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BIGRADED DIFFERENTIAL ALGEBRA FOR VERTEX ALGEBRA COMPLEXES

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Abstract. For the bicomplex structure of grading-restricted vertex algebra cohomology defined in [6], we show that the orthogonality and double grading conditions applied endow it with the structure of a bigraded differential algebra with respect to a natural multiplication. The generators and commutation relations of the bigraded differential algebra form a continual Lie algebra $\mathcal{G}(V)$ with the root space provided by a grading-restricted vertex algebra V. We prove that the differential algebra generates non-vanishing cohomological invariants associated to a vertex algebra V. Finaly, we provide examples associated to various choices of the vertex algebra bicomplex subspaces.

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1. Introduction

The cohomology theory for vertex operator algebras is an important and attractive theme for studies. In [6] the cohomology theory for a grading-restricted vertex algebra [9] was introduced. The definition of bicomplex spaces and coboundary operators uses an interpretation of vertex algebras in terms of rational functions constructed from matrix elements [8] for a grading-restricted vertex algebra [4]. The notion of composability (see Section 3.1) of bicomplex space elements with a number of vertex operators is essentially involved in the formulation. The cohomology of such complexes defines a cohomology of a grading-restricted vertex algebras in the standard way. In this paper we develop ideas concerning algebraic structures following from the cohomology construction [6] for grading-restricted vertex algebras. We show that the orthogonality and double grading conditions applied to the bicomplex associated to a grading-restricted vertex algebra brings about the structure of the bigraded differential algebra with respect the commutator of bicomplex mappings. We show that the orthogonality being applied to the bicomplex spaces leads to relations among mappings and actions of coboundary operators. Using this condition we then find further explicit examples of continual Lie algebras [12] associated to vertex algebras. We derive also the simplest cohomological classes for the bicomplex for a grading-restricted vertex algebra. Such cohomological classes are non-vanishing and independent of the choice of the bicomplex space elements.

As for possible applications of the material presented in this paper, we would like to mention computations of higher cohomologies for grading-restricted vertex

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algebras [7], search for more complicated cohomological invariants, and applications to differential geometry. In particular, since vertex algebras is a useful computational tool, it would be interesting to study possible relations to cohomology of manifolds. One can show that such cohomological invariants possess analytical (with respect to the notion of composability) as well as geometrical meaning. In addition to the natural orthogonality condition, ona can consider variations of multiplications defined for bicomplex spaces, and, therefore, more advanced examples of graded differential algebras. In differential geometry there exist various approaches to the construction of cohomological classes (cf., in particular, [11]). We hope to use these techniques to derive counterparts in the cohomology theory of vertex algebras.

2. The grading-restricted vertex algebra

In this section, we recall [6] properties of grading-restricted vertex algebras and their grading-restricted generalized modules over the base field $\mathbb{C}$ of complex numbers. A vertex algebra $(V, Y_V, \mathbf{1}_V)$, cf. [9], consists of a $\mathbb{Z}$-graded complex vector space

$$V = \bigoplus_{n\in\mathbb{Z}} V_{(n)},$$

$$\dim V_{(n)} < \infty,$$

for each $n \in \mathbb{Z}$, and linear map

$$Y_V : V \to \mathrm{End}\ (V)[[z, z^{-1}]],$$

for a formal parameter z and a distinguished vector $\mathbf{1}_V \in V$. The evaluation of Y_V on $v \in V$ is the vertex operator

$$Y_V(v, z) = \sum_{n\in\mathbb{Z}} v(n) z^{-n-1},$$

with components

$$(Y_V(v))_n = v(n) \in \mathrm{End}\ (V),$$

where

$$Y_V(v, z)\mathbf{1} = v + O(z).$$

A grading-restricted vertex algebra [6], defining is subject to the following conditions:

(1) Grading-restriction condition: $V_{(n)}$ is finite dimensional for all $n \in \mathbb{Z}$, and $V_{(n)} = 0$, for $n \ll 0$.
(2) Lower-truncation condition: For $u, v \in V$, $Y_V(u, z)v$ contains only finitely many negative power terms, i.e., $Y_V(u, z)v \in V((z))$ (the space of formal Laurent series in z with coefficients in V).
(3) Identity property: Let Id_V be the identity operator on V. Then

$$Y_V(\mathbf{1}_V, z) = \mathrm{Id}_V.$$

(4) Creation property: For $u \in V$, $Y_V(u, z)\mathbf{1}_V \in V[[z]]$, and

$$\lim_{z\to 0} Y_V(u, z)\mathbf{1}_V = u.$$

(5) Duality: For $u_1, u_2, v \in V$, $v' \in V' = \coprod_{n\in\mathbb{Z}} V^*_{(n)}$ ($V^*_{(n)}$ denotes the dual vector space to $V_{(n)}$ and $\langle .,. \rangle$ the evaluation pairing $V' \otimes V \to \mathbb{C}$), the series $\langle v', Y_V(u_2, z_2) Y_V(u_1, z_1) v\rangle$, and $\langle v', Y_V(Y_V(u_1, z_1 - z_2) u_2, z_2) v\rangle$, are absolutely convergent in the regions $|z_1| > |z_2| > 0$, $|z_2| > |z_1| > 0$, $|z_2| > |z_1 - z_2| > 0$, respectively, to a common rational function in z_1 and z_2 with the only possible poles at $z_1 = 0 = z_2$ and $z_1 = z_2$.

One assumes the existence of Virasoro vector $\omega \in V$: its vertex operator

$$Y(\omega, z) = \sum_{n\in\mathbb{Z}} L(n) z^{-n-2},$$

is determined by Virasoro operators $L(n) : V \to V$ fulfilling

$$[L(m), L(n)] = (m - n) L(m + n) + \frac{c}{12}(m^3 - m)\, \delta_{m+b,0}\, \mathrm{Id_V},$$

(c is called the central charge of V). The grading operator is given by

$$L(0)u = nu, \quad u \in V_{(n)},$$

(n is called the weight of u and denoted by wt (u)).

(6) $L_V(0)$-bracket formula: Let $L_V(0) : V \to V$ be defined by $L_V(0)v = nv$ for $v \in V_{(n)}$. Then

$$[L_V(0), Y_V(v, z)] = Y_V(L_V(0)v, z) + z\frac{d}{dz} Y_V(v, z),$$

for $v \in V$.

(7) $L_V(-1)$-derivative property: Let $L_V(-1) : V \to V$ be the operator given by

$$L_V(-1)v = \mathrm{Res}_z z^{-2} Y_V(v, z)\mathbf{1} = Y_{(-2)}(v)\mathbf{1},$$

for $v \in V$. Then for $v \in V$,

$$\frac{d}{dz} Y_V(u, z) = Y_V(L_V(-1)u, z) = [L_V(-1), Y_V(u, z)].$$

A grading-restricted generalized V-module is a vector space W equipped with a vertex operator map

$$Y_W : V \otimes W \to W[[z, z^{-1}]],$$

$$u \otimes w \mapsto Y_W(u, w) \equiv Y_W(u, z)w = \sum_{n\in\mathbb{Z}} (Y_W)_n(u, w) z^{-n-1},$$

and linear operators $L_W(0)$ and $L_W(-1)$ on W, satisfying conditions similar as in the definition for a grading-restricted vertex algebra. In particular,

(1) Grading-restriction condition: The vector space W is $\mathbb{C}$-graded, i.e., $W = \coprod_{\alpha\in\mathbb{C}} W_{(\alpha)}$, such that $W_{(\alpha)} = 0$ when the real part of α is sufficiently negative.
(2) Lower-truncation condition: For $u \in V$ and $w \in W$, $Y_W(u, z)w$ contains only finitely many negative power terms, i.e., $Y_W(u, z)w \in W((z))$.
(3) Identity property: Let Id_W be the identity operator on W, $Y_W(\mathbf{1}_V, z) = \mathrm{Id}_W$.

(4) Duality: For u_1, $u_2 \in V$, $w \in W$, $w' \in W' = \coprod_{n\in\mathbb{Z}} W^*_{(n)}$ (W' is the dual V-module to W), the series $\langle w', Y_W(u_1, z_1)Y_W(u_2, z_2)w\rangle$, $\langle w', Y_W(u_2, z_2)Y_W(u_1, z_1)w\rangle$, and $\langle w', Y_W(Y_V(u_1, z_1-z_2)u_2, z_2)w\rangle$, are absolutely convergent in the regions $|z_1| > |z_2| > 0$, $|z_2| > |z_1| > 0$, $|z_2| > |z_1 - z_2| > 0$, respectively, to a common rational function in z_1 and z_2 with the only possible poles at $z_1 = 0 = z_2$ and $z_1 = z_2$.

(5) $L_W(0)$-bracket formula: For $v \in V$, $[L_W(0), Y_W(v, z)] = Y_W(L(0)v, z) + z\frac{d}{dz}Y_W(v, z)$.

(6) $L_W(0)$-grading property: For $w \in W_{(\alpha)}$, there exists $N \in \mathbb{Z}_+$ such that $(L_W(0) - \alpha)^N w = 0$.

(7) $L_W(-1)$-derivative property: For $v \in V$, $\frac{d}{dz}Y_W(u, z) = Y_W(L_V(-1)u, z) = [L_W(-1), Y_W(u, z)]$.

A unique symmetric invertible invariant bilinear form $\langle ., .\rangle$ with normalization

$$\langle \mathbf{1}_V, \mathbf{1}_V\rangle = 1,$$

where [2, 10]

$$\langle Y^\dagger(a, z)\ b, c\rangle = \langle b, Y(a, z)c\rangle\,, \tag{2.1}$$

for

$$\begin{aligned} Y^\dagger(a, z) &= \sum_{n\in\mathbb{Z}} a^\dagger(n)z^{-n-1} \\ &= Y\left(e^{zL_V(1)}\left(-z^{-2}\right)^{L_V(0)} a, z^{-1}\right). \end{aligned} \tag{2.2}$$

3. $\overline{W}$-valued rational functions over torsors

Let us recall the notion of multiple torsors to formulate definitions of chain-cochain double complex structure associated to grading-restricted vertex algebras. First, we recall here the general definition of ordinary torsors [1]. Let G be a group, and S a non–empty set. Then S is called a G-torsor, if it is equipped with a simply transitive right action of G, i.e., given x, $y \in S$, there exists a unique $g \in G$ such that

$$y = x \cdot g,$$

where the right action is given by

$$x \cdot (gh) = (x \cdot g) \cdot h.$$

The choice of any $x \in S$ allows us to identify S with G by sending $x \cdot g$ to g.

