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Product-type classes for vertex algebra cohomology of foliations on complex curves

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PRODUCT-TYPE CLASSES FOR VERTEX ALGEBRA COHOMOLOGY OF FOLIATIONS ON COMPLEX CURVES

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Abstract. We define a product of pairs of double complex spaces $C^n_m(V, \mathcal{W}, \mathcal{F})$ for gradig-restricted vertex algebra cohomology of codimension one foliation on a complex curve. We introduce a vertex algebra counterpart of the classical cohomological class using the orthogonality conditions on elements of double complex spaces with respect to the product we introduced.

1. Introduction

The theory of foliations involves a bunch of approaches [4–7, 11, 12, 15–17, 32] and many others. In certain cases it is useful to express cohomology in terms of connections and use the language of connections in order to study leave spaces of foliations. Connections appear in conformal field theory [3,10] in definitions of many notions and formulas. Vertex algebras, generalizations of ordinary Lie algebras, are essential in conformal field theory. The theory of vertex algebra characters is a rapidly developing field of studies. Algebraic nature of methods applied in this field helps to understand and compute the structure of vertex algebra characters. On the other hand, the geometric side of vertex algebra characters is in associating their formal parameters with local coordinates on a complex variety. Depending on geometry, one can obtain various consequences for a vertex algebra and its space of characters, and vice-versa, one can study geometrical property of a manifold by using algebraic nature of a vertex algebra attached. In this paper we use the vertex algebra cohomology theory [23] in order to construct cohomological invariants for codimension one foliations

There exist a few approches to definition and computation of cohomologies of vertex operator algebras. [23, 31]. Taking into account the above definitions and construction, we aim to consideration of a characteristic classes theory for arbitrary codimension regular and singular foliations vertex operator algebras. Losik [32] defines a smooth structure on the leaf space $M/\mathcal{F}$ of a foliation $\mathcal{F}$ of codimension n on a smooth manifold $\mathcal{M}$ that allows to apply to $M/\mathcal{F}$ the same techniques as to smooth manifolds. In [32] characteristic classes for a foliation as elements of the cohomology of certain bundles over the leaf space $M/\mathcal{F}$ are defined. It would be interesting also to develope intrinsic (i.e., purely coordinate independent) theory of a smooth manifold foliation cohomology involving vertex algebra bundles [3]. Similar to Losik's theory, we use bundles correlation functions) over a foliated space. The idea of studies

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of cohomology of certain bundles on a smooth manifold $\mathcal{M}$ and making connection to a cohomology of $\mathcal{M}$ has first appeared in [6]. This can have a relation with Losik's work [32] proposing a new framework for singular spaces and characteristic classes. In applications, one would be interested in applying techniques of this paper to case of higher-dimensional manifolds of codimension one [1, 2]. In particular, the question of higher non-vanishing invariants, as well as the problem of distinguishing of compact and non-compact leaves for the Reeb foliation of the full torus, are also of high importance. It would be important to establish connection to chiral de-Rham complex on a smooth manifold introduced in [33]. After a modification, one is able to introduce a vertex algebra cohomology of smooth manifolds on a similar basis as in this paper. One can mention a possibility to derive differential equations [22] for characters on separate leaves of foliation. Such equations are derived for various genuses and can be used in frames of Vinogradov theory [37]. The structure of foliation (in our sense) can be also studied from the automorphic function theory point of view. Since on separate leaves one proves automorphic properties of characters, on can think about "global" automorphic properties for the whole foliation.

The plan of the paper is the following. Section 2 contains a description of the transversal basis and vertex algebra interpretation of the local geometry for a foliation on a smooth manifold. In Section 3 we define a product of elements of two $\mathcal{W}_{z_1,\ldots,z_n}$-spaces. Spaces for double chain-cochain complexes associated to a vertex algebra are introduced in Section 4. Product of double complex spaces is defined in Section 5. In Section 6 coboundary operators are defined. It is shown that combining with the double complex spaces they determine chain-cochain double complex. The vertex algebra cohomology of a foliation on a smooth complex curve is introduced. A relation to Crainic-Moerdijk cohomology found. Properties of the product defined on double complex spaces are studied in Section 7. Section 8 describes product-type cohomological invariants for a codimension one foliation on a smooth complex curve. In Appendixes we provide the material required for the construction of the vertex algebra cohomology of foliations. In Appendix 9 we recall the notion of a quasi-conformal grading-restricted vertex algebra and its modules. In Appendix 10 the space of $\mathcal{W}$-valued rational sections of a vertex algebra bundle is described. In Appendix 11 properties of matrix elements for elements of the space $\mathcal{W}$ are listed. In Appendix 12 maps composable with a number of vertex operators are defined. In Appendix 14.2 we describe the approach to cohomology in terms of connections. In Appendix 15 geometrical procedure of sewing two Riemann spheres to form another Riemann sphere is recalled. Appendix 16 contains proofs of Proposition 1, Proposition 2, Proposition 4 Lemma 3, Lemma 1. Appendix 13 contains proofs of Lemmas 4, 5, 6, and Proposition 5.

A consideration of foliations of smooth manifold of arbitrary dimension will be given in [44].

2. Transversal basis description of foliations and vertex algebra interpretation

In this Section we recall [7] the notion of the basis of transversal sections for a foliation, and provide its vertex algebra setup.

2.1. **Basis of transversal sections for a foliation.** This subsection reminds [7] the notion of basis of transversal sections for a codimension one foliation. Let $\mathcal{M}$ be a complex curve equipped with a foliation $\mathcal{F}$ of codimension one.

Definition 1. A transversal section of $\mathcal{F}$ is an embedded one-dimensional submanifold $U \subset M$ which is everywhere transverse to the leaves of foliation.

Definition 2. If α is a path between two points p_1 and p_2 on the same leaf, and U_1 and U_2 are transversal sections through p_1 and p_2, then α defines a transport along the leaves from a neighborhood of p_1 in U_1 to a neighborhood of p_2 in U_2. i.e., a germ of a diffeomorphism

$$hol\ (\alpha) : (U_1, p_1) \hookrightarrow (U_2, p_2),$$

which is called the holonomy of the path α.

Two homotopic paths always define the same holonomy.

Definition 3. If the above transport along α is defined in all of U_1 and embeds U_1 into U_2, this embedding

$$h : U_1 \hookrightarrow U_2,$$

is called the holonomy embedding.

A composition of paths induces an operation of composition on holonomy embeddings. Transversal sections U through p as above should be thought of as neighborhoods of the leaf through p in the leaf space. Then we have

Definition 4. A transversal basis for $\mathcal{M}/\mathcal{F}$ as a family $\mathcal{U}$ of transversal sections $U \subset \mathcal{M}$ with the property that, if U_p is any transversal section through a given point $p \in \mathcal{M}$, there exists a holonomy embedding

$$h : U \hookrightarrow U_p,$$

with $U \in \mathcal{U}$ and $p \in h(U)$.

A transversal section is a one-dimensional disk given by a chart of the foliation. Accordingly, we can construct a transversal basis $\mathcal{U}$ out of a basis $\widetilde{\mathcal{U}}$ of $\mathcal{M}$ by domains of foliation charts

$$\phi_U : \widetilde{U} \tilde{\hookrightarrow} \mathbb{R} \times U,$$

$\widetilde{U} \in \widetilde{\mathcal{U}}$, with $U = \mathbb{R}$.

2.2. **Vertex algebra interpretation of the transversal basis.** Let $\mathcal{U}$ be a family of transversal sections of $\mathcal{F}$. We consider a (n, k)-set of points, $n \geq 1$, $k \geq 1$,

$$(p_1, \ldots, p_n; p'_1, \ldots, p'_k)\,, \tag{2.1}$$

on a smooth complex curve $\mathcal{M}$. Let us denote the set of corresponding local coordinates for $n + k$ points on $\mathcal{M}$ as

$$(c_1(p_1), \ldots, c_n(p_n); c'_1(p'_a), \ldots, c'_k(p'_k))\,.$$

In what follows we consider points (2.1) as points on either the leaf space $\mathcal{M}/\mathcal{F}$ of $\mathcal{F}$, or on transversal sections U_j of the transversal basis $\mathcal{U}$. Since $\mathcal{M}/\mathcal{F}$ is not in general a manifold, one has to be careful in considerations of chains of local coordinates along its leaves [26, 32]. For association of formal parameters of mappings with parameters of vertex operators taken at points of $\mathcal{M}/\mathcal{F}$ we will use in what follows either local coordinates on $\mathcal{M}$ or local coordinates on sections U of a transversal basis $\mathcal{U}$ which are submanifolds of $\mathcal{M}$ of dimension equal to codimenion of foliation $\mathcal{F}$. In case of extremely singular foliations when it is not possible to use local coordinates of $\mathcal{M}$ in order to parameterize a point on $\mathcal{M}/\mathcal{F}$ we still be able to use local coordinates on transversal sections passing through this point on $\mathcal{M}/\mathcal{F}$. In addition to that, note that the complexes considered below are constructed in such a way that one can always use coordinates on transversal sections only, avoiding any possible problems with localization of coordinates on leaves of $\mathcal{M}/\mathcal{F}$.

For a (n, k)-set of a grading-restricted vertex algebra V elements

$$(v_1, \ldots, v_n; v'_1, \ldots, v'_k) , \tag{2.2}$$

we consider linear maps

$$\Phi : V^{\otimes n} \rightarrow \mathcal{W}_{z_1,\ldots,z_n}, \tag{2.3}$$

(see Appendix 10 for the definition of a $\mathcal{W}_{z_1,\ldots,z_n}$ space),

$$\Phi\left(dz_1^{\mathrm{wt}\ v_1} \otimes v_1, c_1(p_1); \ldots; dz_n^{\mathrm{wt}\ v_n} \otimes v_n, c_n(p_1)\right), \tag{2.4}$$

where we identify, as it is usual in the theory of characters for vertex operator algebras on curves [25,40,42,43], n formal parameters $z_1, \ldots, z_n$ of $\mathcal{W}_{z_1,\ldots,z_n}$, with local coordinates $c_i(p_i)$ in vicinities of points p_i, $0 \le i \le n$, on $\mathcal{M}$. Elements $\Phi \in \mathcal{W}_{c_1(p_1),\ldots,c_n(p_n)}$ can be seen as coordinate-independent $\overline{W}$-valued rational sections of a vertex algebra bundle [3] generalization. Note that, according to [3], they can be treated as

$$\left(\mathrm{Aut}\ \mathcal{O}^{(1)}\right)^{\times n}_{z_1,\ldots,z_n} = \mathrm{Aut}_{z_1}\ \mathcal{O}^{(1)} \times \ldots \times \mathrm{Aut}_{z_n}\ \mathcal{O}^{(1)},$$

-torsors of product of groups of independent coordinate transformations. The construction of vertex algebra cohomology of foliations in terms of connections is parallel to ideas of [6]. Such a construction will be explained elsewhere [44].

In what follows, according to definitions of Appendix 10, when we write an element Φ of the space $\mathcal{W}_{z_1,\ldots,z_n}$, we actually have in mind corresponding matrix element $\langle w', \Phi\rangle$ that absolutely converges (on certain domain) to a rational form-valued function

$$\langle w', \Phi\rangle = R(\langle w', \Phi\rangle). \tag{2.5}$$

In notations, we will keep tensor products of vertex algebra elements with wight-powers of z-differentials when it is inevitable only.

In Appendix 13 we prove, that for arbitrary v_i, $v'_j \in V$, $1 \le i \le n$, $1 \le j \le k$, points p'_i with local coordinates $c_i(p'_i)$ on transversal sections $U_i \in \mathcal{U}$ of $\mathcal{M}/\mathcal{F}$, an element (2.4) as well as the vertex operator

$$\omega_W\left(dc_1(p'_i)^{\mathrm{wt}\ (v'_i)} \otimes v'_i, c_1(p'_1)\right) = Y\left(d(c_1(p'_i))^{\mathrm{wt}\ (v'_i)} \otimes v'_i, c_1(p'_i)\right), \tag{2.6}$$

are invariant with respect to the action of $\left(\mathrm{Aut}\ \mathcal{O}^{(1)}\right)^{\times n}_{z_1,\ldots,z_n}$. In (2.6) we mean usual vertex operator (as defined in Appendix 9) not affecting the tensor product with corresponding differential. We assume that the maps (2.3) are composable (according to Definition (31) of Appendix 12), with k vertex operators $\omega_W(v'_i, c_i(p'_i))$, $1 \leq i \leq k$ with k vertex algebra elements from (2.2), and formal parameters associated with local coordinates on k transversal sections of $\mathcal{M}/\mathcal{F}$, of k points from the set (2.1).

The composability of a map Φ with a number of vertex operators consists of two conditions on Φ. The first requires existence of positive integers $N^n_m(v_i, v_j)$ depending on vertex algebra elements v_i and v_j only, while the second condition restricts orders of poles of corresponding sums (12.3) and (16.26). Taking into account these conditions, we will see that the construction of spaces (4.2) depends on the choice of vertex algebra elements (2.2).

In Section 4 we construct spaces for double complexes associated to a grading-restricted vertex algebra and defined for codimension one foliations on complex curves. In order to keep control on the construction, we consider a section U_j of a transversal basis $\mathcal{U}$, and mappings Φ that belong to the space $\mathcal{W}_{c(p_1),\ldots,c(p_n)}$, depending on points $p_1, \ldots, p_n$ of intersection of U_j with leaves of $\mathcal{M}/\mathcal{F}$ of $\mathcal{F}$. It is assumed that local coordinates $c(p_1), \ldots, c(p_n)$ of points p_i, $1 \leq i \leq n$, are taken on $\mathcal{M}$ along these leaves of $\mathcal{M}/\mathcal{F}$. We then consider a collection of k sections of $\mathcal{U}$. In order to define vertex algebra cohomology of $\mathcal{M}/\mathcal{F}$, mappings Φ are supposed to be composable with a number of vertex operators with a number of vertex algebra elements, and formal parameteres identified with local coordinates of points $p'_1, \ldots, p'_k$ on each of k transversal sections U_j, $1 \leq j \leq k$.

3. Product of $\mathcal{W}_{z_1,\ldots,z_n}$-spaces

3.1. **Geometrical interpretation and definition of the ϵ- product for $\mathcal{W}_{z_1,\ldots,z_n}$-valued rational forms.** Recall Definition 28 of $\mathcal{W}_{z_1,\ldots,z_n}$-spaces (Appendix 10). The structure of $\mathcal{W}_{z_1,\ldots,z_n}$-spaces is quite complicated and it is a problem to introduce a product of elements of such spaces algebraically. In order to define an appropriate product of two $\mathcal{W}_{z_1,\ldots,z_n}$-spaces we first have to interpret it geometrically. $\mathcal{W}_{z_1,\ldots,z_n}$-spaces must be associated with certain model spaces. Then a geometric product of such model spaces should be defined, and, finally, an algebraic product of $\mathcal{W}$-spaces should be introduces.

For two $\mathcal{W}_{x_1,\ldots,x_k}$- and $\mathcal{W}_{y_1,\ldots,y_n}$-spaces we first associate formal complex parameters in sets $(x_1, \ldots, x_k)$ and $(y_1, \ldots, y_n)$ to parameters of two auxiliary spaces. Then we describe a geometric procedure to form resulting model space by combining two original model spaces. Formal parameters of algebraic product $\mathcal{W}_{z_1,\ldots,z_{k+n}}$ of $\mathcal{W}_{x_1,\ldots,x_k}$ and $\mathcal{W}_{y_1,\ldots,y_n}$ should be then identified with parameters of resulting auxiliary space. Note that according to our assumption, $(x_1, \ldots, x_k) \in F_k\mathbb{C}$, and $(y_1, \ldots, y_n) \in F_n\mathbb{C}$, i.e., belong to corresponding configuration space (Definition 25, Appendix 10). As it follows from the definition of $F_n\mathbb{C}$, coincidence of two formal parameters is excluded from $F_{k+n}\mathbb{C}$. In general, it might happens that some r formal parameters of $(x_1, \ldots, x_k)$ coincide with formal parameters of $(y_1, \ldots, y_n)$, i.e., $x_{i_l} = y_{j_l}$, $1 \leq i_l, j_l \leq r$.

In Definition 5 of the product of $\mathcal{W}_{x_1,\ldots,x_k}$ and $\mathcal{W}_{y_1,\ldots,y_n}$ below we leave only one of two coinciding formal parameters, i.e., we exclude one formal parameter from each coinciding pair. We require that the set of formal parameters

$$(z_1, \ldots, z_{k+n-r}) = (x_1, \ldots, x_{i_1}, \ldots, x_{i_r}, \ldots, x_k; y_1, \ldots, \widehat{y}_{j_1}, \ldots, \widehat{y}_{j_r}, \ldots, y_n) \quad (3.1)$$

would belong to $F_{k+n-r}\mathbb{C}$ where $\widehat{.}$ denotes the exclusion of corresponding formal parameter for $x_{i_l} = y_{j_l}$, $1 \le l \le r$. We denote this operation of formal parameters exclusion by $\widehat{R}\ \Phi(x_1, \ldots, x_k; y_1, \ldots, y_n; \epsilon)$. Thus, we require that the set of formal parameters $(z_1, \ldots, z_{k+n-r})$ for the resulting product would belong to $F_{k+n-r}\mathbb{C}$. Note that instead of exclusion given by the right hand side of (3.1), we could equivalently omit elements from $(x_1, \ldots, x_k)$ coinsiding with some elements of $(y_1, \ldots, y_n)$.

Recall the notion of intertwining operator (9.14) given in Appendix 9. Let us first give a formal algebraic definition of a product of $\mathcal{W}$-spaces.

Definition 5. For $\Phi(v_1, x_1; \ldots; v_k, x_k) \in \mathcal{W}_{x_1,\ldots,x_k}$, and $\Psi(v'_1, y_1; \ldots; v'_n, y_n) \in \mathcal{W}_{y_1,\ldots,y_n}$ the ϵ-product

$$\begin{aligned} &\Phi(v_1, x_1; \ldots; v_k, x_k) \cdot_\epsilon \Psi(v'_1, y_1; \ldots; v'_n, y_n) \\ &\qquad \mapsto \widehat{R}\ \Theta\left(v_1, x_1; \ldots; v_k, x_k; v'_1, y_1; \ldots; v'_n, y_n; \epsilon\right), \end{aligned} \quad (3.2)$$

is defined by the form via (2.5)

$$\begin{aligned} &\langle w', \widehat{R}\ \Theta(v_1, x_1; \ldots; v_k, x_k; v'_1, y_1; \ldots; v'_n, y_n; \epsilon)\rangle \\ &\quad = \sum_{l \in \mathbb{Z}} \epsilon^l \sum_{u \in V_l} \langle w', Y^W_{WV}\left(\Phi(v_1, x_1; \ldots; v_k, x_k), \zeta_1\right)\ u\rangle \\ &\quad \langle w', Y^W_{WV}\left(\Psi(v'_1, y_1; \ldots; {v'}_{i_1}, \widehat{y}_{i_1}; \ldots; \ldots; {v'}_{j_r}, \widehat{y}_{j_r}; \ldots; v'_n, y_n), \zeta_2\right)\ \overline{u}\rangle, \end{aligned} \quad (3.3)$$

parametrized by ζ_1, $\zeta_2 \in \mathbb{C}$, where all monomials $(x_{i_l} - y_{j_l})$, are exluded for coinciding $x_{i_l} = y_{j_l}$, $1 \le l \le r$, from (3.3). The sum is taken over any V_l-basis $\{u\}$, where $\overline{u}$ is the dual of u with respect to a non-degenerate bilinear form $\langle .\ ,. \rangle_\lambda$, (9.28) over V, (see Appendix 9).

By the standard reasoning [9,43], (3.3) does not depend on the choice of a basis of $u \in V_l$, $l \in \mathbb{Z}$. In the case when multiplied forms Φ and Ψ do not contain V-elements, i.e., for Φ, $\Psi \in \mathcal{W}$, (3.3) defines the product $\Phi \cdot_\epsilon \Psi$

$$\langle w', \Theta(\epsilon)\rangle = \sum_{l \in \mathbb{Z}} \epsilon^l \sum_{u \in V_l} \langle w', Y^W_{WV}\left(\Phi, \zeta_1\right)\ u\rangle \langle w', Y^W_{WV}\left(\Psi, \zeta_2\right)\ \overline{u}\rangle. \quad (3.4)$$

As we will see, Definition 5 is also supported by Lemma 3.

Remark 1. Note that due to (9.14), in Definition 5, it is assumed that $\Phi(v_1, x_1; \ldots; v_k, x_k)$ and $\Psi(v'_1, y_1; \ldots; v'_n, y_n)$ are composable with the V-module W vertex operators $Y_W(u, -\zeta_1)$ and $Y_W(\overline{u}, -\zeta_2)$ correspondingly (cf. Appendix 12 for the definition of composability). The product (3.3) is actually defined by sum of products of matrix elements of ordinary V-module W vertex operators acting on $\mathcal{W}_{z_1,\ldots,z_n}$ elements. Vertex algebra elements $u \in V$ and $\overline{u} \in V'$ are connected by (9.29), and ζ_1 and ζ_2 appear in a relation to each other. The form of the product defined above is natural in terms of the theory of chacaters for vertex operator algebras [10,40,43].

3.2. **Convergence of the ϵ-product.** In order to prove convergence of the product (3.3) of elements of two spaces $\mathcal{W}_{x_1,\dots,x_k}$ and $\mathcal{W}_{y_1,\dots,y_n}$ of rational $\mathcal{W}$-valued forms, we have to use a geometrical interpretation [25, 42]. Recall that a $\mathcal{W}_{z_1,\dots,z_n}$-space is defined by means of matrix elements of the form (2.5). For a vertex algebra V, this corresponds [9] to matrix element of a number of V-vertex operators with formal parameters identified with local coordinates on a Riemann sphere. Geometrically, each space $\mathcal{W}_{z_1,\dots,z_n}$ can be also associated to a Riemann sphere with a few marked points, and local coordinates vanishing at these points [25]. An additional point is identified to annulus center used in order to sew the sphere with another sphere. The product (3.3) has then a geometric interpretation. The resulting model space is then a Riemann sphere formed as a result of sewing procedure. In Appendix 15 we describe explicitly the geometrical procedure of sewing of two spheres [42].

Let us identify (as in [3, 10, 25, 40, 42, 43]) two sets $(x_1, \dots, x_k)$ and $(y_1, \dots, y_n)$ of complex formal parameters, with local coordinates of two sets of points on the first and the second Riemann spheres correspondingly. Complex parameters ζ_1 and ζ_2 of (3.3) play then the roles of coordinates (15.1) of the annuluses (15.3). On identification of annuluses $\mathcal{A}_a$ and $\mathcal{A}_{\overline{a}}$, r coinciding coordinates may occur.

The product (3.3) describes a $\mathcal{W}$-valued rational differential form defined on a sphere formed as a result of geometrical sewing [42] of two initial spheres. Since two initial spaces $\mathcal{W}_{x_1,\dots,x_k}$ and $\mathcal{W}_{y_1,\dots,y_n}$ are defined through rational-valued forms expressed by matrix elements of the form (2.5), it is then proved (see Proposition 1 below), that the resulting product defines a $\mathcal{W}_{x_1,\dots,x_k;y_1,\dots,y_n}$-valued rational form by means of an absolute convergent matrix element on the resulting sphere. The complex sewing parameter parametrizes the module space of sewin spheres as well as the product of $\mathcal{W}$-spaces.

Proposition 1. *The product* (3.3) *of elements of the spaces $\mathcal{W}_{x_1,\dots,x_k}$ and $\mathcal{W}_{y_1,\dots,y_n}$ corresponds to a rational form absolutely converging in ϵ with only possible poles at $x_i = x_j$, $y_{i'} = y_{j'}$, and $x_i = y_j$, $1 \le i, i' \le k$, $1 \le j, j' \le n$.*

Proof of this proposition is contained in Appendix (16).

Next, we formulate

Definition 6. We define the action of an element $\sigma \in S_{k+n-r}$ on the product of $\Phi(v_1, x_1; \dots; v_k, x_k) \in \mathcal{W}_{x_1,\dots,x_k}$, and $\Psi(v'_1, y_1; \dots; v'_n, y_n) \in \mathcal{W}_{y_1,\dots,y_n}$, as

$$\begin{aligned}
&\langle w', \sigma(\widehat{R}\,\Theta)(v_1, x_1; \dots; v_k, x_k; v'_1, y_1; \dots; v'_n, y_n; \epsilon)\rangle \\
&\quad = \langle w', \Theta(\widetilde{v}_{\sigma(1)}, z_{\sigma(1)}; \dots; \widetilde{v}_{\sigma(k+n-r)}, z_{\sigma(k+n-r)}; \epsilon)\rangle \\
&\quad = \sum_{l\in\mathbb{Z}} \epsilon^l \sum_{u \in V_l} \langle w', Y^W_{WV}\left(\Phi(\widetilde{v}_{\sigma(1)}, z_{\sigma(1)}; \dots; \widetilde{v}_{\sigma(k)}, z_{\sigma(k)}), \zeta_1\right)\, u\rangle \\
&\quad \langle w', Y^W_{WV}\left(\Psi(\widetilde{v}_{\sigma(k+1)}, z_{\sigma(k+1)}; \dots; \widetilde{v}_{\sigma(k+n-r)}, z_{\sigma(k+n-r)}), \zeta_2\right)\, \overline{u}\rangle,
\end{aligned} \tag{3.5}$$

where by $(\widetilde{v}_{\sigma(1)}, \dots, \widetilde{v}_{\sigma(k+n-r)})$ we denote a permutation of vertex algebra elements

$$(\widetilde{v}_1, \dots, \widetilde{v}_{k+n-r}) = (v_1, \dots; v_k; \dots, \widehat{v}'_{j_1}, \dots, \widehat{v}'_{j_r}, \dots). \tag{3.6}$$

Next, we have

Lemma 1. *The product* (3.3) *satisfy* (10.3) *for* $\sigma \in S_{k+n-r}$, *i.e.,*

$$\sum_{\sigma \in J^{-1}_{k+n-r;s}} (-1)^{|\sigma|} \widehat{R}\, \Theta \left(v_{\sigma(1)}, x_{\sigma(1)}; \ldots; v_{\sigma(k)}, x_{\sigma(k)}; v'_{\sigma(1)}, y_{\sigma(1)}; \ldots; v'_{\sigma(n)}, y_{\sigma(n)}; \epsilon \right) = 0.$$

The proof of this lemma is contained in Appendix 16.

Next we prove the existence of appropriate $\mathcal{W}_{z_1,\ldots,z_{k+n-r}}$-valued rational form corresponding to the absolute convergent rational form $\mathcal{R}(z_1, \ldots, z_{k+n-r})$ defining the ϵ-product of elements of the spaces $\mathcal{W}_{x_1,\ldots,x_k}$ and $\mathcal{W}_{y_1,\ldots,y_n}$.

