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A note on the weak topology of spaces $C_k(X)$ of continuous functions

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Abstract It is well known that the property of being a bounded set in the class of topological vector spaces E is not a topological property, where a subset $B \subset E$ is called a *bounded* set if every neighbourhood of zero U in E absorbs B. The paper deals with the problem which topological properties of bounded sets for the space $C_k(X)$ (of continuous real-valued functions on a Tychonoff space X with the compact-open topology) endowed with the weak topology of $C_k(X)$ can be transferred to bounded sets of $C_k(Y)$ endowed with the weak topology, assuming that the corresponding weak topologies of both $C_k(X)$ and $C_k(Y)$ are homeomorphic.

Keywords Baire and hereditary Baire space · Bounded subset · Compact and compact scattered · Homeomorphism and linear homeomorphism · Spaces of continuous functions

Mathematics Subject Classification (2000) 54C30 · 46A03 · 54C08

1 Introduction

The problem how to classify topologically locally convex spaces (lcs) attracted several specialists starting from Fréchet [11] and Banach [2]. For infinite dimensional separable Banach spaces this problem has been solved by Kadec [18],

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who proved that each separable Banach space is homeomorphic to a Hilbert space.

Toruńczyk [29] extended this classification for Fréchet lcs, i.e. metrizable and complete lcs, with the same density. Being motivated by this results Banakh [3] posed and studied the following problem.

Problem 1 When Banach spaces with the weak topology are sequentially homeomorphic?

Banakh proved [3, Theorem 1] that for Banach spaces E and F with separable duals, the spaces E_w and F_w are *sequentially homeomorphic* if and only if $W(E) = W(F)$, where $W(E)$ denotes the class of topological spaces homeomorphic to bounded closed subspaces of E_w, and E_w denotes the space E endowed with the weak topology $w = \sigma(E, E')$. Clearly if lcs E and F are linearly homeomorphic, then the corresponding spaces E_w and F_w are linearly homeomorphic.

It is well known that for a Banach space E the dual E' is separable if and only if every bounded set in E_w is metrizable [10, Proposition 3.28].

Fix any hereditary topological property Δ. For example, Δ might be the metrizability, Fréchet–Urysohn property, etc.

Checking the proof of [3, Theorem 1] one can note that for Banach spaces E and F with separable duals and such that E_w and F_w are sequentially homeomorphic one has the following:

Fact 1 *Every bounded set $B \subset E_w$ is homeomorphic to a bounded subset $B' \subset F_w$. Hence B' enjoys the property Δ in F_w, provided B does.*

We will discuss the case when the duals of E and F are not necessarily separable. More precisely, we will discuss the following problem:

When the spaces $C_k(X)_w$ and $C_k(Y)_w$ are homeomorphic for Tychonoff spaces X, Y?

By $C_k(X)$ and $C_p(X)$ denote the space of all continuous real-valued functions on a Tychonoff space X with the compact-open and the pointwise topology, respectively. Although, in general, the above problem is still open, we provide some necessary conditions with possible applications.

Another results (located in the same line of research) have been recently obtained by Krupski and Marciszewski. In [22] they studied the following natural question:

For what infinite compact spaces X and Y are the spaces $C_p(X)$ and $C(Y)_w$ homeomorphic?

Krupski and Marciszewski showed [22, Proposition 3.1] that for infinite compact spaces X and Y there exists no homeomorphism from $C_p(X)$ onto $C(Y)_w$ which is uniformly continuous. Moreover, they proved that spaces $C_p(X)$ and $C_k(Y)_w$ are not homeomorphic provided X, Y are infinite compact spaces and X or Y is scattered [22, Theorem 5.12], see also Corollary 9 below with shorter proofs for both cases.

We prove the following theorem being a consequence of more general results obtained below in the frame of locally convex spaces. Item (3) in Theorem 1 below extends also [22, Corollary 3.2], see also Remark 1, Corollary 8 below. Item (1) provides a necessary condition for the problem posed by Krupski and Marciszewski.