Let V be a grading-restricted vertex algebra, and W a grading-restricted generalized V-module. One defines the configuration spaces [6]:

$$F_n\mathbb{C} = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid z_i \neq z_j, i \neq j\},$$

for $n \in \mathbb{Z}_+$. Let V be a grading-restricted vertex algebra, and W a grading-restricted generalized V-module. By $\overline{W}$ we denote the algebraic completion of W,

$$\overline{W} = \prod_{n\in\mathbb{C}} W_{(n)} = (W')^*.$$

A $\overline{W}$-valued rational function in $(z_1, \ldots, z_n)$ with the only possible poles at $z_i = z_j$, $i \neq j$, is a map

$$\begin{aligned} f : F_n\mathbb{C} &\to \overline{W}, \\ (z_1, \ldots, z_n) &\mapsto f(z_1, \ldots, z_n), \end{aligned}$$

such that for any $w' \in W'$,

$$\langle w', f(z_1, \ldots, z_n)\rangle, \tag{3.1}$$

is a rational function in $(z_1, \ldots, z_n)$ with the only possible poles at $z_i = z_j$, $i \neq j$. Such map one calls $\overline{W}$-valued rational function in $(z_1, \ldots, z_n)$ with possible other poles. Denote the space of all $\overline{W}$-valued rational functions in $(z_1, \ldots, z_n)$ by $\overline{W}_{z_1,\ldots,z_n}$. One defines a left action of S_n on $\overline{W}_{z_1,\ldots,z_n}$ by

$$(\sigma(f))(z_1, \ldots, z_n) = f(z_{\sigma(1)}, \ldots, z_{\sigma(n)}),$$

for $f \in \overline{W}_{z_1,\ldots,z_n}$.

The idea to use torsors [1] is to represent the action of an element ρ of the group of automorphism of a set of formal variables $(z_1, \ldots, z_n)$. by action of V-operators on a set of vertex algebra element $v \in V$. In particular, one consider a vector $(v_i, z_1, \ldots, z_n)$ with $v_i \in V$. Then the same vector equals

$$\left(R^{-1}\left(\rho\right) v_i, w_1, \ldots, w_n\right),$$

i.e., it is identified with

$$R^{-1}\left(\rho\right) v_i \in V,$$

using the transformed formal parameters $(w_1, \ldots, w_n)$. Here $R\left(\rho\right)$ is an operator representing transformation of $z_i \to w_i$, as an action on V. Therefore if we have an operator which is equal to a torsor S of the group of automorphism of $(z_1, \ldots, z_n)$. Then this operator equals

$$R\left(\rho\right)\ S\ R^{-1}\left(\rho\right).$$

Thus, in terms of the coordinates $(v_i, z_1, \ldots, z_n)$, the differential $Y_W(v_i, w_i)\, dw_i^{\mathrm{wt}\ (v_i)}$ becomes

$$Y_W(v_i, z_i)\, dz_i^{\mathrm{wt}\ (v_i)} = R(\rho)\ Y_W\left(v_i, \rho(z_1, \ldots, z_n)\right)\ R^{-1}(\rho)\, dw_i^{\mathrm{wt}\ (v_i)}.$$

In what follows, we will use corresponding formalism of multiple torsors, i.e., we consider vectors $(v_1 \otimes \ldots \otimes v_n)(z_1, \ldots, z_n)$ for $v_i \in V$, $1 \leq i \leq n$. Then, $\overline{W}$-valued function Φ are multiple element $v_i \in V$ torsors with respect to to group of automorphisms of $(z_1, \ldots, z_n)$.

For $w \in W$, the $\overline{W}$-valued function [8] $E^{(n)}_W(v_1 \otimes \cdots \otimes v_n; w)$ is given by

$$E^{(n)}_W(v_1 \otimes \cdots \otimes v_n; w)(z_1, \ldots, z_n) = E(Y_W(v_1, z_1) \cdots Y_W(v_n, z_n)w),$$

where an element $E(.) \in \overline{W}$ is given by

$$\langle w', E(.)\rangle = R(\langle w', .\rangle),$$

and $R(.)$ denotes the rationalization in the sense of [6]. Namely, if a meromorphic function $f(z_1, \ldots, z_n)$ on a region in $\mathbb{C}^n$ can be analytically extended to a rational function in $(z_1, \ldots, z_n)$, then the notation $R(f(z_1, \ldots, z_n))$ is used to denote such rational function. One defines

$$E^{W;(n)}_{WV}(w; v_1 \otimes \cdots \otimes v_n) = E^{(n)}_W(v_1 \otimes \cdots \otimes v_n; w),$$

where $E^{W;(n)}_{WV}(w; v_1 \otimes \cdots \otimes v_n)$ is an element of $\overline{W}_{z_1,\ldots,z_n}$. One defines

$$\Phi \circ \left(E^{(l_1)}_{V;\,\mathbf{1}} \otimes \cdots \otimes E^{(l_n)}_{V;\,\mathbf{1}} \right) : V^{\otimes m+n} \to \overline{W}_{z_1,\ldots,z_{m+n}},$$

by

$$\begin{aligned}
&(\Phi \circ (E^{(l_1)}_{V;\,\mathbf{1}} \otimes \cdots \otimes E^{(l_n)}_{V;\,\mathbf{1}}))(v_1 \otimes \cdots \otimes v_{m+n-1}) \\
&\quad = E(\Phi(E^{(l_1)}_{V;\mathbf{1}}(v_1 \otimes \cdots \otimes v_{l_1}) \otimes \cdots E^{(l_n)}_{V;\mathbf{1}}(v_{l_1+\cdots+l_{n-1}+1} \otimes \cdots \otimes v_{l_1+\cdots+l_{n-1}+l_n}))),
\end{aligned}$$

and

$$E^{(m)}_W \circ_{m+1} \Phi : V^{\otimes m+n} \to \overline{W}_{z_1,\ldots,z_{m+n-1}},$$

is given by

$$(E^{(m)}_W \circ_{m+1} \Phi)(v_1 \otimes \cdots \otimes v_{m+n}) = E(E^{(m)}_W(v_1 \otimes \cdots \otimes v_m; \Phi(v_{m+1} \otimes \cdots \otimes v_{m+n}))).$$

Finally,

$$E^{W;(m)}_{WV} \circ_0 \Phi : V^{\otimes m+n} \to \overline{W}_{z_1,\ldots,z_{m+n-1}},$$

is defined by

$$(E^{W;(m)}_{WV} \circ_0 \Phi)(v_1 \otimes \cdots \otimes v_{m+n}) = E(E^{W;(m)}_{WV}(\Phi(v_1 \otimes \cdots \otimes v_n); v_{n+1} \otimes \cdots \otimes v_{n+m})).$$

In the case that

$$l_1 = \cdots = l_{i-1} = l_{i+1} = 1,$$

$$l_i = m - n - 1,$$

for some $1 \le i \le n$, we will use $\Phi \circ_i E^{(l_i)}_{V;\,\mathbf{1}}$ to denote $\Phi \circ (E^{(l_1)}_{V;\,\mathbf{1}} \otimes \cdots \otimes E^{(l_n)}_{V;\,\mathbf{1}})$.

One defines an action of S_n on the space $\mathrm{Hom}(V^{\otimes n}, \overline{W}_{z_1,\ldots,z_n})$ of linear maps from $V^{\otimes n}$ to $\overline{W}_{z_1,\ldots,z_n}$ by

$$(\sigma(\Phi))(v_1 \otimes \cdots \otimes v_n) = \sigma(\Phi(v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)})),$$

for $\sigma \in S_n$ and $v_1, \ldots, v_n \in V$. We will use the notation $\sigma_{i_1,\ldots,i_n} \in S_n$, to denote the the permutation given by $\sigma_{i_1,\ldots,i_n}(j) = i_j$, for $j = 1, \ldots, n$. In [6] one finds:

Proposition 1. *The subspace of* $\mathrm{Hom}(V^{\otimes n}, \overline{W}_{z_1,\ldots,z_n})$ *consisting of linear maps having the* $L(-1)$*-derivative property, having the* $L(0)$*-conjugation property or being composable with* m *vertex operators is invariant under the action of* S_n*.*

3.1. **Maps composable with vertex operators.** For a V-module

$$W = \coprod_{n \in \mathbb{C}} W_{(n)},$$

and $m \in \mathbb{C}$, let

$$P_m : \overline{W} \to W_{(m)},$$

be the projection from $\overline{W}$ to $W_{(m)}$. Let

$$\Phi : V^{\otimes n} \to \overline{W}_{z_1, \dots, z_n},$$

be a linear map. For $m \in \mathbb{N}$, Φ is said [6, 4] to be composable with m vertex operators if the following conditions are satisfied:

(1) Let $l_1, \dots, l_n \in \mathbb{Z}_+$ such that

$$l_1 + \cdots + l_n = m + n,$$

$v_1, \dots, v_{m+n} \in V$ and $w' \in W'$. Introduce

$$\Psi_i = E_V^{(l_i)}(v_{k_1} \otimes \cdots \otimes v_{k_i}; \mathbf{1}_V)(z_{k_1}, \dots, z_{k_i}),$$

where

$$k_1 = l_1 + \cdots + l_{i-1} + 1, \ \ ..., \ \ v_{k_i} = l_1 + \cdots + l_{i-1} + l_i,$$

for $i = 1, \dots, n$.

Then there exist positive integers $N_m^n(v_i, v_j)$ depending only on v_i and v_j for $i, j = 1, \dots, k$, $i \neq j$ such that the series

$$\sum_{r_1, \dots, r_n \in \mathbb{Z}} \langle w', (\Phi(P_{r_1}\Psi_1 \otimes \cdots \otimes P_{r_n}\Psi_n))(\zeta_1, \dots, \zeta_n)\rangle,$$

is absolutely convergent when

$$|z_{l_1+\cdots+l_{i-1}+p} - \zeta_i| + |z_{l_1+\cdots+l_{j-1}+q} - \zeta_i| < |\zeta_i - \zeta_j|,$$

for i, $j = 1, \dots, k$, $i \neq j$, and for $p = 1, \dots, l_i$ and $q = 1, \dots, l_j$. The sum must be analytically extended to a rational function in $(z_1, \dots, z_{m+n})$, independent of $(\zeta_1, \dots, \zeta_n)$, with the only possible poles at $z_i = z_j$, of order less than or equal to $N_m^n(v_i, v_j)$, for $i, j = 1, \dots, k$, $i \neq j$.

