Statistical solutions to the barotropic Navier–Stokes system

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Statistical solution – general concept

Dynamical system (D)

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{U} = A(t, \mathbf{U})$$

$$\mathbf{U}(0, \cdot) = \mathbf{U}_0$$

Trajectory space

$$\mathbf{U} \in C([0, T]; X) \equiv \mathcal{T}$$

Statistical solution

$$\mathfrak{P} - \text{ probability measure on } \mathcal{T}$$

$$\mathbf{U} - \text{solution of the problem (D)} - \mathfrak{P} \text{ a.s.}$$

Push–forward measure

$$\mathbf{U}(t, \mathbf{U}_0),\ \nu_0 \text{ probability measure on the space of initial data}$$

$$\left\langle \nu_t, G(\widetilde{\mathbf{U}}) \right\rangle = \left\langle \nu_0, G(\mathbf{U}(t, \widetilde{\mathbf{U}}_0) \right\rangle$$

Barotropic Navier–Stokes system

Navier–Stokes system

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}) + \varrho \mathbf{g}$$

$$p(\varrho) \approx a\varrho^\gamma,\ \gamma > 1,\ \mathbb{S} = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{d} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) + \lambda \mathrm{div}_x \mathbf{u} \mathbb{I},\ \mu > 0, \lambda \geq 0$$

Boundary conditions - open system

$$\mathbf{u}|_{\partial\Omega} = \mathbf{u}_B$$

$$\partial\Omega = \Gamma_{\mathrm{in}} \cup \Gamma_{\mathrm{out}}$$

$$\Gamma_{\mathrm{in}} := \left\{ x \in \partial\Omega \ \middle| \ \text{the outer normal } \mathbf{n}(x) \text{ exists, and } \mathbf{u}_B(x) \cdot \mathbf{n}(x) < 0 \right\}.$$

$$\varrho|_{\Gamma_{\mathrm{in}}} = \varrho_B$$

Energy balance

Energy inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left[\frac{1}{2}\varrho\left|\mathbf{u}-\mathbf{u}_B\right|^2 + P(\varrho)\right]\ \mathrm{d}x + \int_\Omega \mathbb{S}(\mathbb{D}_x\mathbf{u}) : \mathbb{D}_x\mathbf{u}\ \mathrm{d}x$$
$$+\int_{\Gamma_{\rm out}} P(\varrho)\mathbf{u}_B\cdot\mathbf{n}\ \mathrm{d}S_x + \int_{\Gamma_{\rm in}} P(\varrho_B)\mathbf{u}_B\cdot\mathbf{n}\ \mathrm{d}S_x$$
$$\leq -\int_\Omega [\varrho\mathbf{u}\otimes\mathbf{u} + p(\varrho)\mathbb{I}] : \nabla_x\mathbf{u}_B\ \mathrm{d}x - \int_\Omega \varrho\mathbf{u}\cdot\mathbf{u}_B\cdot\nabla_x\mathbf{u}_B\ \mathrm{d}x\mathrm{d}t$$
$$+\int_\Omega \mathbb{S}(\mathbb{D}_x\mathbf{u}) : \mathbb{D}_x\mathbf{u}_B\ \mathrm{d}x + \int_\Omega \varrho\mathbf{g}\cdot(\mathbf{u}-\mathbf{u}_B)\ \mathrm{d}x$$

Pressure potential

$$P'(\varrho)\varrho - P(\varrho) = p(\varrho)$$

Phase variables

Energy

$$\mathcal{E} \equiv \int_\Omega \left[\frac{1}{2} \varrho \left| \mathbf{u} - \mathbf{u}_B \right|^2 + P(\varrho) \right] \ \mathrm{d}x, \text{ or rather its càglàd version } \mathcal{E}_{\rm cg}$$

$$\mathcal{E}_{\rm cg} = \int_\Omega \left[\frac{1}{2} \varrho \left| \frac{\mathbf{m}}{\varrho} - \mathbf{u}_B \right|^2 + P(\varrho) \right] (t, \cdot) \ \mathrm{d}x \text{ for any } t \in [0, \infty)$$