Theorem 1 *Let X and Y be infinite Tychonoff spaces. Then:*

1. *If $C_p(X)$ and $C_k(Y)_w$ are homeomorphic and every bounded set in $C_p(X)$ has property Δ, then every bounded set in $C_k(Y)_w$ has property Δ, too.*
2. *For infinite compact scattered X and Y the bounded sets in $C_p(X)$ and $C(Y)_w$ are Fréchet-Urysohn but $C_p(X)$ and $C(Y)_w$ are not homeomorphic.*
3. *If $C_k(Y)$ has a fundamental sequence of bounded sets, then does not exist a continuous homogeneous surjection $T : C_k(Y)_w \to C_p(X)$ such that $T^{-1}(K)$ is compact for each compact $K \subset C_p(X)$.*
4. *If X is pseudocompact, Y is paracompact and locally compact and $C_k(X)_w$, $C_k(Y)_w$ are homeomorphic and every bounded set in $C_k(X)_w$ has property Δ, then the same holds $C_k(Y)_w$.*

Warner [33] proved that $C_k(Y)$ admits a fundamental sequence $(D_n)_n$ of bounded sets, i.e. every D_n is bounded and every bounded set in $C_k(Y)$ is contained in some set D_n, if and only if for each sequence $(G_n)_n$ of pairwise disjoint nonempty open subsets of Y there is compact $K \subset Y$ with $\{n \in \omega : K \cap G_n \neq \emptyset\}$ is infinite. For example, every (DF) (in particular, every normed) space $C_k(Y)$ admits a fundamental sequence of bounded sets.

We refer also to [8] for results when for a Banach space E the closed unit ball with the weak topology may enjoy typical topological properties. Theorem 1 applies to get

Corollary 1 *Let X be pseudocompact, Y paracompact and locally compact. If $C_k(X)_w$ and $C_k(Y)_w$ are homeomorphic and every bounded set in $C_k(X)_w$ is metrizable, then X is compact and countable, Y is countable and admits a weaker compact metrizable topology.*

Indeed, if every bounded set $C_k(X)_w$ has property $\Delta=$ {metrizability}, then X is countable by [13, Theorem 7]. Hence X is compact as being Lindelöf and pseudocompact. By Theorem 1 every bounded set in $C_k(Y)_w$ is metrizable. Again by [13, Theorem 7] Y is countable. By Corollary 3 below the space Y is scattered, hence Y admits a weaker (metrizable) compact topology by [19]. This yields the following

Corollary 2 *Let X and Y be infinite compact metrizable spaces and assume that $C(X)_w$ and $C(Y)_w$ are homeomorphic. Then either both X and Y are uncountable or both X and Y are countable. Conversely, if X and Y are countable and $\chi(X) = \chi(Y)$, or X and Y are uncountable, then $C(X)_w$ and $C(Y)_w$ are linearly homeomorphic.*

The converse implication follows from [5, Theorem 2] and Miljutin's theorem [24], [30]. Nevertheless, the question is whether the lack of the equality

$\chi(X) = \chi(Y)$ for countable compact metrizable spaces X and Y allows, however, to describe a homeomorphism (surely not linear) between weak topologies of $C_k(X)$ and $C_k(Y)$. For the definition of the topological invariant $\chi(X)$ we refer to [5, p.59]. We believe the answer for Problem 2 should be negative.

Problem 2 Let X and Y be compact countable spaces such that $C(X)_w$ and $C(Y)_w$ are homeomorphic. Is it true that $\chi(X) = \chi(Y)$?

In [5, Theorem 3] Bessaga and Pełczyński classified Banach spaces $C(X)$ over zero-dimensional compact metrizable spaces X.

Theorem 2 ([5]) *Let X and Y be infinite compact metrizable zero-dimensional spaces. Then $C(X)$ and $C(Y)$ are linearly homeomorphic if and only if one of the conditions below holds.*

1. *X and Y are countable and $\chi(X) = \chi(Y)$.*
2. *X and Y are uncountable.*

Fréchet-Urysohn property, a possible one in the frame of Δ, is connected with Ruess's [27, Theorem 1] criterion for lcs to contain an isomorphic copy of ℓ_1. Therefore, this might motivate the following theorem. The equivalence between (1) and (2) has been already proved in [15, Lemma 6.3]. To keep the paper self contained we add a direct proof.