(2) For $v_1, \dots, v_{m+n} \in V$, there exist positive integers $N_m^n(v_i, v_j)$, depending only on v_i and v_j, for i, $j = 1, \dots, k$, $i \neq j$, such that for $w' \in W'$, and

$$\mathbf{v}_{n,m} = (v_{1+m} \otimes \cdots \otimes v_{n+m}),$$

$$\mathbf{z}_{n,m} = (z_{1+m}, \dots, z_{n+m}),$$

such that

$$\sum_{q \in \mathbb{C}} \langle w', (E_W^{(m)}(v_1 \otimes \cdots \otimes v_m; P_q((\Phi(\mathbf{v}_{n,m}))(\mathbf{z}_{n,m})))\rangle,$$

is absolutely convergent when $z_i \neq z_j$, $i \neq j$ $|z_i| > |z_k| > 0$ for $i = 1, \dots, m$, and $k = m + 1, \dots, m + n$, and the sum can be analytically extended to a rational function in $(z_1, \dots, z_{m+n})$ with the only possible poles at $z_i = z_j$, of orders less than or equal to $N_m^n(v_i, v_j)$, for $i, j = 1, \dots, k$, $i \neq j$,.

A linear map

$$\Phi : V^{\otimes n} \to \overline{W}_{z_1,\dots,z_n},$$

is said to have the $L(0)$-conjugation property if for $(v_1, \dots, v_n) \in V$, $w' \in W'$, $(z_1, \dots, z_n) \in F_n\mathbb{C}$ and $z \in \mathbb{C}^{\times}$, so that $(zz_1, \dots, zz_n) \in F_n\mathbb{C}$,

$$\begin{aligned} &\langle w', z^{L_W(0)}(\Phi(v_1 \otimes \cdots \otimes v_n))(z_1, \dots, z_n)\rangle \\ &\quad = \langle w', (\Phi(z^{L(0)}v_1 \otimes \cdots \otimes z^{L(0)}v_n))(zz_1, \dots, zz_n)\rangle. \end{aligned} \tag{3.2}$$

For $n \in \mathbb{Z}_+$, a linear map

$$\Phi : V^{\otimes n} \to \overline{W}_{z_1,\dots,z_n},$$

is said to have the $L(-1)$-derivative property if (i)

$$\begin{aligned} &\frac{\partial}{\partial z_i}\langle w', (\Phi(v_1 \otimes \cdots \otimes v_n))(z_1, \dots, z_n)\rangle \\ &\quad = \langle w', (\Phi(v_1 \otimes \cdots \otimes v_{i-1} \otimes L_V(-1)v_i \otimes v_{i+1} \otimes \cdots \otimes v_n))(z_1, \dots, z_n)\rangle, \end{aligned} \tag{3.3}$$

for $i = 1, \dots, n$, $(v_1, \dots, v_n) \in V$, and $w' \in W'$ and (ii)

$$\begin{aligned} &\left(\frac{\partial}{\partial z_1} + \cdots + \frac{\partial}{\partial z_n}\right)\langle w', (\Phi(v_1 \otimes \cdots \otimes v_n))(z_1, \dots, z_n)\rangle \\ &\quad = \langle w', L_W(-1)(\Phi(v_1 \otimes \cdots \otimes v_n))(z_1, \dots, z_n)\rangle, \end{aligned} \tag{3.4}$$

and $(v_1, \dots, v_n) \in V$, $w' \in W'$.

4. Vertex algebra bicomplexes

Let us recall [6] here the definition of shuffles. For $l \in \mathbb{N}$ and $1 \le s \le l-1$, let $J_{l;s}$ be the set of elements of S_l which preserve the order of the first s numbers and the order of the last $l-s$ numbers, i.e.,

$$J_{l,s} = \{\sigma \in S_l \mid \sigma(1) < \cdots < \sigma(s),\ \sigma(s+1) < \cdots < \sigma(l)\}.$$

The elements of $J_{l;s}$ are called shuffles. Let

$$J_{l;s}^{-1} = \{\sigma \mid \sigma \in J_{l;s}\}.$$

Let V be a vertex operator algebra and W a V-module. For $n \in \mathbb{Z}_+$, let $C_0^n(V, W)$ be the vector space of all linear maps from $V^{\otimes n}$ to $\overline{W}_{z_1,\dots,z_n}$ satisfying the $L(-1)$-derivative property and the $L(0)$-conjugation property. For m, $n \in \mathbb{Z}_+$, let $C_m^n(V, W)$ be the vector spaces of all linear maps from $V^{\otimes n}$ to $\overline{W}_{z_1,\dots,z_n}$ composable with m vertex operators, and satisfying the $L(-1)$-derivative property, the $L(0)$-conjugation property, and such that

$$\sum_{\sigma \in J_{l;s}^{-1}} (-1)^{|\sigma|}\sigma\left(\Phi(v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(l)})\right) = 0. \tag{4.1}$$

We also find in [6]

Proposition 2. *Let $C_m^0(V, W) = W$. Then we have*

$$C_m^n(V, W) \subset C_{m-1}^n(V, W),$$

for $m \in \mathbb{Z}_+$.

In [6] the coboundary operator for the bicomplex spaces $C_m^n(V,W)$ was introduced:

$$\delta_m^n : C_m^n(V,W) \to C_{m-1}^{n+1}(V,W). \tag{4.2}$$

For $\Phi \in C_m^n(V,W)$, it is given by

$$\delta_m^n(\Phi) = E_W^{(1)} \circ_2 \Phi + \sum_{i=1}^{n} (-1)^i \Phi \circ_i E_{V;\mathbf{1}}^{(2)} + (-1)^{n+1} \sigma_{n+1,1,\dots,n}(E_W^{(1)} \circ_2 \Phi), \tag{4.3}$$

where $\circ_i$ is defined in Section 1. Explicitly, for $v_1, \dots, v_{n+1} \in V$, $w' \in W'$ and $(z_1, \dots, z_{n+1}) \in F_{n+1}\mathbb{C}$,

$$\begin{aligned}
&\langle w', ((\delta_m^n(\Phi))(v_1 \otimes \cdots \otimes v_{n+1}))(z_1, \dots, z_{n+1})\rangle \\
&\quad = R(\langle w', Y_W(v_1, z_1)(\Phi(v_2 \otimes \cdots \otimes v_{n+1}))(z_2, \dots, z_{n+1})\rangle) \\
&\quad\quad + \sum_{i=1}^{n} (-1)^i R(\langle w', (\Phi(v_1 \otimes \cdots \otimes v_{i-1} \otimes Y_V(v_i, z_i - z_{i+1}) v_{i+1} \\
&\quad\quad\quad\quad \otimes \cdots \otimes v_{n+1}))(z_1, \dots, z_{i-1}, z_{i+1}, \dots, z_{n+1})\rangle) \\
&\quad\quad + (-1)^{n+1} R(\langle w', Y_W(v_{n+1}, z_{n+1})(\Phi(v_1 \otimes \cdots \otimes v_n))(z_1, \dots, z_n)\rangle).
\end{aligned}$$

In the case $n = 2$, there is a subspace of $C_0^2(V,W)$ containing $C_m^2(V,W)$ for all $m \in \mathbb{Z}_+$ such that δ_m^2 is still defined on this subspace. Let $C_{\frac{1}{2}}^2(V,W)$ be the subspace of $C_0^2(V,W)$ consisting of elements Φ such that for $v_1, v_2, v_3 \in V$, $w' \in W'$,

$$\begin{aligned}
\sum_{r \in \mathbb{C}} \big(&\langle w', E_W^{(1)}(v_1; P_r((\Phi(v_2 \otimes v_3))(z_2 - \zeta, z_3 - \zeta)))(z_1, \zeta)\rangle \\
&+ \langle w', (\Phi(v_1 \otimes P_r((E_V^{(2)}(v_2 \otimes v_3; \mathbf{1}))(z_2 - \zeta, z_3 - \zeta))))(z_1, \zeta)\rangle \big),
\end{aligned}$$

$$\begin{aligned}
\sum_{r \in \mathbb{C}} \big(&\langle w', (\Phi(P_r((E_V^{(2)}(v_1 \otimes v_2; \mathbf{1}))(z_1 - \zeta, z_2 - \zeta)) \otimes v_3))(\zeta, z_3)\rangle \\
&+ \langle w', E_{WV}^{W;(1)}(P_r((\Phi(v_1 \otimes v_2))(z_1 - \zeta, z_2 - \zeta)); v_3))(\zeta, z_3)\rangle \big),
\end{aligned}$$

are absolutely convergent in the regions

$$|z_1 - \zeta| > |z_2 - \zeta|,$$

$$|z_2 - \zeta| > 0,$$

$$|\zeta - z_3| > |z_1 - \zeta|,$$

$$|z_2 - \zeta| > 0,$$

respectively, and can be analytically extended to rational functions in z_1 and z_2 with the only possible poles at $z_1, z_2 = 0$ and $z_1 = z_2$. It is clear that

$$C_m^2(V,W) \subset C_{\frac{1}{2}}^2(V,W),$$

for $m \in \mathbb{Z}_+$. The coboundary operator

$$\delta_{\frac{1}{2}}^2 : C_{\frac{1}{2}}^2(V,W) \to C_0^3(V,W), \tag{4.4}$$

is defined in [6] by

$$\delta^2_{\frac{1}{2}}(\Phi) = E^{(1)}_W \circ_2 \Phi + \sum_{i=1}^{2} (-1)^i E^{(2)}_{V,\mathbf{1}_V} \circ_i \Phi + E^{W;(1)}_{WV} \circ_2 \Phi, \tag{4.5}$$

$$\begin{aligned}
&\langle w', ((\delta^2_{\frac{1}{2}}(\Phi))(v_1 \otimes v_2 \otimes v_3))(z_1, z_2, z_3)\rangle \\
&= R(\langle w', (E^{(1)}_W(v_1; \Phi(v_2 \otimes v_3))(z_1, z_2, z_3)\rangle \\
&\quad + \langle w', (\Phi(v_1 \otimes E^{(2)}_V(v_2 \otimes v_3; \mathbf{1})))(z_1, z_2, z_3)\rangle) \\
&\quad - R(\langle w', (\Phi(E^{(2)}_V(v_1 \otimes v_2; \mathbf{1})) \otimes v_3))(z_1, z_2, z_3)\rangle \\
&\quad + \langle w', (E^{W;(1)}_{WV}(\Phi(v_1 \otimes v_2); v_3))(z_1, z_2, z_3)\rangle)
\end{aligned}$$

for $w' \in W'$, $\Phi \in C^2_{\frac{1}{2}}(V, W)$, $v_1, v_2, v_3 \in V$ and $(z_1, z_2, z_3) \in F_3\mathbb{C}$.

