Ring-theoretic (in)finiteness in reduced products of Banach algebras

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(joint work w/ Matthew Daws [UCLan])

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Easy to see: $\sim$ is an equivalence relation on the set of idempotents of $\mathcal{A}$.

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- a commutative unital algebra is DF;
- $\mathcal{A}, \mathcal{B}$ are unital algebras, $\mathcal{A}$ is PI and $\varphi : \mathcal{A} \to \mathcal{B}$ is a unital algebra hom $\implies \mathcal{B}$ is PI.

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Lemma (Rieffel, PLMS, '83 ?)

For a unital Banach algebra $\mathcal{A}$:

$$\mathcal{A} \text{ has stable rank one} \implies \mathcal{A} \text{ is DF.}$$

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- "$\ell^1(Cu_2 \setminus \{\Diamond\})$", where Cu_2 is the second Cuntz semigroup with a zero element $\Diamond$ [folk].

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Proposition (folk, scattered through H.G. Dales' book "Banach Algebras and Automatic Continuity")

Let $\mathcal{A}$ be a unital C^*-algebra. Then

1. *$\mathcal{A}$ is DF as an algebra $\Leftrightarrow$ $\mathcal{A}$ is DF as a C^*-algebra;*
2. *$\mathcal{A}$ is PI as an algebra $\Leftrightarrow$ $\mathcal{A}$ is PI as a C^*-algebra.*

Proof.

(Sketch.) The main ideas used:

- If $p \in \mathcal{A}$ is an idempotent, there is a $q \in \mathcal{A}$ projection with $p \sim q$ and $(pq = q,\ qp = p$ or $pq = p,\ qp = q)$.
- Let $p, q \in \mathcal{A}$ be projections. Then $p \sim q \iff p \approx q$.

$\square$

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$\ell^\infty(\mathcal{A}_n)$ is a unital Banach algebra endowed with pointwise operations and the sup norm

$$\|A\| = \sup_{n\in\mathbb{N}} \|a_n\| \quad (A = (a_n) \in \ell^\infty(\mathcal{A}_n)).$$

In fact, $c_0(\mathcal{A}_n) \trianglelefteq \ell^\infty(\mathcal{A}_n)$ and $c_\mathcal{U}(\mathcal{A}_n) \trianglelefteq \ell^\infty(\mathcal{A}_n)$ with $c_0(\mathcal{A}_n) \subsetneq c_\mathcal{U}(\mathcal{A}_n)$.

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The *asymptotic sequence algebra* and the *ultraproduct* of a sequence of unital Banach algebras $(\mathcal{A}_n)_{n\in\mathbb{N}}$ are defined as

$$\mathrm{Asy}(\mathcal{A}_n) := \ell^\infty(\mathcal{A}_n)/c_0(\mathcal{A}_n), \text{ and} \tag{1}$$

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$$\|\pi(A)\| = \limsup_{n\to\infty} \|a_n\|, \text{ and}$$

$$\|\pi_\mathcal{U}(A)\| = \lim_{n\to\mathcal{U}} \|a_n\| \quad (A = (a_n) \in \ell^\infty(\mathcal{A}_n)).$$

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A word on reduced products

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Both $\mathrm{Asy}(\mathcal{A}_n)$ and $(\mathcal{A}_n)_{\mathcal{U}}$ are special cases of $(\mathcal{A}_n)_{\mathcal{F}}$.

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More on the proofs to follow soon. (Wishful thinking.)

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Corollary (Daws–H.)

Let $(\mathcal{A}_n)_{n\in\mathbb{N}}$ be a sequence of unital Banach algebras such that $\mathrm{Asy}(\mathcal{A}_n)$ is DF. Moreover, suppose that one of the following two conditions hold:

1. $\mathcal{A}_n = \mathcal{A}_m$ *for every* $n, m \in \mathbb{N}$;
2. $\mathcal{A}_n$ *is a* C^**-algebra for each* $n \in \mathbb{N}$.

Then there is $N \in \mathbb{N}$ *such that* $\mathcal{A}_n$ *is DF for* $n \geq N$.

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Then there is $N \in \mathbb{N}$ such that $\mathcal{A}_n$ is DF for $n \geq N$.