Theorem 3 *The following conditions are equivalent for a Tychonoff space X:*

1. *$C_k(X)$ does not contain an isomorphic copy of ℓ_1 (shortly, $\ell_1 \nsubseteq C_k(X)$).*
2. *Every compact subset of X is scattered.*

If additionally X is σ-compact and hereditarily Baire, (1)-(2) are equivalent to:

3. *X is scattered.*

Note that there exist countably compact (hence hereditarily Baire) non-scattered uncountable spaces X for which every compact subset is finite (hence scattered); this shows the item (2) holds while (3) fails. Indeed, V. Tkachuk in [31] proved that there exists a countably compact dense subspace $X \subset \beta\omega \setminus \omega$ whose all compact subsets are finite.

Corollary 3 *For a Čech-complete space X the following assertions are equivalent:*

1. *$\ell_1 \nsubseteq C_k(X)$ and X is Lindelöf.*
2. *X is scattered and Lindelöf.*
3. *X is σ-compact and every compact set in X is scattered.*

2 Proof of Theorem 1 and further corollaries

Let E be a lcs. A subset $B \subset E$ is called a *bounded* set if for every neighbourhood of zero U in E there exists $\lambda > 0$ such that $\lambda B \subset U$.

By result of Ruess [27, Theorem 2.1] we know that: If E is a lcs such that: (i) E is complete; (ii) all bounded sets in E are metrizable, then the following conditions are equivalent:

1. E does not contain any isomorphic copy of ℓ_1.
2. Every bounded sequence in E has a weak Cauchy subsequence, i.e. sequence which is Cauchy in the weak topology of E.

Note that there are non-metrizable lcs which satisfy the above two conditions (i) and (ii). Indeed, let E be a distinguished Fréchet lcs (i.e. the strong dual E'_β of E is barrelled, equivalently is bornological, see [6], [7]). Then clearly E'_β is complete and barrelled, and as proved in [7], every bounded set in E'_β is metrizable. Nevertheless, E'_β, as being a (DF)-space, is metrizable only if E is a Banach space. On the other hand, for the case when $E = C_k(X)$ we have

Corollary 4 *Let X be a Tychonoff space. Then $C_k(X)$ fulfils both conditions (i) and (ii) if and only if $C_k(X)$ is a Fréchet space, i.e. exactly when X is a hemicompact $k_{\mathbb{R}}$-space.*

Indeed, note that condition (ii) holds in $C_k(X)$ if and only if X is hemicompact, see [13, Theorem 2.5], and (i) holds if and only if X is a $k_{\mathbb{R}}$-space, [26, Theorem 10.1.24].

Recall that a topological space X is *Fréchet–Urysohn*, if for every subset $A \subset X$ and every $x \in \overline{A}$ there exists a sequence $(x_n)_n \subset A$ with $x_n \to x$. In [14, Lemma 3.5] we supplemented the above cited result of Ruess as follows.

Lemma 1 *For quasibarrelled E with an ω^ω-base the following assertions are equivalent:*

1. *Each bounded subset of E is weakly Fréchet-Urysohn, i.e. Fréchet-Urysohn in the weak topology of E.*
2. *Any bounded sequence in E has a weakly Cauchy subsequence.*

A lcs E has an ω^ω-base if E admits a base $\{U_\alpha : \alpha \in \omega^\omega\}$ of absolutely convex neighbourhoods of zero with $U_\alpha \subset U_\beta$ if $\beta \leq \alpha$, see [21].

Lemma 1 applies for $C_k(X)$ over Lindelöf Čech-complete spaces X.

Corollary 5 *Let X be a Lindelöf Čech-complete space. Then $C_k(X)$ is a complete barrelled space with an ω^ω-base, so (1) and (2) from Lemma 1 are equivalent for $C_k(X)$.*

Indeed, since X contains a fundamental compact resolution, i.e. a family $\{K_\alpha : \alpha \in \omega^\omega\}$ of compact sets covering X such that $K_\alpha \subset K_\beta$ if $\alpha \leq \beta$, the space $C_k(X)$ has a ω^ω-base, see [12]. The space X is a μ-space, i.e. every functionally bounded set in X is relatively compact, so by [26, Theorem 10.1.20] the space $C_k(X)$ is barrelled and complete (since X is a $k_{\mathbb{R}}$-space). Combining last results we derive

Corollary 6 *For a Fréchet lcs E each bounded subset of E is Fréchet-Urysohn in the weak topology if and only if $\ell_1 \not\subseteq E$.*

The above results imply Schlüchtermann and Wheeler [28, Theorem 5.1]. Next lemma will be used for the proof of Theorem 1.