Using the operators δ^n_m and $\delta^2_{\frac{1}{2}}$, for $m \in \mathbb{Z}_+$ and $n \in \mathbb{N}$, one introduces [6] the n-th cohomology $H^n_m(V, W)$ of a grading-restricted vertex algebra V with coefficient in W, and composable with m vertex operators to be

$$H^n_m(V, W) = \ker \delta^n_m / \mathrm{im}\ \delta^{n-1}_{m+1},$$

$$H^2_{\frac{1}{2}}(V, W) = \ker \delta^2_{\frac{1}{2}} / \mathrm{im}\ \delta^1_2,$$

Then one has [6]

Proposition 3. *For $n \in \mathbb{N}$ and $m \in \mathbb{Z}_+ + 1$, the coboundary operators* (4.3) *and* (4.5) *satisfy the cochain complex conditions, i.e.,*

$$\delta^{n+1}_{m-1} \circ \delta^n_m = 0,$$

$$\delta^2_{\frac{1}{2}} \circ \delta^1_2 = 0.$$

By Proposition 3, we have complexes for $m \geq 0$,

$$0 \longrightarrow C^0_m(V, W) \xrightarrow{\delta^0_m} C^1_{m-1}(V, W) \xrightarrow{\delta^1_{m-1}} \cdots \xrightarrow{\delta^{m-1}_1} C^m_0(V, W) \longrightarrow 0, \tag{4.6}$$

$$0 \longrightarrow C^0_3(V, W) \xrightarrow{\delta^0_3} C^1_2(V, W) \xrightarrow{\delta^1_2} C^2_{\frac{1}{2}}(V, W) \xrightarrow{\delta^2_{\frac{1}{2}}} C^3_0(V, W) \longrightarrow 0. \tag{4.7}$$

The first and last mappings are trivial embeddings and projections. For simplicity, we call derivatives the actions of δ^m_n and $\delta^{\frac{1}{2}}_2$ on the mappings.

5. Bigraded differential algebras associated to a vertex algebra

A natural task now is to introduce an appropriate multiplication on bicomplex spaces of (4.6) and (4.7), and derived an analogue of Leibniz formula. Let us consider two mappings

$$\Phi(v_1 \otimes \ldots \otimes v_n)(z_1, \ldots, z_n) \in C^n_m(V, W),$$

$$\Psi(v_{n+1} \otimes \ldots \otimes v_{n+n'})(z_{n+1}, \ldots, z_{n+n'}) \in C^{n'}_{m'}(V, W),$$

which have r common vertex algebra elements and formal variables, and t common vertex operators that mappings Φ and Ψ are composable with. Note that when applying the coboundary operators (4.3) and (4.5) to a map

$$\chi(v_1 \otimes \ldots \otimes v_n)(z_1, \ldots, z_n) \in C^n_m(V, W),$$

$$\delta^n_m : \chi(v_1 \otimes \ldots \otimes v_n)(z_1, \ldots, z_n) \to \chi(v'_1 \otimes \ldots \otimes v'_n)(z'_1, \ldots, z'_{n+1}) \in C^{n+1}_{m-1}(V, W),$$

one does not necessary assume that we keep the same set of vertex algebra elements and formal parameters, as well as the set of vertex operators composable with for $\delta^n_m \Phi$, though it might happen that some of them could be common with Φ.

For our particular purposes of introduction of a bigraded differential structure associated to double chain-cochain complexes (4.3) and (4.5), we define the product of Φ and Ψ above as

$$\Phi \cdot \Psi : V^{\otimes(n+n')} \to C^{n+n'-r}_{m+m'-t}(V, W), \tag{5.1}$$

$$\Phi \cdot \Psi = [\Phi, \Psi] = \Phi * \Psi - \Psi * \Phi, \tag{5.2}$$

where brackets mean the commutator with respect to the geometrically defined product $*$ of elements of the spaces $C^n_m(V, W)$ and $C^{n'}_{m'}(V, W)$, introduced in Appendix 8. In the case

$$\Psi(v_{n+1} \otimes \ldots \otimes v_{n+n'})(z_{n+1}, \ldots, z_{n+n'}) = \Phi(v_1 \otimes \ldots \otimes v_n)(z_1, \ldots, z_n),$$

we obtain from (5.2) that

$$\Phi(v_1 \otimes \ldots \otimes v_n)(z_1, \ldots, z_n) \cdot \Phi(v_1 \otimes \ldots \otimes v_n)(z_1, \ldots, z_n) = 0. \tag{5.3}$$

The coboundary operators δ^n_m and $\delta^2_{\frac{1}{2}}$ possess a variation of Leibniz law with respect to the product (5.2).

In Appendix 8 we prove the following

Proposition 4. *For arbitrary $w' \in W'$ one has*

$$R\left(\langle w', \delta^{n+n'-r}_{m+m'-t} (\Phi \cdot \Psi)\rangle\right) = R\left(\langle w', \delta^n_m (\Phi) \cdot \Psi\rangle\right) + (-1)^n R\left(\langle w', \Phi \cdot \delta^{n'}_{m'} (\Psi)\rangle\right). \tag{5.4}$$

In this section we provide the main results of the paper by deriving relations for the double graded differential algebras and the cohomological invariants associated to bicomplexes (4.6) and (4.7) for a grading-restricted vertex algebra. Let us give first some further definitions. In this section we skip the dependence on vertex algebra elements and formal parameters in notations for elements of $C^m_n(V, W)$. In analogy with differential forms, we call a map $\Phi \in C^n_m(V, W)$ closed if

$$\delta^n_m \Phi = 0.$$

For $m \geq 1$, we call $\Phi \in C^n_m(V, W)$ exact if there exists $\Psi \in C^{n-1}_{m+1}(V, W)$ such that

$$\Psi = \delta^n_m \Phi.$$

For $\Phi \in C^n_m(V, W)$ we call the cohomology class of mappings $[\Phi]$ the set of all closed forms that differs from Φ by an exact mapping, i.e., for $\chi \in C^{n-1}_{m+1}$,

$$[\Phi] = \Phi + \delta^{n-1}_{m+1} \chi,$$

where it is assumed that both parts of the last formula belongs to the same space $C^n_m(V, W)$.

Under a natural extra condition, the bicomplexes (4.6) and (4.7) allow us to establish relations among elements of bicomplex spaces. By analogy with the notion of integrability for differential forms [5], we introduce here the notion of orthogonality for spaces of a complex. Suppose we consider a complex given by

$$0 \longrightarrow C_0 \xrightarrow{\delta_0} C_1 \xrightarrow{\delta_1} C_2 \xrightarrow{\delta_2} \cdots \xrightarrow{\delta_{n-1}} C_n \xrightarrow{\delta_n} \cdots \tag{5.5}$$

In particular, let us require that for a pair of a bicomplex spaces C_i and C_j, $i, j \geq 0$. there exist subspaces $C'_i \subset C_i$ and such that for $\Phi_i \in C'_i$ and $\Phi_j \in C'_j$,

$$\Phi_i \cdot \delta_j \Phi_j = 0, \tag{5.6}$$

namely, Φ_i supposed to be orthogonal to $\delta_j \Phi_j$ (i.e., commutative with respect to the product (5.2)). We call this the orthogonality condition for mappings of a complex.

It is easy to see that the assumption to belong to the same bicomplex space for both sides of equations following from orthogonality condition applies the double grading condition on bicomplex spaces. Note that in the case of differential forms considered on a smooth manifold, the Frobenius theorem for a distribution provides the orthogonality condition. In this Section we derive algebraic relations occurring from the orthogonality condition on the bicomplex (4.6).

We formulate the first main result of this paper:

Proposition 5. *The orthogonality condition for the bicomplexes* (4.6) *and* (4.7) *brings about the structure of a double graded differential algebra with respect to the multiplication* (5.2).

Proof. Let us consider the most general case. For non-negative n_0, n, n_1, m_0, m, m_1, let $\chi \in C^{n_0}_{m_0}(V, W)$, $\Phi \in C^n_m(V, W)$, and $\alpha \in C^{n_1}_{m_1}(V, W)$. For Φ and α, let r_0 be the number of common vertex algebra elements (and formal parameters), and t_0 be the number of common vertex operators Φ and α are composable to. Note that we assume n, $n_1 \geq r_0$, m, $m_1 \geq t_0$. Applying the ortogonality condition

$$\Phi \cdot \delta^{n_0}_{m_0} \chi = 0,$$

implies that there exist $\alpha_1 \in C^{n_1}_{m_1}(V, W)$, such that

$$\delta^{n_0}_{m_0} \chi = \Phi \cdot \alpha_1.$$

From the last equations we obtain

$$n_0 + 1 = n + n_1 - r_0,$$

$$m_0 - 1 = m + m_1 - t_0.$$

Note that we have extra conditions following from the last identities: $n_0 + 1 \geq 0$, $m_0 - 1 \geq 0$. The conditions above for indexes express the double grading condition for the bicomplexes (4.6) and (4.7). As a result, we have a system in integer variables satisfying the grading conditions above. Consequently applying corresponding derivatives we obtain the full structure of differential relations

$$\begin{aligned}
&\Phi \cdot \delta^{n_0}_{m_0}\chi = 0,\\
&\delta^{n_0}_{m_0}\chi = \Phi \cdot \alpha_1,\\
&0 = \delta^n_m \Phi \cdot \alpha_1 + (-1)^n \Phi \cdot \delta^{n_1}_{m_1}\alpha_1,\\
&0 = \delta^n_m \Phi \cdot \delta^{n_0}_{m_0}\chi,\\
&\delta^{n_0}_{m_0}\chi = \delta^n_m \Phi \cdot \alpha_2,\\
&0 = \delta^n_m \Phi \cdot \delta^{n_i}_{m_i}\alpha_i,\\
&\delta^{n_i}_{m_i}\alpha_i = \delta^n_m \Phi \cdot \alpha_{i+1}, \quad i \geq 2,
\end{aligned} \tag{5.7}$$

where $\alpha_i \in C^{n_i}_{m_i}(V,W)$, and n_i, m_i, $i \geq 2$ satisfy relations

$$n_i = n + n_{i+1} - r_{i+1},$$

$$m_i = m + m_{i+1} - t_{i+1}.$$

The sequence of relations (5.7) does not cancel until the conditions on indeces given above fullfil. $\square$

Thus, we see that the orthogonality condition for the bicomplexes (4.6) and (4.7) together with the action of coboundary operators δ^n_m and $\delta^2_{\frac{1}{2}}$, and the multiplication formulas (5.2)–(5.4), define a differential algebra depending on vertex algebra elements and formal parameters. In particular, in that way we obtain the generators and commutation relations for a continual Lie algebra $\mathcal{G}(V)$ (a generalization of ordinary Lie algebras with continual space of roots, c.f. [12]) with the continual root space represented by a grading-restricted vertex algebra V.

