, \pi_\alpha^\beta\colon \alpha \leq \beta \leq \omega_1\big\rangle$ such that K_{ω_1} is a hereditarily separable Efimov space with various homogeneity properties, defined as follows: for every $\alpha < \omega_1$ the space K_α is homeomorphic to the space 2^ω and there exist:

- closed subsets $A_\alpha^0, A_\alpha^1 \subseteq K_\alpha$ and a point $p_\alpha \in K_\alpha$ such that $K_\alpha = A_\alpha^0 \cup A_\alpha^1$ and $A_\alpha^0 \cap A_\alpha^1 = \big\{p_\alpha\big\}$, and
- a countable group G_α acting on K_α freely, i.e. $gx \neq x$ for any $x \in K_\alpha$ and $g \in G_\alpha \setminus \big\{e_\alpha\big\}$, where $e_\alpha \in G_\alpha$ is the group identity, such that

$$K_{\alpha+1} = \big\{(x, \phi) \in K_\alpha \times 2^{G_\alpha}\colon\ x \in gA_\alpha^{\phi(g)} \text{ for every } g \in G_\alpha\big\}$$

  and $\pi_\alpha^{\alpha+1}\big((x, \phi)\big) = x$.

Let us call such inverse systems *de la Vega systems*. Below, we show that they are based on ω-simple extensions and hence their limits K_{ω_1} have the fsJNP.

Proposition 9.21. *Every de la Vega system is based on ω-simple extensions.*

Proof. Let $\big\langle K_\alpha, \pi_\alpha^\beta\colon \alpha \leq \beta \leq \omega_1\big\rangle$ be a de la Vega system. For each $\alpha < \omega_1$ fix $A_\alpha^0, A_\alpha^1, p_\alpha$ and G_α as in the definition of the system. We need to show that each π_α^β is irreducible and that for each $\alpha < \omega_1$ and closed $U \subseteq K_{\alpha+1}$ the set $\partial\pi_\alpha^{\alpha+1}[U] \setminus \pi_\alpha^{\alpha+1}[\partial U]$ is countable.

(1) For every $\alpha < \beta \leq \omega_1$ the function π^β_α is irreducible.

Since the system is continuous it is enough to show that $\pi^{\alpha+1}_\alpha$ is irreducible. To prove this we will use the following simple characterization of irreducible mappings: a function $f\colon K \to L$ between two totally disconnected compact spaces is irreducible if and only if for every clopen $U \subseteq K$ there is clopen $B \subseteq L$ such that $f^{-1}[B] \subseteq U$.

Thus fix $\alpha < \omega_1$ and a clopen $U \subset K_{\alpha+1}$. Without loss of generality let us assume $K_\alpha = 2^\omega$. Shrinking U if necessary we may assume that $\emptyset \neq U = ([s] \times [t]) \cap K_{\alpha+1}$ for some $s = \langle s(0), \ldots, s(n)\rangle \in 2^{n+1}$ and $t = \langle t(g_0), \ldots, t(g_n)\rangle \in 2^{\{g_i\colon\ i \leq n\}}$, where $[s] = \{x \in 2^\omega\colon\ x \restriction (n+1) = s\}$ and $[t] \subset 2^{G_\alpha}$ is defined analogously. Put:

$$W = \bigcap_{i \leq n} g_i A^{t(g_i)}_\alpha \cap [s].$$

Since $\pi^{\alpha+1}_\alpha[U] = W$, W is non-empty, and $W \setminus \{g_i p\colon\ i \leq n\}$ is open in K_α. Fix any clopen set $B \subset W \setminus \{g_i p : i \leq n\}$ and a pair $(x, \phi) \in B \times 2^{G_\alpha}$ such that $(x, \phi) \in K_{\alpha+1}$, i.e., $(x, \phi) \in (\pi^{\alpha+1}_\alpha)^{-1}(x)$. Since $x \notin \{g_i p : i \leq n\}$, for every $i \leq n$ there is a unique $j_i \in 2$ such that $x \in g_i A^{j_i}_\alpha$, and hence $\phi\big(g_i\big) = t(g_i) = j_i$ for all $i \leq n$. It follows that $(x, \phi) \in [s] \times [t] \cap K_{\alpha+1} \subset U$, so, summarizing, $\big(\pi^{\alpha+1}_\alpha\big)^{-1}[B] \subset U$.