Lemma 2 *Assume that E and F are lcs such that F is a Baire space, i.e. of the second Baire category, and E is covered by a sequence $(S_n)_n$ of bounded sets which have property Δ in E_w. If E_w and F_w are homeomorphic, then every bounded set in F_w has property Δ.*

Proof Fix a property Δ as described in Introduction. Assume that all sets S_n are closed and absolutely convex; hence each S_n is weakly closed.

Let $T\colon E_w \to F_w$ be a homeomorphism and set $K_n := T(S_n)$ for $n \in \omega$. Then each K_n is closed and has property Δ in the weak topology of F. Moreover, each K_n is closed for the original Baire topology ξ in F and $F = \bigcup_{n\in\mathbb{N}} K_n$. By the Baire category theorem, we derive the existence of an $m \in \omega$ such that K_m has a nonempty interior in the topology ξ. Hence there exists an $x_0 \in \operatorname{int} K_m$ and an absolutely convex closed neighbourhood U in ξ such that $x_0 + U \subset K_m$. Take any absolutely convex bounded closed subset B in the space (F, ξ). There exists $\lambda > 0$ such that $\lambda B \subset U$. Since K_m has property Δ in the weak topology of F, the same is true for $x_0 + \lambda B$, and hence also for B. Therefore each ξ-bounded subset of F has property Δ in the weak topology of F.

The following special case of Lemma 2 supplements [22, Theorem 5.12]. Note that $C_p(X)$ is covered by a sequence of bounded sets if, for example, X is compact, and the weak topology of $C_p(X)$ is the original topology of $C_p(X)$.

Corollary 7 *Let X and Y be infinite compact. If $C_p(X)$ and $C(Y)_w$ are homeomorphic, then every bounded set in $C(Y)_w$ has property Δ provided the same holds for $C_p(X)$.*

Consequently, if every bounded set in $C_p(X)$ is Fréchet-Urysohn, then $C_p(X)$ is Fréchet-Urysohn by [13, Proposition 2.1], and every bounded set in $C(Y)_w$ is Fréchet-Urysohn due to Corollary 7.

Corollary 8 *[22, Theorem 5.12, Corollary 5.11] Let X and Y be infinite and compact. If X or Y is scattered, then $C_p(X)$ and $C(Y)_w$ are not homeomorphic.*

Proof Assume there exists a homeomorphism $T : C_p(X) \to C(Y)_w$. Assume first that X is scattered. Then $C(Y)_w$ is Fréchet-Urysohn (since $C_p(X)$ is Fréchet-Urysohn by [1, Theorem III.1.2]). But the weak topology of any normed space E is Fréchet-Urysohn if and only if E is finite-dimensional, see for example, [21, Lemma 14.6].

Now assume Y is scattered. By Theorem 3 and Corollary 6 the unit ball S in $C(Y)_w$ is Fréchet-Urysohn. Set $D_n = T^{-1}(nS)$ for all $n \in \omega$. Then $C_p(X)$ is covered by a sequence of Fréchet-Urysohn spaces, and we apply [32, Problem 450] to get that $C_p(X)$ is Fréchet-Urysohn. Hence X is scattered by [1, Theorem III.1.2], and by the previous case we get a contradiction.

Problem 3 Let X and Y be infinite compact spaces and $C_p(X)$, $C(Y)_w$ are homeomorphic. Assume that $C_p(X)$ contains bounded subset without property Δ. Is the same statement true for $C(Y)_w$?

The following corollary extends [22, Corollary 3.2].