Proposition 6. *For the bicomplex* (4.6) *the generators*

$$\left\{\chi\ , \Phi, \alpha_i,\ \delta^{n_0}_{m_0}\chi, \delta^{n_1}_{m_1}\Phi,\ \delta^{n_i}_{m_i}\alpha_i,\ \right\},$$

with $i \geq 0$, and commutation relations (5.7) *form a continual Lie algebra $\mathcal{G}(V)$ with a root space provided by the grading-restricted vertex algebra V.*

Proof. With the redefinition (we suppress here the dependence on vertex algebra elements and formal parameters),

$$H_0 = \delta^0_3\chi,$$

$$H^*_0 = \chi,$$

$$X_{+1} = \Phi,$$

$$X_{-i} = \alpha_i,$$

$$Y_{+1} = \delta^1_2\Phi,$$

$$Y_{-i} = \delta^1_t\alpha_i,$$

we arrive at the commutation relations:

$$\begin{aligned}
&[H_0, X_{+1}] = 0,\\
&[X_{+1}, X_{-1}] = H_0,\\
&[Y_{+1}, X_{-1}] = (-1)^n \, [Y_{-1}, X_{+1}],\\
&[Y_{+1}, X_{-2}] = 0,\\
&[Y_{+1}, Y_{-i}] = 0,\\
&\left[Y_{+1}, X_{-(i+1)}\right] = Y_{-i}.
\end{aligned}$$

One easily checks Jacobi identities for generators. □

From the above proposition we see that, under the orthogonality and grading conditions, the bicomplexes (4.6) and (4.7) provide the set of relations among mappings and their derivatives.

Proposition 7. *The set of commutation relations generates a sequence of non-vanishing cohomological classes:*

$$\left[\left(\delta^{n_0}_{m_0}\chi\right)\cdot\chi\right], \ \left[\left(\delta^{n}_{m}\Phi\right)\cdot\Phi\right], \ \left[\left(\delta^{n_i}_{n_i}\alpha_i\right)\cdot\alpha_i\right],$$

for $i = 1, \ldots, L$, for some $L \in \mathbb{N}$, with non-vanishing $\left(\delta^{n_0}_{m_0}\chi\right)\cdot\chi$, $\left(\delta^{n}_{m}\Phi\right)\cdot\Phi$, and $\left(\delta^{n_i}_{m_i}\alpha_i\right)\cdot\alpha_i$. These classes are independent on the choices of $\chi \in C^{n_0}_{m_0}(V,W)$, $\Phi \in C^{n}_{m}(V,W)$, and $\alpha_i \in C^{n_i}_{m_i}(V,W)$.

Proof. Let ϕ be one of generators $\chi, \Phi, \alpha_i, \beta$, $1 \le i \le L$. Let us show now the non-vanishing property of $\left(\left(\delta^1_2\phi\right)\cdot\phi\right)$. Indeed, suppose

$$\left(\delta^n_m\phi\right)\cdot\phi = 0.$$

Then there exists $\gamma \in C^{n'}_{m'}(V,W)$, such that

$$\delta^n_m\phi = \gamma\cdot\phi.$$

Both sides of the last equality should belong to the same bicomplex space but one can see that it is not possible since we obtain $m' = t-1$, i.e., the number of common vertex operators for the last equation is greater than for one of multipliers. Thus, $(\delta^n_m\phi)\cdot\phi$ is non-vanishing.

Now let us show that $[(\delta^n_m\phi)\cdot\phi]$ is invariant, i.e., it does not depend on the choice of $\Phi \in C^n_m(V,W)$. Substitute ϕ by $(\phi+\eta) \in C^n_m(V,W)$. We have

$$\begin{aligned}
\left(\delta^n_m\left(\phi+\eta\right)\right)\cdot(\phi+\eta) \quad &= \quad (\delta^n_m\phi)\cdot\phi + \left((\delta^n_m\phi)\cdot\eta - \phi\cdot\delta^n_m\eta\right)\\
&+ \quad \left(\phi\cdot\delta^n_m\eta + \delta^n_m\eta\cdot\phi\right) + (\delta^n_m\eta)\cdot\eta.
\end{aligned} \tag{5.8}$$

Since

$$\left(\phi\cdot\delta^n_m\eta + (\delta^n_m\eta)\cdot\phi\right) = \phi\delta^n_m\eta - (\delta^n_m\eta)\phi + (\delta^n_m\eta)\,\phi - \phi\,\delta^n_m\eta = 0,$$

then (5.8) represents the same cohomology class $[(\delta^n_m\phi)\cdot\phi]$. □

6. Examples

In this section we consider particularly interesting examples of algebras described in Proposition 6. The orthogonality condition for a bicomplex sequence (4.7), together with the action of coboundary operators δ^n_m and $\delta^2_{\frac{1}{2}}$, and the multiplication formulas (5.2)–(5.4), define a differential bigraded algebra depending on vertex algebra elements and formal parameters. In particular, for the bicomplex (4.6), we obtain in this way the generators and commutation relations for a continual Lie algebra $\mathcal{G}(V)$ (a generalization of ordinary Lie algebras with continual space of roots, c.f. [12]) with the continual root space represented by a grading-restricted vertex algebra V.

6.1. **Invariants associated with $C^2_{\frac{1}{2}}(V,W)$.** Due to non-trivial action of the derivative

$$\delta^2_{\frac{1}{2}} : C^2_{\frac{1}{2}}(V,W) \to C^3_0(V,W),$$

the case when $\Phi \in C^2_{\frac{1}{2}}$ is exceptional among the relations coming from the double grading condition for a vertex algebra bicomplex, and allow to reconstruct classical invariants. Let us consider this case. Let $\Phi \in C^2_{\frac{1}{2}}(V,W)$, and $\chi \in C^n_m(V,W)$. Then we require the orthogonality:

$$\chi \cdot \delta^2_{\frac{1}{2}} \Phi = 0.$$

Thus there exist $\beta \in C^{n'}_{m'}(V,W)$ such that

$$\delta^2_{\frac{1}{2}} \Phi = \chi \cdot \beta.$$

We then get

$$3 = n + n' - r,$$
$$0 = m + m' - t,$$

Let $n = r+\alpha$, $0 \le \alpha \le 3$, $n' = 3-\alpha$; $m' = t-m \ge 0$, i.e., $t = m$, thus $m = t = m' = 0$. Thus, $\chi \in C^{r+\alpha}_0(V,W)$, $\beta \in C^{3-\alpha}_0(V,W)$. For $r + \alpha = 3 - \alpha = 2$ we obtain $\alpha = 1$, $r = 1$. If we require $\chi = \Phi \in C^2_k(V,W) \subset C^2_{\frac{1}{2}}(V,W)$, $k > 0$, then the equation

$$\delta^2_{\frac{1}{2}} \Phi = \Phi \cdot \beta,$$

corresponds to a generalization of Gadbillon-Vey invariant [5] for differential forms. In general, we obtain the commutations

$$[H, X_{+2}] = 0,$$
$$[H, Y_-] = X_{+2},$$
$$[X_{-2}, X_{+2}] = H,$$
$$[Y_+, H_0] = X_{+2},$$

for generators

$$H = \chi,$$
$$X_{+1} = \Phi,$$
$$X_{+2} = \delta^2_{\frac{1}{2}} \Phi,$$
$$Y_- = \beta.$$

It is easy to see that since all mappings have zero operators composable with, then all further actions of the derivatives vanish. Nevertheless, recall that

$$C^2_k(V,W) \subset C^2_{\frac{1}{2}}(V,W),$$

$k > 0$, thus we can consider the most general case when $\chi \in C^{r+\alpha}_{m_0}(V,W)$, $\Phi \in C^2_{m_1}(V,W)$, $\beta \in C^{3-\alpha}_{m_2}(V,W)$. Then the grading condition requires $m_1 - 1 = m_0 + m_2 - t'$, where t' is the number of common vertex operators for $\chi \in C^{r+\alpha}_{m_0}(V,W)$ and $\beta \in C^{3-\alpha}_{m_2}(V,W)$. Thus on applying derivatives we obtain further commutation relations of the form (5.7).