(2) for every $\alpha < \omega_1$ and closed $U \subseteq K_{\alpha+1}$, $\Big|\partial \pi^{\alpha+1}_\alpha[U] \setminus \pi^{\alpha+1}_\alpha[\partial U]\Big| \leq \omega$.

We shall show that

$$\partial \pi^{\alpha+1}_\alpha[U] \setminus \pi^{\alpha+1}_\alpha[\partial U] \subseteq \{gp\colon\ g \in G_\alpha\}.$$

This has been almost done in the previous paragraph. Indeed, suppose that

$$x \in \pi^{\alpha+1}_\alpha[U] \setminus \big(\pi^{\alpha+1}_\alpha[\partial U] \cup \{gp\colon\ g \in G_\alpha\}\big).$$

Then, $x \in \pi^{\alpha+1}_\alpha[U^\circ] \setminus \{gp\colon\ g \in G_\alpha\}$, and, by the same argument as in (1), we get a clopen set $B \subset K_\alpha$ containing x and such that $\big(\pi^{\alpha+1}_\alpha\big)^{-1}(x) \subseteq \big(\pi^{\alpha+1}_\alpha\big)^{-1}[B] \subseteq U^\circ$, and therefore $x \in B \subseteq \big(\pi^{\alpha+1}_\alpha\big)[U]^\circ$. But this implies that $x \notin \partial \pi^{\alpha+1}_\alpha[U]$, which completes the proof.

□

Recall that a space X is called *rigid* if it has no non-trivial autohomeomorphisms, i.e. every homeomorphism $f\colon X \to X$ is the identity. Combining Theorem 9.19 with Proposition 9.21 and de la Vega's [22, Theorems 5.1 and 5.2], we get the following corollary, important in the view of Kąkol and Śliwa [48, Example 15].

Corollary 9.22. *Assume $\diamondsuit$.*

(1) There exists a hereditarily separable totally disconnected rigid Efimov space satisfying the fsJNP.

(2) There exists a hereditarily separable totally disconnected Efimov space K satisfying the fsJNP and such that any two non-empty clopen subsets of F are homeomorphic.

Furthermore, using the generalizations of de la Vega [23] obtained in Backé [4], we get the next corollary.

Corollary 9.23. *Under $\diamondsuit$ there exists a hereditarily separable totally disconnected Efimov space K satisfying the fsJNP and such that there are no disjoint infinite closed homeomorphic subspaces of K.*

Let us not here that we do not know however whether the classes of compact spaces obtained by simple extensions and τ-simple extensions for $\tau \in [\omega, \mathfrak{c}]$ are essentially different. Thus, we ask the following crucial questions.

Question 9.24.
1. *Does there exist a compact space which is the limit of an inverse system based on ω-simple extensions but not the limit of any inverse system based on simple extensions?*
2. *Assume that $\mathfrak{c} > \omega_1$. Does there exist a compact space which is the limit of an inverse system based on ω_1-simple extensions but not the limit of any inverse system based on simple extensions?*

The following problem is a special case of Question 9.24.

Question 9.25. *Does there (consistently) exist a de la Vega system whose limit cannot be represented as the limit of an inverse system based on simple extensions?*

10. l-Equivalence

It is a well-known fact that given two spaces X and Y if an operator $L\colon C_p(X) \to C_p(Y)$ is a linear homeomorphism, then the operator $L^*\colon \Delta(Y) \to \Delta(X)$ given by the formula $L^*(\mu) = \mu \circ L$ is also a linear homeomorphism (see Tkachuk [83, Problem 237]). Since for every measure $\mu \in \Delta(X)$, written $\mu = \sum_{i=1}^n \alpha_i \cdot \delta_{x_i}$, we have $L^*(\mu) = \mu \circ L = \sum_{i=1}^n \alpha_i \cdot \big(\delta_{x_i} \circ L\big)$, we obtain the following proposition.