Lemma 2. *For all choices of elements of the spaces* $\mathcal{W}_{x_1,\ldots,x_k}$ *and* $\mathcal{W}_{y_1,\ldots,y_n}$ *there exists an element* $\widehat{R}\ \Theta(v_1, x_1; \ldots; v_k, x_k; v'_1, y_1; \ldots; v'_n, y_n;\ \epsilon) \in \mathcal{W}_{z_1,\ldots,z_{k+n-r}}$*-valued rational form such that the product* (3.3) *converges to*

$$R(x_1, \ldots, x_k; y_1, \ldots, y_n; \epsilon) = \langle w', \widehat{R}\ \Theta(v_1, x_1; \ldots; v_k, x_k; v'_1, y_1; \ldots; v'_n, y_n; \epsilon)\rangle.$$

Proof. In the proof of Proposition 1 we showed the absolute convergence of the product (3.3) to a rational form $R(x_1, \ldots, x_k; y_1, \ldots, y_n; \epsilon)$. The lemma follows from completeness of $\overline{W}_{x_1,\ldots,x_k;y_1,\ldots,y_n}$ and density of the space of rational differential forms. □

We formulate

Definition 7. For $\Phi(v_1, x_1; \ldots; v_k, x_k) \in \mathcal{W}_{x_1,\ldots,x_k}$ and $\Psi(v'_1, y_1; \ldots; v'_n, y_k) \in \mathcal{W}_{y_1,\ldots,y_n}$, with r coinciding formal parameters $x_{i_q} = y_{j_q}$, $1 \le q \le r$, we define the action of $\partial_s = \partial_{z_s} = \partial/\partial_{z_s}$, $1 \le s \le k+n-r$, $1 \le i \le k$, $1 \le j \le n$ the differentiation of $\widehat{R}\ \Theta(v_1, x_1; \ldots; v_k, x_k;\ v'_1, y_1; \ldots; v'_n, y_n; \epsilon)$ with respect to the s-th entry of $(z_1, \ldots, z_{k+n-r})$, as follows

$$\begin{aligned} &\langle w', \partial_s \widehat{R}\ \Theta(v_1, x_1; \ldots; v_k, x_k; v'_1, y_1; \ldots; v'_n, y_n; \epsilon)\rangle \\ &= \sum_{l \in \mathbb{Z}} \epsilon^l \sum_{u \in V_l} \langle w', \partial^{\delta_{s,i}}_{x_i} Y^W_{WV} \left(\Phi(v_1, x_1; \ldots; v_k, x_k), \zeta_1\right)\ u\rangle \\ &\qquad \langle w', \partial^{\delta_{s,j} - \delta_{i_q, j_q}}_{y_j} Y^W_{WV} \left(\Psi(v'_1, y_1; \ldots; v'_n, y_n), \zeta_2\right) \overline{u}\rangle. \end{aligned} \tag{3.7}$$

Remark 2. As we see in the last expressions, the $L_V(0)$-conjugation property (11.6) for the product (3.3) includes the action of $z^{L_V(0)}$-operator on complex parameters ζ_a, $a = 1, 2$.

Proposition 2. *The product* (3.3) *satisfies the* $L_V(-1)$*-derivative* (11.1) *and* $L_V(0)$*-conjugation* (11.6) *properties.*

Proof of this propositions in Appendix 16.

Summing up the results of Proposition (1), Lemma (1), Lemma (2), and Proposition (2), we obtain the main statement of this section:

Proposition 3. *The product* (3.3) *provides a map*

$$\cdot_\epsilon : \mathcal{W}_{x_1,\ldots,x_k} \times \mathcal{W}_{y_1,\ldots,y_n} \to \mathcal{W}_{z_1,\ldots,z_{k+n-r}}.$$

□

We then have

Definition 8. For fixed sets $(v_1, \ldots, v_k)$, $(v'_1, \ldots, v'_n) \in V$, $(x_1, \ldots, x_k) \in \mathbb{C}$, $(y_1, \ldots, y_n) \in \mathbb{C}$, we call the set of all $\mathcal{W}_{z_1,\ldots,z_{k+n-r}}$ -valued rational forms $\widehat{R}\ \Theta(v_1, x_1; \ldots; v_k, x_k ; v'_1, y_1; \ldots; v'_n, y_n; \epsilon)$ defined by (3.3) with the parameter ϵ exhausting all possible values, the complete product of the spaces $\mathcal{W}_{x_1,\ldots,x_k}$ and $\mathcal{W}_{y_1,\ldots,y_n}$.

3.3. **Properties of the $\mathcal{W}_{z_1,\ldots,z_n}$-product.** In this subsection we study properties of the product $\widehat{R}\ \Theta(v_1, x_1; \ldots; v_k, x_k; v'_1, y_1; \ldots; v'_n, y_n; \epsilon)$ of (3.3). We have

Proposition 4. *For generic elements v_i, $v'_j \in V$, $1 \le i \le k$, $1 \le j \le n$, of a quasi-conformal grading-restricted vertex algebra, the product* (3.3) *is canonincal with respect to the action of the group* $(\mathrm{Aut}\ \mathcal{O})^{\times(k+n-r)}_{z_1,\ldots,z_{k+n-r}}$ *of independent $k+n-r$-dimensional changes*

$$(z_1, \ldots, z_{k+n-r}) \mapsto (z'_1, \ldots, z'_{k+n-r}) = (\rho(z_1), \ldots, \rho(z_{k+n-r})), \tag{3.8}$$

of formal parameters.

Proof of this proposition is in Appenedix 16.

In the geometric interpretation in terms of auxiliary Riemann spheres, the definition (3.3) depends on the choice of insertion points p_i, $1 \le i \le k$, with local coordinated x_i on $\widehat{\Sigma}^{(0)}_1$, and p'_i, $1 \le j \le k$, with local coordinates y_j on $\widehat{\Sigma}^{(0)}_2$. Suppose we change the distribution of points among two initial Riemann spheres. We then formulate the following

Lemma 3. *In the setup above, for a fixed set $(\widetilde{v}_1, \ldots, \widetilde{v}_{k+n}) \in V$, of vertex algebra elements, the ϵ-product $\Theta(\widetilde{v}_1, z_1; \ldots; \widetilde{v}_{k+n}, z_{k+n}; \epsilon) \in \mathcal{W}_{z_1,\ldots,z_{k+n}}$,*

$$\cdot_\epsilon : \mathcal{W}_{z_1,\ldots,z_s} \times \mathcal{W}_{z_{s+1},\ldots,z_{k+n}} \to \mathcal{W}_{z_1,\ldots,z_{k+n}}, \tag{3.9}$$

remains the same for elements $\Phi(\widetilde{v}_1, z_1; \ldots; \widetilde{v}_s, z_s) \in \mathcal{W}_{z_1,\ldots,z_s}$ and $\Psi(\widetilde{v}_{s+1}, z_{s+1}; \ldots ; \widetilde{v}_{k+n}, z_{k+n}) \in \mathcal{W}_{z_{k+1},\ldots,z_{k+n}}$, for any $0 \le s \le k+n$.

Proof of this lemma is in Appendix 16.

Remark 3. This Lemma is important for the formulation of cohomological invariants associated to grading-restricted vertex algebras on smooth manifolds [44]. In the case $s = 0$, we obtain from (3.10),

$$\cdot_\epsilon : \mathcal{W} \times \mathcal{W}_{z_1,\ldots,z_{k+n}} \to \mathcal{W}_{z_1,\ldots,z_{k+n}}. \tag{3.10}$$

4. Spaces for double complexes

In this section we introduce the definition of spaces for a double complex suitable for the construction a grading-restricted vertex algebra cohomology for codimension one foliations on complex curves. We first introduce

Definition 9. Let $(v_1, \ldots, v_n)$, $(v'_1, \ldots, v'_k)$ be two sets ot vertex algebra V elements, and $(p_1, \ldots, p_n)$ be points with local coordinates $(c_1(p_1), \ldots, c_n(p_n))$ taken on the same transversal section $U_j \in \mathcal{U}$, $j \ge 1$ of the foliation $\mathcal{F}$ transversal basis $\mathcal{U}$ on a complex curve. Assuming $k \ge 1$, $n \ge 0$, we denote by $C^n(V, \mathcal{W}, \mathcal{F})(U_j)$, $0 \le j \le k$, the space of all linear maps (2.3)

$$\Phi : V^{\otimes n} \to \mathcal{W}_{c_1(p_1),\ldots,c_n(p_n)}, \tag{4.1}$$

composable with a k of vertex operators (2.6) with formal parameters identified with local coordinates $c'_j(p'_j)$ functions around points p'_j on each of transversal sections U_j, $1 \le j \le k$.

The set of vertex algebra elements (2.2) plays the role of parameters in our further construction of the vertex algebra cohomology associated with the foliation F. According to considerations of Subsection 2.1, we assume that each transversal section of a transversal basis $\mathcal{U}$ possess a coordinate chart which is induced by a coordinate chart of $\mathcal{M}$ (cf. [7]).

Recall the notion of a holonomy embedding (cf. Subsection 2.1, cf. [7]) which maps a section into another section of a transversal basis, and a coordinate chart on the first section into a coordinate chart on the second transversal section. Motivated by the definition of spaces for the Čech-de Rham complex in [7] (see Subsection 2.1), let us now introduce the following spaces:

Definition 10. For $n \ge 0$, and $1 \le m \le k$, with Definition 9, we define the space

$$C^n_m(V, \mathcal{W}, \mathcal{U}, \mathcal{F}) = \bigcap_{\substack{U_1 \overset{h_1}{\hookrightarrow} \ldots \overset{h_{m-1}}{\hookrightarrow} U_m \\ 1 \le j \le m}} C^n(V, \mathcal{W}, \mathcal{F})(U_j), \tag{4.2}$$

where the intersection ranges over all possible $(m-1)$-tuples of holonomy embeddings h_j, $j \in \{1, \ldots, m-1\}$, between transversal sections of the baisis $\mathcal{U}$ for $\mathcal{F}$.

First, we have the following

Lemma 4. (4.2) *is non-empty.*

Lemma 5. *The double complex* (4.2) *does not depend on the choice of transversal basis $\mathcal{U}$.*

Thus we then denote $C^n_m(V, \mathcal{W}, \mathcal{U}, \mathcal{F})$ as $C^n_m(V, \mathcal{W}, \mathcal{F})$. Recall the notation of a quasi-conformal grading-restricted vertex algebra given in Appendix 9. The main statement of this section is contained in the following

Proposition 5. *For a quasi-conformal grading-restricted vertex algebra V and its module W, the construction* (4.2) *is canonical, i.e., does not depend on foliation preserving choice of local coordinates on $\mathcal{M}/\mathcal{F}$.*

The proofs of Lemmas 4, 5, and Proposition 5 is contained in Appendix 13.

Remark 4. The condition of quasi-conformality is necessary in the proof of invariance of elements of the space $\mathcal{W}_{z_1, \ldots, z_n}$ with respect to a vertex algebraic representation (cf. Appendix 9) of the group $\left(\mathrm{Aut}\ \mathcal{O}^{(1)}\right)^{\times n}_{z_1, \ldots, z_n}$. In what follows, when it concerns the spaces (4.2) we will always assume the quasi-conformality of V.

Proofs of generalizations of Lemmas 4, 5, 6 and Proposition 5 for the case of an arbitrary codimension foliation on a smooth n-dimensional manifold will be given in [44]. The proof of Proposition 5 is contained in Appendix 13.

Let W be a grading-restricted V module. Since for $n = 0$, Φ does not include variables, and due to Definition 31 of the composability, we can put:

$$C^0_k(V, \mathcal{W}, \mathcal{F}) = W, \tag{4.3}$$

for $k \geq 0$. Nevertheless, according to our Definition 4.2, mappings that belong to (4.3) are assumed to be composable with a number of vertex operators with depending on local coordinates of k points on k transversal sections.

We observe

Lemma 6.

$$C_m^n(V, \mathcal{W}, \mathcal{F}) \subset C_{m-1}^n(V, \mathcal{W}, \mathcal{F}). \tag{4.4}$$

The proof of this Lemma is contained in Appendix 13.

5. The product of $C_m^n(V, \mathcal{W}, \mathcal{F})$-spaces

In this section we consider an application of the material of Section 3 to double complex spaces $C_m^n(V, \mathcal{W}, \mathcal{F})$ (Definition 10, Section 4) for a foliation $\mathcal{F}$ on a complex curve. We introduce the product of two double complex spaces with the image in another double complex space coherent with respect to the original differentials (6.3) and (6.7), and the symmetry property (10.3). We prove the canonicity of the product, and derive an analogue of Leibniz formula.

5.1. **Geometrical adaptation of the ϵ-product to a foliation.** In this subsection we show how the definition of the product of $\mathcal{W}_{z_1,\ldots,z_n}$-spaces can be extended to the case of $C_n^k(V, \mathcal{W}, \mathcal{F})$-spaces for a codimension one foliation of a complex curve. Recall Definition 4.2 of $C_n^k(V, \mathcal{W}, \mathcal{F})$-spaces in Section 4. We use again the geometrical scheme of sewing of two Riemann surfaces in order to introduce the product of two elements $\Phi \in C_m^k(V, \mathcal{W}, \mathcal{F})$ and $\Psi \in C_{m'}^n(V, \mathcal{W}, \mathcal{F})$ which belong to two double complex spaces (4.2) for the same foliation $\mathcal{F}$. The construction is again local, thus we assume that both spaces $C_m^k(V, \mathcal{W}, \mathcal{F})$ and $C_{m'}^n(V, \mathcal{W}, \mathcal{F})$ are considered on the same fixed transversal basis $\mathcal{U}$. Moreover, we assume that marked points used in Definition 4.2 of the spases $C_m^k(V, \mathcal{W}, \mathcal{F})$ and $C_{m'}^n(V, \mathcal{W}, \mathcal{F})$ are choosen on the same transversal section.

Let us recall again the setup for two double complex spaces $C_m^k(V, \mathcal{W}, \mathcal{F})$ and $C_{m'}^n(V, \mathcal{W}, \mathcal{F})$. For $(p_1, \ldots, p_k)$, $(\widetilde{p}_1, \ldots, \widetilde{p}_n)$ being two sets of points with local coordinates $(c_1(p_1), \ldots, c_k(p_k))$ and $(\widetilde{c}_1(\widetilde{p}_1), \ldots, \widetilde{c}_n(\widetilde{p}_n))$ taken on the j-th transversal section $U_j \in \mathcal{U}$, $j \geq 1$, of the transversal basis $\mathcal{U}$. For $k \geq 0$, $n \geq 0$, $C^k(V, \mathcal{W}, \mathcal{F})(U_j)$ and $C^n(V, \mathcal{W}, \mathcal{F})(U_j)$, $0 \leq j \leq l$, be as before the spaces of all linear maps (2.3)

$$\begin{aligned} \Phi &: V^{\otimes k} \to \mathcal{W}_{c_1(p_1),\ldots,c_k(p_k)}, \\ \Psi &: V^{\otimes n} \to \mathcal{W}_{\widetilde{c}_1(\widetilde{p}_1),\ldots,\widetilde{c}_n(\widetilde{p}_n)}, \end{aligned} \tag{5.1}$$

composable with l_1 and l_2 vertex operators (2.6) with formal parameters identified with local coordinate functions $c'_j(p'_j)$ and $\widetilde{c}'_j(p'_{j'})$ around points p_j, $p'_{j'}$, on each of transversal sections U_j, $1 \leq j \leq l_1$, and $U_{j'}$, $1 \leq j' \leq l_2$, correspondingly. Then, for $k \geq 0$, $1 \leq m \leq l_1$, and $n \geq 0$, and $1 \leq m' \leq l_2$, according to Definition 4.2, the spaces $C_m^k(V, \mathcal{W}, \mathcal{F})$ and $C_{m'}^n(V, \mathcal{W}, \mathcal{F})$ are:

$$C_m^k(V, \mathcal{W}, \mathcal{F}) = \bigcap_{\substack{U_1 \overset{h_1}{\hookrightarrow} \ldots \overset{h_{m-1}}{\hookrightarrow} U_m \\ 1 \leq i \leq m}} C^k(V, \mathcal{W}, \mathcal{F})(U_i), \tag{5.2}$$

$$C^n_{m'}(V,\mathcal{W},\mathcal{U},\mathcal{F}) = \bigcap_{\substack{U_1 \overset{h'_1}{\hookrightarrow} \ldots \overset{h'_{m'-1}}{\hookrightarrow} U_{m'} \\ 1\le i'\le m'}} C^n(V,\mathcal{W},\mathcal{F})(U_{i'}), \tag{5.3}$$

where the intersection ranges over all possible m- and m'-tuples of holonomy embeddings h_i, $i \in \{1,\ldots,m-1\}$, and $h'_{i'}$, $i' \in \{1,\ldots,m'-1\}$, between transversal sections $(U_1,\ldots,U_m)$ and $(U_1,\ldots,U_{m'})$ of the baisis $\mathcal{U}$ for $\mathcal{F}$. In the setup above, we then formulate

Definition 11. For $\Phi(v_1,x_1;\ldots;v_k,x_k) \in C^k_m(V,\mathcal{W},\mathcal{F})$, and $\Psi(v'_1,y_1;\ldots;v'_n,y_n) \in C^n_{m'}(V,\mathcal{W},\mathcal{F})$ the product

$$\begin{aligned}\Phi(v_1,x_1;\ldots;v_k,x_k) \cdot_\epsilon \Psi(v'_1,y_1;\ldots;v'_n,y_n) \\ \mapsto \widehat{R}\,\Theta\left(v_1,x_1;\ldots;v_k,x_k;v'_1,y_1;\ldots;v'_n,y_n;\epsilon\right),\end{aligned} \tag{5.4}$$

is a $\mathcal{W}_{z_1,\ldots,z_{k+n-r}}$ -valued rational form

$$\begin{aligned}&\langle w', \widehat{R}\,\Theta(v_1,x_1;\ldots;v_k,x_k;v'_1,y_1;\ldots;v'_n,y_n;\epsilon)\rangle \\ &\quad = \langle w', \Phi(v_1,x_1;\ldots;v_k,x_k) \cdot_\epsilon \Psi(v'_1,y_1;\ldots;v'_n,y_n)\rangle \\ &\quad = \sum_{l\in\mathbb{Z}} \epsilon^l \sum_{u\in V_l} \langle w', Y^W_{WV}\left(\Phi(v_1,x_1;\ldots;v_k,x_k),\zeta_1\right)\ u\rangle \\ &\qquad\qquad \langle w', Y^W_{WV}\left(\Psi(v'_1,y_1;\ldots;v'_n,y_n),\zeta_2\right)\ \overline{u}\rangle,\end{aligned} \tag{5.5}$$

defined by (3.3).

Let t be the number of common vertex operators the mappings $\Phi(v_1,x_1;\ \ldots;\ v_k,x_k) \in C^k_m(V,\mathcal{W},\mathcal{F})$ and $\Psi(v'_1,y_1;\ldots;v'_n,y_n) \in C^n_{m'}(V,\mathcal{W},\mathcal{F})$, are composable with.

Proposition 6. *For $\Phi(v_1,x_1;\ldots;v_k,x_k) \in C^k_m(V,\mathcal{W},\mathcal{F})$ and $\Psi(v'_1,y_1;\ldots;v'_n,y_n) \in C^n_{m'}(V,\mathcal{W},\mathcal{F})$, the product $\widehat{R}\,\Theta\left(v_1,x_1;\ldots;v_k,x_k;v'_1,y_1;\ldots;v'_n,y_n;\epsilon\right)$ (5.5) belongs to the space $C^{k+n-r}_{m+m'-t}(V,\mathcal{W},\mathcal{F})$, i.e.,*

$$\cdot_\epsilon : C^k_m(V,\mathcal{W},\mathcal{F}) \times C^n_{m'}(V,\mathcal{W},\mathcal{F}) \to C^{k+n-r}_{m+m'-t}(V,\mathcal{W},\mathcal{F}). \tag{5.6}$$

Proof. In Proposition 1 we proved that $\widehat{R}\,\Theta\left(v_1,x_1\ ;\ \ldots;v_k,x_k;v'_1,y_1;\ldots;v'_n,y_n;\epsilon\right) \in \mathcal{W}_{x_1;,\ldots,x_k;y_1,\ldots,y_n}$. Namely, the rational form corresponding to the ϵ-product $\widehat{R}\,\Theta(v_1,x_1;\ldots;v_k,x_k;v'_1,y_1;\ldots;v'_n,y_n;\epsilon)$ converges in ϵ, and satisfies (10.3), $L_V(0)$-conjugation (11.6) and $L_V(-1)$-derivative (11.1) properties. The action of $\sigma \in S_{k+n-r}$ on the product $\Theta(v_1,x_1;\ldots;v_k,x_k;v'_{k+1},y_1;\ldots;\ v'_n,y_n;\epsilon)$ (5.5) is given by (10.1). Then we see that for the sets of points $(p_1,\ldots,p_k;p'_1,\ldots,p'_n)$, taken on the same transversal section $U_j \in \mathcal{U}$, $j \ge 1$, by Proposition 1 we obtain a map

$$\begin{aligned}\widehat{R}\,\Theta\left(v_1,x_1;\ldots;v_k,x_k;v'_1,y_1;\ldots;v'_n,y_n;\epsilon\right) \\ : V^{\otimes(k+n)} \to \mathcal{W}_{c''_1(p''_1),\ldots,c''_1(p''_{k+n-r})},\end{aligned} \tag{5.7}$$

with formal parameters $(z_1,\ldots,z_{k+n-r})$ identified with local coordinates $(c''_1(p''_1),\ldots,c''_1(p''_{k+n-r}))$ of points

$$(p''_1,\ldots,p''_{k+n-r}) = (p_1,\ldots,p_k;p_1,\ldots,\widehat{p'_{i_l}},\ldots,p'_n),$$

for coinciding points $p_{i_l} = p'_{j_l}$, $1 \le l \le r$. Next, we prove

Proposition 7. *The product* $\Theta\,(v_1, x_1; \ldots; v_k, x_k; v'_1, y_1; \ldots; v'_n, y_n; \epsilon)$ (5.5) *is composable with* $m + m' - t$ *vertex operators.*

The proof of this proposition is contained in Appendix 16.

Since we have proved that the product $\widehat{R}\;\Theta\,(v_1, x_1; \ldots; v_k, x_k; v'_1, y_1; \ldots; v'_n, y_n; \epsilon)$ is composable with a $m + m' - t$ of vertex operators (2.6) with formal parameters identified with local coordinates $c_j(p''_j)$ functions around points $(p_1, \ldots, p_k; p'_1, \ldots, p'_n)$ on each of transversal sections U_j, $1 \le j \le l$, we conclude that according to Definition 9, the product $\widehat{R}\;\Theta\,(v_1, x_1; \ldots; v_k, x_k; v'_1, y_1; \ldots; v'_n, y_n; \epsilon)$ belongs to the space $C^{k+n-r}(V, \mathcal{W}, \mathcal{F})(U_j)$, $0 \le j \le l$, for $l \ge 0$, on one of transversal sections $U_j \in \mathcal{U}$, $j \ge 1$.

$$\Theta : V^{\otimes(k+n-r)} \to \mathcal{W}_{c_1(p_1), \ldots, c_k(p_k); c'_1(p'_1), \ldots, c'_n(p'_n)}, \tag{5.8}$$

and the intersection ranges over all possible $m+m'-t$-tuples of holonomy embeddings h_i, $i \in \{1, \ldots, m + m' - t - 1\}$, between transversal sections $U_1, \ldots, U_l$ of the baisis $\mathcal{U}$ for $\mathcal{F}$. The product $\Theta\,(v_1, x_1; \ldots; v_k, x_k; v'_1, y_1; \ldots; v'_n, y_n; \epsilon)$ belongs to the space

$$C^{k+n-r}_{m+m'-t}(V, \mathcal{W}, \mathcal{U}, \mathcal{F}) = \bigcap_{\substack{U_1 \overset{h_1}{\hookrightarrow} \ldots \overset{h_{m+m'-1}}{\hookrightarrow} U_{m+m'} \\ 1 \le j \le m+m'-t}} C^{k+n-r}(V, \mathcal{W}, \mathcal{F})(U_j), \tag{5.9}$$

where the intersection ranges over all possible $m + m' - t$-tuples of holonomy embeddings h_i, $i \in \{1, \ldots, m + m' - t - 1\}$, between transversal sections $U_1, \ldots, U_{m+m'-t-1}$ of the baisis $\mathcal{U}$ for $\mathcal{F}$. This finishes the proof of the proposition. □

6. Coboundary operators and cohomology of codimension one foliations

In this Section we recall the definition of differential operators acting on double complex spaces. Recall the definitions of E-elements given in Appendix 11. Consider the vector of E-elements:

$$\mathcal{E} \quad = \quad \left(E^{(1)}_W,\; \sum_{i=1}^{n} (-1)^i\; E^{(2)}_{V;\mathbf{1}_V},\; E^{W;(1)}_{WV} \right). \tag{6.1}$$

Then we formulate

Definition 12. The coboundary operator δ^n_m acting on elements $\Phi \in C^n_m(V, \mathcal{W}, \mathcal{F})$ of the spaces (4.2), is defined by

$$\delta^n_m \Phi = \mathcal{E} \cdot_\epsilon \Phi, \tag{6.2}$$

Remark 5. The action of δ^n_m has the orthogonality condititION form (cf. Section 8) [14] with respect to the $\cdot_\epsilon$ product (5.5). Note that it is assumed that the coboundary operator does not affect $dc(p)^{\mathrm{wt}\;(v_i)}$-tensor multipliers in Φ.

Then we obtain

Lemma 7. *The definition (6.3) is equivalent to a multi-point vertex algebra connection (cf. Definition 14.1 in Section 14.2):*

$$\delta^n_m \Phi = G(p_1, \ldots, p_{n+1}), \tag{6.3}$$

where

$$\begin{aligned} G(p_1, \ldots, p_{n+1}) \quad &= \quad \langle w', \sum_{i=1}^{n} (-1)^i \; \Phi \left(\omega_V \left(v_i, c_i(p_i) - c_{i+1}(p_{i+1})\right) v_{i+1}\right)\rangle, \qquad (6.4) \\ &+ \quad \langle w', \omega_W \left(v_1, c_1(p_1)\right) \; \Phi(v_2, c_2(p_2); \ldots ; v_{n+1}, c_n(p_{n+1}))\rangle \\ &+ \quad (-1)^{n+1} \langle w', \omega_W(v_{n+1}, c_{n+1}(p_{n+1})) \; \Phi(v_1, c_1(p_2); \ldots ; v_n, c_n(p_n))\rangle, \end{aligned}$$

for arbitrary $w' \in W'$ (dual to W).