6.2. **Algebra associated with the short bicomplex** (4.7)**.** One can also develop another example associated to the bicomplex (4.7). Consider $\chi \in C^0_3(V,W)$, $\Phi \in C^1_2(V,W)$. The orthogonality condition gives

$$\Phi \cdot \delta^0_3 \chi = 0.$$

Thus, there exists $\alpha \in C^n_m(V,W)$, such that

$$\delta^0_3 \chi = \Phi \cdot \alpha,$$

which gives $\alpha \in C^1_t(V,W)$. For the short sequence (4.7) we get a continual Lie algebra $\mathcal{G}(V)$ with generators

$$\left\{ \Phi(v_1), \chi,\ \alpha(v_2),\ \delta^1_2 \Phi(v_1)\ , \delta^0_3 \chi,\ \delta^1_t \alpha(v_2),\, 0 \le t \le 2 \right\},$$

and commutation relations for a continual Lie algebra $\mathcal{G}(V)$

$$\Phi \cdot \delta^1_t \alpha = \alpha \cdot \delta^1_2 \Phi \neq 0,$$
$$\delta^0_3 \chi = \Phi \cdot \alpha.$$

With the redefinition

$$H = \delta^0_3 \chi,$$
$$H^* = \chi,$$
$$X_+(v_1) = \Phi(v_1),$$
$$X_-(v_2) = \alpha(v_2),$$
$$Y_+(v_1) = \delta^1_2 \Phi(v_1),$$
$$Y_-(v_2) = \delta^1_t \alpha(v_2),$$

the commutation relations become:

$$[X_+(v_1), X_-(v_2)] = H,$$
$$[X_+(v_1), Y_-(v_1)] = [X_-(v_2), Y_+(v_1)]\,,$$

i.e., the orthogonality condition brings about a representation of an affinization [9] of continual counterpart of the Lie algebra sl_2. We prove here that that the cohomological classes:

$$\left[\left(\delta^1_2 \Phi \right) \cdot \Phi \right],\ \left[\left(\delta^0_3 \chi \right) \cdot \chi \right],\ \left[\left(\delta^1_t \alpha \right) \cdot \alpha \right],$$

for $0 \le t \le 2$, with non-vanishing $\left(\delta^1_2 \Phi \right) \cdot \Phi$, $\left(\delta^0_3 \chi \right) \cdot \chi$, and $\left(\delta^1_t \alpha \right) \cdot \alpha$, are independent on the choice of $\Phi \in C^1_2(V,W)$, $\chi \in C^0_3(V,W)$, and $\alpha \in C^1_t(V,W)$. Vertex algebra elements play the role of roots belonging to continual non-commutative root space given by a vertex algebra V.

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7. Appendix: Continual Lie algebras

Continual Lie algebras were introduced in [12] and then studied in [13, 14]. Suppose $\mathcal{E}$ is an associative algebra (which we call the *base algebra*) over $\mathbb{R}$ or $\mathbb{C}$, and

$$K_0,\ K_\pm,\ K_{0,0} : \mathcal{E} \times \mathcal{E} \to \mathcal{E},$$

are bilinear mappings. The local Lie part of a continual Lie algebra is defined as

$$\widehat{\mathcal{G}} = \mathcal{G}_{-1} \oplus \mathcal{G}_0 \oplus \mathcal{G}_{+1},$$

where $\mathcal{G}_i$, $i = 0, \pm 1$, are isomorphic to $\mathcal{E}$ and parametrized by its elements. The subspaces $\mathcal{G}_i$ consist of the elements

$$\{X_i(\phi), \phi \in \mathcal{E}\}, i = 0, \pm 1.$$

The generators $X_i(\phi)$ are subject to the commutation relations

$$[X_0(\phi), X_0(\psi)] = X_0(K_{0,0}(\phi,\psi)),$$

$$[X_0(\phi), X_{\pm 1}(\psi)] = X_{\pm 1}(K_\pm(\phi,\psi)),$$

$$[X_{+1}(\phi), X_{-1}(\psi)] = X_0(K_0(\phi,\psi)),$$

for all ϕ, $\psi \in \mathcal{E}$. It is also assumed that Jacobi identities are satisfied. Then the conditions on mappings $K_{0,0}$, $K_{0,\,\pm}$ follow:

$$K_\pm(K_{0,0}(\phi,\psi),\chi) = K_\pm(\phi, K_\pm(\psi,\chi)) - K_\pm(\psi, K_\pm(\phi,\chi)),$$

$$K_{0,0}(\psi, K_0(\phi,\chi)) = K_0(K_+(\psi,\phi),\chi)) + K_0(\phi, K_-(\psi,\chi)),$$

for all ϕ, ψ, $\chi \in \mathcal{E}$. An infinite dimensional algebra

$$\mathcal{G}(\mathcal{E};K) = \mathcal{G}'(\mathcal{E};K)/J,$$

is called a *continual contragredient Lie algebra*, where $\mathcal{G}'(\mathcal{E};K)$ is a Lie algebra freely generated by $\widehat{\mathcal{G}}$, and J is the largest homogeneous ideal with trivial intersection with $\mathcal{G}_0$ (consideration of the quotient is equivalent to imposing the Serre relations in an ordinary Lie algebra case) [13, 14].

8. Appendix: a multiplication of $C^n_m(V,W)$-spaces

In this appendix we recall the definition of the simplest variant of multiplication $*$ of elements of two double complex spaces with the image in another double complex space coherent with respect to the original differential (4.2), and satisfying the symmetry (4.1), $L_V(0)$-conjugation (3.2), and $L_V(-1)$-derivative (3.3)–(3.4) properties described in Section 1. We prove also an analogue of Leibniz formula (5.4).

8.1. **Geometrical products of $C^n_m(V,W)$-spaces.** Let us first clarify the geometrical origin of the multiplication. The structure of $C^n_m(V,W)$-spaces is quite complicated and it is difficult to introduce algebraically a product of its elements. In order to define an appropriate product of two $C^n_m(V,W)$-spaces we first have to interpret them geometrically. Basically, a $C^n_m(V,W)$-space must be associated with a certain model space, the algebraic $\overline{W}$-language should be transferred to a geometrical one, two model spaces should be "connected" appropriately, and, finally, a product should be defined.

For two spaces $C^k_m(V,W)$ and $C^n_{m'}(V,W)$ we first associate formal complex parameters in the sets $(x_1,\ldots,x_k)$ and $(y_1,\ldots,y_n)$ to parameters of two auxiliary spaces. Then we describe a geometric procedure to form a resulting model space by combining two original model spaces. Formal parameters of $W_{z_1,\ldots,z_{k+n}}$ should be then identified with parameters of the resulting space.

Note that according to our assumption, $(x_1,\ldots,x_k)\in F_k\mathbb{C}$, and $(y_1,\ldots,y_n)\in F_n\mathbb{C}$. As it follows from the definition of the configuration space $F_n\mathbb{C}$ in Subsection 1, in the case of coincidence of two formal parameters they are excluded from $F_{k+n}\mathbb{C}$. In general, it may happen that a number r of formal parameters $(x_1,\ldots,x_k)$ of $C^k_m(V,W)$ coincides with r formal parameters $(y_1,\ldots,y_n)$ of $C^n_{m'}(V,W)$ on the whole $\mathbb{C}$ (or on a domain of definition). Then, we exclude one formal parameter from each coinciding pair. We require that the set of formal parameters

$$(z_1,\ldots,z_{k+n-r})=(x_1,\ldots,x_{i_l},\ldots x_k;y_1,\ldots,\widehat{y}_{i_l},\ldots,y_n), \tag{8.1}$$

for $1\le i\le r$, where $\widehat{.}$ denotes the exclusion of corresponding formal parameter for $x_{i_l}=y_{j_l}$, $1\le l\le r$, for the resulting model space would belong to $F_{k+n-r}\mathbb{C}$. We denote this operation of formal parameters exclusion by

$$\widehat{R}\;\Phi(v_1\otimes\ldots\otimes v_k;v'_1\otimes\ldots\otimes v'_n)(x_1,\ldots,x_k;y_1,\ldots,y_n).$$

8.2. **Product of matrix elements.** The simplest possible product of elements of two $C^n_m(V,W)$-spaces is defined by products of matrix elements of the form (3.1) summed over a $V_{(l)}$-basis for $l\in\mathbb{Z}$. In geometrical language it corresponds to sewing of two Riemann spheres associated to $C^n_m(V,W)$-spaces. We have the following definition. For $\Phi(v_1\otimes\ldots\otimes v_k)(x_1,\ldots,x_k)\in C^k_m(V,W)$, and $\Psi(v'_1\otimes\ldots\otimes v'_n)(y_1,\ldots,y_n)\in C^n_{m'}(V,W)$, the product

$$\begin{aligned}\Phi(v_1\otimes\ldots\otimes v_k)(x_1,\ldots,x_k)*\Psi(v'_1\otimes\ldots\otimes v'_n)(y_1,\ldots,y_n)&\\ \mapsto\widehat{R}\;\Theta\left(v_1\otimes\ldots\otimes v_k\otimes v'_1\otimes\ldots\otimes v'_n\right)(x_1,\ldots,x_k;y_1,\ldots,y_n)&,\end{aligned} \tag{8.2}$$

is a $\overline{W}_{z_1,\ldots,z_{k+n-r}}$-valued rational form

$$\begin{aligned}&\langle w',\widehat{R}\;\Theta\left(v_1\otimes\ldots\otimes v_k\otimes v'_1\otimes\ldots\otimes v'_n\right)(x_1,\ldots,x_k;y_1,\ldots,y_n)\rangle\\ &\quad=\sum_{l\in\mathbb{Z}}\epsilon^l\sum_{u_l\in V_{(l)}}\langle w',Y^W_{WV}\left(\Phi(v_1\otimes\ldots\otimes v_k)(x_1,\ldots,x_k),\zeta_1\right)\;u_l\rangle\\ &\quad\langle w',Y^W_{WV}\left(\Psi(v'_1\otimes\ldots\otimes {v'}_{i_1}\otimes\ldots {v'}_{j_r}\otimes\ldots\otimes v'_n)\right.\\ &\qquad\qquad\left.(y_1\otimes\ldots\otimes y_{i_1}\otimes\ldots y_{j_r}\otimes\ldots\otimes y_n),\zeta_2\right)\;\overline{u}_l\rangle,\end{aligned} \tag{8.3}$$

parametrized by ζ_1, $\zeta_2 \in \mathbb{C}$. The sum is taken over any V_l-basis $\{u_l\}$, where $\overline{u}_l$ is the dual of u_l with respect to a non-degenerate bilinear form $\langle . , . \rangle$, (2.1) over V. The operation $\widehat{R}$ eliminates r formal parameters from Θ from the set $(y_1, \ldots, y_n)$ coinciding with r formal parameter of the set $(x_1, \ldots, x_k)$, and excludes all monomials $(x_{i_l} - y_{j_l})$, $1 \le l \le r$, from (8.3). By the standard reasoning [2, 18], (8.3) does not depend on the choice of a basis of $u \in V_l$, $l \in \mathbb{Z}$. The form of the product defined above is natural in terms of the theory of charaters for vertex operator algebras [15, 3, 18]. Due to the symmetry of the geometrical interpretation described above, we could exclude r formal parameters from the set $(x_1, \ldots, x_k)$ in (8.3) which belong to coinciding pairs resulting to the same definition of the $*$-product.