Proposition 10.1. *Let X and Y be two spaces. Assume that X has the fsJNP. If $C_p(X)$ and $C_p(Y)$ are linearly homeomorphic, then $C_p(Y)$ has the fsJNP, too.* □

Let K be a compact space. Recall that *the Alexandrov Duplicate $AD(K)$* of K is the space defined as follows: $AD(K) = K \times \{0,1\}$ where for each $x \in K$ the point $(x,1)$ is isolated and the basic open neighborhoods of $(x,0)$ are the sets of the form $(U \times \{0\}) \cup \big((U \setminus \{x\}) \times \{1\}\big)$ for every open basic neighborhood U of x in K. It follows that $AD(K)$ is compact (cf. [82, Problem 364]) and $C_p(AD(K))$ is linearly homoemorphic to $C_p\big(K \cup \alpha(|K|)\big)$, where $\alpha(|K|)$ denotes the one-point compactification of the cardinal number $|K|$ (see [83, Problem 267]). Since $\alpha(|K|)$ contains a non-trivial convergent sequence, $K \cup \alpha(|K|)$ has the fsJNP and hence $AD(K)$ does, too.

Proposition 10.2. *Let K be a compact space. Then, its Alexandrov Duplicate $AD(K)$ has the fsJNP.* □

It follows that $AD(\beta\omega)$ has the fsJNP, although $\beta\omega$ does not have. This is interesting in the context of Sections 5.1 and 8, where examples of compact spaces with the fsJNP and containing many copies of $\beta\omega$ were given—$AD(\beta\omega)$ is another such example but of completely different kind.

11. Products and the JNP

Khurana [49, Theorem 2], Cembranos [20, Corollaries 2–3] and Freniche [36, Corollary 2.6] proved that given two infinite compact spaces K and L the space $C(K\times L)$ does not have the Grothendieck property. In this section we will strengthen their result by proving that the product $K\times L$ does not even have the ℓ_1-Grothendieck property (Theorem 11.3 and Corollary 11.4)—our proof is totally constructive and does not require any techniques from Banach space theory. As a corollary, we obtain that the space $C_p(K\times L)$ has always the quotient isomorphic to $(c_0)_p$.

Let us start with some additional notation. For every $n\in\omega_+$ put $\Omega_n=\{-1,1\}^n$ and $\Sigma_n=n\times\{n\}$ (so $\left|\Omega_n\right|=2^n$ and $\left|\Sigma_n\right|=n$). To simplify the notation, we will usually write $i\in\Sigma_n$ instead of $(i,n)\in\Sigma_n$—this should cause no confusion. Put also $\Omega=\bigcup_{n\in\omega_+}\Omega_n$ and $\Sigma=\bigcup_{n\in\omega_+}\Sigma_n$, and endow these two sets with the discrete topology. This way, we can think of the product space $\Omega\times\Sigma$ as a countable union of pairwise disjoint discrete rectangles $\Omega_k\times\Sigma_m$ of size $m2^k$—the rectangles $\Omega_n\times\Sigma_n$, lying along the diagonal, will bear a special meaning, namely, they will be the supports of measures from an fsJN-sequence $\langle\mu_n\colon\ n\in\omega_+\rangle$ on the space $\beta\Omega\times\beta\Sigma$ defined as follows ($n\in\omega_+$):

$$\mu_n=\sum_{\substack{s\in\Omega_n\\ i\in\Sigma_n}}\frac{s(i)}{n2^n}\delta_{(s,i)}.$$

Then, $\operatorname{supp}\left(\mu_n\right)=\Omega_n\times\Sigma_n$, so $\left|\operatorname{supp}\left(\mu_n\right)\right|=n2^n$, $\left\|\mu_n\right\|=1$, and

$$\pi_i\left[\operatorname{supp}\left(\mu_n\right)\right]\cap\pi_i\left[\operatorname{supp}\left(\mu_{n'}\right)\right]=\emptyset$$

for every $n\neq n'$ and $i\in\{0,1\}$ (here π_i denotes the projection on the i-th coordinate). Note that for each $n\in\omega_+$ and any two sets $A\in\wp(\Omega)$ and $B\in\wp(\Sigma)$ we have:

$$(\dagger)\qquad \left|\mu_n\big([A]\times[B]\big)\right|\le\frac{\left|A\cap\Omega_n\right|}{2^n}\cdot\frac{\left|B\cap\Sigma_n\right|}{n},$$

where $[A]$ and $[B]$ always denote the clopen subsets of $\beta\Omega$ and $\beta\Sigma$ corresponding in the sense of the Stone duality to A and B, respectively—since $\beta\Omega$ and $\beta\Sigma$ are extremely disconnected, we have $[A]=\overline{A}^{\beta\Omega}$ and $[B]=\overline{B}^{\beta\Sigma}$.