Data

$$\mathbf{d}_B \equiv [\varrho_B, \mathbf{u}_B, \mathbf{g}].$$

Semiflow selection

$$U : \Big(t; (\varrho_0, \mathbf{m}_0, \mathcal{E}_0, \mathbf{d}_B) \Big) \mapsto [\varrho(t, \cdot), \mathbf{m}(t, \cdot), \mathcal{E}_{\rm cg}(t), \mathbf{d}_B]$$

$$U\Big(t + s; (\varrho_0, \mathbf{m}_0, \mathcal{E}_0, \mathbf{d}_B) \Big) = U \Big(t; U\Big(s; (\varrho_0, \mathbf{m}_0, \mathcal{E}_0, \mathbf{d}_B) \Big) \Big) \text{ for any } t, s \geq 0$$

Semiflow selection

Existence of weak solutions

- Lions, Pierre-Louis **Mathematical topics in fluid mechanics. Vol. 2. Compressible models** Oxford University Press, 1998
- Plotnikov, Pavel ; Sokolowski, Jan **Compressible Navier-Stokes equations. Theory and shape optimization** Instytut Matematyczny Polskiej Akademii Nauk. Monografie Matematyczne (New Series) 2012. xvi+457 pp.
- Chang, T. and Jin, B. J. and Novotný, A. **Compressible Navier-Stokes system with general inflow-outflow boundary data** SIAM Journal on Mathematical Analysis, 2019

Measurable semiflow selection

- Cardona, Jorge E. and Kapitanski, Lev **Semiflow selection and Markov selection theorems**, Topol. Methods Nonlinear Anal., 2020
- Basarić, Danica **Semiflow selection for the compressible Navier-Stokes system** J. Evol. Equ., 2021

Construction of semiflow selection

Semiflow

$$U[t, \varrho_0, \mathbf{m}_0, E_0] \mapsto [\varrho(t), \mathbf{m}(t), E(t-)],\ \ t > 0$$

Semigroup property

$$U[t_1 + t_2, \varrho_0, \mathbf{m}_0, E_0] = U\left[t_2, U[t_1, \varrho_0, \mathbf{m}_0, E_0]\right] \text{ for any } 0 \leq t_1 \leq t_2$$

Dissipative solution

$$\varrho \in C_{\rm weak,loc}([0,\infty); L^{\gamma}(Q))$$

$$\mathbf{m} \in C_{\rm weak,loc}([0,\infty); L^{\frac{2\gamma}{\gamma+1}}(Q; R^N))$$

$$E \in BV_{\rm loc}([0,\infty); R),\ \text{(non–increasing)}$$

Initial data

$$\varrho(0,\cdot) = \varrho_0,\ \mathbf{m}(0,\cdot) = \mathbf{m}_0,\ E(0+) \leq E_0$$

Abstract setting

Phase space

$$X = W^{-\ell,2}(Q) \times W^{-\ell,2}(Q; R^N) \times R$$

Data space

$$D = \left\{ [\varrho_0, \mathbf{m}_0, E_0] \in X \ \middle| \ \varrho_0 \geq 0, \ \int_\Omega \left[\frac{1}{2} \frac{|\mathbf{m}_0|^2}{\varrho_0} + \frac{a}{\gamma - 1} \varrho_0^\gamma \right] \ \mathrm{d}x \leq E_0 \right\}.$$

Trajectory space

$$\Omega = C_{\mathrm{loc}}([0, \infty); W^{-\ell,2}(Q)) \times C_{\mathrm{loc}}([0, \infty); W^{-\ell,2}(Q; R^N)) \times L^1_{\mathrm{loc}}(0, \infty)$$

Method by Krylov adapted by Cardona and Kapitanski

Multi–valued solution mapping

$$\mathcal{U} : [\varrho_0, \mathbf{m}_0, E_0] \mapsto [\varrho, \mathbf{m}, E] \in 2^{\Omega}$$

Time shift

$$S_T \circ \xi,\ \ S_T \circ \xi(t) = \xi(T + t),\ \ t \geq 0.$$

Continuation

$$\xi_1 \cup_T \xi_2(\tau) = \left\{ \begin{array}{l} \xi_1(\tau) \text{ for } 0 \leq \tau \leq T, \\ \\ \xi_2(\tau - T) \text{ for } \tau > T. \end{array} \right.$$