Proof. The statement follows from the intertwining operator (cf. Appendix 9) representation of the definition (6.3) in the form

$$\delta^n_m \Phi = \sum_{i=1}^{3} \langle w', e^{\xi_i L_W(-1)} \; \omega^W_{WV} \left(\Phi_i\right) u_i\rangle,$$

for some $\xi_i \in \mathbb{C}$, and $u_i \in V$, and Φ_i obvious from (6.3). Namely,

$$\begin{aligned} &\delta^n_m \Phi = \langle w', e^{c_1(p_1) L(-1)_W} \; \omega^W_{WV} \left(\Phi \left(v_2, c_2(p_2); \ldots ; v_n, c_{n+1}(p_{n+1}), -c_1(p_1)\right) v_1\right\rangle \\ &+ \sum_{i=1}^{n} (-1)^i e^{\zeta L_W(-1)} \langle w', \omega^W_{WV} \left(\Phi \left(\omega_V \left(v_i, c_i(p_i) - c_{i+1}(p_{i+1})\right), -\zeta\right) \mathbf{1}_V\rangle\right) \\ &+ \langle w', e^{c_{n+1}(p_{n+1}) L(-1)_W} \; \omega^W_{WV} \left(\Phi \left(v_1, c_1(p_1); \ldots ; v_n, c_n(p_n), -c_{n+1}(p_{n+1})\right) v_{n+1}\right)\rangle, \end{aligned}$$

for an arbitrary $\zeta \in \mathbb{C}$. □

Remark 6. Inspecting construction of the double complex spaces (4.2) we see that the action (6.4) of the δ^n_m on an element of $C^n_m(V, \mathcal{W}, \mathcal{F})$ provides a coupling (in terms of $\mathcal{W}_{z_1,\ldots,z_n}$-valued rational functions) of vertex operators taken at the local coordinates $c_i(z_{p_i})$, $0 \le i \le k$, at the vicinities of the same points p_i taken on transversal sections for $\mathcal{F}$, with elements of $C^n_{m-1}(V, \mathcal{W}, \mathcal{F})$ taken at points at the local coordinates $c_i(z_{p_i})$, $0 \le i \le n$ on $\mathcal{M}$ for points p_i considered on the leaves of $\mathcal{M}/\mathcal{F}$.

6.1. **Complexes on transversal connections.** In addition to the double complex $(C^n_m(V,\ \mathcal{W},\ \mathcal{F}),\ \delta^n_m)$ provided by (4.2) and (6.3), there exists an exceptional short double complex which we call transversal connection complex. We have

Lemma 8. *For $n = 2$, and $k = 0$, there exists a subspace $C^0_{ex}(V, \mathcal{W}, \mathcal{F})$*

$$C^2_m(V, \mathcal{W}, \mathcal{F}) \subset C^0_{ex}(V, \mathcal{W}, \mathcal{F}) \subset C^2_0(V, \mathcal{W}, \mathcal{F}),$$

for all $m \ge 1$, with the action of coboundary operator δ^2_m defined.

Proof. Let us consider the space $C^2_0(V, \mathcal{W}, \mathcal{F})$. vertex operators composable. Indeed, the space $C^2_0(V, \mathcal{W}, \mathcal{F})$ contains elements of $\mathcal{W}_{c_1(p_1), c_2(p_2)}$ so that the action of δ^2_0 is zero. Nevertheless, as for $\mathcal{J}^n_m(\Phi)$ in (16.26), Definition 31, let us consider sum of projections

$$P_r : \mathcal{W}_{z_i, z_j} \to W_r,$$

for $r \in \mathbb{C}$, and $(i,j) = (1,2), (2,3)$, so that the condition (16.26) is satisfied for some connections similar to the action (16.26) of δ_0^2. Separating the first two and the second two summands in (6.4), we find that for a subspace of $C_0^2(V, \mathcal{W}, \mathcal{F})$, which we denote as $C_{ex}^2(V, \mathcal{W}, \mathcal{F})$, consisting of three-point connections Φ such that for v_1, v_2, $v_3 \in V$, $w' \in W'$, and arbitrary $\zeta \in \mathbb{C}$, the following forms of connections

$$\begin{aligned}
&G_1(c_1(p_1), c_2(p_2), c_3(p_3)) \\
&= \sum_{r \in \mathbb{C}} \Big(\langle w', E_W^{(1)} \left(v_1, c_1(p_1); P_r \left(\Phi\left(v_2, c_2(p_2) - \zeta; v_3, c_3(p_3) - \zeta\right)\right)\right\rangle \\
&\quad + \langle w', \Phi \Big(v_1, c_1(p_1); P_r \Big(E_V^{(2)} \left(v_2, c_2(p_2) - \zeta; v_3, c_3(z_3) - \zeta; \mathbf{1}_V \Big), \zeta\Big)\right\rangle \Big) \\
&= \sum_{r \in \mathbb{C}} \left(\langle w', \omega_W \left(v_1, c_1(p_1)\right)\ P_r \left(\Phi\left(v_2, c_2(p_2) - \zeta; v_3, c_3(p_3) - \zeta\right)\right)\right\rangle \\
&\quad + \langle w', \Phi \left(v_1, c_1(p_1); P_r \left(\omega_V \left(v_2, c_2(p_2) - \zeta\right) \omega_V \left(v_3, c_3(z_3) - \zeta\right) \mathbf{1}_V\right), \zeta\right)\rangle \big),
\end{aligned} \tag{6.5}$$

and

$$\begin{aligned}
&G_2(c_1(p_1), c_2(p_2), c_3(p_3)) \\
&\quad = \sum_{r \in \mathbb{C}} \Big(\langle w', \Phi \Big(P_r \Big(E_V^{(2)} \left(v_1, c_1(p_1) - \zeta, v_2, c_2(p_2) - \zeta; \mathbf{1}_V\right)\Big), \zeta; v_3, c_3(p_3)\Big)\Big\rangle \\
&\quad + \langle w', E_{WV}^{W;(1)} \left(P_r \left(\Phi\left(v_1, c_1(p_1) - \zeta; v_2, c_2(p_2) - \zeta\right), \zeta; v_3, c_3(p_3)\right)\right)\rangle \Big) \\
&= \sum_{r \in \mathbb{C}} \left(\langle w', \Phi \left(P_r(\omega_V \left(v_1, c_1(p_1) - \zeta\right) \omega_V \left(v_2, c_2(p_2) - \zeta\right) \mathbf{1}_V, \zeta)\right); v_3, c_3(p_3))\right\rangle \\
&\quad + \langle w', \omega_V \left(v_3, c_3(p_3)\right)\ P_r \left(\Phi\left(v_1, c_1(p_1) - \zeta; v_2, c_2(p_2) - \zeta\right)\right)\rangle \big),
\end{aligned} \tag{6.6}$$

are absolutely convergent in the regions

$$|c_1(p_1) - \zeta| > |c_2(p_2) - \zeta|,$$

$$|c_2(p_2) - \zeta| > 0,$$

$$|\zeta - c_3(p_3)| > |c_1(p_1) - \zeta|,$$

$$|c_2(p_2) - \zeta| > 0,$$

where c_i, $1 \le i \le 3$ are coordinate functions, respectively, and can be analytically extended to rational form-valued functions in $c_1(p_1)$ and $c_2(p_2)$ with the only possible poles at $c_1(p_1)$, $c_2(p_2) = 0$, and $c_1(p_1) = c_2(p_2)$. Note that (6.5) and (6.6) constitute the first two and the last two terms of (6.4) correspondingly. According to Proposition 14 (cf. Appendix 12), $C_m^2(V, \mathcal{W}, \mathcal{F})$ is a subspace of $C_{ex}^2(V, \mathcal{W}, \mathcal{F})$, for $m \ge 0$, and $\Phi \in C_m^2(V, \mathcal{W}, \mathcal{F})$ are composable with m vertex operators. Note that (6.5) and (6.6) represent sums of forms $G_{tr}(p, p')$ of transversal connections (14.4) (cf. Section 14). □

Remark 7. It is important to mention that, according to the general principle, abserved in [1], for non-vanishing connection $G(c(p), c(p'), c(p''))$, there exists an invariant structure, e.g., a cohomological class. In our case, it appears as a non-empty subspaces $C^2_m(V, \mathcal{W}, \mathcal{F}) \subset C^2_{ex}(V, \mathcal{W}, \mathcal{F})$ in $C^2_0(V, \mathcal{W}, \mathcal{F})$.

Then we have

Definition 13. The coboundary operator

$$\delta^2_{ex} : C^2_{ex}(V, \mathcal{W}, \mathcal{F}) \to C^3_0(V, \mathcal{W}, \mathcal{F}), \tag{6.7}$$

is defined by three point connection of the form

$$\delta^2_{ex}\Phi = \mathcal{E}_{ex} \cdot_\epsilon \Phi = G_{ex}(p_1, p_2, p_3), \tag{6.8}$$

where

$$\mathcal{E}_{ex} = \left(E^{(1)}_W,\ \sum_{i=1}^{2} (-1)^n E^{(2)}_{V;\mathbf{1}_V},\ E^{W;(1)}_{WV} \right), \tag{6.9}$$

$$\begin{aligned} G_{ex}(p_1, p_2, p_3) \quad = \quad & \langle w', \omega_W\,(v_1, c_1(p_1))\ \Phi\,(v_2, c_2(p_2); v_3, c_3(p_3)))\rangle) \\ & -\langle w', \Phi\,(\omega_V(v_1, c_1(p_1))\ \ \omega_V(v_2, c_2(p_2))\mathbf{1}_V; v_3, c_3(p_3)))\rangle \\ & +\langle w', \Phi(v_1, c_1(p_1);\ \omega_V(v_2, c_2(p_2))\ \omega_V(v_3, c_3(p_3))\mathbf{1}_V)\rangle \\ & +\langle w', \omega_W(v_3, c_3(p_3))\ \Phi\,(v_1, c_1(p_1); v_2, c_2(p_2))\rangle, \end{aligned} \tag{6.10}$$

for $w' \in W'$, $\Phi \in C^2_{ex}(V, \mathcal{W}, \mathcal{F})$, $v_1, v_2, v_3 \in V$ and $(z_1, z_2, z_3) \in F_3\mathbb{C}$.

Then we have

Proposition 8. *The operators* (6.3) *and* (6.7) *provide the chain-cochain complexes*

$$\delta^n_m : C^n_m(V, \mathcal{W}, \mathcal{F}) \to C^{n+1}_{m-1}(V, \mathcal{W}, \mathcal{F}), \tag{6.11}$$

$$\delta^{n+1}_{m-1} \circ \delta^n_m = 0, \tag{6.12}$$

$$\delta^2_{ex} \circ \delta^1_2 = 0,$$

$$0 \longrightarrow C^0_m(V, \mathcal{W}, \mathcal{F}) \xrightarrow{\delta^0_m} C^1_{m-1}(V, \mathcal{W}, \mathcal{F}) \xrightarrow{\delta^1_{m-1}} \cdots \xrightarrow{\delta^{m-1}_1} C^m_0(V, \mathcal{W}, \mathcal{F}) \longrightarrow 0, \tag{6.13}$$

$$0 \longrightarrow C^0_3(V, \mathcal{W}, \mathcal{F}) \xrightarrow{\delta^0_3} C^1_2(V, \mathcal{W}, \mathcal{F}) \xrightarrow{\delta^1_2} C^2_{ex}(V, \mathcal{W}, \mathcal{F}) \xrightarrow{\delta^2_{ex}} C^3_0(V, \mathcal{W}, \mathcal{F}) \longrightarrow 0, \tag{6.14}$$

on the spaces (4.2).

Since

$$\delta^1_2\, C^1_2(V, \mathcal{W}, \mathcal{F}) \subset C^2_1(V, \mathcal{W}, \mathcal{F}) \subset C^2_{ex}(V, \mathcal{W}, \mathcal{F}),$$

the second formula follows from the first one, and

$$\delta^2_{ex} \circ \delta^1_2 = \delta^2_1 \circ \delta^1_2 = 0.$$

Proof. The proof of this proposition is analogous to that of Proposition (4.1) in [23] for chain-cochain complex of a grading-restricted vertex algebra. The only difference is that we work with the space $\mathcal{W}_{c_1(p_1), \ldots, c_n(p_n)}$ instead of $W_{z_1, \ldots, z_n}$. □

6.2. **Vertex algebra cohomology and relation to Crainic and Moerdijk construction.** Now let us define the cohomology of the leaf space $\mathcal{M}/\mathcal{F}$ for codimension one foliation $\mathcal{F}$ associated with a grading-restricted vertex algebra V.

Definition 14. We define the n-th cohomology $H_k^n(V, \mathcal{W}, \mathcal{F})$ of $\mathcal{M}/\mathcal{F}$ with coefficients in $\mathcal{W}_{z_1,\ldots,z_n}$ (containing maps composable k vertex operators on k transversal sections) to be the factor space of closed multi-point connections by the space of connection forms:

$$H_k^n(V, \mathcal{W}, \mathcal{F}) = \mathcal{C}on^n_{k;\, cl}/G^{n-1}_{k+1}. \tag{6.15}$$

Note that due to (6.4), (6.10), and Definitions 14.1 and 14.2 (cf. Section 6), it is easy to see that (6.15) is equivalent to the standard cohomology definition

$$H_k^n(V, \mathcal{W}, \mathcal{F}) = \ker \delta^n_k / \mathrm{im}\, \delta^{n-1}_{k+1}. \tag{6.16}$$

Recall the construction of the Čech-de Rham cohomology of a foliation [7]. Consider a foliation $\mathcal{F}$ of codimension one defined on a smooth complex curve $\mathcal{M}$. Consider the double complex

$$C^{k,l} = \prod_{U_0 \overset{h_1}{\hookrightarrow} \cdots \overset{h_k}{\hookrightarrow} U_k} \Omega^l(U_0), \tag{6.17}$$

where $\Omega^l(U_0)$ is the space of differential l-forms on U_0, and the product ranges over all k-tuples of holonomy embeddings between transversal sections from a fixed transversal basis $\mathcal{U}$. Component of $\varpi \in C^{k,l}$ are denoted by $\varpi(h_1, \ldots, h_l) \in \Omega^l(\mathcal{U}_0)$. The vertical differential is defined as

$$(-1)^k d : C^{k,l} \to C^{k,l+1},$$

where d is the usual de Rham differential. The horizontal differential

$$\delta : C^{k,l} \to C^{k+1,l},$$

is given by

$$\delta = \sum_{i=1}^{k} (-1)^i \delta_i,$$

$$\delta_i \varpi(h_1, \ldots, h_{k+1}) = G(h_1, \ldots, h_{k+1}), \tag{6.18}$$

where $G(h_1, \ldots, h_{k+1})$ is the multi-point connection of the form (14.1), i.e.,

$$\delta_i \varpi(h_1, \ldots, h_{p+1}) = \begin{cases} h_1^* \varpi(h_2, \ldots, h_{p+1}), & \text{if } i = 0, \\ \varpi(h_1, \ldots, h_{i+1} h_i, \ldots, h_{p+1}), & \text{if } 0 < i < p+1, \\ \varpi(h_1, \ldots, h_p), & \text{if } i = p+1. \end{cases} \tag{6.19}$$

This double complex is actually a bigraded differential algebra, with the usual product

$$(\varpi \cdot \eta)(h_1, \ldots, h_{k+k'}) = (-1)^{kk'} \varpi(h_1, \ldots, h_k)\; h_1^* \ldots h_k^*\, .\eta(h_{k+1}, \ldots h_{k+k'}), \tag{6.20}$$

for $\varpi \in C^{k,l}$ and $\eta \in C^{k',l'}$, thus $(\varpi \cdot \eta)(h_1, \ldots, h_{k+k'}) \in C^{k+k', l+l'}$.

Definition 15. The cohomology $\check{H}^*_{\mathcal{U}}(M/\mathcal{F})$ of this complex is called the Čech-de Rham cohomology of the leaf space $\mathcal{M}/\mathcal{F}$ with respect to the transversal basis $\mathcal{U}$. It is defined by

$$\check{H}^*_{\mathcal{U}}(M/\mathcal{F}) = \mathcal{C}on^{k+1}_{cl}(h_1, \ldots, h_{k+1})/G^k(h_1, \ldots, h_k),$$

where $\mathcal{C}on^{k+1}_{cl}(h_1, \ldots, h_{k+1})$ is the space of closed multi-point connections, and G^k $(h_1, \ldots, h_k)$ is the space of k-point connection forms.

In this subsection we show the following

Lemma 9. *In the case of codimension one foliation on a smooth complex curve, the construction of the double complex* $\left(C^{k,l}, \delta\right)$*,* (6.17)*,* (6.18) *results from the construction of the double complexes* $(C^n_m(V, \mathcal{W}, \mathcal{F}), \delta^n_m)$ *of* (6.13) *and* (6.14)*.*

Proof. One constructs the space of differential forms of degree k

$$\langle w', \Phi\left(dc_1(p_1)^{\mathrm{wt}\ (v_1)} \otimes v_1, c_1(p_1); \ldots; dc_n(p_n)^{\mathrm{wt}\ (v_n)} v_n, c_n(p_n)\right)\rangle, \tag{6.21}$$

by elements Φ of $C^n_m(V, \mathcal{W}, \mathcal{F})$ such that $n = k$ the total degree

$$\sum_{i=1}^{n} \mathrm{wt}\ (v_i) = l,$$

$v_i \in V$. The condition of composability of Φ with m vertex operators allows us make the association of the differential form $\varpi(h_1, \ldots, h_n)$ with (6.21) $(h^*_1, \ldots, h^*_k)$ with $(v_i, \ldots, v_k)$, and to represent a sequence of holomorphic embeddings $h_1, \ldots, h_p$ for $U_0, \ldots, U_p$ in (6.17) by vertex operators ω_W, i.e,

$$\left(h(h^*_1) \ldots h(h^*_n)\right)(z_1, \ldots, z_n)) = \omega_W\left(v_1, t_1(p_1)\right) \ldots \omega_W\left(v_l, t(p_n)\right).$$

Then, by using Definitions of coboundary operator (6.3), we see that the definition of the coboundary operator of [7] is parallel to the definition (6.3). □

7. Properties of the ϵ-product of $C^k_m(V, \mathcal{W}, \mathcal{F})$-spaces

Since the product of $\Phi(v_1, x_1; \ldots; v_k, x_k) \in C^k_m(V, \mathcal{W}, \mathcal{F})$ and $\Psi(v'_1, y_1; \ldots; v'_n, y_n) \in C^n_{m'}(V, \mathcal{W}, \mathcal{F})$ results in an element of $C^{k+n-r}_{m+m'-t}(V, \mathcal{W}, \mathcal{F})$, then, similar to [23], the following corollary follows directly from Proposition (6) and Definition 6:

Corollary 1. *For the spaces* $\mathcal{W}_{x_1,\ldots,x_k}$ *and* $\mathcal{W}_{y_1,\ldots,y_n}$ *with the product* (5.5) $\Theta \in \mathcal{W}_{z_1,\ldots,z_{k+n-r}}$*, the subspace of* $\mathrm{Hom}(V^{\otimes n}, \mathcal{W}_{z_1,\ldots,z_{k+n-r}})$ *consisting of linear maps having the* $L_W(-1)$*-derivative property, having the* $L_V(0)$*-conjugation property or being composable with* m *vertex operators is invariant under the action of* S_{k+n-r}*.* □

We also have

Corollary 2. *For a fixed set* $(v_1, \ldots v_k; v_{k+1}, \ldots, v_{k+n-r}) \in V$ *of vertex algebra elements, and fixed* $k+n-r$*, and* $m+m'-t$*, the* ϵ*-product* $\widehat{R}\ \Theta(v_1, z_1; \ldots; v_k, z_k; v_{k+1}, z_{k+1}; \ldots; v_{k+n-r}, y_{k+n-r}; \epsilon)$*,*

$$\cdot_\epsilon : C^k_m(V, \mathcal{W}, \mathcal{F}) \times C^n_{m'}(V, \mathcal{W}, \mathcal{F}) \to C^{k+n-r}_{m+m'-t}(V, \mathcal{W}, \mathcal{F}),$$

of the spaces $C^k_m(V, \mathcal{W}, \mathcal{F})$ *and* $C^n_{m'}(V, \mathcal{W}, \mathcal{F})$*, for all choices of* k*,* n*,* m*,* $m' \geq 0$*, is the same element of* $C^{k+n-r}_{m+m'-t}(V, \mathcal{W}, \mathcal{F})$ *for all possible* $k \geq 0$*.* □

Proof. In Proposition 1 we have proved that the result of the maps belongs to $W_{z_1,\ldots,z_{k+n-r}}$, for all k, $n \geq 0$, and fixed $k+n-r$. As in proof of Proposition 6, by checking conditions for the forms (16.14) and (16.18), we see by Proposition 14, the product $\widehat{R}\,\Theta(v_1,x_1;\ldots;v_k,x_k;v'_1,y_1;\ldots;v'_n,y_n)$ is composable with fixed $m+m'-t$ number of vertex operators. □

By Proposition 4, elements of the space $\mathcal{W}_{z_1,\ldots,z_{k+n-r}}$ resulting from the ϵ-product (3.3) are invariant with respect to independent changes of formal parameters of the group $(\mathrm{Aut}\,\mathcal{O})^{\times(k+n-r)}_{z_1,\ldots,z_{k+n-r}}$. Now we prove the following

Corollary 3. *For* $\Phi(v_1,x_1;\ldots;v_k,x_k) \in C^k_m(V,\mathcal{W},\mathcal{F})$ *and* $\Psi(v'_1,y_1;\ldots;v'_n,y_n) \in C^n_{m'}(V,\ \mathcal{W},\mathcal{F})$*, the product*

$$\begin{aligned}\widehat{R}\,\Theta\,(v_1,x_1;\ldots;v_k,x_k;v'_1,y_1;\ldots;v'_n,y_n;\epsilon)&\\ =\Phi(v_1,x_1;\ldots;v_k,x_k)\cdot_\epsilon\Psi(v'_1,y_1;\ldots;v'_n,y_n),&\end{aligned} \tag{7.1}$$

is canonical with respect to the action

$$(z_1,\ldots,z_{k+n-r})\mapsto(z'_1,\ldots,z'_{k+n-r})=(\rho(z_1),\ldots,\rho(z_{k+n-r})), \tag{7.2}$$

of elements the group $(\mathrm{Aut}\,\mathcal{O})^{\times(k+n-r)}_{z_1,\ldots,z_{k+n-r}}$.

Proof. In Subsection 11 we have proved that the product (3.3) belongs to $W_{z_1,\ldots,z_{k+n-r}}$, and is invariant with respect to the group $(\mathrm{Aut}\,\mathcal{O})^{\times(k+n-r)}_{z_1,\ldots,z_{k+n-r}}$. Similar as in the proof of Proposition 5, vertex operators $\omega_V(v_i,x_i)$, $1\leq i\leq m$, composable with $\Phi(v_1,x_1;\ldots;v_k,x_k)$, and vertex operators $\omega_V(v_j,y_j)$, $1\leq j\leq m'$, composable with $\Psi(v'_1,y_1;\ldots;v'_n,y_n)$, are also invariant with respect to independent changes of coordinates $(\rho(z_1),\ldots,\rho(z_{k+n-r}))\in(\mathrm{Aut}\,\mathcal{O})^{\times(k+n-r)}_{z_1,\ldots,z_{k+n-r}}$. □

7.1. **Coboundary operator acting on the product of elements of $C^n_m(V,\mathcal{W},\mathcal{F})$-spaces.** In Proposition 6 we proved that the product (5.5) of elements of spaces $C^k_m(V,\mathcal{W},\mathcal{F})$ and $C^n_{m'}(V,\mathcal{W},\mathcal{F})$ belongs to $C^{k+n-r}_{m+m'-t}(V,\mathcal{W},\mathcal{F})$. Thus, the product admits the action ot the differential operators $\delta^{k+n-r}_{m+m'-t}$ and δ^{2-r}_{ex-t} defined in (6.3) and (6.7). The co-boundary operators (6.3) and (6.7) possesse a variation of Leibniz law with respect to the product (5.5). Indeed, we state here

Proposition 9. *For* $\Phi(v_1,x_1;\ldots;v_k,x_k)\in C^k_m(V,\mathcal{W},\mathcal{F})$ *and* $\Psi(v'_1,y_1;\ldots;v'_n,y_n)\in C^n_{m'}(V,\mathcal{W},\mathcal{F})$*, the action of the differential* $\delta^{k+n-r}_{m+m'-t}$ (6.3) *(and* δ^{2-r}_{ex-t} (6.7)*) on the* ϵ*-product* (5.5) *is given by*

$$\begin{aligned}&\delta^{k+n-r}_{m+m'-t}\left(\Phi(v_1,x_1;\ldots;v_k,x_k)\cdot_\epsilon\Psi(v'_1,y_1;\ldots;v'_n,y_n)\right)\\ &\quad=\left(\delta^k_m\Phi(\widetilde{v}_1,z_1;\ldots;\widetilde{v}_k,z_k)\right)\cdot_\epsilon\Psi(\widetilde{v}_{k+1},z_{k+1};\ldots;\widetilde{v}_{k+n-r},z_{k+n-r})\\ &\quad+(-1)^k\Phi(\widetilde{v}_1,z_1;\ldots;\widetilde{v}_k,z_k)\cdot_\epsilon\left(\delta^{n-r}_{m'-t}\Psi(\widetilde{v}_1,z_{k+1};\ldots;\widetilde{v}_{k+n-r},z_{k+n-r})\right),\end{aligned} \tag{7.3}$$

where we use the notation as in (3.1) *and* (3.6)*.*

The proof of this proposition is in Appendix 9.

Remark 8. Checking (6.3) we see that an extra arbitrary vertex algebra element $v_{n+1} \in V$, as well as corresponding extra arbitrary formal parameter z_{n+1} appear as results of the action of δ^n_m on $\Phi \in C^n_m(V, \mathcal{W}, \mathcal{F})$ mapping it to $C^{n+1}_{m-1}(V, \mathcal{W}, \mathcal{F})$. In application to the ϵ-product (5.5) these extra arbitrary elements are involved in the definition of the action of $\delta^{k+n-r}_{m+m'-t}$ on $\Phi(v_1, x_1; \ldots; v_k, x_k) \cdot_\epsilon \Psi(v'_1, y_1; \ldots; v'_n, y_n)$.

Note that both sides of (7.3) belong to the space $C^{n+n'-r-1}_{m+m'-t+1}(V, \mathcal{W}, \mathcal{F})$. The co-boundary operators δ^n_m and $\delta^{n'}_{m'}$ in (7.3) do not include the number of common vertex algebra elements (and formal parameters), neither the number of common vertex operators corresponding mappings composable with. The dependence on common vertex algebra elements, parameters, and composable vertex operators is taken into account in mappings multiplying the action of co-boundary operators on Φ.

We have the following

Corollary 4. *The product* (5.5) *and the differential operators* (6.3), (6.7) *endow the space* $C^k_m(V, \mathcal{W}, \mathcal{F}) \times C^n_{m'}(V, \mathcal{W}, \mathcal{F})$, k, $n \geq 0$, m, $m' \geq 0$, *with the structure of a double graded differential algebra* $\mathcal{G}\left(V, \mathcal{W}, \cdot_\epsilon, \delta^{k+n-r}_{m+m'-t}\right)$. □

Finally, we prove the following

Proposition 10. *The multiplication* (5.5) *extends the chain-cochain complexes* (6.13) *and* (6.14) *property* (6.12) *to all products* $C^k_m(V, \mathcal{W}, \mathcal{F}) \times C^n_{m'}(V, \mathcal{W}, \mathcal{F})$, k, $n \geq 0$, m, $m' \geq 0$.