Before we state and prove the main proposition of this section, we need to provide a bit of explanation of probability tools we use in the proof. For every $n\in\omega_+$ and $i\in n$ define the function $X_i\colon\Omega_n\to\{0,1\}$ as follows: $X_i(r)=1$ if and only if $r(i)=1$, where $r\in\Omega_n$. Put $S_n=\sum_{i=0}^{n-1}X_i$, so $S_n\colon\Omega_n\to n$ is the function computing the number of 1's in the argument sequence $r\in\Omega_n$. For a finite set $A\in[\omega]^{<\omega}$, let P_A denotes the standard product probability on 2^A (assigning $1/2^{|A|}$ to each elementary event, i.e. $P_A(\{r\})=1/2^{|A|}$ for each $r\in 2^{|A|}$). Recall that for every $k\le n$ it holds:

$$P_n(S_n=k)=P_n\big(\{r\in\Omega_n\colon\ S_n(r)=k\}\big)=\binom{n}{k}1/2^n.$$

We will need the following fact estimating the probability that $S_n(r)$ has value "far" (with respect to ε) from $n/2$, i.e. that "r contains *significantly* more (with respect to ε) 1's than -1's, or *vice versa*".

Fact 11.1. *If $n \in \omega_+$ and $\varepsilon \in \big(0, 1/12\big]$ are such numbers that $n \geq 3/\varepsilon$, then:*

$$P_n\big(\big|S_n - n/2\big| \geq \varepsilon n/2\big) \leq \frac{\sqrt{2}}{\varepsilon\sqrt{n}}.$$

Proof. See Bollobás [12, Theorem 1.7.(i)]. □

We are ready to prove the main result of this section.

Proposition 11.2. *The sequence $\big\langle \mu_n\colon\ n \in \omega_+\big\rangle$ defined above is an fsJN-sequence. Consequently, $\beta\Omega \times \beta\Sigma$ has the fsJNP.*

Proof. Since $\beta\Omega \times \beta\Sigma$ is a totally disconnected compact space, to prove that $\big\langle \mu_n\colon\ n \in \omega_+\big\rangle$ is weakly* null it is enough to show that it converges to 0 on every clopen subset of the form $[A] \times [B]$, where $A \in \wp(\Omega)$ and $B \in \wp(\Sigma)$. So let us fix two such sets A and B.

Fix $\varepsilon \in \big(0, 1/12\big]$ and put:

$$I_0 = \big\{n \in \omega_+\colon\ \big|B \cap \Sigma_n\big| < 2/\varepsilon^4\big\}$$

and

$$I_1 = \omega_+ \setminus I_0 = \big\{n \in \omega_+\colon \big|B \cap \Sigma_n\big| \geq 2/\varepsilon^4\big\}.$$

For each $i \in \{0, 1\}$ we will find $N_i \in \omega$ such that for every $n \geq N_i$, $n \in I_i$, it holds

$$\big|\mu_n\big([A] \times [B]\big)\big| = \big|\mu_n\big(\big[A \cap \Omega_n\big] \times \big[B \cap \Sigma_n\big]\big)\big| < 2\varepsilon.$$