Basic ansatz

- **(A1) Compactness:** For any $[\varrho_0, \mathbf{m}_0, E_0] \in D$, the set $\mathcal{U}[\varrho_0, \mathbf{m}_0, E_0]$ is a non–empty compact subset of Ω
- **(A2)** The mapping

$$D \ni [\varrho_0, \mathbf{m}_0, E_0] \mapsto \mathcal{U}[\varrho_0, \mathbf{m}_0, E_0] \in 2^{\Omega}$$

  is **Borel measurable**, where the range of $\mathcal{U}$ is endowed with the Hausdorff metric on the subspace of compact sets in 2^{Ω}
- **(A3) Shift invariance:** For any

$$[\varrho, \mathbf{m}, E] \in \mathcal{U}[\varrho_0, \mathbf{m}_0, E_0],$$

  we have

$$S_T \circ [\varrho, \mathbf{m}, E] \in \mathcal{U}[\varrho(T), \mathbf{m}(t), E(T-)] \text{ for any } T > 0.$$

- **(A4) Continuation:** If $T > 0$, and

$$[\varrho^1, \mathbf{m}^1, E^1] \in \mathcal{U}[\varrho_0, \mathbf{m}_0, E_0],\ [\varrho^2, \mathbf{m}^2, E^2] \in \mathcal{U}[\varrho^1(T), \mathbf{m}^1(T), E^1(T-)],$$

  then

$$[\varrho^1, \mathbf{m}^1, E^1] \cup_T [\varrho^2, \mathbf{m}^2, E^2] \in \mathcal{U}[\varrho_0, \mathbf{m}_0, E_0].$$

Induction argument

System of functionals

$$I_{\lambda,F}[\varrho, \mathbf{m}, E] = \int_0^\infty \exp(-\lambda t) F(\varrho, \mathbf{m}, E)\ \mathrm{d}t,\ \ \lambda > 0$$

where

$$F : X = W^{-\ell,2}(Q) \times W^{-\ell,2}(Q; R^N) \times R \to R$$

is a bounded and continuous functional

Semiflow reduction

$$\begin{aligned} & I_{\lambda,F} \circ \mathcal{U}[\varrho_0, \mathbf{m}_0, E_0] \\ & = \Big\{ [\varrho, \mathbf{m}, E] \in \mathcal{U}[\varrho_0, \mathbf{m}_0, E_0] \ \Big| \\ & I_{\lambda,F}[\varrho, \mathbf{m}, E] \leq I_{\lambda,F}[\widetilde{\varrho}, \widetilde{\mathbf{m}}, \widetilde{E}] \text{ for all } [\widetilde{\varrho}, \widetilde{\mathbf{m}}, \widetilde{E}] \in \mathcal{U}[\varrho_0, \mathbf{m}_0, E_0] \Big\} \end{aligned}$$

Induction argument

Statistical solution

Data space

$$\mathcal{D} = \left\{ [\varrho_0, \mathbf{m}_0, \varrho_B, \mathbf{u}_B, \mathbf{g}] \ \middle| \ \varrho_0 \in L^{\gamma}(\Omega), \mathbf{m}_0 \in L^{\frac{2\gamma}{\gamma+1}}(\Omega; R^d), \ \int_\Omega E\left(\varrho_0, \mathbf{m}_0 \ \middle| \ \mathbf{u}_B \right) \ \mathrm{d}x < \infty \right.$$

$$\left. \varrho_B \in C(\partial\Omega), \ \varrho_B \geq \underline{\varrho} > 0, \ \mathbf{u}_B \in C^1(\overline{\Omega}; R^d), \ \mathbf{g} \in C(\overline{\Omega}; R^d) \right\}$$

$$\mathfrak{P}[\mathcal{D}] = \left\{ \nu \ \middle| \ \nu \text{ a complete Borel probability measure on } X_{\mathcal{D}}, \ \mathrm{supp}[\nu] \subset \mathcal{D} \right\}.$$

Statistical solution

Family of (Markov) operators

$$M_t : \mathfrak{P}[\mathcal{D}] \to \mathfrak{P}[\mathcal{D}] \text{ for any } t \geq 0,$$