Corollary 9 *Let X and Y be Tychonoff spaces and assume that $T : C_p(X) \to C_k(Y)_w$ is a linear homeomorphism. If Y is hemicompact, then the weak topology of $C_k(Y)$ is metrizable. Hence, if Y is compact, then Y is finite.*

Proof Let $\{U_n : n \in \omega\}$ be a countable decreasing base of neighbourhoods of zero $C_k(Y)$ (since Y is hemicompact). For each $\alpha = (\alpha(n)) \in \omega^\omega$ set $A_\alpha = \bigcap_n \alpha(n)U_n$. Then $\{A_\alpha : \alpha \in \omega^\omega\}$ is a fundamental bounded resolution in $C_k(Y)$, i.e. each set A_α is bounded, every bounded set in $C_k(Y)$ is contained in some A_α, and $A_\alpha \subset A_\beta$ if $\alpha \leq \beta$.

Since bounded sets in $C_k(Y)$ and $C_k(Y)_w$ are the same, the assumption on the map T applies to deduce that $C_p(X)$ admits a fundamental bounded resolution. But this holds if and only if X is countable, see [16, Theorem 3.3], so $C_p(X)$ is metrizable.

Since $T : C_p(X) \to C_k(Y)_w$ is a homeomorphism, the space $C_k(Y)_w$ is metrizable. Clearly, if Y is compact, the last note implies that the weak topology of $C_k(Y)$ coincides with original normed topology of $C_k(Y)$, hence Y is finite.

Proof (Theorem 1)

Item (1): By [13, Proposition 2.1] the space $C_p(X)$ has property Δ if and only if every bounded set in the space $C_p(X)$ has property Δ. Indeed, the open interval $(-1, 1)$ is homeomorphic to the real line $\mathbb{R}$, so the subspace $C(X, (-1, 1))$ of $C_p(X)$ is homeomorphic to $C_p(X)$. Since the set $C_p(X, (-1, 1))$ is a bounded set in $C_p(X)$, the conclusion easily follows.

Item (2): The spaces $C_p(X)$ and $C(Y)_w$ are not homeomorphic: In fact, if X is scattered, the space $C_p(X)$ is Fréchet-Urysohn [1, Theorem III.1.2]. Then $C(Y)_w$ is Fréchet-Urysohn, which implies that Y must be finite. Indeed, since every Fréchet-Urysohn lcs E is *bornological*, i.e. every linear map from E into a lcs F sending bounded sets into bounded sets is continuous, see [21, Lemma 14.6], the weak topology of $C(Y)$ coincides with the Banach topology of $C(Y)$. Hence $C(Y)$ is finite dimensional, so Y is finite, indeed.

On the other hand, in general, if Y is a compact and scattered space, the weak topology of $C(Y)$ coincides with the pointwise topology of $C_p(Y)$ on the unit ball of the Banach space $C(Y)$. Hence every bounded set in $C(Y)_w$ is Fréchet-Urysohn.

Item (3): Assume, on the contrary, that there exists a continuous homogeneous surjection $T : C_k(Y)_w \to C_p(X)$ such that $T^{-1}(K)$ is compact for each compact set $K \subset C_p(X)$. By the assumption $C_k(Y)$ admits a fundamental sequence of bounded absolutely convex sets $(D_n)_n$.

Since $T(tf) = tT(f)$ for each scalar $t \in \mathbb{R}$ and $f \in C(Y)$, the space $C_p(X)$ is covered by a sequence of bounded sets $S_n = T(D_n)$, where $n \in \omega$. Since $(D_n)_n$ is a fundamental sequence of bounded sets in $C_k(Y)$, we may assume that $S_n \subset S_{n+1}$ for all $n \in \omega$. If $C_n = \overline{\text{absconv} S_n}$, $n \in \omega$, is the absolutely convex closed envelope of the set S_n, every set C_n is absolutely convex, bounded and closed in $C_p(X)$.

Let $K \subset C_p(X)$ be a compact set. By assumption $T^{-1}(K)$ is compact, hence as being bounded, $T^{-1}(K)$ is contained in some D_n, and then $K \subset C_n$. But this implies that $(2^n C_n)_n$ is a fundamental sequence of bounded sets in $C_p(X)$. If fact, assume that there exists a bounded set $B \subset C_p(X)$ such that $B \not\subseteq 2^n C_n$ for all $n \in \omega$. Then for each $n \in \omega$ select $2^{-n} x_n \notin C_n$. Since the set $G = \{2^{-n} x_x : n \in \omega\} \cup \{0\}$ is compact in $C_p(X)$, there exists $m \in \omega$ such that $G \subset C_m$. We reach to a contradiction.