Proof. For $\Phi \in C^k_m(V, \mathcal{W}, \mathcal{F})$ and $\Psi \in C^n_{m'}(V, \mathcal{W}, \mathcal{F})$ we proved in Proposition 6 that the product $\Phi \cdot_\epsilon \Psi$ belongs to the space $C^{k+n-r}_{m+m'-t}(V, \mathcal{W}, \mathcal{F})$. Using (7.3) and chain-cochain property for Φ and Ψ we also check that

$$\begin{aligned} &\delta^{k+n+1-r}_{m+m'-1-t} \circ \delta^{k+n-r}_{m+m'-t} \left(\Phi \cdot_\epsilon \Psi\right) = 0. \\ &\delta^{2-r}_{ex-t} \circ \delta^{1-r}_{2-t} \left(\Phi \cdot_\epsilon \Psi\right) = 0. \end{aligned} \tag{7.4}$$

Thus, the the chain-cochain property extends to the product $C^k_m(V, \mathcal{W}, \mathcal{F}) \times C^n_{m'}(V, \mathcal{W}, \mathcal{F})$. □

7.2. **The exceptional complex.** For elements of the spaces $C^2_{ex}(V, \mathcal{W}, \mathcal{F})$ we have the following

Corollary 5. *The product of elements of the spaces* $C^2_{ex}(V, \mathcal{W}, \mathcal{F})$ *and* $C^n_m(V, \mathcal{W}, \mathcal{F})$ *is given by* (5.5),

$$\cdot_\epsilon : C^2_{ex}(V, \mathcal{W}, \mathcal{F}) \times C^n_m(V, \mathcal{W}, \mathcal{F}) \to C^{n+2-r}_m(V, \mathcal{W}, \mathcal{F}), \tag{7.5}$$

and, in particular,

$$\cdot_\epsilon : C^2_{ex}(V, \mathcal{W}, \mathcal{F}) \times C^2_{ex}(V, \mathcal{W}, \mathcal{F}) \to C^{4-r}_0(V, \mathcal{W}, \mathcal{F}).$$

Proof. The fact that the number of formal parameters is $n + 2 - r$ in the product (5.5) follows from Proposition (1). Consider the product (5.5) for $C^2_{ex}(V, \mathcal{W}, \mathcal{F})$ and $C^n_m(V, \mathcal{W}, \mathcal{F})$. It is clear that, similar to considerations of the proof of Proposition 6, the total number m of vertex operators the product Θ is composable to remains the same. □

8. Product-type cohomological classes

8.1. **The commutator multiplication.** In this subsection we define further product of pair of elements of spaces $C^k_m(V, \mathcal{W}, \mathcal{F})$ and $C^n_{m'}(V, \mathcal{W}, \mathcal{F})$, suitable for formulation of cohomological invariants. Let us consider the mappings

$$\Phi(v_1, z_1; \ldots; v_n, z_k) \in C^k_m(V, \mathcal{W}, \mathcal{F}),$$

$$\Psi(v_{k+1}, z_{k+1}; \ldots; v_{k+n}, z_{k+n}) \in C^n_{m'}(V, \mathcal{W}, \mathcal{F}),$$

with r common vertex algebra elements (and, correspondingly, r formal variables), and t common vertex operators mappings Φ and Ψ are composable with. Note that when applying the co-boundary operators (6.3) and (6.7) to a map $\Phi(v_1, z_1; \ldots; v_n, z_n) \in C^n_m(V, \mathcal{W}, \mathcal{F})$,

$$\delta^n_m : \Phi(v_1, z_1; \ldots; v_n, z_n) \to \Phi(v'_1, z'_1; \ldots; v'_{n+1}, z'_{n+1}) \in C^{n+1}_{m-1}(V, \mathcal{W}, \mathcal{F}),$$

one does not necessary assume that we keep the same set of vertex algebra elements/formal parameters and vertex operators composable with for $\delta^n_m \Phi$, though it might happen that some of them could be common with Φ. Then we have

Definition 16. Let us define extra product of Φ and Ψ,

$$\Phi \cdot \Psi : V^{\otimes(k+n-r)} \to \mathcal{W}_{z_1,\ldots,z_{k+n-r}}, \tag{8.1}$$

$$\Phi \cdot \Psi = [\Phi, \cdot_\epsilon \Psi] = \Phi \cdot_\epsilon \Psi - \Psi \cdot_\epsilon \Phi, \tag{8.2}$$

where brackets denote ordinary commutator in $\mathcal{W}_{z_1,\ldots,z_{k+n-r}}$.

Due to the properties of the maps $\Phi \in C^k_m(V, \mathcal{W}, \mathcal{F})$ and $\Psi \in C^n_{m'}(V, \mathcal{W}, \mathcal{F})$, we obtain

Lemma 10. *The product $\Phi \cdot \Psi$ belongs to the space $C^{k+n-r}_{m+m'-t}(V, \mathcal{W}, \mathcal{F})$. For $k = n$ and*

$$\Psi(v_{n+1}, z_{n+1}; \ldots; v_{2n}, z_{2n}) = \Phi(v_1, z_1; \ldots; v_n, z_n),$$

we obtain from (3) *and* (5.5) *that*

$$\Phi(v_1, z_1; \ldots; v_n, z_n) \cdot \Phi(v_1, z_1; \ldots; v_n, z_n) = 0. \tag{8.3}$$

□

The product (8.1) will be used in the next subsection in order to introduce cohomological invariants.

8.2. **Cohomological invariants.** In this subsection, using the vertex algebra double complex construction (6.11)–(6.12), we provide invariants for the grading-restricted vertex algebra cohomology of codimension one foliations on complex curves. Recall (Appendix 14) definitions of cohomological classes associated to grading-restricted vertex algebras. In this subection we consider the general classes of cohomological invariants which arrise from Definition 11 of a product of pairs of $C^n_m(V, \mathcal{W}, \mathcal{F})$-spaces. Under a natural extra condition, the double complexes (6.13) and (6.14) allow us to establish relations among elements of $C^n_m(V, \mathcal{W}, \mathcal{F})$ spaces. By analogy with the notion of integrability for differential forms [14], we use here the notion of orthogonality for spaces of a complex.

Definition 17. For the double complexes (6.13) and (6.14) let us require that for a pair of double complex spaces $C^k_m(V, \mathcal{W}, \mathcal{F})$ and $C^n_{m'}(V, \mathcal{W}, \mathcal{F})$, there exist subspaces

$$\widetilde{C}^k_m(V, \mathcal{W}, \mathcal{F}) \subset C^k_m(V, \mathcal{W}, \mathcal{F}),$$

$$\widetilde{C}^n_{m'}(V, \mathcal{W}, \mathcal{F}) \subset C^n_{m'}(V, \mathcal{W}, \mathcal{F}),$$

such that for all $\Phi \in \widetilde{C}^k_m(V, \mathcal{W}, \mathcal{F})$ and all $\Psi \in \widetilde{C}^n_{m'}(V, \mathcal{W}, \mathcal{F})$,

$$\Phi \cdot \delta^n_{m'} \Psi = 0, \tag{8.4}$$

namely, Φ supposed to be orthogonal to $\delta^n_{m'}\Psi$ with respect to the product (3). We call this the orthogonality condition for mappings of double complexes (6.13) and (6.14).

Note that in the case of differential forms considered on a smooth manifold, the Frobenius theorem for a distribution provides the orthogonality condition [14]. The fact that both sides of (8.6) (see below) belong to the same double complex space, apply limitations to possible combinations of (k, m) and (n, m'). Below we derive algebraic relations occurring from the orthogonality condition on the double bicomplexes (6.13) and (6.14).

Taking into account the correspondence (see Subsection 14) with the Čech-de Rham complex due to [7], we reformulate the derivation of product-type invariant in vertex algebra terms. Recall that the Godbillon–Vey cohomological class [14] is considered on codimension one foliations of three-dimensional smooth manifolds. In this paper, we supply its analogue for complex curves. According to Definition 4.2 we have k-tuples of one-dimesional transversal sections. In each section we attach one vertex operator $Y_W(u_k, w_k)$, $u_k \in V$, $w_k \in U_k$. Similarly to differential forms setup, a mapping $\Phi \in C^m_k(V, \mathcal{W}, \mathcal{F})$ defines a codimension one foliation. As we see from (6.3), (8.3), and (7.3) it satisfyies properties simmilar to differential forms.

Now we show that the analog of the integrability condition provide a generalization of product-type invariant for codimension one foliations on complex curves. Here we formulate the main statement of this paper:

Proposition 11. *The set of commutation relations The product* (5.5)*, the differential operators* (6.3)*,* (6.7)*, and the orthogonality condition* (8.4) *applied to the double complexes* (6.13) *and* (6.14) *generate non-vanishing cohomology classes*

$$\left[\left(\delta^1_2 \Phi\right) \cdot \Phi\right], \quad \left[\left(\delta^0_3 \Lambda\right) \cdot \Lambda\right], \quad \left[\left(\delta^1_t \Psi\right) \cdot \Psi\right],$$

independent on the choice of $\Phi \in C^1_2(V, \mathcal{W}, \mathcal{F})$*,* $\Lambda \in C^0_3(V, \mathcal{W}, \mathcal{F})$*, and* $\Psi \in C^1_t(V, \mathcal{W}, \mathcal{F})$*.*

Remark 9. In this paper we provide results concerning complex curves, i.e., the case $n \le 1$, $n_0 \le 1$, $n_i \le 1$. They generalize to the case of higher dimensional complex manifolds.

Proof. Let us consider two maps $\Phi(v_1) \in C^1_2(V, \mathcal{W}, \mathcal{F})$ and $\Lambda \in C^0_3(V, \mathcal{W}, \mathcal{F})$. We require them to be orthogonal, i.e.,

$$\Phi \cdot \delta^0_3 \Lambda = 0. \tag{8.5}$$

Thus, there exists $\Psi(v_2) \in C^n_m(V, \mathcal{W}, \mathcal{F})$, such that

$$\delta^0_3 \Lambda = \Phi \cdot \Psi, \tag{8.6}$$

and $1 = 1 + n - r$, $2 = 2 + m - t$, i.e., $n = r$, which leads to $r = 1$; $m = t$, $0 \le t \le 2$, i.e., $\Psi \in C^1_t(V, \mathcal{W}, \mathcal{F})$. Here r and t are correspondingly numbers of common vertex algebra element (and formal parameteres) and vertex operators a map composable with. All other orthogonality conditions for the short sequence (6.14) does not allow relations of the form (8.6).

Consider now (8.5). We obtain, using (7.3)

$$\delta^{2-r'}_{4-t'}(\Phi \cdot \delta^0_3 \Lambda) = \left(\delta^1_2 \Phi\right) \cdot \delta^0_3 \Lambda + \Phi \cdot \delta^1_2 \delta^0_3 \Lambda = \left(\delta^1_2 \Phi\right) \cdot \delta^0_3 \Lambda = \left(\delta^1_2 \Phi\right) \cdot \Phi \cdot \Psi.$$

Thus

$$0 = \delta^{3-r'}_{3-t'} \delta^{2-r'}_{4-t'}(\Phi \cdot \delta^0_3 \Lambda) = \delta^{3-r'}_{3-t'} \left(\left(\delta^1_2 \Phi\right) \cdot \Phi \cdot \Psi. \right),$$

and $\left(\left(\delta^1_2 \Phi\right) \cdot \Phi \cdot \Psi\right)\big)$ is closed. At the same time, from (8.5) it follows that

$$0 = \delta^1_2 \Phi \cdot \delta^0_3 \Lambda - \Phi \cdot \delta^1_2 \delta^0_3 \Lambda = \left(\Phi \cdot \delta^0_3 \Lambda\right).$$

Thus

$$\delta^1_2 \Phi \cdot \delta^0_3 \Lambda = \delta^1_2 \Phi \cdot \Phi \cdot \Psi = 0.$$

Consider (8.6). Acting by δ^1_2 and substituting back we obtain

$$0 = \delta^1_2 \delta^0_3 \Lambda = \delta^1_2 (\Phi \cdot \Psi) = \delta^1_2(\Phi) \cdot \Psi - \Phi \cdot \delta^1_t \Psi.$$

thus

$$\delta^1_2(\Phi) \cdot \Psi = \Phi \cdot \delta^1_t \Psi.$$

The last equality trivializes on applying δ^3_{t+1} to both sides.

Let us show now the non-vanishing property of $\left(\left(\delta^1_2 \Phi\right) \cdot \Phi\right)$. Indeed, suppose

$$\left(\delta^1_2 \Phi\right) \cdot \Phi = 0.$$

Then there exists $\Gamma \in C^n_m(V, \mathcal{W}, \mathcal{F})$, such that

$$\delta^1_2 \Phi = \Gamma \cdot \Phi.$$

Both sides of the last equality should belong to the same double complex space but one can see that it is not possible. Thus, $\left(\delta^1_2 \Phi\right) \cdot \Phi$ is non-vanishing. One proves in the same way that $\left(\delta^0_3 \Lambda\right) \cdot \Lambda$ and $\left(\delta^1_t \Psi\right) \cdot \Psi$ do not vanish too.

Now let us show that $\left[\left(\delta^1_2 \Phi\right) \cdot \Phi\right]$ is invariant, i.e., it does not depend on the choice of $\Phi \in C^1_2(V, \mathcal{W}, \mathcal{F})$. Substitute Φ by $(\Phi + \eta) \in C^1_2(V, \mathcal{W}, \mathcal{F})$. We have

$$\begin{aligned} \left(\delta^1_2 \left(\Phi + \eta\right)\right) \cdot (\Phi + \eta) \quad &= \quad \left(\delta^1_2 \Phi\right) \cdot \Phi + \left(\left(\delta^1_2 \Phi\right) \cdot \eta - \Phi \cdot \delta^1_2 \eta\right) \\ &+ \quad \left(\Phi \cdot \delta^1_2 \eta + \delta^1_2 \eta \cdot \Phi\right) + \left(\delta^1_2 \eta\right) \cdot \eta. \end{aligned} \tag{8.7}$$

Since

$$\left(\Phi \cdot \delta^1_2 \eta + \left(\delta^1_2 \eta\right) \cdot \Phi\right) = \Phi \cdot_\epsilon \delta^1_2 \eta - (\delta^1_2 \eta) \cdot_\epsilon \Phi + \left(\delta^1_2 \eta\right) \cdot_\epsilon \Phi - \Phi \cdot_\epsilon \delta^1_2 \eta = 0,$$

then (8.7) represents the same cohomology class $\left[\left(\delta^1_2 \Phi\right) \cdot \Phi\right]$. The same folds for $\left[\left(\delta^0_3 \Lambda\right) \cdot \Lambda\right]$, and $\left[\left(\delta^1_t \Psi\right) \cdot \Psi\right]$. □

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9. Appendix: Grading-restricted vertex algebras and their modules

In this Section, following [23] we recall basic properties of grading-restricted vertex algebras. and their grading-restricted generalized modules, useful for our purposes in later sections. We work over the base field $\mathbb{C}$ of complex numbers.

9.1. Grading-restricted vertex algebras.

Definition 18. A vertex algebra $(V, Y_V, \mathbf{1}_V)$, (cf. [27]), consists of a $\mathbb{Z}$-graded complex vector space

$$V = \coprod_{n \in \mathbb{Z}} V_{(n)}, \quad \dim V_{(n)} < \infty,$$

for each $n \in \mathbb{Z}$, and linear map

$$Y_V : V \to \mathrm{End}\ (V)[[z, z^{-1}]],$$

for a formal parameter z and a distinguished vector $\mathbf{1}_V \in V$. The evaluation of Y_V on $v \in V$ is the vertex operator

$$Y_V(v) \equiv Y_V(v, z) = \sum_{n \in \mathbb{Z}} v(n) z^{-n-1}, \tag{9.1}$$

with components $(Y_V(v))_n = v(n) \in \mathrm{End}\ (V)$, where $Y_V(v, z)\mathbf{1}_V = v + O(z)$.

Definition 19. A grading-restricted vertex algebra satisfies the following conditions:

(1) Grading-restriction condition: $V_{(n)}$ is finite dimensional for all $n \in \mathbb{Z}$, and $V_{(n)} = 0$ for $n \ll 0$;

(2) Lower-truncation condition: For u, $v \in V$, $Y_V(u, z)v$ contains only finitely many negative power terms, that is,

$$Y_V(u, z)v \in V((z)),$$

(the space of formal Laurent series in z with coefficients in V);

(3) Identity property: Let Id_V be the identity operator on V. Then

$$Y_V(\mathbf{1}_V, z) = \mathrm{Id}_V;$$

(4) Creation property: For $u \in V$,

$$Y_V(u, z)\mathbf{1}_V \in V[[z]],$$

and

$$\lim_{z \to 0} Y_V(u, z)\mathbf{1}_V = u;$$

(5) Duality: For $u_1, u_2, v \in V$,

$$v' \in V' = \coprod_{n \in \mathbb{Z}} V^*_{(n)},$$

where $V^*_{(n)}$ denotes the dual vector space to $V_{(n)}$ and $\langle\, .,. \rangle$ the evaluation pairing $V' \otimes V \to \mathbb{C}$, the series

$$\langle v', Y_V(u_1, z_1) Y_V(u_2, z_2) v \rangle, \tag{9.2}$$

$$\langle v', Y_V(u_2, z_2) Y_V(u_1, z_1) v \rangle, \tag{9.3}$$

$$\langle v', Y_V(Y_V(u_1, z_1 - z_2) u_2, z_2) v \rangle, \tag{9.4}$$

are absolutely convergent in the regions

$$|z_1| > |z_2| > 0,$$

$$|z_2| > |z_1| > 0,$$

$$|z_2| > |z_1 - z_2| > 0,$$

respectively, to a common rational function in z_1 and z_2 with the only possible poles at $z_1 = 0 = z_2$ and $z_1 = z_2$;

(6) $L_V(0)$-bracket formula: Let $L_V(0) : V \to V$, be defined by

$$L_V(0) v = nv, \qquad n = \text{wt }(v),$$

for $v \in V_{(n)}$. Then

$$[L_V(0), Y_V(v, z)] = Y_V(L_V(0)v, z) + z \frac{d}{dz} Y_V(v, z),$$

for $v \in V$.

(7) $L_V(-1)$-derivative property: Let

$$L_V(-1) : V \to V,$$

be the operator given by

$$L_V(-1) v = \text{Res}_z z^{-2} Y_V(v, z) \mathbf{1}_V = Y_{(-2)}(v) \mathbf{1}_V,$$

for $v \in V$. Then for $v \in V$,

$$\frac{d}{dz} Y_V(u, z) = Y_V(L_V(-1)u, z) = [L_V(-1), Y_V(u, z)]. \tag{9.5}$$

In addition to that, we recall here the following definition (cf. [3]):

Definition 20. A grading-restricted vertex algebra V is called conformal of central charge $c \in \mathbb{C}$, if there exists a non-zero conformal vector (Virasoro vector) $\omega \in V_{(2)}$ such that the corresponding vertex operator

$$Y_V(\omega, z) = \sum_{n \in \mathbb{Z}} L_V(n) z^{-n-2},$$

is determined by modes of Virasoro algebra $L_V(n) : V \to V$ satisfying

$$[L_V(m), L_V(n)] = (m - n) L(m + n) + \frac{c}{12}(m^3 - m) \delta_{m+b,0} \,\text{Id}_\text{V}.$$

Definition 21. A vector A which belongs to a module W of a quasi-conformal grading-restricted vertex algebra V is called primary of conformal dimension $\Delta(A) \in \mathbb{Z}_+$ if

$$\begin{aligned} L_W(k)A &= 0, \ k > 0, \\ L_W(0)A &= \Delta(A)A. \end{aligned}$$

9.2. **Grading-restricted generalized V-module.** In this subsection we describe the grading-restricted generalized V-module for a grading-restricted vertex algebra V.

Definition 22. A grading-restricted generalized V-module is a vector space W equipped with a vertex operator map

$$\begin{aligned} Y_W : V \otimes W &\to W[[z, z^{-1}]], \\ u \otimes w &\mapsto Y_W(u, w) \equiv Y_W(u, z)w = \sum_{n \in \mathbb{Z}} (Y_W)_n(u, w) z^{-n-1}, \end{aligned}$$

and linear operators $L_W(0)$ and $L_W(-1)$ on W satisfying the following conditions:

(1) Grading-restriction condition: The vector space W is $\mathbb{C}$-graded, that is,
$$W = \coprod_{\alpha \in \mathbb{C}} W_{(\alpha)},$$
such that $W_{(\alpha)} = 0$ when the real part of α is sufficiently negative;

(2) Lower-truncation condition: For $u \in V$ and $w \in W$, $Y_W(u, z)w$ contains only finitely many negative power terms, that is, $Y_W(u, z)w \in W((z))$;

(3) Identity property: Let Id_W be the identity operator on W. Then
$$Y_W(\mathbf{1}_V, z) = \mathrm{Id}_W;$$

(4) Duality: For $u_1, u_2 \in V$, $w \in W$,
$$w' \in W' = \coprod_{n \in \mathbb{Z}} W^*_{(n)},$$
W' denotes the dual V-module to W and $\langle \, .,. \rangle$ their evaluation pairing, the series
$$\langle w', Y_W(u_1, z_1) Y_W(u_2, z_2) w \rangle, \tag{9.6}$$
$$\langle w', Y_W(u_2, z_2) Y_W(u_1, z_1) w \rangle, \tag{9.7}$$
$$\langle w', Y_W(Y_V(u_1, z_1 - z_2)u_2, z_2) w \rangle, \tag{9.8}$$
are absolutely convergent in the regions
$$|z_1| > |z_2| > 0,$$
$$|z_2| > |z_1| > 0,$$
$$|z_2| > |z_1 - z_2| > 0,$$
respectively, to a common rational function in z_1 and z_2 with the only possible poles at $z_1 = 0 = z_2$ and $z_1 = z_2$.

(5) $L_W(0)$-bracket formula: For $v \in V$,

$$[L_W(0), Y_W(v,z)] = Y_W(L_V(0)v, z) + z\frac{d}{dz}Y_W(v,z); \tag{9.9}$$

(6) $L_W(0)$-grading property: For $w \in W_{(\alpha)}$, there exists $N \in \mathbb{Z}_+$ such that

$$(L_W(0) - \alpha)^N w = 0; \tag{9.10}$$

(7) $L_W(-1)$-derivative property: For $v \in V$,

$$\frac{d}{dz}Y_W(u,z) = Y_W(L_V(-1)u, z) = [L_W(-1), Y_W(u,z)]. \tag{9.11}$$

The translation property of vertex operators

$$Y_W(u,z) = e^{-z' L_W(-1)} Y_W(u, z+z') e^{z' L_W(-1)}, \tag{9.12}$$

for $z' \in \mathbb{C}$, follows from from (9.11). For $v \in V$, and $w \in W$, the intertwining operator

$$\begin{aligned} Y^W_{WV} &: V \to W, \\ v &\mapsto Y^W_{WV}(w,z)v, \end{aligned} \tag{9.13}$$

is defined by

$$Y^W_{WV}(w,z)v = e^{zL_W(-1)} Y_W(v,-z)w. \tag{9.14}$$

For $a \in \mathbb{C}$, the conjugation property with respect to the grading operator $L_W(0)$ is given by

$$a^{L_W(0)}\, Y_W(v,z)\, a^{-L_W(0)} = Y_W(a^{L_W(0)}v, az). \tag{9.15}$$

9.3. **Generators of Virasoro algebra and the group of automorphisms.** Let us recall some further facts from [3] relating generators of Virasoro algebra with the group of automorphisms in complex dimension one. Let us represent an element of $\mathrm{Aut}_z\, \mathcal{O}^{(1)}$ by the map

$$z \mapsto \rho = \rho(z), \tag{9.16}$$

given by the power series

$$\rho(z) = \sum_{k\geq 1} a_k z^k, \tag{9.17}$$

$\rho(z)$ can be represented in an exponential form

$$f(z) = \exp\left(\sum_{k>-1} \beta_k\ z^{k+1}\partial_z\right) (\beta_0)^{z\partial_z} .z, \tag{9.18}$$

where we express $\beta_k \in \mathbb{C}$, $k \geq 0$, through combinations of a_k, $k \geq 1$. A representation of Virasoro algebra modes in terms of differenatial operators is given by [27]

$$L_W(m) \mapsto -\zeta^{m+1}\partial_\zeta, \tag{9.19}$$

for $m \in \mathbb{Z}$. By expanding (9.18) and comparing to (9.17) we obtain a system of equations which, can be solved recursively for all β_k. In [3], $v \in V$, they derive the formula

$$[L_W(n), Y_W(v,z)] = \sum_{m\geq -1} \frac{1}{(m+1)!} \left(\partial_z^{m+1} z^{m+1}\right)\ Y_W(L_V(m)v, z), \tag{9.20}$$

of a Virasoro generator commutation with a vertex operator. Given a vector field

$$\beta(z)\partial_z = \sum_{n\geq -1} \beta_n z^{n+1}\partial_z, \tag{9.21}$$

which belongs to local Lie algebra of $\mathrm{Aut}_z\ \mathcal{O}^{(1)}$, one introduces the operator

$$\beta = -\sum_{n\geq -1} \beta_n L_W(n).$$

We conlclude from (9.21) with the following

Lemma 11.

$$[\beta, Y_W(v,z)] = \sum_{m\geq -1} \frac{1}{(m+1)!} \left(\partial_z^{m+1}\beta(z)\right)\ Y_W(L_V(m)v, z). \tag{9.22}$$

The formula (9.22) is used in [3] (Chapter 6) in order to prove invariance of vertex operators multiplied by conformal weight differentials in case of primary states, and in generic case.

Let us give some further definition:

Definition 23. A grading-restricted vertex algebra V-module W is called quasi-conformal if it carries an action of local Lie algebra of $\mathrm{Aut}_z\ \mathcal{O}$ such that commutation formula (9.22) holds for any $v \in V$, the element

$$L_W(-1) = -\partial_z,$$

as the translation operator T,

$$L_W(0) = -z\partial_z,$$

acts semi-simply with integral eigenvalues, and the Lie subalgebra of the positive part of local Lie algebra of $\mathrm{Aut}_z\ \mathcal{O}^{(n)}$ acts locally nilpotently.