We define the action of an element $\sigma \in S_{k+n-r}$ on the product of $\Phi(v_1 \otimes \ldots \otimes v_k)(x_1, \ldots, x_k) \in C^k_m(V, W)$ and $\Psi(v'_1 \otimes \ldots \otimes v'_n)(y_1, \ldots, y_n) \in C^n_{m'}(V, W)$, as

$$\begin{aligned}
&\langle w', \sigma(\widehat{R}\, \mathcal{F})(v_1 \otimes \ldots \otimes v_k \otimes v'_1 \otimes \ldots \otimes v'_n)(x_1, \ldots, x_k; y_1, \ldots, y_n))\rangle \\
&\quad = \langle w', \Theta(\widetilde{v}_{\sigma(1)} \otimes \ldots \otimes \widetilde{v}_{\sigma(k+n-r)})(z_{\sigma(1)}, \ldots, z_{\sigma(k+n-r)})\rangle \\
&\quad = \sum_{l \in \mathbb{Z}} \sum_{u_l \in V_{(l)}} \langle w', Y^W_{WV}\left(\Phi(\widetilde{v}_{\sigma(1)} \otimes \ldots \otimes \widetilde{v}_{\sigma(k)})(z_{\sigma(1)}, \ldots, z_{\sigma(k)}), \zeta_1\right)\ u_l\rangle \\
&\quad \langle w', Y^W_{WV}\left(\Psi(\widetilde{v}_{\sigma(k+1)} \otimes \ldots \otimes \widetilde{v}_{\sigma(k+n-r)}) z_{\sigma(k+1)}, \ldots, z_{\sigma(k+n-r)}), \zeta_2\right)\ \overline{u}_l\rangle,
\end{aligned} \tag{8.4}$$

where by $(\widetilde{v}_{\sigma(1)}, \ldots, \widetilde{v}_{\sigma(k+n-r)})$ we denote a permutation of

$$(\widetilde{v}_1, \ldots, \widetilde{v}_{k+n-r}) = (v_1, \ldots; v_k; v'_1, \ldots, \widehat{v}'_{j_1}, \ldots, \widehat{v}'_{j_r}, \ldots, v'_n). \tag{8.5}$$

Let t be the number of common vertex operators the mappings $\Phi(v_1 \otimes \ldots v_k)(x_1, \ldots, x_k) \in C^k_m(V, W)$ and $\Psi(v'_1 \otimes \ldots, \otimes v'_n)(y_1, \ldots, y_n) \in C^n_{m'}(V, W)$, are composable with. Using the definition of $C^n_m(V, W)$-space and the definition of mappings composable with vertex operators, we then have

Proposition 8. *For $\Phi(v_1 \otimes \ldots \otimes v_k)(x_1, \ldots, x_k) \in C^k_m(V, W)$ and $\Psi(v'_1 \otimes \ldots \otimes v'_n)(y_1, \ldots, y_n) \in C^n_{m'}(V, W)$, the product*

$$\widehat{R}\, \Theta\left(v_1 \otimes \ldots \otimes v_k \otimes v'_1 \otimes \ldots \otimes v'_n\right)\left(x_1, \ldots, x_k; y_1, \ldots, y_n\right),$$

(8.3) *belongs to the space $C^{k+n-r}_{m+m'-t}(V, W)$, i.e.,*

$$* : C^k_m(V, W) \times C^n_{m'}(V, W) \to C^{k+n-r}_{m+m'-t}(V, W). \tag{8.6}$$

The proof of this proposition is as follows. Using the geometrical construction of sewing two Riemann spheres to form another Riemann sphere, we prove that the product (8.3) belongs to the space $C^{k+n-r}_{m+m'-t}(V, W)$. We show that (8.3) converges to a $\overline{W}$-valued rational function defined on the configuration space $F\mathbb{C}_{k+n-r}$, for formal variables with only possible poles at

$$(z_1, \ldots, z_{k+n-r}) = (x_1, \ldots, x_k; y_1, \ldots, \widehat{y}_{i_l}, \ldots, y_n),$$

satisfies (4.1), $L_V(0)$-symmetry (3.2), and $L_V(-1)$-derivative (3.3)–(3.4) conditions, and composable with $m + m' - t$ vertex operators.

In order to prove convergence of a product of elements of two spaces $C^k_m(V, W)$ and $C^n_{m'}(V, W)$ we use a geometrical interpretation [8, 17]. Recall that a $C^k_m(V, W)$

-space is defined by means of matrix elements [2] of the form (3.1), and satisfying $L(0)$-conjugation, $L(-1)$-derivsative conditions, (4.1), and composable with m vertex operators. For a vertex algebra V, and it module W, satisying certain extra conditions [16], one associate elements of a space $C^k_m(V,W)$ with the data on the Riemann sphere. In particular, formal parameters of $C^k_m(V,W)$-elements and vertex operators they are composable to, are identified with local coordinates of marked points on a sphere. For a pair of spaces $C^k_m(V,W)$ and $C^n_{m'}(V,W)$, we consider data on two Riemann spheres. Two extra points are chosen for centers of annuli used in order to sew spheres [17, 8] to obtain another sphere. The resulting product (8.3) represents a sum of products of matrix elements originated from two original Riemann spheres. Two complex parameters ζ_1, ζ_2 of (8.3) are identified with coordinates on annuli. After identification of annuli r coinciding coordinates may occur. This takes into account case of coinciding formal parameters.

The sewing parameter condition is [17] $\zeta_1\zeta_2 = \epsilon$. In two sphere ϵ-sewing formulation, the complex parameters ζ_a, $a = 1, 2$ are coordinates inside identified annuluses, and $|\zeta_a| \le r_a$. The product (8.3) converges for various cases of V and W. In particular, it is converges in case of a vertex algebra decomposable into Heisenberg vertex operator algebras [16]. Some further converging examples of V and it modules W will be considered elsewhere. The matrix elements in (8.3) are absolutely convergent in powers of ϵ with some radii of convergence $R_a \le r_a$, with $|\zeta_a| \le R_a$. By expanding the product (8.3) as power series in ϵ for $|\zeta_a| \le R_a$, where $|\epsilon| \le r$ for $r < r_1r_2$. By applying Cauchy's inequality to coefficients for x- and y-depending parts of the product we find that (8.3) is absolute convergent as a formal series in ϵ is defined for $|\zeta_a| \le r_a$, and $|\epsilon| \le r$ for $r < r_1r_2$, with extra poles only at z_i, $1 \le i \le k + n - r$.

When (8.3) is convergent, using geometrical procedure [8, 17] of sewing of two Riemann spheres, we prove that the limiting function is analytically extendable to a $\overline{W}$-valued function defined on the configuration space

$$F\mathbb{C}_{k+n-r} = \{(z_1, ..., z_{k+n-r}) : z_i \ne z_j, i \ne j\}.$$

Element $\Phi \in C^k_m(V,W)$ and $\Psi \in C^n_{m'}(V,W)$ are defined on the configuration spaces $F\mathbb{C}_k$ and $F\mathbb{C}_n$ correspondingly. Due to the construction of the product (8.3), for every $u_l \in V_{(l)}$, each summand in (8.3) defines a rational function on the configuration space $F\mathbb{C}_{k+n-r}$. Recall the sewing relation $\zeta_1\zeta_2 = \epsilon$ for two Riemann spheres.

By construction, it is assumed that annular regions are not intersecting. After the identification of annular regions the analytic extension of the limiting function of (8.3) is given by the the sewing geometrical procedure of [8, 17]. The extension is from two original Riemann spheres to the sphere formed by sewing. The construction of (8.3) provides that it gives the $\overline{W}$-valued rational function on the configuration space $F\mathbb{C}_{k+n-r}$. Similar, the construction of the product (8.3) provides that the limiting function is a $\overline{W}$-valued rational function with the only possible poles at $z_i = z_j$, $1 \le i < j \le k + n - r$.

We define the action of

$$\partial_p = \partial_{z_p} = \partial/\partial_{z_p},$$

for $1 \le p \le k+n-r$, the differentiation of

$$\Theta(v_1 \otimes \ldots \otimes v_k \otimes v'_1 \otimes \ldots \otimes \widehat{v}'_{i_l} \otimes \ldots \otimes v'_n)(x_1, \ldots, x_k; y_1, \ldots; \widehat{y}_{i_l}, \ldots, y_n),$$

for $1 \le l \le r$, with respect to the p-th entry of $(x_1, \ldots, x_k; y_1, \ldots, \widehat{y}_{i_l}, \ldots, y_n)$ as follows

$$\begin{aligned}
&\langle w', \partial_p \Theta(v_1 \otimes \ldots \otimes v_k \otimes v'_1 \otimes \ldots \otimes \widehat{v}'_{i_l} \otimes \ldots \otimes v'_n) \\
&\qquad\qquad\qquad (x_1, \ldots, x_k; y_1, \ldots, \widehat{y}_{i_l}, \ldots, y_n)\rangle \\
&= \sum_{q \in \mathbb{Z}} \epsilon^q \sum_{u_q \in V_q} \langle w', \partial^{\delta_{p,i}}_{x_i} Y^W_{WV} \left(\Phi(v_1 \otimes \ldots \otimes v_k)(x_1, \ldots, x_k), \zeta_1\right)\ u_q\rangle \\
&\qquad \langle w', \partial^{\delta_{p,j}}_{y_j} Y^W_{WV} \left(\Phi(v'_1 \otimes \ldots \otimes \widehat{v}'_{i_l} \otimes \ldots \otimes v'_n)(y_1, \ldots, \widehat{y}_{i_l}, \ldots, y_n), \zeta_2\right) \overline{u}_q\rangle.
\end{aligned} \tag{8.7}$$

By direct substitution we prove that the product (8.3) satisfies the $L_V(-1)$-derivative (3.3)–(3.4) and $L_V(0)$-conjugation (3.2) properties. Using the definition of the action of an element $\sigma \in S_{k+n-r}$ on the product (8.3), we prove (4.1) for (8.3).