This shows that $(2^n C_n)_n$ is a fundamental sequence of bounded sets in $C_p(X)$. Next we apply [21, Proposition 2.13] to derive that $\mathbb{R}^X = \overline{C_p(X)} = \overline{\bigcup_n 2^n C_n} = \bigcup_n \overline{2^n C_n}$, where the closure is taken in $\mathbb{R}^X$. By the Baire category theorem, some $\overline{C_n}$ is a bounded neighbourhood of zero in $\mathbb{R}^X$. This implies that X is finite, and we reach a contradiction.

Item (4): If X is pseudocompact, every $f \in C(X)$ is bounded on X. Hence the space $C_k(X)$ is covered by a sequence of bounded sets $(S_n)_n$, where

$$S_n = nS, \text{ and } S = \{f \in C(X) : \sup_{x \in X} |f(x)| \leq 1\}, \ \ n \in \omega.$$

On the other hand, if Y is paracompact and locally compact, the space $C_k(Y)$ is a Baire space. It is enough to apply Lemma 2.

Note also that if X is not pseudocompact, $C_k(X)$ is not covered by a sequence $(S_n)_n$ of bounded sets. In fact, $C_k(X)$ contains a (complemented) copy of $\mathbb{R}^\omega$, [20], and if $C_k(X)$ is covered by a sequence of bounded sets, the Baire category theorem applies to get a contradiction, since $\mathbb{R}^\omega$ is not normed.

From Lemma 2 and Corollary 6 we have

Corollary 10 *Suppose that E is a Banach space and F a Fréchet lcs. Assume that E_w and F_w are homeomorphic. If E does not contain any isomorphic copy of ℓ_1 then neither does F.*

Remark 1 Checking the proof of [22, Proposition 3.1] one can get also the following stronger version: If X and Y are infinite Tychonoff spaces and $C_k(Y)$ admits a fundamental sequence of bounded sets, then does not exist a uniformly continuous surjection $T : C_k(Y)_w \to C_p(X)$ such that $T^{-1}(K)$ is compact for each compact $K \subset C_p(X)$.

3 Proof of Theorem 3

Proof (Theorem 3) (2) $\Rightarrow$ (1): Assume that every compact subset of X is scattered. Let $\mathcal{K}(X)$ be the family of all compact subset of X. Note that the space $C_k(X)$ is isomorphic to a subspace of the product $\prod_{K \in \mathcal{K}(X)} C_k(K)$ of Banach spaces $C_k(K)$.

To get a contradiction assume that the space $C_k(X)$ contains an isomorphic copy E of ℓ_1. Observe that for E there exists a finite family $\mathcal{F} \subset \mathcal{K}(X)$ such that E is isomorphic to a subspace of the finite product $\prod_{j \in F} C_k(K_j)$ for

$\mathcal{F} = \{K_j : j \in F\}$ for some finite set F. In fact, let B be the unit (bounded) ball of the normed space E. There exists a finite set F such that

$$\bigcap_{j\in F} \pi_j^{-1}(U_j) \cap \prod_{K\in\mathcal{K}(X)} C_k(K) \subset B,$$

where U_j are balls in spaces $C_k(K_j)$, $j \in F$, and π_j are natural projections from E onto $C_k(K_j)$. Denote by π_F the (continuous) projection from $\prod_{K\in\mathcal{K}(X)} C_k(K)$ onto $\prod_{j\in F} C_k(K_j)$. Then $\pi_F|E$ is an injective continuous and open map from E onto $(\pi_F|E)(E) \subset \prod_{j\in F} C_k(K_j)$. The injectivity of $\pi_F|E$ is deduced from the fact that B is a bounded neighbourhood of zero in E. On the other hand, one shows that the image $(\pi_F|E)(B)$ is an open neighbourhood of zero in $\prod_{j\in F} C_k(K_j)$. The product $\prod_{j\in F} C_k(K_j)$ is isomorphic to the space $C_k(\bigoplus_{j\in F} K_j)$. Moreover, the compact space $\bigoplus_{j\in F} K_j$ is scattered. Hence we apply [10, Theorem 12.29] to get that E has separable dual which is impossible.