Recall [3] the exponential form $f(\zeta)$ (9.18) of the coordinate transformation (9.16) $\rho(z) \in \mathrm{Aut}_z\ \mathcal{O}^{(1)}$. A quasi-conformal vertex algebra posseses the formula (9.22), thus it is possible by using the identification (9.19), to introduce the linear operator representing $f(\zeta)$ (9.18) on $\mathcal{W}_{z_1,\ldots,z_n}$,

$$P\left(f(\zeta)\right) = \exp\left(\sum_{m>0} (m+1)\ \beta_m\ L_V(m)\right) \beta_0^{L_W(0)}, \tag{9.23}$$

(note that we have a different normalization in it). In [3] (Chapter 6) it was shown that the action of an operator similar to (9.23) on a vertex algebra element $v \in V_n$ contains finitely meny terms, and subspaces

$$V_{\leq m} = \bigoplus_{n\geq K}^{m} V_n,$$

are stable under all operators $P(f)$, $f \in \mathrm{Aut}_z\ \mathcal{O}^{(1)}$. In [3] they proved the following

Lemma 12. *The assignment*

$$f \mapsto P(f),$$

defines a representation of $\mathrm{Aut}_z\, \mathcal{O}^{(1)}$ *on* V,

$$P(f_1 * f_2) = P(f_1)\, P(f_2),$$

which is the inductive limit of the representations $V_{\le m}$, $m \ge K$.

Similarly, (9.23) provides a representation operator on $\mathcal{W}_{z_1,\dots,z_n}$.

9.4. **Non-degenerate invariant bilinear form on V.** The subalgebra

$$\{L_V(-1), L_V(0), L_V(1)\} \cong SL(2,\mathbb{C}),$$

associated with Möbius transformations on z naturally acts on V, (cf., e.g. [27]). In particular,

$$\gamma_\lambda = \begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix} : z \mapsto w = -\frac{\lambda^2}{z}, \tag{9.24}$$

is generated by

$$T_\lambda = \exp\left(\lambda L_V(-1)\right)\, \exp\left(\lambda^{-1} L_V(1)\right)\, \exp\left(\lambda L_V(-1)\right),$$

where

$$T_\lambda Y(u,z) T_\lambda^{-1} = Y\left(\exp\left(-\frac{z}{\lambda^2} L_V(1)\right)\left(-\frac{z}{\lambda}\right)^{-2L_V(0)} u, -\frac{\lambda^2}{z}\right). \tag{9.25}$$

In our considerations (cf. Appendix 15) of Riemann sphere sewing, we use in particular, the Möbius map

$$z \mapsto z' = \epsilon/z,$$

associated with the sewing condition (15.4) with

$$\lambda = -\xi \epsilon^{\frac{1}{2}}, \tag{9.26}$$

with $\xi \in \{\pm\sqrt{-1}\}$. The adjoint vertex operator [9,27] is defined by

$$Y^\dagger(u,z) = \sum_{n\in\mathbb{Z}} u^\dagger(n) z^{-n-1} = T_\lambda Y(u,z) T_\lambda^{-1}. \tag{9.27}$$

A bilinear form $\langle .,. \rangle_\lambda$ on V is invariant if for all a, b, $u \in V$, if

$$\langle Y(u,z)a, b\rangle_\lambda = \langle a, Y^\dagger(u,z) b\rangle_\lambda, \tag{9.28}$$

i.e.

$$\langle u(n)a, b\rangle_\lambda = \langle a, u^\dagger(n) b\rangle_\lambda.$$

Thus it follows that

$$\langle L_V(0)a, b\rangle_\lambda = \langle a, L_V(0) b\rangle_\lambda, \tag{9.29}$$

so that

$$\langle a, b\rangle_\lambda = 0, \tag{9.30}$$

if $wt(a) \neq wt(b)$ for homogeneous a, b. One also finds

$$\langle a, b\rangle_\lambda = \langle b, a\rangle_\lambda.$$

The form $\langle ., . \rangle_\lambda$ is unique up to normalization if $L_V(1)V_1 = V_0$. Given any V basis $\{u^\alpha\}$ we define the dual V basis $\{\overline{u}^\beta\}$ where

$$\langle u^\alpha, \overline{u}^\beta \rangle_\lambda = \delta^{\alpha\beta}.$$

10. Appendix: $\mathcal{W}_{z_1,\dots,z_n}$-valued rational functions

Recall the definition of shuffles.

Definition 24. Let S_q be the permutation group. For $l \in \mathbb{N}$ and $1 \le s \le l-1$, let $J_{l;s}$ be the set of elements of S_l which preserve the order of the first s numbers and the order of the last $l-s$ numbers, that is,

$$J_{l,s} = \{\sigma \in S_l \mid \sigma(1) < \cdots < \sigma(s),\ \sigma(s+1) < \cdots < \sigma(l)\}.$$

The elements of $J_{l;s}$ are called shuffles, and we use the notation

$$J_{l;s}^{-1} = \{\sigma \mid \sigma \in J_{l;s}\}.$$

Definition 25. We define the configuration spaces:

$$F_n\mathbb{C} = \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid z_i \neq z_j, i \neq j\},$$

for $n \in \mathbb{Z}_+$.

Let V be a grading-restricted vertex algebra, and W a a grading-restricted generalized V-module. By $\overline{W}$ we denote the algebraic completion of W,

$$\overline{W} = \prod_{n\in\mathbb{C}} W_{(n)} = (W')^*.$$

Definition 26. A $\overline{W}$-valued rational function in $(z_1, \dots, z_n)$ with the only possible poles at $z_i = z_j$, $i \neq j$, is a map

$$\begin{array}{rcl} f : F_n\mathbb{C} & \to & \overline{W}, \\ (z_1, \dots, z_n) & \mapsto & f(z_1, \dots, z_n), \end{array}$$

such that for any $w' \in W'$,

$$R(z_1, \dots, z_n) = \langle w', f(z_1, \dots, z_n) \rangle,$$

is a rational function in $(z_1, \dots, z_n)$ with the only possible poles at $z_i = z_j$, $i \neq j$. In this paper, such a map is called $\overline{W}$-valued rational function in $(z_1, \dots, z_n)$ with possible other poles. The space of $\overline{W}$-valued rational functions is denoted by $\overline{W}_{z_1,\dots,z_n}$.

Definition 27. One defines an action of S_n on the space $\mathrm{Hom}(V^{\otimes n}, \overline{W}_{z_1,\dots,z_n})$ of linear maps from $V^{\otimes n}$ to $\overline{W}_{z_1,\dots,z_n}$ by

$$\sigma(\Phi)(v_1, z_1; \cdots; v_n, z_n) = \Phi(v_{\sigma(1)}, v_{\sigma(1)}; \cdots v_{\sigma(n)}, z_{\sigma(n)}), \tag{10.1}$$

for $\sigma \in S_n$, and $v_1, \dots, v_n \in V$.

We will use the notation $\sigma_{i_1,\dots,i_n} \in S_n$, to denote the the permutation given by $\sigma_{i_1,\dots,i_n}(j) = i_j$, for $j = 1, \dots, n$. In [23] one finds:

Proposition 12. *The subspace of* $\mathrm{Hom}(V^{\otimes n}, \mathcal{W}_{z_1,\dots,z_n})$ *consisting of linear maps having the* $L(-1)$*-derivative property, having the* $L(0)$*-conjugation property or being composable with* m *vertex operators is invariant under the action of* S_n*.*

Let us introduce another definition:

Definition 28. We define the space $\mathcal{W}_{z_1,\dots,z_n}$ of $\overline{W}_{z_1,\dots,z_n}$-valued rational forms Φ with each vertex algebra element entry v_i, $1 \le i \le n$ of a quasi-conformal grading-restricted vertex algebra V tensored with power wt (v_i)-differential of corresponding formal parameter z_i, i.e.,

$$\Phi\left(dz_1^{\mathrm{wt}\ (v_1)} \otimes v_1, z_1; \cdots ; dz_n^{\mathrm{wt}\ (v_n)} \otimes v_n, z_n\right) \in \mathcal{W}_{z_1,\dots,z_n}. \tag{10.2}$$

We assume also that (10.2) satisfy $L_V(-1)$-derivative (11.1), $L_V(0)$-conjugation (11.6) properties, and the symmetry property with respect to action of the symmetric group S_n:

$$\sum_{\sigma \in J_{l;s}^{-1}} (-1)^{|\sigma|} \left(\Phi(v_{\sigma(1)}, z_{\sigma(1)}; \cdots ; v_{\sigma(l)}, z_{\sigma(1)})\right) = 0. \tag{10.3}$$

In Section 4 we prove that (10.2) is invariant with respect to changes of formal parameters $(z_1, \dots, z_n)$.

11. Appendix: Properties of matrix elements for a grading-restricted vertex algebra

Let V be a grading-restricted vertex algebra and W a grading-restricted generalized V-module. Let us recall some definitions and facts about matrix elements for a grading-restricted vertex algebra [23]. If a meromorphic function $f(z_1, \dots, z_n)$ on a domain in $\mathbb{C}^n$ is analytically extendable to a rational function in $z_1, \dots, z_n$, we denote this rational function by $R(f(z_1, \dots, z_n))$. Let us recall a few definitions from [23]

Definition 29. For $n \in \mathbb{Z}_+$, a linear map

$$\Phi(v_1, z_1; \dots ; v_n, z_n) = V^{\otimes n} \to \mathcal{W}_{z_1,\dots,z_n},$$

is said to have the $L(-1)$-derivative property if

$$(i) \qquad \langle w', \partial_{z_i} \Phi(v_1, z_1; \dots ; v_n, z_n)\rangle = \langle w', \Phi(v_1, z_1; \dots ; L_V(-1) v_i, z_i; \dots ; v_n, z_n)\rangle, \tag{11.1}$$

for $i = 1, \dots, n$, $v_1, \dots, v_n \in V$, $w' \in W$, and

$$(ii) \qquad \sum_{i=1}^{n} \partial_{z_i} \langle w', \Phi(v_1, z_1; \dots ; v_n, z_n)\rangle = \langle w', L_W(-1).\Psi(v_1, z_1; \dots ; v_n, z_n)\rangle, \tag{11.2}$$

with some action . of $L_W(-1)$ on $\Phi(v_1, z_1; \dots ; v_n, z_n)$, and and $v_1, \dots, v_n \in V$.

Note that since $L_W(-1)$ is a weight-one operator on W, for any $z \in \mathbb{C}$, $e^{zL_W(-1)}$ is a well-defined linear operator on $\overline{W}$.

In [23] we find the following

Proposition 13. *Let Φ be a linear map having the $L(-1)$-derivative property. Then for $v_1, \ldots, v_n \in V$, $w' \in W'$, $(z_1, \ldots, z_n) \in F_n\mathbb{C}$, $z \in \mathbb{C}$ such that $(z_1+z, \ldots, z_n+z) \in F_n\mathbb{C}$,*

$$\langle w', e^{zL_W(-1)} \Phi\left(v_1, z_1; \ldots ; v_n, z_n\right)\rangle = \langle w', \Phi(v_1, z_1+z; \ldots ; v_n, z_n+z)\rangle, \qquad (11.3)$$

and for $v_1, \ldots, v_n \in V$, $w' \in W'$, $(z_1, \ldots, z_n) \in F_n\mathbb{C}$, $z \in \mathbb{C}$, and $1 \le i \le n$ such that

$$(z_1, \ldots, z_{i-1}, z_i+z, z_{i+1}, \ldots, z_n) \in F_n\mathbb{C},$$

the power series expansion of

$$\langle w', \Phi(v_1, z_1; \ldots ; v_{i-1}, z_{i-1}; v_i, z_i+z; v_{v+1}, z_{i+1}; \ldots v_n, z_n)\rangle, \qquad (11.4)$$

in z is equal to the power series

$$\langle w', \Phi(v_1 z_1; \ldots ; v_{i-1}, z_{i-1}; e^{zL(-1)} v_i, z_i; v_{i+1}, z_{i+1}; \ldots ; v_n, z_n)\rangle, \qquad (11.5)$$

in z. In particular, the power series (11.5) in z is absolutely convergent to (11.4) in the disk $|z| < \min_{i\neq j}\{|z_i - z_j|\}$.

Finally, we have

Definition 30. A linear map

$$\Phi : V^{\otimes n} \to \mathcal{W}_{z_1, \ldots, z_n}$$

has the $L(0)$-conjugation property if for $v_1, \ldots, v_n \in V$, $w' \in W'$, $(z_1, \ldots, z_n) \in F_n\mathbb{C}$ and $z \in \mathbb{C}^\times$ so that $(zz_1, \ldots, zz_n) \in F_n\mathbb{C}$,

$$\langle w', z^{L_W(0)} \Phi\left(v_1, z_1; \cdots ; v_n, z_n\right)\rangle = \langle w', \Phi(z^{L(0)} v_1, zz_1; \cdots ; z^{L(0)} v_n, zz_n)\rangle. \qquad (11.6)$$

11.1. ***E*-elements.** For $w \in W$, the $\overline{W}$-valued function is given by

$$E^{(n)}_W(v_1, z_1; \cdots ; v_n, z_n; w) = E(\omega_W(v_1, z_1) \ldots \omega_W(v_n, z_n) w),$$

where an element $E(.) \in \overline{W}$ is given by (see notations for ω_W in Section 4)

$$\langle w', E(.)\rangle = R(\langle w', .\rangle),$$

and $R(.)$ denotes the following (cf. [23]). Namely, if a meromorphic function $f(z_1, \ldots, z_n)$ on a region in $\mathbb{C}^n$ can be analytically extended to a rational function in $(z_1, \ldots, z_n)$, then the notation $R(f(z_1, \ldots, z_n))$ is used to denote such rational function. One defines

$$E^{W;(n)}_{WV}(w; v_1, z_1; \ldots ; v_n, z_n) = E^{(n)}_W(v_1, z_1; \ldots ; v_n, z_n; w),$$

where $E^{W;(n)}_{WV}(w; v_1, z_1; \ldots ; v_n, z_n)$ is an element of $\overline{W}_{z_1, \ldots, z_n}$. One defines

$$\Phi \circ \left(E^{(l_1)}_{V;\,\mathbf{1}} \otimes \cdots \otimes E^{(l_n)}_{V;\,\mathbf{1}}\right) : V^{\otimes m+n} \to \overline{W}_{z_1, \ldots, z_{m+n}},$$

by

$$\begin{aligned}(\Phi \circ (E^{(l_1)}_{V;\,\mathbf{1}} \otimes \cdots \otimes E^{(l_n)}_{V;\,\mathbf{1}}))(v_1 \otimes \cdots \otimes v_{m+n-1})\\ = E(\Phi(E^{(l_1)}_{V;\mathbf{1}}(v_1 \otimes \cdots \otimes v_{l_1}) \otimes \cdots\\ \otimes E^{(l_n)}_{V;\mathbf{1}}(v_{l_1+\cdots+l_{n-1}+1} \otimes \cdots \otimes v_{l_1+\cdots+l_{n-1}+l_n}))),\end{aligned}$$

and
$$E_W^{(m)} \circ_0 \Phi : V^{\otimes m+n} \to \overline{W}_{z_1,\dots,z_{m+n-1}},$$
is given by
$$(E_W^{(m)} \circ_0 \Phi)(v_1 \otimes \cdots \otimes v_{m+n})$$
$$= E(E_W^{(m)}(v_1 \otimes \cdots \otimes v_m; \Phi(v_{m+1} \otimes \cdots \otimes v_{m+n}))).$$
Finally,
$$E_{WV}^{W;(m)} \circ_{m+1} \Phi : V^{\otimes m+n} \to \overline{W}_{z_1,\dots,z_{m+n-1}},$$
is defined by
$$(E_{WV}^{W;(m)} \circ_{m+1} \Phi)(v_1 \otimes \cdots \otimes v_{m+n}) = E(E_{WV}^{W;(m)}(\Phi(v_1 \otimes \cdots \otimes v_n); v_{n+1} \otimes \cdots \otimes v_{n+m})).$$
In the case that $l_1 = \cdots = l_{i-1} = l_{i+1} = 1$ and $l_i = m - n - 1$, for some $1 \le i \le n$, we will use $\Phi \circ_i E_{V;\,\mathbf{1}}^{(l_i)}$ to denote $\Phi \circ (E_{V;\,\mathbf{1}}^{(l_1)} \otimes \cdots \otimes E_{V;\,\mathbf{1}}^{(l_n)})$. Note that our notations differ with that of [23].

12. Appendix: Maps composable with vertex operators

In the construction of double complexes in Section 6 we would like to use linear maps from tensor powers of V to the space $\mathcal{W}_{z_1,\dots,z_n}$ to define cochains in vertex algebra cohomology theory. For that purpose, in particular, to define the coboundary operator, we have to compose cochains with vertex operators. However, as mentioned in [23], the images of vertex operator maps in general do not belong to algebras or thier modules. They belong to corresponding algebraic completions which constitute one of the most subtle features of the theory of vertex algebras. Because of this, we might not be able to compose vertex operators directly. In order to overcome this problem [25], we first write a series by projecting an element of the algebraic completion of an algebra or a module to its homogeneous components. Then we compose these homogeneous components with vertex operators, and take formal sums. If such formal sums are absolutely convergent, then these operators can be composed and can be used in constructions.

Another question that appears is the question of associativity. Compositions of maps are usually associative. But for compositions of maps defined by sums of absolutely convergent series the existence of does not provide associativity in general. Nevertheless, the requirement of analyticity provides the associativity [23].

Definition 31. For a V-module
$$W = \coprod_{n \in \mathbb{C}} W_{(n)},$$
and $m \in \mathbb{C}$, let
$$P_m : \overline{W} \to W_{(m)},$$
be the projection from $\overline{W}$ to $W_{(m)}$. Let
$$\Phi : V^{\otimes n} \to \mathcal{W}_{z_1,\dots,z_n},$$
be a linear map. For $m \in \mathbb{N}$, Φ is called [23, 38] to be composable with m vertex operators if the following conditions are satisfied:

1) Let $l_1, \dots, l_n \in \mathbb{Z}_+$ such that $l_1 + \cdots + l_n = m + n$, $v_1, \dots, v_{m+n} \in V$ and $w' \in W'$. Set

$$\Xi_i = E^{(l_i)}_V(v_{k_1}, z_{k_1} - \zeta_i; \dots; v_{k_i}, z_{k_i} - \zeta_i; \mathbf{1}_V), \tag{12.1}$$

where

$$k_1 = l_1 + \cdots + l_{i-1} + 1, \quad \dots, \quad k_i = l_1 + \cdots + l_{i-1} + l_i, \tag{12.2}$$

for $i = 1, \dots, n$. Then there exist positive integers $N^n_m(v_i, v_j)$ depending only on v_i and v_j for $i, j = 1, \dots, k$, $i \neq j$ such that the series

$$\mathcal{I}^n_m(\Phi) = \sum_{r_1,\dots,r_n \in \mathbb{Z}} \langle w', \Phi(P_{r_1}\Xi_1; \zeta_1; \dots; P_{r_n}\Xi_n, \zeta_n)\rangle, \tag{12.3}$$

is absolutely convergent when

$$|z_{l_1+\cdots+l_{i-1}+p} - \zeta_i| + |z_{l_1+\cdots+l_{j-1}+q} - \zeta_i| < |\zeta_i - \zeta_j|, \tag{12.4}$$

for $i, j = 1, \dots, k$, $i \neq j$ and for $p = 1, \dots, l_i$ and $q = 1, \dots, l_j$. The sum must be analytically extended to a rational function in $(z_1, \dots, z_{m+n})$, independent of $(\zeta_1, \dots, \zeta_n)$, with the only possible poles at $z_i = z_j$, of order less than or equal to $N^n_m(v_i, v_j)$, for $i, j = 1, \dots, k$, $i \neq j$.

2) For $v_1, \dots, v_{m+n} \in V$, there exist positive integers $N^n_m(v_i, v_j)$, depending only on v_i and v_j, for $i, j = 1, \dots, k$, $i \neq j$, such that for $w' \in W'$, and

$$\mathbf{v}_{n,m} = (v_{1+m} \otimes \cdots \otimes v_{n+m}),$$

$$\mathbf{z}_{n,m} = (z_{1+m}, \dots, z_{n+m}),$$

such that

$$\mathcal{J}^n_m(\Phi) = \sum_{q \in \mathbb{C}} \langle w', E^{(m)}_W\Big(v_1 \otimes \cdots \otimes v_m; P_q(\Phi(\mathbf{v}_{n,m})(\mathbf{z}_{n,m})\Big)\rangle, \tag{12.5}$$

is absolutely convergent when

$$\begin{aligned} z_i \neq z_j, \quad i \neq j, \\ |z_i| > |z_k| > 0, \end{aligned} \tag{12.6}$$

for $i = 1, \dots, m$, and $k = m+1, \dots, m+n$, and the sum can be analytically extended to a rational function in $(z_1, \dots, z_{m+n})$ with the only possible poles at $z_i = z_j$, of orders less than or equal to $N^n_m(v_i, v_j)$, for $i, j = 1, \dots, k$, $i \neq j$,.

In [23], we the following useful proposition is proven:

Proposition 14. *Let $\Phi : V^{\otimes n} \to \overline{W}_{z_1,\dots,z_n}$ be composable with m vertex operators. Then we have:*

(1) *For $p \leq m$, Φ is composable with p vertex operators and for $p, q \in \mathbb{Z}_+$ such that $p + q \leq m$ and $l_1, \dots, l_n \in \mathbb{Z}_+$ such that $l_1 + \cdots + l_n = p + n$, $\Phi \circ (E^{(l_1)}_{V;\,\mathbf{1}} \otimes \cdots \otimes E^{(l_n)}_{V;\,\mathbf{1}})$ and $E^{(p)}_W \circ_{p+1} \Phi$ are composable with q vertex operators.*

(2) *For $p, q \in \mathbb{Z}_+$ such that $p+q \le m$, $l_1, \dots, l_n \in \mathbb{Z}_+$ such that $l_1+\dots+l_n = p+n$ and $k_1, \dots, k_{p+n} \in \mathbb{Z}_+$ such that $k_1 + \dots + k_{p+n} = q+p+n$, we have*

$$(\Phi \circ (E^{(l_1)}_{V;\,\mathbf{1}} \otimes \cdots \otimes E^{(l_n)}_{V;\,\mathbf{1}})) \circ (E^{(k_1)}_{V;\,\mathbf{1}} \otimes \cdots \otimes E^{(k_{p+n})}_{V;\,\mathbf{1}})$$
$$= \Phi \circ (E^{(k_1+\cdots+k_{l_1})}_{V;\,\mathbf{1}} \otimes \cdots \otimes E^{(k_{l_1+\cdots+l_{n-1}+1}+\cdots+k_{p+n})}_{V;\,\mathbf{1}}).$$

(3) *For $p, q \in \mathbb{Z}_+$ such that $p+q \le m$ and $l_1, \dots, l_n \in \mathbb{Z}_+$ such that $l_1+\dots+l_n = p+n$, we have*

$$E^{(q)}_W \circ_{q+1} (\Phi \circ (E^{(l_1)}_{V;\,\mathbf{1}} \otimes \cdots \otimes E^{(l_n)}_{V;\,\mathbf{1}})) = (E^{(q)}_W \circ_{q+1} \Phi) \circ (E^{(l_1)}_{V;\,\mathbf{1}} \otimes \cdots \otimes E^{(l_n)}_{V;\,\mathbf{1}}).$$

(4) *For $p, q \in \mathbb{Z}_+$ such that $p+q \le m$, we have*

$$E^{(p)}_W \circ_{p+1} (E^{(q)}_W \circ_{q+1} \Phi) = E^{(p+q)}_W \circ_{p+q+1} \Phi.$$

13. Appendix: Proofs of Lemmas 4, 5, 6 and Proposiiton 5

In this Appendix we provide proofs of Lemma 5 and Proposiiton 5

13.1. **Proof of Lemma 4.** We start with the proof of Lemma 4.

Proof. From the construction of spaces for double complex for a grading-restricted vertex algebra cohomology, it is clear that the spaces $C^n(V, \mathcal{W}, \mathcal{U}, \mathcal{F})(U_j)$, $1 \le s \le m$ in Definition 9 are non-empty. On each transversal section U_s, $1 \le s \le m$, $\Phi(v_1, c_j(p_1); \dots; v_n, c_j(p_n))$ belongs to the space $\mathcal{W}_{c_j(p_1),\dots,c_j(p_n)}$, and satisfy the $L(-1)$-derivative (11.1) and $L(0)$-conjugation (11.6) properties. A map $\Phi(v_1, c_j(p_1); \dots; v_n, c_j(p_n))$ is composable with m vertex operators with formal parameters identified with local coordinates $c_j(p'_j)$, on each transversal section U_j. Note that on each transversal section, n and m the spaces (4.2) remain the same. The only difference may be constituted by the composibility conditions (12.3) and (16.26) for Φ.

In particular, for $l_1, \dots, l_n \in \mathbb{Z}_+$ such that $l_1 + \dots + l_n = n+m$, $v_1, \dots, v_{m+n} \in V$ and $w' \in W'$, recall (16.16) that

$$\Xi_i \quad = \quad \omega_V(v_{k_1}, c_{k_1}(p_{k_1}) - \zeta_i) \dots \omega_V(v_{k_i}, c_{k_i}(p_{k_i}) - \zeta_i)\, \mathbf{1}_V, \tag{13.1}$$

where k_i is defined in (16.21), for $i = 1, \dots, n$, depend on coordinates of points on transversal sections. At the same time, in the first composibility condition (12.3) depends on projections $P_r(\Xi_i)$, $r \in \mathbb{C}$, of $\mathcal{W}_{c(p_1),\dots,c(p_n)}$ to W, and on arbitrary variables ζ_i, $1 \le i \le m$. On each transversal connection U_s, $1 \le s \le m$, the absolute convergency is assumed for the series (12.3) (cf. Appendix 12). Positive integers $N^n_m(v_i, v_j)$, (depending only on v_i and v_j) as well as ζ_i, for i, $j = 1, \dots, k$, $i \ne j$, may vary for transversal sections U_s. Nevertheless, the domains of convergency determined by the conditions (16.15) which have the form

$$|c_{m_i}(p_{m_i}) - \zeta_i| + |c_{n_i}(p_{n_i}) - \zeta_i| < |\zeta_i - \zeta_j|, \tag{13.2}$$

for $m_i = l_1 + \dots + l_{i-1} + p$, $n = l_1 + \dots + l_{j-1} + q$, i, $j = 1, \dots, k$, $i \ne j$ and for $p = 1, \dots, l_i$ and $q = 1, \dots, l_j$, are limited by $|\zeta_i - \zeta_j|$ in (13.2) from above. Thus, for the intersection variation of sets of homology embeddings in (4.2), the absolute convergency condition for (12.3) is still fulfilled. Under intersection in (4.2) by choosing appropriate $N^n_m(v_i, v_j)$, one can analytically extend (12.3) to a rational function in

$(c_1(p_1), \ldots, c_{n+m}(p_{n+m}))$, independent of $(\zeta_1, \ldots, \zeta_n)$, with the only possible poles at $c_i(p_i) = c_j(p_j)$, of order less than or equal to $N^n_m(v_i, v_j)$, for $i, j = 1, \ldots, k$, $i \neq j$.