If for some $i \in \{0, 1\}$ the set I_i is finite, then let immediately $N_i = 1 + \max I_i$. If I_0 is infinite, then by (†) for every $n \in I_0$ we have:

$$\big|\mu_n\big([A] \times [B]\big)\big| \leq \frac{\big|A \cap \Omega_n\big|}{2^n} \cdot \frac{\big|B \cap \Sigma_n\big|}{n} \leq \frac{\big|B \cap \Sigma_n\big|}{n} < \frac{2}{n\varepsilon^4},$$

so there exists $N_0 \in \omega$ such that for every $n \geq N_0$, $n \in I_0$, we have:

$$\big|\mu_n\big([A] \times [B]\big)\big| < 2\varepsilon.$$

Let us now assume that I_1 is infinite. For every $n \in I_1$ define also the set $\Delta_{n,\varepsilon}$ as follows:

$$\Delta_{n,\varepsilon} = \Big\{s \in \Omega_n\colon\ \Big|\big|\big\{i \in B \cap \Sigma_n\colon\ s(i) = 1\big\}\big| - \frac{\big|B \cap \Sigma_n\big|}{2}\Big| \geq \varepsilon\frac{\big|B \cap \Sigma_n\big|}{2}\Big\},$$

so $\Delta_{n,\varepsilon}$ denotes the event that $s \in \Omega_n$ is "far" (with respect to ε) from having the same numbers of 1's and -1's when restricted to the set B. If we put similarly:

$$\Gamma_{n,\varepsilon} = \Big\{s \in 2^{B\cap\Sigma_n}\colon\ \Big|\big|\big\{i \in B \cap \Sigma_n\colon\ s(i) = 1\big\}\big| - \frac{\big|B \cap \Sigma_n\big|}{2}\Big| \geq \varepsilon\frac{\big|B \cap \Sigma_n\big|}{2}\Big\},$$

then we trivially have:

$$(\times) \qquad \Delta_{n,\varepsilon} = \Gamma_{n,\varepsilon} \times 2^{\Sigma_n \setminus B}.$$

Using this, for every $n \in I_1$ we will estimate the values of measures (see (2) and (3)):

$$(*) \qquad \left|\mu_n\big(\big[A \cap \Delta_{n,\varepsilon}\big] \times \big[B \cap \Sigma_n\big]\big)\right| = \Big| \sum_{\substack{s \in A \cap \Delta_{n,\varepsilon} \\ i \in B \cap \Sigma_n}} \frac{s(i)}{n2^n} \Big|$$

and

$$(**) \qquad \left|\mu_n\big(\big[A \cap \big(\Omega_n \setminus \Delta_{n,\varepsilon}\big)\big] \times \big[B \cap \Sigma_n\big]\big)\right| = \Big| \sum_{\substack{s \in A \cap (\Omega_n \setminus \Delta_{n,\varepsilon}) \\ i \in B \cap \Sigma_n}} \frac{s(i)}{n2^n} \Big|.$$

Note that:

$$\left|\mu_n\big([A] \times [B]\big)\right| \le \left|\mu_n\big(\big[A \cap \Delta_{n,\varepsilon}\big] \times \big[B \cap \Sigma_n\big]\big)\right| + \left|\mu_n\big(\big[A \cap \big(\Omega_n \setminus \Delta_{n,\varepsilon}\big)\big] \times \big[B \cap \Sigma_n\big]\big)\right|,$$

so obtaining "good" estimations of $(*)$ and $(**)$ will finish the proof.

Fix $n \in I_1$ and let us start with the estimation of $(*)$. Note that $\left|B \cap \Sigma_n\right| \ge 3/\varepsilon$, so recall $(\times)$ and apply Fact 11.1 with the set $B \cap \Sigma_n$ instead of the set n to get that:

$$(0) \qquad P_n\big(\Delta_{n,\varepsilon}\big) = P_{B \cap \Sigma_n}\big(\Gamma_{n,\varepsilon}\big) \cdot P_{\Sigma_n \setminus B}\big(2^{\Sigma_n \setminus B}\big) = P_{B \cap \Sigma_n}\big(\Gamma_{n,\varepsilon}\big) \le \frac{\sqrt{2}}{\varepsilon\sqrt{\left|B \cap \Sigma_n\right|}}.$$