Next, we show that (8.3) is composable with $m+m'-t$ vertex operators. Recall that $\Phi(v_1 \otimes \ldots \otimes v_k)(x_1, \ldots, x_k)$ is composable with m vertex operators, and $\Psi(v'_1 \otimes \ldots \otimes v'_n)(y_1, \ldots, y_n)$ is composable with m' vertex operators. Let us consider the first condition of composability for the product (8.3) with a number of vertex operators. We redefine the notations for the set

$$\begin{aligned}
&(v''_1, \ldots, v''_k; v''_{k+1}, \ldots, v''_{k+m}; v''_{k+m+1}, \ldots, v''_{k+n-r+m+m'-t}; v''_{n-r+1}, \ldots, v''_{n-r+m'-t}) \\
&\qquad = (v_1, \ldots, v_k; v_{k+1}, \ldots, v_{k+m}; v'_1, \ldots, v'_n; v'_{n-r+1}, \ldots, v'_{n-r+m'-t}), \\
&(z_1, \ldots, z_k; z_{k+1}, \ldots, z_{k+n-r}) = (x_1, \ldots, x_k; y_1, \ldots, \widehat{y}_{i_l} \ldots, y_n),
\end{aligned}$$

of vertex algebra V elements. Introduce $l''_1, \ldots, l''_{k+n-r} \in \mathbb{Z}_+$, such that $l''_1 + \ldots + l''_{k+n-r} = k+n-r+m+m'-t$. Define

$$\Psi''_i = E^{(l''_{i''})}_V(v''_{k''_1} \otimes \ldots \otimes v''_{k''_{i''}}; \mathbf{1}_V)(z_{k''_1} - \zeta''_{i''}, \ldots, z_{k''_{i''}} - \zeta''_{i''}), \tag{8.8}$$

where

$$k''_1 = l''_1 + \ldots + l''_{i''-1} + 1, \quad \ldots, \quad k''_{i''} = l''_1 + \ldots + l''_{i''-1} + l''_{i''}, \tag{8.9}$$

for $i'' = 1, \ldots, k+n-r$, and we take

$$(\zeta''_1, \ldots, \zeta''_{k+n-r}) = (\zeta_1, \ldots, \zeta_k; \zeta'_1, \ldots, \zeta'_n).$$

Then we consider

$$\begin{aligned}
\mathcal{I}^{k+n-r}_{m+m'-t}(\widehat{R}\ \Theta) = \sum_{r''_1, \ldots, r''_{k+n-r} \in \mathbb{Z}} &\langle w', \widehat{R}\ \Theta(P_{r''_1}\Psi''_1 \otimes \ldots \otimes P_{r''_{k+n-r}}\Psi''_{k+n-r}) \\
&\qquad (\zeta''_1, \ldots, \zeta''_{k+n-r})\rangle,
\end{aligned} \tag{8.10}$$

and prove it is absolutely convergent with some conditions. The condition

$$|z_{l''_1+\ldots+l''_{i-1}+p''} - \zeta''_i| + |z_{l''_1+\ldots+l''_{j-1}+q''} - \zeta''_i| < |\zeta''_i - \zeta''_j|, \tag{8.11}$$

of absolute convergence for (8.10) for i'', $j''=1,\dots,k+n-r$, $i\neq j$ and for $p''=1,\dots,l''_i$ and $q''=1,\dots,l''_j$, follows from corresponding conditions for Φ and Ψ. We obtain

$$\left|\mathcal{I}^{k+n-r}_{m+m'-t}(\widehat{R}\,\Theta)\right|\le\left|\mathcal{I}^k_m(\Phi)\right|\;\left|\mathcal{I}^n_{m'}(\Psi)\right|.$$

Thus, we infer that (8.10) is absolutely convergent. Recall that the maximal orders of possible poles of (8.10) are $N^k_m(v_i,v_j)$, $N^n_{m'}(v'_{i'},v'_{j'})$ at $x_i=x_j$, $y_{i'}=y_{j'}$. From the last expression we deduce that there exist positive integers $N^{k+n-r}_{m+m'-t}(v''_{i''},v''_{j''})$ for i, $j=1,\dots,k$, $i\neq j$, i', $j'=1,\dots,n$, $i'\neq j$, depending only on $v''_{i''}$ and $v''_{j''}$ for i'', $j''=1,\dots,k+n$, $i''\neq j''$ such that the series (8.10) can be analytically extended to a rational function in $(x_1,\dots,x_k;y_1,\dots,y_n)$, independent of $(\zeta''_1,\dots,\zeta''_{k+n-r})$, with extra possible poles at and $x_i=y_j$, of order less than or equal to $N^{k+n-r}_{m+m'-t}(v''_{i''},v''_{j''})$, for i'', $j''=1,\dots,n$, $i''\neq j''$.

Let us proceed with the second condition of composability. For the product (8.3) we obtain $(v''_1,\dots,v''_{k+n-r+m+m'-t})\in V$, and $(z_1,\dots\,,\,z_{k+n-r+m+m'-t})\in\mathbb{C}$, we find positive integers $N^{k+n-r}_{m+m'-t}(v'_i,v'_j)$, depending only on v'_i and v''_j, for i'', $j''=1,\dots,k+n-r$, $i''\neq j''$, such that for arbitrary $w'\in W'$. Under conditions

$$\begin{aligned}z_{i''}\neq z_{j''},\quad i''\neq j'',\\ |z_{i''}|>|z_{k'''}|>0,\end{aligned}\tag{8.12}$$

for $i''=1,\dots,m+m'-t$, and $k'''=m+m'-t+1,\dots,m+m'-t+k+n-r$, let us introduce

$$\begin{aligned}\mathcal{J}^{k+n-r}_{m+m'-t}(\widehat{R}\,\Theta)=\sum_{q\in\mathbb{C}}\langle w',E^{(m+m'-t)}_W\Big(&v''_1\otimes\ldots\otimes v''_{m+m'-t};\\ P_q(\widehat{R}\,\Theta(v''_{m+m'-t+1}&\otimes\ldots\otimes v''_{m+m'-t+k+n-r})\\ &(z_{m+m'-t+1},\dots,z_{m+m'-t+k+n-r}))\Big)(z_1,\dots,z_{m+m'-t})\rangle.\end{aligned}\tag{8.13}$$

We then obtain

$$|\mathcal{J}^{k+n}_{m+m'}(\widehat{R}\,\Theta)|\le\left|\mathcal{J}^k_m(\Phi)\right|\;\left|\mathcal{J}^n_{m'}(\Psi)\right|,$$

where we have used the invariance of (8.3) with respect to $\sigma\in S_{m+m'-t+k+n-r}$. $\mathcal{J}^k_m(\Phi)$ and $\mathcal{J}^n_{m'}(\Psi)$ in the last expression are absolute convergent. Thus, we infer that $\mathcal{J}^{k+n-r}_{m+m'-t}(\widehat{R}\,\Theta)$ is absolutely convergent, and the sum (8.10) is analytically extendable to a rational function in $(z_1,\dots,z_{k+n-r+m+m'-t})$ with the only possible poles at $x_i=x_j$, $y_{i'}=y_{j'}$, and at $x_i=y_{j'}$, i.e., the only possible poles at $z_{i''}=z_{j''}$, of orders less than or equal to $N^{k+n-r}_{m+m'-t}(v''_{i''},v''_{j''})$, for i'', $j''=1,\dots,k'''$, $i''\neq j''$. This finishes the proof of the proposition.

8.3. **Coboundary operator acting on the product space.** The product admits the action ot the differential operator $\delta^{k+n-r}_{m+m'-t}$ defined in (4.3) and (4.5) where r is the number of common formal parameters, and t the number of commpon composable vertex operators for $\Phi\in C^k_m(V,W)$ and $\Psi\in C^n_{m'}(V,W)$. The co-boundary operators (4.3) and (4.5) possesse a variation of Leibniz law with respect to the $*$-product. By direct computation we check

Proposition 9. *For $\Phi(v_1\otimes\ldots\otimes v_k)(x_1\ldots x_k)\in C^k_m(V,W)$ and $\Psi(v'_1\otimes\ldots\otimes v'_n)(y_1,\ldots,y_n)\in C^n_{m'}(V,W)$, the action of $\delta^{k+n-r}_{m+m'-t}$ on their product* (8.3) *is given by*

$$\begin{aligned}
&\delta^{k+n-r}_{m+m'-t}(\Phi(v_1\otimes\ldots\otimes v_k)(x_1,\ldots,x_k)*\Psi(v'_1\otimes\ldots\otimes v'_n)(y_1,\ldots,y_n))\\
&\quad=\left(\delta^k_m\Phi(\widetilde{v}_1\otimes\ldots\otimes\widetilde{v}_k)(z_1,\ldots,z_k)\right)*\Psi(\widetilde{v}_{k+1}\otimes\ldots\otimes\widetilde{v}_{k+n-r})(z_{k+1},\ldots,z_{k+n-r})\\
&+(-1)^k\Phi(\widetilde{v}_1\otimes\ldots\otimes\widetilde{v}_k)(z_1,\ldots,z_k)\\
&\qquad*\left(\delta^{n-r}_{m'-t}\Psi(\widetilde{v}_1\otimes\ldots\otimes\widetilde{v}_{k+n-r})(z_{k+1},\ldots,z_{k+n-r})\right).
\end{aligned}\tag{8.14}$$

□

Finally, we have the following

Corollary 1. *The multiplication* (8.3) *extends the chain-cochain complex structure of Proposition 3 to all products $C^k_m(V,W)\times C^n_{m'}(V,W)$, k, $n\geq 0$, m, $m'\geq 0$.* □

For elements of the spaces $C^2_{ex}(V,W)$ we have the following

Corollary 2. *The product of elements of the spaces $C^2_{ex}(V,W)$ and $C^n_m(V,W)$ is given by* (8.3),

$$*:C^2_{ex}(V,W)\times C^n_m(V,W)\to C^{n+2-r}_m(V,W),\tag{8.15}$$

and, in particular,

$$*:C^2_{ex}(V,W)\times C^2_{ex}(V,W)\to C^{4-r}_0(V,W).$$

□

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