As for the second condition in Definition of composibility, we note that, on each transversal section, the domains of absolute convergensy $c_i(p_i) \neq c_j(p_j)$, $i \neq j$

$$|c_i(p_i)| > |c_k(p_j)| > 0,$$

for $i = 1, \ldots, m$, and $k = 1 + m, \ldots, n + m$, for

$$\mathcal{J}^n_m(\Phi) = \sum_{q \in \mathbb{C}} \langle w', \omega_W(v_1, c_1(p_1)) \ldots \omega_W(v_m, c_m(p_m)) \\ P_q(\Phi(v_{1+m}, c_{1+m}(p_{1+m}); \ldots; v_{n+m}, c_{n+m}(p_{n+m}))\rangle, \quad (13.3)$$

are limited from below by the same set ot absolute values of local coordinates on transversal section. Thus, under intersection in (4.2) this condition is preserved, and the sum (16.26) can be analytically extended to a rational function in $(c_1(p_1), \ldots, c_{m+n}(p_{m+n}))$ with the only possible poles at $c_i(p_i) = c_j(p_j)$, of orders less than or equal to $N^n_m(v_i, v_j)$, for $i, j = 1, \ldots, k$, $i \neq j$. Thus, we proved the lemma. □

13.2. **Proof of Lemma 5.** Next we give proof of Lemma 5.

Proof. Suppose we consider another transversal basis $\mathcal{U}'$ for $\mathcal{F}$. According to the definition, for each transversal section U_i which belong to the original basis $\mathcal{U}$ in (4.2) there exists a holonomy embedding

$$h'_i : U_i \hookrightarrow U'_j,$$

i.e., it embeds U_i into a section U'_j of our new transversal basis $\mathcal{U}'$. Then consider the sequnce of holonomy embeddings $\{h'_k\}$ such that

$$U'_0 \overset{h'_1}{\hookrightarrow} \ldots \overset{h'_k}{\hookrightarrow} U'_k.$$

For the combination of embeddings $\{h'_i, i \geq 0\}$ and

$$U_0 \overset{h_1}{\hookrightarrow} \ldots \overset{h_k}{\hookrightarrow} U_k,$$

we obtain commutative diagrams. Since the intersection in (4.2) is performed over all sets of homology mappings, then it is independent on the choice of a transversal basis. □

13.3. **Proof of Proposition 5.** Next, we prove Proposition 5.

Proof. Here we prove that for generic elements of a quasi-conformal grading-restricted vertex algebra Φ and $\omega_W \in \mathcal{W}_{z_1, \ldots, z_n}$ and are canonical, i.e., independent on changes

$$z_i \mapsto w_i = \rho(z_i), \quad 1 \leq i \leq n, \quad (13.4)$$

of local coordinates of $c_i(p_i)$ and $c_j(p'_j)$ at points p_i and p'_j, $1 \leq i \leq n$, $1 \leq j \leq k$. Thus the construction of the double complex spaces (4.2) is proved to be canonical too. Let us denote by

$$\xi_i = \left(\beta_0^{-1} \, dw_i\right)^{\mathrm{wt} \ (v_i)}.$$

Recall the linear operator (2.6) (cf. Appendix 9). Define introduce the action of the transformations (13.4) as

$$\Phi\left(dw_1^{\mathrm{wt}\ (v_1)}\otimes v_1, w_1;\ldots;dw_n^{\mathrm{wt}\ (v_n)}\otimes v_n, w_n\right)$$
$$=\left(\frac{df(\zeta)}{d\zeta}\right)^{-L_W(0)}\ P(f(\zeta))\ \Phi\left(\xi_1\otimes v_1, z_1;\ldots;\xi_n\otimes v_n, z_n\right). \quad (13.5)$$

We then obtain

Lemma 13. *An element* (10.2)

$$\Phi\left(dz_1^{\mathrm{wt}\ (v_1)}\otimes v_1, z_1;\ldots;dz_n^{\mathrm{wt}\ (v_n)}\otimes v_n, z_n\right),$$

of $\mathcal{W}_{z_1,\ldots,z_n}$ *is canonical is invariant under transformations* (13.4) *of* $\left(\mathrm{Aut}\ \mathcal{O}^{(1)}\right)^{\times n}_{z_1,\ldots,z_n}$.

Proof. Consider (13.5). First, note that

$$f'(\zeta)=\frac{df(\zeta)}{d\zeta}=\sum_{m\geq 0}(m+1)\ \beta_m\zeta^m.$$

By using the identification (9.19) and and the $L_W(-1)$-properties (11.1) and (11.6) we obtain

$$\langle w', \Phi\left(dw_1^{\mathrm{wt}\ (v_1)}\otimes v_1, w_1;\ldots;dw_n^{\mathrm{wt}\ (v_n)}\otimes v_n, w_n\right)\rangle$$
$$=\langle w', f'(\zeta)^{-L_W(0)}\ P(f(\zeta))\ \Phi\left(\xi_1\otimes v_1, z_1;\ldots;\xi_n\otimes v_n, z_n\right)\rangle$$

$$=\langle w', \left(\frac{df(\zeta)}{d\zeta}\right)^{-L_W(0)}\ \Phi\left(dw_1^{\mathrm{wt}\ (v_1)}\otimes v_1, \sum_{m\geq 0}(m+1)\ \beta_m z_1^{m+1};\ldots;\right.$$
$$\left. dw_n^{\mathrm{wt}\ (v_n)}\otimes v_n, \sum_{m\geq 0}(m+1)\ \beta_m z_n^{m+1}\right)\rangle$$

$$=\langle w', \left(\frac{df(\zeta)}{d\zeta}\right)^{-L_W(0)}\ \Phi\left(dw_1^{\mathrm{wt}\ (v_1)}\otimes v_1, \left(\frac{df(z_1)}{dz_1}\right)z_1;\right.$$
$$\left.\ldots;dw_n^{\mathrm{wt}\ (v_n)}\otimes v_n, \left(\frac{df(z_n)}{dz_n}\right)z_n\right)\rangle$$

$$=\langle w', \Phi\left(\left(\frac{df(z_1)}{dz_i}\ dw_1\right)^{-\mathrm{wt}\ (v_1)}\otimes v_1, z_1;\right.$$
$$\left.\ldots;\left(\frac{df(z_n)}{dz_n}\ dw_n\right)^{-\mathrm{wt}\ (v_n)}\otimes v_n, z_n\right)\rangle$$

$$=\langle w', \Phi\left(dz_1^{\mathrm{wt}\ (v_1)}\otimes v_1, z_1;\ldots;dz_n^{\mathrm{wt}\ (v_n)}\otimes v_n, z_n\right)\rangle.$$

Thus we proved the Lemma. □

The elements $\Phi(v_1, z_1; \ldots; v_n, z_n)$ of $C^n_k(V, \mathcal{W}, \mathcal{F})$ belong to the space $\mathcal{W}_{z_1,\ldots,z_n}$ and assumed to be composable with a set of vertex operators $\omega_W(v'_j, c_j(p'_j))$, $1 \le j \le k$. Vertex operators $\omega_W(dc(p)^{\mathrm{wt}\,(v')} \otimes v'_j, c_j(p'_j))$ constitute particular examples of mapping of $C^1_\infty(V, \mathcal{W}, \mathcal{F})$ and, therefore, are invariant with respect to (13.4). Thus, the construction of spaces (4.2) is invariant under the action of the group $(\mathrm{Aut}\ \mathcal{O})^{\times n}_{z_1,\ldots,z_n}$. □

13.4. **Proof of Lemma 6.** Finally, we give a proof of Lemma 6.

Proof. Since n is the same for both spaces in (4.4), it only remains to check that the conditions for (12.3) and (16.26) for $\Phi(v_1, c_j(p_1); \ldots; v_n, c_j(p_n))$ of composibility Definition 12 with vertex operators are stronger for $C^n_m(V, \mathcal{W}, \mathcal{U}, \mathcal{F})$ then for $C^n_{m-1}(V, \mathcal{W}, \mathcal{U}, \mathcal{F})$. In particular, in the first condition for (12.3) in definition of composability 31 the difference between the spaces in (4.4) is in indeces. Consider (13.1). For $C^n_{m-1}(V, \mathcal{W}, \mathcal{U}, \mathcal{F})$, the summations in idexes

$$k_1 = l_1 + \cdots + l_{i-1} + 1,\ ...,\ k_i = l_1 + \cdots + l_{i-1} + l_i,$$

for the coordinates $c_j(p_1)$, ..., $c_j(p_n)$ with $l_1, \ldots, l_n \in \mathbb{Z}_+$, such that $l_1 + \cdots + l_n = n + (m-1)$, and vertex algebra elements $v_1, \ldots, v_{n+(m-1)}$ are included in summation for indexes for $C^n_m(V, \mathcal{W}, \mathcal{U}, \mathcal{F})$. The conditions for the domains of absolute convergency for $\mathcal{M}$, i.e.,

$$|c_{l_1+\cdots+l_{i-1}+p} - \zeta_i| + |c_{l_1+\cdots+l_{j-1}+q} - \zeta_i| < |\zeta_i - \zeta_j|,$$

for i, $j = 1, \ldots, k$, $i \ne j$, and for $p = 1, \ldots, l_i$ and $q = 1, \ldots, l_j$, for the series (12.3) are more restrictive then for $m-1$. The conditions for $\mathcal{I}^n_{m-1}(\Phi)$ to be extended analytically to a rational function in $(c_1(p_1), \ldots, c_{n+(m-1)}(p_{n+(m-1)}))$, with positive integers $N^n_{m-1}(v_i, v_j)$, depending only on v_i and v_j for $i, j = 1, \ldots, k$, $i \ne j$, are included in the conditions for $\mathcal{I}^n_m(\Phi)$.

Similarly, the second condition for (16.26), of is absolute convergency and analytical extension to a rational function in $(c_1(p_1), \ldots, c_{m+n}(p_{m+n}))$, with the only possible poles at $c_i(p_i) = c_j(p_j)$, of orders less than or equal to $N^n_m(v_i, v_j)$, for $i, j = 1, \ldots, k$, $i \ne j$, for (16.26) when

$$c_i(p_i) \ne c_j(p_j),\ i \ne j,\ |c_i(p_i)| > |c_k(p_k)| > 0,$$

for $i = 1, \ldots, m$, and $k = m+1, \ldots, m+n$ includes the same condition for $\mathcal{J}^n_{m-1}(\Phi)$. Thus we obtain the conclusion of Lemma. □

14. Appendix: Cohomological classes and connections

14.1. **Classes of grading-restricted vertex alebra cohomology.** In this section we describe certain classes associated to the first and the second vertex algebra cohomologies for codimension one foliations. Let us give some further definitions. Usually, the cohomology classes for codimension one foliations [7, 14, 28] are introduced by means of an extra condition (in particular, the orthogonality condition) applied to differential forms, and leading to the integrability condition. As we mentioned in Section 6, it is a separate problem to introduce a product defined on one or among various spaces $C^n_m(V, \mathcal{W}, \mathcal{F})$ of (4.2). Note that elements of $\mathcal{E}$ in (6.3) and $\mathcal{E}_{ex}$ in (6.9) can be seen as elements of spaces $C^1_\infty(V, \mathcal{W}, \mathcal{F})$, i.e., maps composable with an infinite number of vertex operators. Though the actions of coboundary operators δ^n_m and δ^2_{ex} in (6.3) and (6.7) are written in form of a product (as in Frobenius theorem [14]), and, in contrast to the case of differential forms, it is complicated to use these products for further formulation of cohomological invariants and derivation of analogues of the product-type invariants. Nevertheless, even with such a product yet missing, it is possible to introduce the lower-level cohomological classes of the form $[\delta\eta]$ which are counterparts of the Godbillon class [13]. Let us give some further definitions. By analogy with differential forms, let us introduce

Definition 32. We call a map

$$\Phi \in C^n_k(V, \mathcal{W}, \mathcal{F}),$$

closed if it is a closed connection:

$$\delta^n_k \Phi = G(\Phi) = 0.$$

For $k \geq 1$, we call it exact if there exists

$$\Psi \in C^{n+1}_{k-1}(V, \mathcal{W}, \mathcal{F}),$$

such that

$$\Psi = \delta^n_k \Phi,$$

i.e., Ψ is a form of connection.

Definition 33. For $\Phi \in C^n_k(V, \mathcal{W}, \mathcal{F})$ we call the cohomology class of mappings $[\Phi]$ the set of all closed forms that differ from Φ by an exact mapping, i.e., for $\Lambda \in C^{n-1}_{k+1}(V, \mathcal{W}, \mathcal{F})$,

$$[\Phi] = \Phi + \delta^{n-1}_{k+1}\Lambda.$$

As we will see in this section, there are cohomological classes, (i.e., $[\Phi]$, $\Phi \in C^1_m(V, \mathcal{W}, \mathcal{F})$, $m \geq 0$), associated with two-point connections and the first cohomology $H^1_m(V, \mathcal{W}, \mathcal{F})$, and classes (i.e., $[\Phi]$, $\Phi \in C^2_{ex}(V, \mathcal{W}, \mathcal{F})$), associated with transversal connections and the second cohomology $H^2_{ex}(V, \mathcal{W}, \mathcal{F})$, of $\mathcal{M}/\mathcal{F}$. The cohomological classes we obtain are vertex algebra cohomology counterparts of the Godbillon class [13, 28] for codimension one foliations.

Remark 10. As it was discovered in [1,2], it is a usual situation when the existence of a connection (affine of projective) for codimension one foliations on smooth manifolds prevents corresponding cohomology classes from vanishing. Note also, that for a few examples of codimension one foliations, the cohomology class $[d\eta]$ is always zero.

Remark 11. In contrast to [1], our cohomological class is a functional of $v \in V$. That means that the actual functional form of $\Phi(v, z)$ (and therefore $\langle w', \Phi\rangle$, for $w' \in W'$) varies with various choices of $v \in V$. That allows one to use it in order to distinguish types of leaves of $\mathcal{M}/\mathcal{F}$.

14.2. **Cohomology in terms of connections.** In various situations it is sometimes effective to use an interpretation of cohomology in terms of connections. In particular in our supporting example of vertex algebra cohomology of codimension one foliations. It is convenient to introduce multi-point connections over a graded space and to express coboundary operators and cohomology in terms of connections:

$$\delta^n \phi \in G^{n+1}(\phi),$$

$$\delta^n \phi = G(\phi).$$

Then the cohomology is defined as the factor space

$$H^n = \mathcal{C}on^n_{cl}/G^{n-1},$$

of closed multi-point connections with respect to the space of connection forms defined below.

14.3. **Multi-point holomorphic connections.** We start this section with definitions of holomorphic multi-point connections on a smooth complex variety. Let $\mathcal{X}$ be a smooth complex variety and $\mathcal{V} \to \mathcal{X}$ a holomorphic vector bundle over $\mathcal{X}$. Let E be the sheaf of holomorphic sections of $\mathcal{V}$. Denote by Ω the sheaf of differentials on $\mathcal{X}$. A holomorphic connection ∇ on E is a $\mathbb{C}$-linear map

$$\nabla : E \to E \otimes \Omega,$$

satisfying the Leibniz formula

$$\nabla(f\phi) = \nabla f \phi + \phi \otimes dz,$$

for any holomorphic function f. Motivated by the definition of the holomorphic connection ∇ defined for a vertex algebra bundle (cf. Section 6, [3]) over a smooth complex variety $\mathcal{X}$, we introduce the definition of the multiple point holomorphic connection over $\mathcal{X}$.

Definition 34. Let $\mathcal{V}$ be a holomorphic vector bundle over $\mathcal{X}$, and $\mathcal{X}_0$ its subvariety. A holomorphic multi-point connection $\mathcal{G}$ on $\mathcal{V}$ is a $\mathbb{C}$-multi-linear map

$$\mathcal{G} : E \to E \otimes \Omega,$$

such that for any holomorphic function f, and two sections $\phi(p)$ and $\psi(p')$ at points p and p' on $\mathcal{X}_0$ correspondingly, we have

$$\sum_{q, q' \mathcal{X}_0 \subset \mathcal{X}} \mathcal{G}\left(f(\psi(q)).\phi(q')\right) = f(\psi(p'))\, \mathcal{G}\left(\phi(p)\right) + f(\phi(p))\, \mathcal{G}\left(\psi(p')\right), \tag{14.1}$$

where the summation on left hand side is performed over a locus of points q, q' on $\mathcal{X}_0$. We denote by $\mathcal{C}on_{\mathcal{X}_0}(\mathcal{S})$ the space of such connections defined over a smooth complex variety $\mathcal{X}$. We will call $\mathcal{G}$ satisfying (14.1), a closed connection, and denote the space of such connections by $\mathcal{C}on^n_{\mathcal{X}_0;cl}$.

Geometrically, for a vector bundle $\mathcal{V}$ defined over a complex variety $\mathcal{X}$, a multi-point holomorphic connection (14.1) relates two sections ϕ and ψ of E at points p and p' with a number of sections at a subvariety $\mathcal{X}_0$ of $\mathcal{X}$.

Definition 35. We call

$$G(\phi, \psi) = f(\phi(p))\, \mathcal{G}\left(\psi(p')\right) + f(\psi(p'))\, \mathcal{G}\left(\phi(p)\right) - \sum_{q, q' \mathcal{X}_0 \subset \mathcal{X}} \mathcal{G}\left(f(\psi(q')).\phi(q)\right), \quad (14.2)$$

the form of a holomorphic connection $\mathcal{G}$. The space of form for n-point holomorphic connection forms will be denoted by $G^n(p, p', q, q')$.

Let us formulate another definition which we use in the next section:

Definition 36. We call a multi-point holomorphic connection $\mathcal{G}$ the transversal connection, i.e., when it satisfies

$$f(\psi(p'))\, \mathcal{G}(\phi(p)) + f(\phi(p))\, \mathcal{G}(\psi(p')) = 0. \quad (14.3)$$

We call

$$G_{tr}(p, p') = (\psi(p'))\, \mathcal{G}(\phi(p)) + f(\phi(p))\, \mathcal{G}(\psi(p')), \quad (14.4)$$

the form of a transversal connection. The space of such connections is denoted by G^2_{tr}.

15. Appendix: A sphere formed from sewing of two spheres

The matrix element for a number of vertex operators of a vertex algebra is usually associated [9, 10, 40] with a vertex algebra character on a sphere. We extrapolate this notion to the case of $\mathcal{W}_{z_1,\dots,z_n}$ spaces. In Section 3 we explained that a space $\mathcal{W}_{z_1,\dots,z_n}$ can be associated with a Riemann sphere with marked points, while the product of two such spaces is then associated with a sewing of such two spheres with a number of marked points and extra points with local coordinates identified with formal parameters of $\mathcal{W}_{x_1,\dots,x_k}$ and $\mathcal{W}_{y_1,\dots,y_n}$. In order to supply an appropriate geometric construction for the product, we use the ϵ-sewing procedure (described in this Appendix) for two initial spheres to obtain a matrix element associated with (3.2).

Remark 12. In addition to the ϵ-sewing procedure of two initial spheres, one can alternatively use the self-sewing procedure [42] for the sphere to get, at first, the torus, and then by sending parameters to appropriate limit by shrinking genus to zero. As a result, one obtains again the sphere but with a different parameterization. In the case of spheres, such a procedure consideration of the product of $\mathcal{W}$-spaces so we focus in this paper on the ϵ-formalizm only.

In our particular case of $\mathcal{W}$-values rational functions obtained from matrix elements (2.5) two initial auxiliary spaces we take Riemann spheres $\Sigma^{(0)}_a$, $a = 1$, 2, and the resulting space is formed by the sphere $\Sigma^{(0)}$ obtained by the procedure of sewing $\Sigma^{(0)}_a$. The formal parameters $(x_1, \dots, x_k)$ and $(y_1, \dots, y_n)$ are identified with local coordinates of k and n points on two initial spheres $\Sigma^{(0)}_a$, $a = 1, 2$ correspondingly. In the ϵ sewing procedure, some r points among $(p_1, \dots, p_k)$ may coincide with points among $(p'_1, \dots, p'_n)$ when we identify the annuluses (15.3). This corresponds to the singular case of coincidence of r formal parameters.

Consider the sphere formed by sewing together two initial spheres in the sewing scheme referred to as the ϵ-formalism in [42]. Let $\Sigma_a^{(0)}$, $a=1$, 2 be to initial spheres. Introduce a complex sewing parameter ϵ where

$$|\epsilon| \leq r_1 r_2,$$

Consider k distinct points on $p_i \in \Sigma_1^{(0)}$, $i=1,\dots,k$, with local coordinates $(x_1,\dots,x_k) \in F_k\mathbb{C}$, and distinct points $p_j \in \Sigma_2^{(0)}$, $j=1,\dots,n$, with local coordinates $(y_1,\dots,y_n) \in F_n\mathbb{C}$, with

$$|x_i| \geq |\epsilon|/r_2,$$
$$|y_i| \geq |\epsilon|/r_1.$$

Choose a local coordinate $z_a \in \mathbb{C}$ on $\Sigma_a^{(0)}$ in the neighborhood of points $p_a \in \Sigma_a^{(0)}$, $a=1$, 2. Consider the closed disks

$$|\zeta_a| \leq r_a,$$

and excise the disk

$$\{\zeta_a,\ |\zeta_a| \leq |\epsilon|/r_{\overline{a}}\} \subset \Sigma_a^{(0)}, \tag{15.1}$$

to form a punctured sphere

$$\widehat{\Sigma}_a^{(0)} = \Sigma_a^{(0)} \backslash \{\zeta_a, |\zeta_a| \leq |\epsilon|/r_{\overline{a}}\}.$$

We use the convention

$$\overline{1}=2, \quad \overline{2}=1. \tag{15.2}$$

Define the annulus

$$\mathcal{A}_a = \{\zeta_a, |\epsilon|/r_{\overline{a}} \leq |\zeta_a| \leq r_a\} \subset \widehat{\Sigma}_a^{(0)}, \tag{15.3}$$

and identify $\mathcal{A}_1$ and $\mathcal{A}_2$ as a single region $\mathcal{A}=\mathcal{A}_1 \simeq \mathcal{A}_2$ via the sewing relation

$$\zeta_1\zeta_2 = \epsilon. \tag{15.4}$$

In this way we obtain a genus zero compact Riemann surface

$$\Sigma^{(0)} = \left\{\widehat{\Sigma}_1^{(0)} \backslash \mathcal{A}_1\right\} \cup \left\{\widehat{\Sigma}_2^{(0)} \backslash \mathcal{A}_2\right\} \cup \mathcal{A}.$$

This sphere form a suitable geometrical model for the construction of a product of $\mathcal{W}$-valued rational forms in Section 3.

16. Proofs of Proposition 1, Proposition 2, Proposition 4 Lemma 3, Lemma 1

16.1. **Proof of Proposition 1.**

Proof. In order to prove this proposition we use the geometrical interpretation of the product (3.3) in terms of Riemann spheres with marked points (see Appendix 15). We consider two sets of vertex algebra elements $(v_1,\dots,v_k)$ and $(v'_1,\dots,v'_k)$, and two sets of formal complex parameters $(x_1,\dots,x_k)$, $(y_1,\dots,y_n)$. Formal parameters are identified with local coordinates of k points on the Riemann sphere $\widehat{\Sigma}_1^{(0)}$, and n points on $\widehat{\Sigma}_2^{(0)}$, with excised annuluses $\mathcal{A}_a$ (see definitions and notations in Appendix 15). Recall the sewing parameter condition (15.4)

$$\zeta_1\zeta_2 = \epsilon,$$

of the sewing procedure. Then, for (3.3) we obtain

$$\begin{aligned}
&\langle w', \Theta(v_1, x_1; \ldots; v_k, x_k; v'_1, y_1; \ldots; v'_n, y_n; \epsilon)\rangle \\
&= \sum_{l\in\mathbb{Z}} \epsilon^l \sum_{u\in V_l} \langle w', Y^W_{WV}\left(\Phi(v_1, x_1; \ldots; v_k, x_k), \zeta_1\right)\ u\rangle \\
&\qquad \langle w', Y^W_{WV}\left(\Psi(v'_1, y_1; \ldots; v'_n, y_n), \zeta_2\right)\ \overline{u}\rangle \\
&= \sum_{l\in\mathbb{Z}} \epsilon^l \sum_{u\in V_l} \langle w', e^{\zeta_1\ L_W(-1)}\ Y_W\left(u, -\zeta_1\right)\ \Phi(v_1, x_1; \ldots; v_k, x_k)\rangle \\
&\qquad \langle w', e^{\zeta_2\ L_W(-1)}\ Y_W\left(\overline{u}, -\zeta_2\right)\ \Psi(v'_1, y_1; \ldots; v'_n, y_n)\rangle.
\end{aligned}$$

Recall from (15.1) (see Appendix 15) that in two sphere ϵ-sewing formulation, the complex parameters ζ_a, $a = 1, 2$ are coordinates inside identified annuluses $\mathcal{A}_a$, and $|\zeta_a| \le r_a$. Therefore, due to Proposition 13 the matrix elements

$$\widetilde{\mathcal{R}}(x_1, \ldots, x_k; \zeta_1) = \langle w', e^{\zeta_1\ L_W(-1)}\ Y_W\left(u, -\zeta_1\right)\ \Phi(v_1, x_1; \ldots; v_k, x_k)\rangle, \tag{16.1}$$

$$\widetilde{\mathcal{R}}(y_1, \ldots, y_n; \zeta_2) = \langle w', e^{\zeta_2\ L_W(-1)}\ Y_W\left(\overline{u}, -\zeta_2\right)\ \Psi(v'_1, y_1; \ldots; v'_n, y_n)\rangle, \tag{16.2}$$

are absolutely convergent in powers of ϵ with some radia of convergence $R_a \le r_a$, with $|\zeta_a| \le R_a$. The dependence of (16.1) and (16.2) on ϵ is expressed via ζ_a, $a = 1$, 2. Let us rewrite the product (3.3) as

$$\begin{aligned}
&\langle w', \Theta(v_1, x_1; \ldots; v_k, x_k; v'_1, y_1; \ldots; v'_n, y_n; \epsilon)\rangle \\
&= \sum_{l\in Z} \epsilon^l \left(\langle w', \Theta(v_1, x_1; \ldots; v_k, x_k; v'_1, y_1; \ldots; v'_n, y_n)\rangle\right)_l \\
&= \sum_{l\in\mathbb{Z}} \sum_{u\in V_l} \sum_{m\in\mathbb{C}} \epsilon^{l-m-1}\ \widetilde{\mathcal{R}}_m(x_1, \ldots, x_k; \zeta_1)\ \widetilde{\mathcal{R}}_m(y_1, \ldots, y_n; \zeta_2),
\end{aligned} \tag{16.3}$$

as a formal series in ϵ for $|\zeta_a| \le R_a$, where and $|\epsilon| \le r$ for $r < r_1 r_2$. Then we apply Cauchy's inequality to coefficient forms (16.1) and (16.2) to find

$$\left|\widetilde{\mathcal{R}}_m(x_1, \ldots, x_k; \zeta_1)\right| \le M_1 R_1^{-m}, \tag{16.4}$$

with

$$M_1 = \sup_{|\zeta_1|\le R_1, |\epsilon|\le r} \left|\widetilde{\mathcal{R}}(x_1, \ldots, x_k; \zeta_1)\right|.$$