(1) $\Rightarrow$ (2): If (1) holds, then every compact set in X is scattered. Indeed, assume that there exists a compact set $K \subset X$ which is not scattered. Then there exists a continuous surjection $T : K \to [0, 1]$, see [30]. Applying the Tietze–Urysohn theorem one gets an extension $g : X \to [0, 1]$. Therefore the adjoint map $h \mapsto h \circ g$ embeds $C[0, 1]$ into $C_k(X)$, a contradiction since ℓ_1 embeds into $C[0, 1]$. So, every compact subset of X is scattered.

Now assume that $(X_n)_n$ is a sequence of compact subsets of X covering X and X is hereditarily Baire. Assume (2): Hence every X_n is scattered. We prove now that X is scattered. Let $D \subset X$ be a non-empty subset of X. We show that D contains an isolated point. Let K be the closure of D in X. Then (by assumption) K is Baire and $K = \bigcup_n K \cap X_n$. Then there exists $m \in \omega$ and an open non-empty set U in K such that $U \subset K \cap X_m$. Hence $U = U \cap X_m$, and by assumption, there exists an isolated point $y \in U \cap X_m$. Then there exists an open set $V_y \ni y$ in K with $V_y \cap U = V_y \cap U \cap X_m = \{y\}$. Next, chose an open set $W_y \ni y$ in X such that $W_y \cap K = V_y \cap U$, so $W_y \cap K = \{y\}$. Hence y is isolated in K and $y \in D$.

Proof (Corollary 3) (1) $\Rightarrow$ (2): From Theorem 3 it follows that every compact set in X is scattered. Since X is Čech-complete, the whole X is scattered, see [4, Theorem 1].

(2) $\Rightarrow$ (3): Note that X is σ-compact. Indeed, there exists a Polish space Y which is an image of X under a perfect map, see [9, 5.5.9(a)]. Since X is scattered, Y is scattered, too. Hence Y is countable by [30, 8.8.5], so X must be σ-compact.

(3) $\Rightarrow$ (1): From Theorem 3.

Corollary 11 *Let $C(X)$ and $C(Y)$ be Banach spaces. Assume $C(X)_w$ and $C(Y)_w$ are homeomorphic. Then X is scattered if and only Y is scattered.*

A regular topological space X is a cosmic (resp. $\aleph_0$)-space if and only if X is a continuous (resp. continuous compact-covering) image of a separable metric space, see [23].

We know that if E and F are two infinite dimensional Banach spaces for which E_w and F_w are homeomorphic, then E_w has all bounded sets weakly Fréchet-Urysohn if and only if the same holds and F_w. The converse implication fails.

Example 1 Consider two examples:

1. There exist separable Banach spaces E and F such that E_w and F_w are not homeomorphic but both E and F do not contain an isomorphic copy of ℓ_1.
2. The spaces ℓ_1 and $C[0,1]$ contain isomorphic copies of ℓ_1 but $(\ell_1)_w$ and $C[0,1]_w$ are not homeomorphic.

Proof (1): Let E be the *James tree space*, see [10]. The dual E' is not separable yet $\ell_1 \nsubseteq E$. By [14, Theorem 1.6] the space E_w is not an $\aleph_0$-space (see [23]). On the other hand, any separable Banach space F whose dual is separable does not contain an isomorphic copy of ℓ_1 and F_w is an $\aleph_0$-space (again by [14, Theorem 1.6]). Consequently, E_w and F_w are not homeomorphic.

(2) It is known that every separable Banach space E with the Schur property is an $\aleph_0$-space in the weak topology, see [15, Remark 4.5]. But $C[0,1]_w$ is not $\aleph_0$-space by [23, Proposition 10.8].

Problem 4 It is known that $C_p(\mathbb{R})$ and $C_p[0,1]$ are homeomorphic [1], as well as $C_k(\mathbb{R})$ and $C[0,1]$. Are the spaces $C_k(\mathbb{R})_w$ and $C[0,1]_w$ homeomorphic? Recall that $C_k(\mathbb{R})$ and $C[0,1]$ contain isomorphic copies of ℓ_1.

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