It follows that for every $s \in \Omega_n \setminus \Delta_{n,\varepsilon}$ we have:

$$\Big| \sum_{i \in B \cap \Sigma_n} s(i) \Big| = \Big| \left|\big\{i \in B \cap \Sigma_n \colon\ s(i) = 1\big\}\right| - \left|\big\{i \in B \cap \Sigma_n \colon\ s(i) = -1\big\}\right| \Big| \le$$

$$\Big| \left|\big\{i \in B \cap \Sigma_n \colon\ s(i) = 1\big\}\right| - \frac{\left|B \cap \Sigma_n\right|}{2} \Big| +$$

$$\Big| \left|\big\{i \in B \cap \Sigma_n \colon\ s(i) = -1\big\}\right| - \frac{\left|B \cap \Sigma_n\right|}{2} \Big| <$$

$$2 \cdot \varepsilon \frac{\left|B \cap \Sigma_n\right|}{2} = \varepsilon \left|B \cap \Sigma_n\right|,$$

so:

$$(1) \qquad \Big| \sum_{i \in B \cap \Sigma_n} s(i) \Big| < \varepsilon \left|B \cap \Sigma_n\right|.$$

Finally, it holds that:

$$\Big| \sum_{\substack{s \in A \cap \Delta_{n,\varepsilon} \\ i \in B \cap \Sigma_n}} \frac{s(i)}{n2^n} \Big| \le \sum_{\substack{s \in A \cap \Delta_{n,\varepsilon} \\ i \in B \cap \Sigma_n}} \frac{1}{n2^n} = \frac{\left|A \cap \Delta_{n,\varepsilon}\right| \cdot \left|B \cap \Sigma_n\right|}{n2^n} \le$$

$$\frac{\left|\Delta_{n,\varepsilon}\right| \cdot n}{n2^n} = P_n\big(\Delta_{n,\varepsilon}\big),$$

so by (0):

$$\Big|\sum_{\substack{s\in A\cap\Delta_{n,\varepsilon}\\ i\in B\cap\Sigma_n}}\frac{s(i)}{n2^n}\Big|\le\frac{\sqrt{2}}{\varepsilon\sqrt{\big|B\cap\Sigma_n\big|}}. \tag{2}$$

We estimate $(**)$ in a similar way:

$$\Big|\sum_{\substack{s\in A\cap(\Omega_n\setminus\Delta_{n,\varepsilon})\\ i\in B\cap\Sigma_n}}\frac{s(i)}{n2^n}\Big|\le\frac{1}{n2^n}\Big|\sum_{\substack{s\in A\cap(\Omega_n\setminus\Delta_{n,\varepsilon})\\ i\in B\cap\Sigma_n}}s(i)\Big|=$$

$$\frac{1}{n2^n}\Big|\sum_{s\in A\cap(\Omega_n\setminus\Delta_{n,\varepsilon})}\ \sum_{i\in B\cap\Sigma_n}s(i)\Big|\le\frac{1}{n2^n}\sum_{s\in A\cap(\Omega_n\setminus\Delta_{n,\varepsilon})}\Big|\sum_{i\in B\cap\Sigma_n}s(i)\Big|\le$$

$$\frac{1}{n}\max\Big\{\Big|\sum_{i\in B\cap\Sigma_n}s(i)\Big|\colon\ s\in\Omega_n\setminus\Delta_{n,\varepsilon}\Big\},$$

so by (1):

$$\Big|\sum_{\substack{s\in A\cap(\Omega_n\setminus\Delta_{n,\varepsilon})\\ i\in B\cap\Sigma_n}}\frac{s(i)}{n2^n}\Big|<\frac{\varepsilon\big|B\cap\Sigma_n\big|}{n}\le\frac{\varepsilon n}{n}=\varepsilon. \tag{3}$$

We are ready to finish the proof—using (2), (3) and the fact that $\big|B\cap\Sigma_n\big|>2/\varepsilon^4$, we conclude that:

$$\big|\mu_n\big([A]\times[B]\big)\big|=\Big|\sum_{\substack{s\in A\cap\Omega_n\\ i\in B\cap\Sigma_n}}\frac{s(i)}{n2^n}\Big|\le\Big|\sum_{\substack{s\in A\cap\Delta_{n,\varepsilon}\\ i\in B\cap\Sigma_n}}\frac{s(i)}{n2^n}\Big|+\Big|\sum_{\substack{s\in A\cap(\Omega_n\setminus\Delta_{n,\varepsilon})\\ i\in B\cap\Sigma_n}}\frac{s(i)}{n2^n}\Big|<$$

$$\frac{\sqrt{2}}{\varepsilon\sqrt{\big|B\cap\Sigma_n\big|}}+\varepsilon<\varepsilon+\varepsilon=2\varepsilon.$$