Similarly,

$$\left|\widetilde{\mathcal{R}}_m(y_1, \ldots, y_n; \zeta_2)\right| \le M_2 R_2^{-m}, \tag{16.5}$$

for

$$M_2 = \sup_{|\zeta_2|\le R_2, |\epsilon|\le r} \left|\widetilde{\mathcal{R}}(y_1, \ldots, y_n; \zeta_2)\right|.$$

Using (16.4) and (16.5) we obtain for (16.3)

$$\begin{aligned}
&|(\langle w', \mathcal{F}(v_1, x_1; \ldots; v_k, x_k; v'_1, y_1; \ldots; v'_n, y_n\rangle)_l| \\
&\le \left|\widetilde{\mathcal{R}}_m(x_1, \ldots, x_k; \zeta_1)\right|\ \left|\widetilde{\mathcal{R}}_m(y_1, \ldots, y_n; \zeta_2)\right| \\
&\le M_1\ M_2\ \left(R_1 R_2\right)^{-m}.
\end{aligned} \tag{16.6}$$

Thus, for $M = \min\{M_1, M_2\}$ and $R = \max\{R_1, R_2\}$, such that

$$|\mathcal{R}_l(x_1; \ldots, x_k; y_1, \ldots, y'_n; \zeta_1, \zeta_2)| \le MR^{-l+m+1}. \tag{16.7}$$

Thus, we see that (3.3) is absolute convergent as a formal series in ϵ is defined for $|\zeta_a| \le r_a$, and $|\epsilon| \le r$ for $r < r_1 r_2$, with extra poles only at $x_i = y_j$, $1 \le i \le k$, $1 \le j \le n$. ☐

16.2. **Proof of Proposition 2.**

Proof. By using (11.1) for $\Phi(v_1, x_1; \ldots; v_k, x_k)$ and $\Psi(v'_1, y_1; \ldots; v'_n, y_n)$, we consider

$$\begin{aligned}
&\langle w', \partial_l \Theta(v_1, x_1; \ldots; v_k, x_k; v'_1, y_1; \ldots; v'_n, y_n; \epsilon)\rangle \\
&= \sum_{l \in \mathbb{Z}} \epsilon^l \sum_{u \in V_l} \langle w', \partial_{x_i}^{\delta_{l,i}} Y_{WV}^W \left(\Phi(v_1, x_1; \ldots; v_k, x_k), \zeta_1\right)\ u\rangle \\
&\qquad \langle w', \partial_{y_j}^{\delta_{l,j}} Y_{WV}^W \left(\Psi(v'_1, y_1; \ldots; v'_n, y_n), \zeta_2\right) \overline{u}\rangle \\
&= \sum_{l \in \mathbb{Z}} \epsilon^l \sum_{u \in V_l} \langle w', \partial_{x_i}^{\delta_{l,i}} Y_W \left(u, -\zeta_1\right) \Phi(v_1, x_1; \ldots; v_k, x_k))\ u\rangle \\
&\qquad \langle w', \partial_{y_j}^{\delta_{l,j}} Y_W \left(\overline{u}, -\zeta_2\right) \Psi(v'_1, y_1; \ldots; v'_n, y_n)\rangle \\
&= \sum_{l \in \mathbb{Z}} \epsilon^l \sum_{u \in V_l} \langle w', Y_{WV}^W \left(\partial_{x_i}^{\delta_{l,i}} \Phi(v_1, x_1; \ldots; v_k, x_k), \zeta_1\right)\ u\rangle \\
&\qquad \langle w', Y_{WV}^W \left(\partial_{y_j}^{\delta_{l,j}} \Psi(v'_1, y_1; \ldots; v_n, y_n), \zeta_2\right) \overline{u}\rangle \\
&= \sum_{l \in \mathbb{Z}} \epsilon^l \sum_{u \in V_l} \langle w', Y_{WV}^W \left(\Phi(v_1, x_1; \ldots; (L_V(-1))^{\delta_{l,i}}\, v_i, x_i; \ldots; v_k, x_k), \zeta_1\right)\ u\rangle \\
&\qquad \langle w', Y_{WV}^W \left(\Psi(v'_1, y_1; \ldots; (L_V(-1))^{\delta_{l,j}}\, v'_j, y_j; \ldots; v'_n, y_n), \zeta_2\right) \overline{u}\rangle \\
&= \langle w', \Theta(v_1, x_1; \ldots; (L_V(-1))_l\,; \ldots; v'_n, y_n; \epsilon)\rangle,
\end{aligned} \tag{16.8}$$

where $(L_V(-1))_l$ acts on the l-th entry of $(v_1, \ldots; v_k; v'_1, \ldots, v'_n)$. Summing over l we obtain

$$\begin{aligned}
&\sum_{l=1}^{k+n} \partial_l \Theta(v_1, x_1; \ldots; v_k, x_k; v'_1, y_1; \ldots; v'_n, y_n; \epsilon)\rangle \\
&= \sum_{l=1}^{k+n} \langle w', \Theta(v'_1, x_1; \ldots; (L_V(-1))\,; \ldots; v'_n, y_n; \epsilon)\rangle \\
&= \langle w', L_W(-1).\Theta(v_1, x_1; \ldots; v_k, x_k; v'_1, y_1; \ldots; v'_n, y_n; \epsilon)\rangle.
\end{aligned} \tag{16.9}$$

Due to (11.6), (9.9), (9.29), (9.30), and (9.15), we have

$$\langle w', \Theta(z^{L_V(0)} v_1, z\, x_1; \ldots; z^{L_V(0)} v_k, z\, x_k; z^{L_V(0)} v'_1, z\, y_1; \ldots; z^{L_V(0)} v'_n, z\, y_n; \epsilon)\rangle$$

$$= \sum_{l\in\mathbb{Z}} \epsilon^l \sum_{u\in V_l} \langle w', Y_{WV}^W \left(\Phi(z^{L_V(0)} v_1, z\ x_1; \dots; z^{L_V(0)} v_k, z\ x_k), \zeta_1 \right)\ u\rangle$$
$$\langle w', Y_{WV}^W \left(\Psi(z^{L_V(0)} v'_1, z\ y_1; \dots; z^{L_V(0)} v'_n, z\ y_n), \zeta_2 \right) \overline{u}\rangle$$

$$= \sum_{l\in\mathbb{Z}} \epsilon^l \sum_{u\in V_l} \langle w', Y_{WV}^W \left(z^{L_V(0)} \Phi(v_1, x_1; \dots; v_k, x_k), \zeta_1 \right)\ u\rangle$$
$$\langle w', Y_{WV}^W \left(z^{L_V(0)} \Psi(v'_1, y_1; \dots; v'_n, y_n), \zeta_2 \right) \overline{u}\rangle$$

$$= \sum_{l\in\mathbb{Z}} \epsilon^l \sum_{u\in V_l} \langle w', e^{\zeta_1 L_W(-1)} Y_W\left(u, -\zeta_1\right) z^{L_V(0)} \Phi(v_1, x_1; \dots; v_k, x_k)\rangle$$
$$\langle w', e^{\zeta_2 L_W(-1)}\ Y_W\left(\overline{u}, -\zeta_2\right) z^{L_V(0)} \Psi(v'_1, y_1; \dots; v'_n, y_n)\rangle$$

$$= \sum_{l\in\mathbb{Z}} \epsilon^l \sum_{u\in V_l} \langle w', e^{\zeta_1 L_W(-1)} z^{L_V(0)} Y_W \left(z^{-L_V(0)} u, -z\ \zeta_1 \right) \Phi(v_1, x_1; \dots; v_k, x_k)\rangle$$
$$\langle w', e^{\zeta_2 L_W(-1)}\ z^{L_W(0)}\ Y_W \left(z^{-L_V(0)} \overline{u}, -z\ \zeta_2 \right) \Psi(v'_1, y_1; \dots; v'_n, y_n)\rangle$$
$$= \sum_{l\in\mathbb{Z}} \epsilon^l \sum_{u\in V_l} \langle w', e^{\zeta_1 L_W(-1)} z^{L_W(0)} z^{-\mathrm{wt} u}\ Y_W\left(u, -z\ \zeta_1\right) \Phi(v_1, x_1; \dots; v_k, x_k)\rangle$$
$$\langle w', e^{\zeta_2 L_W(-1)}\ z^{L_W(0)}\ z^{-\mathrm{wt}\overline{u}}\ Y_W\left(\overline{u}, -z\ \zeta_2\right) \Psi(v'_1, y_1; \dots; v'_n, y_n)\rangle$$

$$= \sum_{l\in\mathbb{Z}} \epsilon^l \sum_{u\in V_l} \langle w', z^{L_W(0)} e^{\zeta_1 L_W(-1)} Y_W\left(u, -z\zeta_1\right) \Phi(v_1, x_1; \dots; v_k, x_k)\rangle$$
$$\langle w', z^{L_W(0)} e^{\zeta_2 L_W(-1)} Y_W\left(\overline{u}, -z\zeta_2\right) \Psi(v'_1, y_1; \dots; v'_n, y_n), \rangle$$

$$= \sum_{l\in\mathbb{Z}} \epsilon^l \sum_{u\in V_l} \langle w', z^{L_W(0)}\ Y_{WV}^W\left(\Phi(v_1, x_1; \dots; v_k, x_k), z\zeta_1\right)\ u\rangle$$
$$\langle w', z^{L_W(0)}\ Y_{WV}^W\left(\Psi(v'_1, y_1; \dots; v'_n, y_n), z\zeta_2\right) \overline{u}\rangle$$

$$= \sum_{l\in\mathbb{Z}} \epsilon^l \sum_{u\in V_l} \langle w', z^{L_W(0)}\ Y_{WV}^W\left(\Phi(v_1, x_1; \dots; v_k, x_k), \zeta'_1\right)\ u\rangle$$
$$\langle w', z^{L_W(0)}\ Y_{WV}^W\left(\Psi(v'_1, y_1; \dots; v'_n, y_n), \zeta'_2\right) \overline{u}\rangle$$

$$= \langle w', \left(z^{L_W(0)} \right) .\Theta(v_1, x_1; \dots; v_k, x_k; v'_1, y_1; \dots; v'_n, y_n; \epsilon)\rangle.$$

With (15.4), we obtain (11.6) for (3.3). □

16.3. **Proof of Proposition 4.**

Proof. Note that due to Proposition 4

$$\begin{aligned} \Phi(v_1, x'_1; \dots; v_k, x'_k) &= \Phi(v_1, x_1; \dots; v_k, x_k), \\ \Psi(v_1, y'_1; \dots; v_n, y'_n) &= \Psi(v_1, y_1; \dots; v_n, y_n). \end{aligned}$$

Thus,

$$\begin{aligned}
&\langle w', \Theta(v_1, x_1'; \ldots; v_k, x_k'; v_1', y_1'; \ldots; v_n', y_n'; \epsilon)\rangle \\
&= \sum_{l\in\mathbb{Z}} \epsilon^l \sum_{u\in V_l} \langle w', Y^W_{WV}\left(\Phi(v_1, x_1'; \ldots; v_k, x_k'), \zeta_1\right)\ u\rangle \\
&\qquad\qquad \langle w', Y^W_{WV}\left(\Psi(v_1', y_1'; \ldots; v_n', y_n'), \zeta_2\right)\overline{u}\rangle \\
&= \sum_{l\in\mathbb{Z}} \epsilon^l \sum_{u\in V_l} \langle w', Y^W_{WV}\left(\Phi(v_1, x_1; \ldots; v_k, x_k), \zeta_1\right)\ u\rangle \\
&\qquad\qquad \langle w', Y^W_{WV}\left(\Psi(v_1', y_1; \ldots; v_n', y_n), \zeta_2\right)\overline{u}\rangle \\
&= \langle w', \Theta(v_1, x_1; \ldots; v_k, x_k; v_1', y_1; \ldots; v_n', y_n; \epsilon)\rangle.
\end{aligned}$$

Thus, the product (3.3) is invariant under (13.4). ☐

16.4. **Proof of Lemma 3.**

Proof. Let $\widetilde{v}_i \in V$, $1 \le i \le k$, $\widetilde{v}_j \in V$, $1 \le j \le k$, and z_i, z_j are corresponding formal parameters. We show that the ϵ-product of $\Phi(\widetilde{v}_1, z_1; \ldots; \widetilde{v}_k, z_k)$ and $\Psi(\widetilde{v}_{k+1}, z_{k+1}; \ldots; \widetilde{v}_n, z_n)$, i.e., the $\mathcal{W}_{z_1,\ldots,z_{k+n-r}}$-valued differential form

$$\Theta((\widetilde{v}_1, z_1; \ldots; \widetilde{v}_k, z_k); (\widetilde{v}_{k+1}, z_{k+1}; \ldots; \widetilde{v}_n, z_n); \zeta_1, \zeta_2; \epsilon) \tag{16.10}$$

is independent of the choice of $0 \le k \le n$. Consider

$$\begin{aligned}
&\langle w', \Theta(\widetilde{v}_1, z_1; \ldots; \widetilde{v}_k, z_k; \widetilde{v}_{k+1}, z_{k+1}; \ldots; \widetilde{v}_n, z_n; \zeta_1, \zeta_2; \epsilon)\rangle \\
&\quad = \sum_{l\in\mathbb{Z}} \epsilon^l \sum_{u\in V_l} \langle w', Y^W_{WV}\left(\Phi(\widetilde{v}_1, z_1; \ldots; \widetilde{v}_k, z_k), \zeta_1\right)\ u\rangle \\
&\qquad\qquad \langle w', Y^W_{WV}\left(\Psi(\widetilde{v}_{k+1}, z_{k+1}; \ldots; \widetilde{v}_n, z_n), \zeta_2\right)\overline{u}\rangle.
\end{aligned} \tag{16.11}$$

On the other hand, for $0 \le m \le k$, consider

$$\begin{aligned}
&\sum_{l\in\mathbb{Z}} \epsilon^l \sum_{u\in V_l} \langle w', Y^W_{WV}\left(\Phi(\widetilde{v}_1, z_1; \ldots; \widetilde{v}_m, z_m), \zeta_1\right)\ u\rangle \\
&\qquad\qquad \langle w', Y^W_{WV}\left(\Psi(\widetilde{v}_{m+1}, z_{m+1}'; \ldots; \widetilde{v}_k, z_k'; \widetilde{v}_{k+1}, z_1; \ldots; \widetilde{v}_n, z_n), \zeta_2\right)\overline{u}\rangle \\
&\quad = \langle w', \Theta(\widetilde{v}_1, z_1; \ldots; \widetilde{v}_m, z_m; \widetilde{v}_{m+1}, z_{m+1}'; \ldots; \widetilde{v}_k, z_k'; \widetilde{v}_{k+1}, z_{k+1}; \ldots; \widetilde{v}_n, z_n)\rangle.
\end{aligned}$$

The last is the ϵ-product (3.3) of $\Phi(\widetilde{v}_1, z_1; \ldots; \widetilde{v}_m, z_m) \in \mathcal{W}_{z_1,\ldots,z_m}$ and $\Psi(\widetilde{v}_{m+1}, z_{m+1}';$ $\ldots;$ $\widetilde{v}_k, z_k';$ $\widetilde{v}_{k+1}, z_1;$ $\ldots;$ $\widetilde{v}_n, z_n) \in \mathcal{W}_{z_{m+1}',\ldots,z_k';z_1,\ldots,z_n}$. Let us apply the invariance with respect to a subgroup of $\left(\mathrm{Aut}\ \mathcal{O}^{(1)}\right)^{\times(k+n)}_{z_1,\ldots,z_{k+n}}$, with $(z_1, \ldots, z_m)$ and $(z_{k+1}, \ldots, z_n)$ remaining unchanged. Then we obtain the same product (16.11). ☐

16.5. **Proof of Lemma 1.**

Proof. For arbitrary $w' \in W'$, we have

$$\sum_{\sigma \in J^{-1}_{k+n;s}} (-1)^{|\sigma|} \langle w', \Theta\left(v_{\sigma(1)}, x_{\sigma(1)}; \ldots; v_{\sigma(k)}, x_{\sigma(k)}; v'_{\sigma(1)}, y_{\sigma(1)}; \ldots; v'_{\sigma(n)}, y_{\sigma(n)})\right)\rangle$$

$$= \sum_{\sigma \in J^{-1}_{k+n;s}} (-1)^{|\sigma|} \sum_{l \in \mathbb{Z}} \epsilon^l \sum_{u \in V_l} \langle w', Y^W_{WV}\left(\Phi(v_{\sigma(1)}, x_{\sigma(1)}; \ldots; v_{\sigma(k)}, x_{\sigma(k)}), \zeta_1\right) u\rangle$$

$$\langle w', Y^W_{WV}\left(\Psi(v'_{\sigma(1)}, y_{\sigma(1)}; \ldots; v'_{\sigma(n)}, y_{\sigma(n)}), \zeta_2\right) \overline{u}\rangle$$

$$= \sum_{l \in \mathbb{Z}} \epsilon^l \sum_{u \in V_l} \sum_{\sigma \in J^{-1}_{k+n;s}} (-1)^{|\sigma|} \langle w', e^{\zeta_1 L_W(-1)}\, Y_W(u, -\zeta_1)\, \Phi(v_{\sigma(1)}, x_{\sigma(1)}; \ldots; v_{\sigma(k)}, x_{\sigma(k)})\rangle$$

$$\langle w', e^{\zeta_2 L_W(-1)}\, Y_W(\overline{u}, -\zeta_2)\, \Psi(v'_{\sigma(1)}, y_{\sigma(1)}; \ldots; v'_{\sigma(n)}, y_{\sigma(n)})\rangle$$

$$= \sum_{l \in \mathbb{Z}} \epsilon^l \sum_{u \in V_l} \langle w', e^{\zeta_1 L_W(-1)}\, Y_W(u, -\zeta_1) \sum_{\sigma \in J^{-1}_{k;s}} (-1)^{|\sigma|} \Phi(v_{\sigma(1)}, x_{\sigma(1)}; \ldots; v_{\sigma(k)}, x_{\sigma(k)})\rangle$$

$$\langle w', e^{\zeta_2 L_W(-1)}\, Y_W(\overline{u}, -\zeta_2)\, \Psi(v'_{\sigma(1)}, y_{\sigma(1)}; \ldots; v'_{\sigma(n)}, y_{\sigma(n)})\rangle$$

$$+ \sum_{l \in \mathbb{Z}} \epsilon^l \sum_{u \in V_l} \langle w', e^{\zeta_1 L_W(-1)}\, Y_W(u, -\zeta_1)\, \Phi(v_{\sigma(1)}, x_{\sigma(1)}; \ldots; v_{\sigma(k)}, x_{\sigma(k)})\rangle$$

$$\langle w', e^{\zeta_2 L_W(-1)}\, Y_W(\overline{u}, -\zeta_2) \sum_{\sigma \in J^{-1}_{n;s}} (-1)^{|\sigma|} \Psi(v'_{\sigma(1)}, y_{\sigma(1)}; \ldots; v'_{\sigma(n)}, y_{\sigma(n)})\rangle = 0,$$

since, $J^{-1}_{k+n;s} = J^{-1}_{k;s} \times J^{-1}_{n;s}$, and due to the fact that $\mathcal{F}(v_1, x_1; \ldots; v_k, x_k)$ and $\mathcal{F}(v'_1, y_1; \ldots; v'_n, y_n)$ satisfy (10.1). □

16.6. **Proof of Proposition 7.**

Proof. Recall that $\Phi(v_1, x_1; \ldots; v_k, x_k)$ is composable with m vertex operators, and $\Psi(v'_1, y_1; \ldots; v'_n, y_n)$ is composable with m' vertex operators. For $\Phi(v_1, x_1; \ldots; v_k, x_k)$ we have:

1) Let $l_1, \ldots, l_k \in \mathbb{Z}_+$ such that $l_1 + \ldots + l_k = k + m$, and $v_1, \ldots, v_{k+m} \in V$, and arbitrary $w' \in W'$. Set

$$\Xi_i \quad = \quad E^{(l_i)}_V(v_{k_1}, x_{k_1} - \zeta_i; \ldots; v_{k_i}, x_{k_i} - \zeta_i; \mathbf{1}_V), \tag{16.12}$$

where

$$k_1 = l_1 + \ldots + l_{i-1} + 1, \quad \ldots, \quad k_i = l_1 + \ldots + l_{i-1} + l_i, \tag{16.13}$$

for $i = 1, \ldots, k$. Then the series

$$\mathcal{I}^k_m(\Phi) = \sum_{r_1, \ldots, r_k \in \mathbb{Z}} \langle w', \Phi(P_{r_1}\Xi_1; \zeta_1; \ldots; P_{r_k}\Xi_k, \zeta_k)\rangle, \tag{16.14}$$

is absolutely convergent when

$$|x_{l_1+\ldots+l_{i-1}+p} - \zeta_i| + |x_{l_1+\ldots+l_{j-1}+q} - \zeta_i| < |\zeta_i - \zeta_j|, \tag{16.15}$$

for i, $j = 1, \ldots, k$, $i \neq j$ and for $p = 1, \ldots, l_i$ and $q = 1, \ldots, l_j$. There exist positive integers $N^k_m(v_i, v_j)$, depending only on v_i and v_j for $i, j = 1, \ldots, k$, $i \neq j$, such that the sum is analytically extended to a rational function in $(x_1,\ldots,x_{k+m})$, independent of $(\zeta_1,\ldots,\zeta_k)$, with the only possible poles at $x_i=x_j$, of order less than or equal to $N^k_m(v_i,v_j)$, for i, $j=1,\ldots,k$, $i\neq j$.

For $\Psi(v'_1,y_1;\ldots;v'_n,y_n)$ we have:

1') Let $l'_1,\ldots,l'_n\in\mathbb{Z}_+$ such that $l'_1+\ldots+l'_n=n+m'$, $v'_1,\ldots,v_{n+m'}\in V$ and arbitrary $w'\in W'$. Set

$$\Xi'_{i'} \quad = \quad E_V^{(l'_{i'})}(v'_{k'_1},y_{k'_1}-\zeta'_{i'};\ldots;v'_{k'_{i'}},y_{k'_{i'}}-\zeta'_{i'};\mathbf{1}_V), \tag{16.16}$$

where

$$k'_1=l'_1+\ldots+l'_{i'-1}+1,\quad \ldots,\quad k'_{i'}=l'_1+\ldots+l'_{i'-1}+l'_{i'}, \tag{16.17}$$

for $i'=1,\ldots,n$. Then the series

$$\mathcal{I}^n_{m'}(\Psi)=\sum_{r'_1,\ldots,r'_n\in\mathbb{Z}}\langle w',\Psi(P_{r'_1}\Psi'_1;\zeta'_1;\ldots;P_{r'_n}\Psi'_n,\zeta'_n)\rangle, \tag{16.18}$$

is absolutely convergent when

$$|y_{l'_1+\ldots+l'_{i'-1}+p'}-\zeta'_{i'}|+|y_{l'_1+\ldots+l'_{j'-1}+q'}-\zeta'_{i'}|<|\zeta'_{i'}-\zeta'_{j'}|, \tag{16.19}$$

for i', $j'=1,\ldots,n$, $i'\neq j'$ and for $p'=1,\ldots,l'_i$ and $q'=1,\ldots,l'_j$. There exist positive integers $N^n_{m'}(v'_{i'},v'_{j'})$, depending only on $v'_{i'}$ and $v'_{j'}$ for i, $j=1,\ldots,n$, $i'\neq j'$, such that the sum is analytically extended to a rational function in $(y_1,\ldots,y_{n+m'})$, independent of $(\zeta'_1,\ldots,\zeta'_n)$, with the only possible poles at $y_{i'}=y_{j'}$, of order less than or equal to $N^n_{m'}(v'_{i'},v'_{j'})$, for i', $j'=1,\ldots,n$, $i'\neq j'$.

Now let us consider the first condition of Definition 31 of composability for the product (5.5) of $\Phi(v_1,x_1;\ldots;v_k,x_k)$ and $\Psi(v'_1,y_1;\ldots;v'_n,y_n)$ with a number of vertex operators. Then we obtain for $\Theta\left(v_1,x_1;\ldots;v_k,x_k;v'_1,y_1;\ldots;v'_n,y_n;\epsilon\right)$ the following. We redefine the notations for the set

$$\begin{aligned}&(v''_1,\ldots,v''_k;v''_{k+1},\ldots,v''_{k+m};v''_{k+m+1},\ldots,v''_{k+n+m+m'};v_{n+1},\ldots,v'_{n+m'})\\ &\qquad=(v_1,\ldots,v_k;v_{k+1},\ldots,v_{k+m};v'_1,\ldots,v'_n;v'_{n+1},\ldots,v'_{n+m'}),\\ &(z_1,\ldots,z_k;z_{k+1},\ldots,z_{k+n-r})=(x_1,\ldots,x_k;y_1,\ldots,y_n),\end{aligned}$$

of vertex algebra V elements. Introduce $l''_1,\ldots,l''_{k+n}\in\mathbb{Z}_+$, such that $l''_1+\ldots+l''_{k+n}=k+n+m+m'$. Define

$$\Xi''_i \quad = \quad E_V^{(l''_{i''})}(v''_{k''_1},z_{k''_1}-\zeta''_{i''};\ldots;v''_{k''_{i''}},z_{k''_{i''}}-\zeta''_{i''};\mathbf{1}_V), \tag{16.20}$$

where

$$k''_1=l''_1+\ldots+l''_{i''-1}+1,\quad \ldots,\quad k''_{i''}=l''_1+\ldots+l''_{i''-1}+l''_{i''}, \tag{16.21}$$

for $i''=1,\ldots,k+n$, and we take

$$(\zeta''_1,\ldots,\zeta''_{k+n})=(\zeta_1,\ldots,\zeta_k;\zeta'_1,\ldots,\zeta'_n).$$

Then we consider

$$\mathcal{I}^{k+n}_{m+m'}(\Theta)=\sum_{r''_1,\ldots,r''_{k+n}\in\mathbb{Z}}\langle w',\Theta(P_{r''_1}\Psi''_1;\zeta''_1;\ldots;P_{r''_{k+n}}\Psi''_{k+n},\zeta''_{k+n})\rangle, \tag{16.22}$$

and prove it is absolutely convergent with some conditions.