Properties of statistical solution

- **Initial data**

$$M_0(\nu) = \nu \text{ for any } \nu \in \mathfrak{P}[\mathcal{D}]$$

- **Convexity**

$$M_t\left(\sum_{i=1}^{N} \alpha_i \nu_i\right) = \sum_{i=1}^{N} \alpha_i M_t(\nu_i), \text{ for any } \alpha_i \geq 0, \ \sum_{i=1}^{N} \alpha_i = 1, \ \nu_i \in \mathfrak{P}(\mathcal{D})$$

- **Semigroup property**

$$M_{t+s} = M_t \circ M_s \text{ for any } t \geq 0 \text{ and a.a. } s \in (0, \infty)$$

- **(Weak) continuity**

$$t \mapsto M_t(\nu) \text{ is continuous with respect to the weak topology on } \mathfrak{P}(\mathcal{D})$$

Compatibility with the Navier–Stokes system

$$M_t\left(\delta_{[\varrho_0,\mathbf{m}_0,\mathbf{d}_B]}\right) = \delta_{[\varrho(t,\cdot),\mathbf{m}(t,\cdot),\mathbf{d}_B]} \text{ for any } t \geq 0$$

$$M_t\left(\nu\right) = \int_{\mathcal{D}} \delta_{[\varrho(t,\cdot),\mathbf{m}(t,\cdot),\mathbf{d}_B]} \mathrm{d}\nu(\varrho_0,\mathbf{m}_0,\mathbf{d}_B) \text{ for any } \nu \in \mathfrak{P}[\mathcal{D}],$$

where $[\varrho, \mathbf{m}]$ is a finite energy weak solution with the data

$$(\varrho_0, \mathbf{m}_0, \mathbf{d}_B)$$

$$\mathcal{E}_0 = \int_{\Omega} E\left(\varrho_0, \mathbf{m}_0 \,\middle|\, \mathbf{u}_B\right)\ \mathrm{d}x$$

Continuity with respect to the initial data

Weak–strong uniqueness property

$$[\varrho_0, \mathbf{m}_0, \mathbf{d}_B],\ \varrho_0 > 0,\ \mathcal{E}_0 = \int_\Omega E\left(\varrho_0, \mathbf{m}_0 \middle| \mathbf{u}_B\right)\ \mathrm{d}x$$

$$\Rightarrow$$

Lipchitz solution $\widetilde{\varrho}, \widetilde{\mathbf{m}}$ for $[0, T_{\max})$

Stability:

$$M_t(\delta_{[\varrho_0, \mathbf{m}_0, \mathbf{d}_B]}) = \delta_{[\widetilde{\varrho}(t,\cdot), \widetilde{\mathbf{m}}(t,\cdot), \mathbf{d}_B]} \text{ for all } t \in [0, T_{\max}).$$

Distance

Relative energy

$$E\left(\varrho, \mathbf{m} \middle| \widetilde{\varrho}, \widetilde{\mathbf{m}}\right) \equiv \frac{1}{2}\varrho \left| \frac{\mathbf{m}}{\varrho} - \frac{\widetilde{\mathbf{m}}}{\widetilde{\varrho}} \right|^2 + \left(P(\varrho) - P'(\widetilde{\varrho})(\varrho - \widetilde{\varrho}) - P(\widetilde{\varrho}) \right)$$

$$\mathcal{E}\left(\varrho, \mathbf{m} \middle| \widetilde{\varrho}, \widetilde{\mathbf{m}}\right) \equiv \int_\Omega E\left(\varrho, \mathbf{m} \middle| \widetilde{\varrho}, \widetilde{\mathbf{m}}\right) \ \mathrm{d}x.$$

Bregman–Wasserstein distance

$$W_E(\nu_1, \nu_2) \equiv \inf_{\mu \in \Pi(\nu_1;\nu_2)} \int_{\mathcal{D}\times\mathcal{D}} \mathcal{E}\left(\varrho, \mathbf{m} \middle| \widetilde{\varrho}, \widetilde{\mathbf{m}}\right) \mathrm{d}\mu([\varrho, \mathbf{m}, \mathbf{d}_B; \widetilde{\varrho}, \widetilde{\mathbf{m}}, \mathbf{d}_B]),$$

$$\Pi(\nu_1; \nu_2) \equiv \left\{ \mu \in \mathfrak{P}(\mathcal{D} \times \mathcal{D}) \ \middle| \ \pi_1(\mu) = \nu_1, \ \pi_2(\mu) = \nu_2 \right\}.$$