It follows that if for N_1 we take any number from I_1, then for every $n\ge N_1$ we have:

$$\big|\mu_n\big([A]\times[B]\big)\big|<2\varepsilon.$$

We finish the proof by denoting $N=\max\big(N_0,N_1\big)$ and seeing that for every $n\ge N$ we obviously have the same inequality, i.e.:

$$\big|\mu_n\big([A]\times[B]\big)\big|<2\varepsilon.$$

Since $\varepsilon\in\big(0,1/12\big]$ is arbitrary, it holds that $\lim_{n\to\infty}\mu_n\big([A]\times[B]\big)=0$ and hence $\big\langle\mu_n\colon\ n\in\omega_+\big\rangle$ is weakly* null. □

Theorem 11.3. *For every two infinite compact spaces K and L, their product $K\times L$ has the fsJNP.*

Proof. First, notice that ω is homeomorphic to both Ω and Σ, so $\beta\omega$, $\beta\Omega$ and $\beta\Sigma$ are mutually homeomorphic. Consequently, by Proposition 11.2, $(\beta\omega)^2$ has the fsJNP and an fsJN-sequence witnessing this fact may be defined with supports contained completely in ω^2 (as measures μ_n's defined above on $\beta\Omega\times\beta\Sigma$ have supports contained in $\Omega\times\Sigma$).

Let D and E be discrete countable subsets of K and L, respectively. Let $\varphi\colon\omega\to D$ and $\psi\colon\omega\to E$ be bijections. By the Stone Extension Property of $\beta\omega$, there are continuous maps $\Phi\colon\beta\omega\to K$ and $\Psi\colon\beta\omega\to L$ such that $\Phi\restriction\omega=\varphi$ and $\Psi\restriction\omega=\psi$. Let $\langle\mu_n\colon\ n\in\omega\rangle$ be an fsJN-sequence of measures on $(\beta\omega)^2$ with supports in ω^2. For each $n\in\omega$ define a measure ν_n on $K\times L$ as follows:

$$\nu_n=\sum_{(x,y)\in\operatorname{supp}(\mu_n)}\mu_n\big(\{(x,y)\}\big)\cdot\delta_{(\varphi(x),\psi(y))},$$

it follows that $\|\nu_n\|=1$ and $\operatorname{supp}(\nu_n)$ is finite. Since $\langle\mu_n\colon\ n\in\omega\rangle$ is weakly* null, for every $f\in C(K\times L)$ we have:

$$\lim_{n\to\infty}\nu_n(f)=\lim_{n\to\infty}\mu_n\big(f(\Phi,\Psi)\big)=0,$$

where $f(\Phi,\Psi)(x,y)=f(\Phi(x),\Psi(y))\in C(\beta\omega\times\beta\omega)$, so $\langle\nu_n\colon\ n\in\omega\rangle$ is also weakly* null. This proves that $K\times L$ has also the fsJNP. $\square$

Theorem 11.3 yields a strengthening of the result of Khurana, Cembranos and Freniche.

Corollary 11.4. *For every two infinite compact spaces K and L, their product $K\times L$ does not have the ℓ_1-Grothendieck property.* $\square$

Schachermayer [72, Proposition 5.3] and Cembranos [20, Corollary 2] proved that a $C(K)$-space is Grothendieck if and only if it does not contain any complemented copy of c_0. This implies Cembranos' result [20, Corollary 3] stating that the space $C(K\times L)$ admits a complemented copy of c_0 for any two infinite compact spaces K and L. Corollary 11.4, together with the characterization of Banakh, Kąkol and Śliwa [7, Theorem 1], implies the following stronger result.

Corollary 11.5. *For every two infinite compact spaces K and L, the space $C_p(K\times L)$ admits $(c_0)_p$ as a quotient.* $\square$

Let us note here that the results presented in this section were also studied and generalized by Kąkol, Marciszewski, Sobota and Zdomskyy [47] in the class of pseudocompact spaces.

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