The C^*-case is very well known.

The situation regarding proper infiniteness is “reversed”.

Recall that a unital algebra $\mathcal{A}$ is PI if there exist idempotents $p, q \in \mathcal{A}$ such that $p \sim 1 \sim q$ and $p \perp q$.

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Both of these results are somewhat harder to prove than their respective DF-counterparts.

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- If $\mathcal{A}$ is a C^*-algebra, then $\mathcal{A}$ has stable rank one $\Leftrightarrow$ $\operatorname{Asy}(\mathcal{A})$ has stable rank one. [follows from work of *e.g.* Farah–Rørdam]

Ideas & tools behind some of the simpler proofs

We went to prove:

Theorem

Assume $(\mathcal{A}_n)_{n\in\mathbb{N}}$ *is a sequence of DF Banach algebras. Then* $\mathrm{Asy}(\mathcal{A}_n)$ *is DF.*

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Very simple but very important fact

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The Approximate Idempotent Lemma:

Proposition (folk)

Let $a \in \mathcal{A}$ be such that $\nu := \|a^2 - a\| < 1/4$. Then there is an idempotent $p \in \mathcal{A}$ such that $\|p - a\| \leq f_{\|a\|}(\nu)$ holds. Moreover, if $y \in \mathcal{A}$ is such that $ay = ya$ then $yp = py$.

In the Proposition above, $f_M : [0, 1/4) \to \mathbb{R}$ is some monotone increasing, non-negative continuous function for each $M > 0$. Also, $f_M \leq f_N$ when $N > M > 0$.

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By continuity of $f_{\|X\|}$, it follows that $\lim_{n\geq N} f_{\|X\|}(\nu_n) = 0$; consequently $\lim_{n\geq N} \|x_n - p_n'\| = 0$.

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$$\lim_{n\to\infty} \|1_n - a_n b_n\| = 0 \text{ and } \lim_{n\to\infty} \|p_n - b_n a_n\| = 0.$$

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Let $K \geq M$ be such that for every $n \geq K$

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\|1_n - p_n\| &= \|q_n - p'_n\| \\
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&\leq \|b_n u_n^{-1} a_n - b_n a_n\| + \|b_n a_n - x_n\| + \|x_n - p'_n\|
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Then for every $n \geq K$ we have $p_n = p'_n$ and $1_n = q_n$, thus

$$\begin{aligned}
\|1_n - p_n\| &= \|q_n - p'_n\| \\
&= \|b_n u_n^{-1} a_n - p'_n\| \\
&\leq \|b_n u_n^{-1} a_n - b_n a_n\| + \|b_n a_n - x_n\| + \|x_n - p'_n\| \\
&\leq \|b_n\| \|u_n^{-1} - 1_n\| \|a_n\| + \|b_n a_n - x_n\| + \|x_n - p'_n\| \\
&\leq \|A\| \|B\| \delta/(1-\delta) + 2\delta \\
&< 1.
\end{aligned}$$

This concludes the proof. □

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Clearly $p_n \in \mathcal{A}$ is an idempotent for each $n \in \mathbb{N}$, but $\|p_n\| = n$ and hence $(p_n) \notin \ell^\infty(\mathcal{A})$.

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Theorem (Daws–H.)

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- $\ell^1(I,\nu) = \overline{\mathrm{span}\{\delta_s : s \in I\}}^{\|\cdot\|_\nu}$, hence

$$f = \sum_{s\in I} f(s)\delta_s \quad (f \in \ell^1(I,\nu))$$

where the sum converges in the norm $\|\cdot\|_\nu$.

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The *convolution product* on $\ell^1(S, \omega)$ is defined by

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$(\ell^1(S, \omega), *)$ is a unital Banach algebra.

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For our counter-example, we need the *bicyclic monoid* BC. That is, the free monoid generated by elements p, q subject to the single relation that $pq = e$:

$$BC = \langle p, q : pq = e \rangle.$$

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1. $\mathcal{A}_n$ is DI for each $n \in \mathbb{N}$; and
2. $\mathrm{Asy}(\mathcal{A}_n)$ is DF. (Follows from Prop. and some actual work.)

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Sources

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