Distance to regular solutions

Regular trajectories

$$\mathcal{T}_{L,T} = \Big\{ [\varrho, \mathbf{m}, \mathbf{d}_B] \ \Big| \ [\varrho, \mathbf{m}] \text{ is a Lipschitz solution in } [0,T] \times \Omega, \\ \text{with the boundary data } \mathbf{d}_B, \\ \inf_{(0,T)\times\Omega} \varrho \geq L^{-1}, \ \| [\varrho, \mathbf{m}] \|_{W^{1,\infty}(0,T)\times\Omega; R^{d+1})} \leq L \Big\}$$

Regular data

$$\mathcal{D}_{L,T} := \Big\{ [\varrho(0), \mathbf{m}(0), \mathbf{d}_B] \ \Big| \ [\varrho, \mathbf{m}, \mathbf{d}_B] \in \mathcal{T}_{L,T} \Big\} \subset \mathcal{D}_R \subset \mathcal{D}.$$

Regular data stability

- **Regular data**

$$\widetilde{\mathbf{d}}_B = [\varrho_B, \mathbf{u}_B, \mathbf{g}] \in C(\partial\Omega) \times C^1(\overline{\Omega}; R^d) \times C(\overline{\Omega}; R^d), \ \inf_{\partial\Omega} \varrho_B > 0$$

- **Family of measures**

$$\{\nu_n\}_{n=1}^{\infty}, \ \nu_n \in \mathfrak{P}(\mathcal{D}), \ \nu \in \mathfrak{P}(\mathcal{D})$$

$$\mathrm{supp}[\nu] \subset \mathcal{D}_{L,T} \text{ for some } L, T > 0$$

$$\nu_n \left\{ \mathbf{d}_B = \widetilde{\mathbf{d}}_B \right\} = \nu \left\{ \mathbf{d}_B = \widetilde{\mathbf{d}}_B \right\} = 1$$

- **Initial data convergence**

$$W_E\left(\nu_n, \nu\right) \longrightarrow 0 \qquad \text{as} \quad n \to \infty$$

- **Conclusion**

$$\sup_{t\in[0,T]} W_E\left(M_t(\nu_n), M_t(\nu)\right) \longrightarrow 0 \qquad \text{as} \quad n \to \infty$$

Maximal solutions

Maximal energy dissipation

$$[\varrho_0, \mathbf{m}_0],\ \mathcal{E}_0 = \int_\Omega E\left(\varrho_0, \mathbf{m}_0 \Big| \mathbf{u}_B\right)\ \mathrm{d}x,\ \text{with the boundary data } \mathbf{d}_B,$$

$$[\varrho_1, \mathbf{m}_1] \prec [\varrho_2, \mathbf{m}_2] \ \Leftrightarrow\ \int_\Omega E\left(\varrho_1, \mathbf{m}_1 \Big| \mathbf{u}_B\right)(t,\cdot)\ \mathrm{d}x \leq \int_\Omega E\left(\varrho_2, \mathbf{m}_2 \Big| \mathbf{u}_B\right)(t,\cdot)\ \mathrm{d}x$$

Principle of maximal dissipation

$$[\varrho_0, \mathbf{m}_0],\ \mathcal{E}_0 = \int_\Omega E\left(\varrho_0, \mathbf{m}_0 \Big| \mathbf{u}_B\right)\ \mathrm{d}x,\ \text{with the boundary data } \mathbf{d}_B$$

$$\mathcal{E}_{\rm cg}(t) \to E_\infty < \infty \text{ as } t \to \infty$$

$$[\varrho, \mathbf{m}] \text{ maximal } \ \Rightarrow\ \int_\Omega E\left(\varrho, \mathbf{m} \Big| \mathbf{u}_B\right)(t,\cdot)\ \mathrm{d}x \to E_\infty \text{ as } t \to \infty$$