The condition

$$|z_{l''_1+\ldots+l''_{i-1}+p''}-\zeta''_i|+|z_{l''_1+\ldots+l''_{j-1}+q''}-\zeta''_i|<|\zeta''_i-\zeta''_j|, \qquad (16.23)$$

of absolute convergence for (16.22) for i'', $j''=1,\ldots,k+n$, $i\neq j$ and for $p''=1,\ldots,l''_i$ and $q''=1,\ldots,l''_j$, follows from the conditions (16.15) and (16.29). The action of $e^{\zeta L_W(-1)}\, Y_W(.,.)$, $a=1,\,2$, in

$$\langle w', e^{\zeta_1 L_W(-1)}\, Y_W(u,-\zeta) \sum_{r_1,\ldots,r_k\in\mathbb{Z}} \Phi(P_{r_1}\Xi_1;\zeta_1;\ldots;P_{r_k}\Xi_k,\zeta_k)\rangle,$$

$$\langle w', e^{\zeta_2 L_W(-1)}\, Y_W(\overline{u},-\widetilde{\zeta}) \sum_{r'_1,\ldots,r'_n\in\mathbb{Z}} \Psi(P_{r'_1}\Xi'_1;\zeta_1;\ldots;P_{r'_k}\Xi'_n,\zeta'_n)\rangle,$$

does not affect the absolute convergency of (16.14) and (16.18). We obtain

$$\begin{aligned}
&\left|\mathcal{I}^{k+n}_{m+m'}(\Theta)\right| = \\
&= \left| \sum_{r''_1,\ldots,r''_{k+n}\in\mathbb{Z}} \langle w', \Theta(P_{r''_1}\Xi''_1;\zeta''_1;\ldots;P_{r''_{k+n}}\Xi''_{k+n},\zeta''_{k+n})\rangle \right| \\
&= \left| \sum_{l\in\mathbb{Z}} \epsilon^l \sum_{u\in V_l} \langle w', Y^W_{VW}(\sum_{r_1,\ldots,r_k\in\mathbb{Z}} \Phi(P_{r_1}\Xi_1;\zeta_1;\ldots;P_{r_k}\Xi_k,\zeta_k),\zeta)u\rangle \right. \\
&\qquad \left. \langle w', Y^W_{VW}(\sum_{r'_1,\ldots,r'_n\in\mathbb{Z}} \Psi(P_{r'_1}\Xi'_1;\zeta'_1;\ldots;P_{r'_n}\Xi'_n,\zeta'_n),\widetilde{\zeta})\overline{u}\rangle \right| \\
&\qquad \le \left|\mathcal{I}^k_m(\Phi)\right|\ \left|\mathcal{I}^n_{m'}(\Psi)\right|.
\end{aligned}$$

Thus, we infer that (16.22) is absolutely convergent. Recall that the maximal orders of possible poles of (16.22) are $N^k_m(v_i,v_j)$, $N^n_{m'}(v'_{i'},v'_{j'})$ at $x_i=x_j$, $y_{i'}=y_{j'}$. From the last expression we infer that there exist positive integers $N^{k+n}_{m+m'}(v''_{i''},v''_{j''})$ for i, $j=1,\ldots,k$, $i\neq j$, i', $j'=1,\ldots,n$, $i'\neq j$, depending only on $v''_{i''}$ and $v''_{j''}$ for i'', $j''=1,\ldots,k+n$, $i''\neq j''$ such that the series (16.22) can be analytically extended to a rational function in $(x_1,\ldots,x_k;y_1,\ldots,y_n)$, independent of $(\zeta''_1,\ldots,\zeta''_{k+n})$, with extra possible poles at and $x_i=y_j$, of order less than or equal to $N^{k+n}_{m+m'}(v''_{i''},v''_{j''})$, for i'', $j''=1,\ldots,n$, $i''\neq j''$.

Let us proceed with the second condition of composability. For $\Phi(v_1,x_1;\ldots;v_k,x_k)\in C^k_m(V,\mathcal{W},\mathcal{F})$, and $(v_1,\ldots,v_{k+m})\in V$, $(x_1,\ldots,x_{k+m})\in\mathbb{C}$, we have

2) For arbitrary $w'\in W'$, the series

$$\mathcal{J}^k_m(\Phi)=\sum_{q\in\mathbb{C}}\langle w', E^{(m)}_W\Big(v_1,x_1;\ldots;v_m,x_m;P_q(\Phi(v_{m+1},x_{m+1};\ldots;v_{m+k},x_{m+k}\Big)\rangle, \qquad (16.24)$$

is absolutely convergent when

$$\begin{aligned} x_i \neq x_j, \quad i \neq j, \\ |x_i| > |x_{k'}| > 0, \end{aligned} \tag{16.25}$$

for $i = 1, \ldots, m$, and $k' = m+1, \ldots, k+m$, and the sum can be analytically extended to a rational function in $(x_1, \ldots, x_{k+m})$ with the only possible poles at $x_i = x_j$, of orders less than or equal to $N^k_m(v_i, v_j)$, for $i, j = 1, \ldots, k$, $i \neq j$.

2') For $\Psi(v'_1, y_1; \ldots; v'_n, y_n) \in C^n_{m'}(V, \mathcal{W}, \mathcal{F})$, $(v'_1, \ldots, v'_{n+m'}) \in V$, and $(y_1, \ldots, y_{n+m'}) \in \mathbb{C}$, the series

$$\begin{aligned} \mathcal{J}^n_{m'}(\Psi) = \sum_{q \in \mathbb{C}} \langle w', E^{(m')}_W \Big(v'_1, y_1; \ldots; v'_{m'}, y_{m'}; \\ P_q(\Psi(v'_{m'+1}, y_{m'+1}; \ldots; v'_{m'+n}, y_{m'+n})) \Big) \rangle, \end{aligned} \tag{16.26}$$

is absolutely convergent when

$$\begin{aligned} y_{i'} \neq y_{j'}, \quad i' \neq j', \\ |y_{i'}| > |y_{k''}| > 0, \end{aligned} \tag{16.27}$$

for $i' = 1, \ldots, m'$, and $k'' = m'+1, \ldots, n+m'$, and the sum can be analytically extended to a rational function in $(y_1, \ldots, y_{n+m'})$ with the only possible poles at $y_{i'} = y_{j'}$, of orders less than or equal to $N^n_{m'}(v'_{i'}, v'_{j'})$, for i', $j' = 1, \ldots, n$, $i' \neq j'$.

2") Thus, for the product (5.5) we obtain $(v''_1, \ldots, v''_{k+n+m+m'}) \in V$, and $(z_1, \ldots, z_{k+n+m+m'}) \in \mathbb{C}$, we find positive integers $N^{k+n}_{m+m'}(v'_i, v'_j)$, depending only on v'_i and v''_j, for i'', $j'' = 1, \ldots, k+n$, $i'' \neq j''$, such that for arbitrary $w' \in W'$. First we note

Lemma 14.

$$\begin{aligned} &\sum_{q \in \mathbb{C}} \langle w', E^{(m+m')}_W \Big(v''_1, z_1; \ldots; v''_{m+m'}, z_{m+m'}; \\ &\qquad P_q\Big(\Theta(v''_{m+m'+1}, z_{m+m'+1}; \ldots; v''_{m+m'+k+n}, z_{m+m'+k+n} \Big) \Big) \rangle \\ &= \sum_{l \in \mathbb{Z}} \epsilon^l \sum_{u \in V_l} \langle w', E^{(m)}_W \Big(v_{k+1}, x_{k+1}; \ldots; v_{k+m}, x_{k+m}; \\ &\qquad P_q\Big(Y^W_{WV}\left(\Phi(v_1, x_1; \ldots; v_k, x_k), \zeta_1\right)\ u \Big) \Big) \rangle \\ &\quad \langle w', E^{(m')}_W \Big(v'_{n+1}, y_{n+1}; \ldots; v'_{n+m'}, y_{n+m'}; \\ &\qquad P_q\Big(Y^W_{WV}\left(\Psi(v'_1, y_1; \ldots; v'_n, y_n), \zeta_2\right)\ \overline{u} \Big) \Big) \rangle. \end{aligned}$$

Proof. Consider

$$\sum_{l\in\mathbb{Z}} \epsilon^l \sum_{u\in V_l} \langle w', E_W^{(m+m')}\Big(v''_1, z_1; \ldots; v''_{m+m'}, z_{m+m'};$$
$$P_q\Big(Y_{WV}^W\left(\Phi(v''_{m+m'+1}, z_{m+m'+1}; \ldots; v''_{m+m'+k}, z_{m+m'+k}), \zeta_1\right)\ u\Big)\Big)\rangle$$
$$\langle w', E_W^{(m+m')}\Big(v''_1, z_1; \ldots; v''_{m+m'}, z_{m+m'};$$
$$P_q\Big(Y_{WV}^W\left(\Psi(v''_{m+m'+k+1}, z_{m+m'+k+1}; \ldots;\right.$$
$$\left. v''_{m+m'+k+n}, z_{m+m'+k+n}), \zeta_2\right)\ \overline{u}\Big)\Big)\rangle$$

$$= \sum_{q\in\mathbb{C}} \sum_{l\in\mathbb{Z}} \epsilon^l \sum_{u\in V_l} \langle w', E_W^{(m+m')}\Big(v''_1, z_1; \ldots; v''_{m+m'}, z_{m+m'};$$
$$P_q\Big(e^{\zeta_1 L_W(-1)}\ Y_W\left(u, -\zeta_1\right)\ \Phi(v''_{m+m'+1}, z_{m+m'+1}; \ldots; v''_{m+m'+k}, z_{m+m'+k}))\ \Big)\rangle$$
$$\langle w', E_W^{(m+m')}\Big(v''_1, z_1; \ldots; v''_{m+m'}, z_{m+m'};$$
$$P_q\Big(e^{\zeta_2 L_W(-1)}\ Y_W\left(\overline{u}, -\zeta_2\right)\ \Psi(v''_{m+m'+k+1}, z_{m+m'+k+1}; \ldots;$$
$$v''_{m+m'+k+n}, z_{m+m'+k+n}))\ \Big)\rangle.$$

The action of exponentials $e^{\zeta_a L_W(-1)}$, $a = 1, 2$, of the differential operator $L_W(-1)$, and W-module vertex operators $Y_W(u, -\zeta_1)$, $Y_W(u, -\zeta_2)$ shifts the grading index q of W_q-subspaces by $\alpha \in \mathbb{C}$ which can be later rescaled to q. Thus, we can rewrite the last expression as

$$= \sum_{q\in\mathbb{C}} \sum_{l\in\mathbb{Z}} \epsilon^l \sum_{u\in V_l} \langle w', E_W^{(m+m')}\Big(v''_1, z_1; \ldots; v''_{m+m'}, z_{m+m'};$$
$$e^{\zeta_1 L_W(-1)}\ Y_W\Big(u, -\zeta_1\Big)\ P_{q+\alpha}\Big(\Phi(v''_{m+m'+1}, z_{m+m'+1}; \ldots; v''_{m+m'+k}, z_{m+m'+k})\Big)\ \Big)\rangle$$
$$\langle w', E_W^{(m+m')}\Big(v''_1, z_1; \ldots; v''_{m+m'}, z_{m+m'};$$
$$e^{\zeta_2 L_W(-1)}\ Y_W\Big(\overline{u}, -\zeta_2\Big)$$
$$P_{q+\alpha}\Big(\Psi(v''_{m+m'+k+1}, z_{m+m'+k+1}; \ldots; v''_{m+m'+k+n}, z_{m+m'+k+n})\Big)\ \Big)\rangle$$

$$= \sum_{q\in\mathbb{C}} \sum_{l\in\mathbb{Z}} \epsilon^l \sum_{u\in V_l} \langle w', E_W^{(m+m')}\Big(v''_1, z_1; \ldots; v''_{m+m'}, z_{m+m'};$$
$$Y_{WV}^W\ \Big(P_{q+\alpha}\Big(\Phi(v''_{m+m'+1}, z_{m+m'+1}; \ldots; v''_{m+m'+k}, z_{m+m'+k})\Big)\ \Big), \zeta_1\Big)\ u\rangle$$
$$\langle w', E_W^{(m+m')}\Big(v''_1, z_1; \ldots; v''_{m+m'}, z_{m+m'};$$
$$Y_{WV}^W\ \Big(P_{q+\alpha}\Big(\Psi(v''_{m+m'+k+1}, z_{m+m'+k+1}; \ldots; v''_{m+m'+k+n}, z_{m+m'+k+n}), -\zeta_2)\ \overline{u}\Big)\ \Big)\rangle$$

$$= \sum_{q\in\mathbb{C}} \sum_{\widetilde{w}\in W} \langle w', E^{(m+m')}_W \Big(v''_1, z_1; \ldots; v''_{m+m'}, z_{m+m'}; \widetilde{w} \Big) \rangle$$

$$\sum_{l\in\mathbb{Z}} \epsilon^l \sum_{u\in V_l} \langle w', Y^W_{WV} \Big(\ P_{q+\alpha} \Big(\Phi(v''_{m+m'+1}, z_{m+m'+1}; \ldots; v''_{m+m'+k}, z_{m+m'+k}), -\zeta_1) \ u \Big) \ \Big) \rangle$$

$$\langle \widetilde{w}', E^{(m+m')}_W \Big(v''_1, z_1; \ldots; v''_{m+m'}, z_{m+m'}; \widetilde{w} \Big) \rangle$$

$$\langle w', Y^W_{WV} \ \Big(P_{q+\alpha} \Big(\Psi(v''_{m+m'+k+1}, z_{m+m'+k+1}; \ldots; v''_{m+m'+k+n}, z_{m+m'+k+n}), -\zeta_2) \ \overline{u} \Big) \ \Big) \rangle$$

$$= \sum_{q\in\mathbb{C}} \langle w', E^{(m+m')}_W \Big(v''_1, z_1; \ldots; v''_{m+m'}, z_{m+m'};$$
$$P_{q+\alpha} \Big(\Theta(v''_{m+m'+1}, z_{m+m'+1}; \ldots; v''_{m+m'+k}, z_{m+m'+k};$$
$$v''_{m+m'+k+1}, z_{m+m'+k+1}; \ldots; v''_{m+m'+k+n}, z_{m+m'+k+n}) \Big) \Big) \rangle.$$

Now note that, according to Proposition 4, as an element of $\mathcal{W}_{z_1,\ldots,z_{k+n+m+m'}}$

$$\langle w', E^{(m+m')}_W \Big(v''_1, z_1; \ldots; v''_{m+m'}, z_{m+m'};$$
$$P_{q+\alpha} \Big(\Theta(v''_{m+m'+1}, z_{m+m'+1}; \ldots; v''_{m+m'+k}, z_{m+m'+k};$$
$$v''_{m+m'+k+1}, z_{m+m'+k+1}; \ldots; v''_{m+m'+k+n}, z_{m+m'+k+n}) \Big) \Big) \rangle, \quad (16.28)$$

is invariant with respect to the action of $\sigma \in S_{k+n+m+m'}$. Thus we are able to use this invariance to show that (16.28) is reduced to

$$\langle w', E^{(m+m')}_W \Big(v''_{k+1}, z_{k+1}; \ldots; v''_{k+1+m}, z_{k+1+m}; v''_{n+1}, z_{n+1}; \ldots; v''_{n+1+m'}, z_{n+1+m'};$$
$$P_{q+\alpha} \Big(\Theta(v''_1, z_1; \ldots; v''_k, z_k; v''_{k+1}, z_{k+1}; \ldots; v''_{k+n}, z_{k+n}) \Big) \Big) \rangle$$
$$= \langle w', E^{(m+m')}_W \Big(v_{k+1}, x_{k+1}; \ldots; v_{k+1+m}, x_{k+1+m}; v'_{n+1}, y_{n+1}; \ldots; v'_{n+1+m'}, y_{n+1+m'};$$
$$P_{q+\alpha} \Big(\Theta(v_1, x_1; \ldots; v_k, x_k; v'_1, y_1; \ldots; v'_n, y_n) \Big) \Big) \rangle.$$

Similarly, since

$$\langle w', E^{(m)}_W \Big(v''_1, z_1; \ldots; v''_{m+m'}, z_{m+m'};$$
$$P_q \Big(Y^W_{WV} \left(\mathcal{F}(v''_{m+m'+1}, z_{m+m'+1}; \ldots; v''_{m+m'+k}, z_{m+m'+k}), \zeta_1 \right) \ u \Big) \Big) \rangle,$$

$$\langle w', E^{(m')}_W \Big(v''_1, z_1; \ldots; v''_{m+m'}, z_{m+m'};$$
$$P_q \Big(Y^W_{WV} \left(\mathcal{F}(v''_{m+m'+k+1}, z_{m+m'+k+1}; \ldots; v''_{m+m'+k+n}, z_{m+m'+k+n}), \zeta_2 \right) \ \overline{u} \Big) \Big) \rangle.$$

correspond to elements of $\mathcal{W}_{z_1,\ldots,z_{m+m'+k}}$ and $\mathcal{W}_{z_{m+m'+k+1},\ldots,z_{m+m'+k+n}}$, we use Proposition 4 again and obtain

$$\langle w', E^{(m)}_W \Big(v_{k+1}, x_{k+1}; \ldots; v_{k+m}, x_{k+m}; P_q \Big(Y^W_{WV} \left(\mathcal{F}(v_1, x_1; \ldots; v_k, x_k), \zeta_1 \right) \ u \Big) \Big) \rangle$$

$$\langle w', E^{(m')}_W\Big(v'_{n+1}, y_{n+1}; \ldots; v'_{n+m'}, y_{n+m'}; P_q\Big(Y^W_{WV}\left(\mathcal{F}(v'_1, y_1; \ldots; v'_n, y_n), \zeta_2\right)\ \overline{u}\Big)\Big)\rangle,$$

correspondingly. Thus, the assertion of Lemma follows. ☐

Under conditions

$$z_{i''} \neq z_{j''}, \quad i'' \neq j'',$$
$$|z_{i''}| > |z_{k'''}| > 0, \tag{16.29}$$

for $i'' = 1, \ldots, m+m'$, and $k''' = m+m'+1, \ldots, m+m'+k+n$, let us introduce

$$\mathcal{J}^{k+n}_{m+m'}(\Theta) = \sum_{q\in\mathbb{C}} \langle w', E^{(m+m')}_W\Big(v''_1, z_1; \ldots; v''_{m+m'}, z_{m+m'};$$
$$P_q\Big(\Theta(v''_{m+m'+1}, z_{m+m'+1}; \ldots; v''_{m+m'+k+n}, z_{m+m'+k+n}); \epsilon\Big)\Big)\rangle. \tag{16.30}$$

Using Lemma 14 we obtain

$$|\mathcal{J}^{k+n}_{m+m'}(\Theta)|$$
$$= \Bigg|\sum_{q\in\mathbb{C}} \langle w', E^{(m+m')}_W\Big(v''_1, z_1; \ldots; v''_{m+m'}, z_{m+m'};$$
$$P_q\Big(\Theta(v''_{m+m'+1}, z_{m+m'+1}; \ldots; v''_{m+m'+k+n}, z_{m+m'+k+n}); \epsilon)\Big)\rangle\Bigg|$$

$$= \Bigg|\sum_{q\in\mathbb{C}}\sum_{l\in\mathbb{Z}} \epsilon^l \sum_{u\in V_l} \langle w', E^{(m)}_W\Big(v_{k+1}, x_{k+1}; \ldots; v_{k+m}, x_{k+m};$$
$$P_q\Big(Y^W_{WV}\left(\Phi(v_1, x_1; \ldots; v_k, x_k), \zeta_1\right)\ u\Big)\Big)\rangle$$
$$\langle w', E^{(m')}_W\Big(v'_{n+1}, y_{n+1}; \ldots; v'_{n+m'}, y_{n+m'};$$
$$P_q\Big(Y^W_{WV}\left(\Psi(v'_1, y_1; \ldots; v'_n, y_n), \zeta_2\right)\ \overline{u}\Big)\Big)\rangle\Bigg|$$
$$\leq \left|\mathcal{J}^k_m(\mathcal{F})\right|\ \left|\mathcal{J}^n_{m'}(\mathcal{F})\right|,$$

where we have used the invariance of (5.5) with respect to $\sigma \in S_{m+m'+k+n}$. According to Definitions 31 $\mathcal{J}^k_m(\Phi)$ and $\mathcal{J}^n_{m'}(\Psi)$ in the last expression are absolute convergent. Thus, we infer that $\mathcal{J}^{k+n}_{m+m'}(\Theta)$ is absolutely convergent, and the sum (16.22) is analytically extendable to a rational function in $(z_1, \ldots, z_{k+n+m+m'})$ with the only possible poles at $x_i = x_j$, $y_{i'} = y_{j'}$, and at $x_i = y_{j'}$, i.e., the only possible poles at $z_{i''} = z_{j''}$, of orders less than or equal to $N^{k+n}_{m+m'}(v''_{i''}, v''_{j''})$, for i'', $j'' = 1, \ldots, k'''$, $i'' \neq j''$. ☐

16.7. **Proof of Proposition 9.**

Proof. For a vertex operator $Y_{V,W}(v, z)$ let us introduce a notation

$$\omega_{V,W} = Y_{V,W}(v, z)\ dz^{\mathrm{wt} v}.$$

Let us use notations (3.1) and (3.6). According to (6.3), the action of $\delta^{k+n-r}_{m+m'-t}$ on $\widehat{R}\Theta(v_1, x_1; \dots; v_k, x_k; v'_1, y_1; \dots; v'_k, y_n; \epsilon)$ is given by

$$\langle w', \delta^{k+n-r}_{m+m'-t} \widehat{R}\, \Theta(v_1, x_1; \dots; v_k, x_k; v'_1, y_1; \dots; v'_n, y_n; \epsilon)\rangle$$

$$= \langle w', \sum_{i=1}^{k} (-1)^i\ \widehat{R}\ \Theta(\widetilde{v}_1, z_1; \dots; \widetilde{v}_{i-1}, z_{i-1};\ \omega_V(\widetilde{v}_i, z_i - z_{i+1})\widetilde{v}_{i+1}, z_{i+1};\ \widetilde{v}_{i+2}, z_{i+2};$$
$$\dots; \widetilde{v}_k, z_k; \widetilde{v}_{k+1}, z_{k+1}; \dots; \widetilde{v}_{k+n}, z_{k+n}; \epsilon)\rangle$$

$$+ \sum_{i=1}^{n-r} (-1)^i\ \langle w', \Theta\left(\widetilde{v}_1, z_1; \dots; \widetilde{v}_k, z_k; \widetilde{v}_{k+1}, z_{k+1}; \dots; \widetilde{v}_{k+i-1}, z_{k+i-1};\right.$$
$$\omega_V\left(\widetilde{v}_{k+i}, z_{k+i} - z_{k+i+1}\right) \widetilde{v}_{k+i+1}, z_{k+i+1};$$
$$\left.\widetilde{v}_{k+i+2}, z_{k+i+2}; \dots; \widetilde{v}_{k+n-r}, z_{k+n-r}; \epsilon\right)\rangle$$

$$+ \langle w', \omega_W\left(\widetilde{v}_1, z_1\right)\ \Theta(\widetilde{v}_2, z_2; \dots; \widetilde{v}_k, z_k; \widetilde{v}_{k+1}, z_{k+1}; \dots; \widetilde{v}_{k+n-r}, z_{k+n-r}; \epsilon)\rangle$$

$$+ \langle w, (-1)^{k+n+1-r} \omega_W(\widetilde{v}_{k+n-r+1}, z_{k+n-r+1})$$
$$\mathcal{F}(\widetilde{v}_1, z_1; \dots; \widetilde{v}_k, z_k; \widetilde{v}_{k+1}, z_{k+1}; \dots; \widetilde{v}_{k+n-r}, z_{k+n-r}; \epsilon)\rangle$$

$$= \sum_{l \in \mathbb{Z}} \epsilon^l \langle w', \sum_{i=1}^{k} (-1)^i\ Y^W_{VW}(\Theta(\widetilde{v}_1, z_1; \dots; \widetilde{v}_{i-1}, z_{i-1};\ \omega_V(\widetilde{v}_i, z_i - z_{i+1})\widetilde{v}_{i+1}, z_{i+1};$$
$$\widetilde{v}_{i+2}, z_{i+2}; \dots; \widetilde{v}_k, z_k), \zeta_1)u\rangle$$
$$\langle w', Y^W_{VW}(\Theta(\widetilde{v}_{k+1}, z_{k+1}; \dots; \widetilde{v}_{k+n-r}, z_{k+n-r}), \zeta_2)\overline{u}\rangle$$

$$+ \sum_{l \in \mathbb{Z}} \epsilon^l \sum_{i=1}^{n-r} (-1)^i\ \langle w', Y^W_{VW}(\Phi\left(\widetilde{v}_1, z_1; \dots; \widetilde{v}_k, z_k\right), \zeta_1)\, u\rangle$$
$$\langle w', Y^W_{VW}(\Psi(\widetilde{v}_{k+1}, z_{k+1}; \dots; \widetilde{v}_{k+i-1}, z_{k+i-1};$$
$$\omega_V\left(\widetilde{v}_i, z_{k+i} - z_{k+i+1}\right)\ \widetilde{v}_{k+i+1}, z_{k+i+1}; \widetilde{v}_{k+i+2}, z_{k+i+2};$$
$$\dots; \widetilde{v}_{k+n-r}, z_{k+n-r}), \zeta_2)\overline{u}\rangle$$

$$+ \sum_{l \in \mathbb{Z}} \epsilon^l \langle w', Y^W_{VW}(\omega_W\left(\widetilde{v}_1, z_1\right)\ \Phi(\widetilde{v}_2, z_2; \dots; \widetilde{v}_k, z_k), \zeta_1)u\rangle$$
$$\langle w', Y^W_{VW}(\Psi(\widetilde{v}_{k+1}, z_{k+1}; \dots; \widetilde{v}_{k+n-r}, z_{k+n-r}), \zeta_2)\overline{u}\rangle$$

$$+ \sum_{l \in \mathbb{Z}} \epsilon^l \langle w', Y^W_{VW}((-1)^{k+1} \omega_W\left(\widetilde{v}_{k+1}, z_{k+1}\right)\ \Phi(\widetilde{v}_1, z_1; \dots; \widetilde{v}_k, z_k), \zeta_1)u\rangle$$
$$\langle w', Y^W_{VW}(\Psi(\widetilde{v}_{k+2}, z_{k+2}; \dots; \widetilde{v}_{k+n-r}, z_{k+n-r}), \zeta_2)\overline{u}\rangle$$

$$
\begin{aligned}
&- \sum_{l \in \mathbb{Z}} \epsilon^l \langle w', (-1)^{k+1} \langle w', Y^W_{VW}(\omega_W\,(\widetilde{v}_{k+1}, z_{k+1})\ \ \Phi(\widetilde{v}_1, z_1; \ldots; \widetilde{v}_k, z_k), \zeta_1) u \rangle \\
&\qquad\qquad \langle w', Y^W_{VW}(\Psi(\widetilde{v}_{k+2}, z_{k+2}; \ldots; \widetilde{v}_{k+n-r}, z_{k+n-r}), \zeta_2)\overline{u}\rangle \\
&+ \sum_{l \in \mathbb{Z}} \epsilon^l \langle w', Y^W_{VW}(\Phi(\widetilde{v}_1, z_1; \ldots; \widetilde{v}_k, z_k), \zeta_1) u \rangle \\
&\quad \langle w', Y^W_{VW}(\omega_W(\widetilde{v}_{k+n-r+1}, z_{k+n-r+1}) \\
&\qquad\qquad \Psi(\widetilde{v}_{k+1}, z_{k+1}; \ldots; \widetilde{v}_{k+n-r}, z_{k+n-r}), \zeta_2)\overline{u}\rangle \\
&- \sum_{l \in \mathbb{Z}} \epsilon^l \langle w', Y^W_{VW}(\Phi(\widetilde{v}_1, z_1; \ldots; \widetilde{v}_k, z_k), \zeta_1) \rangle \\
&\quad \langle w', Y^W_{VW}(\omega_W(\widetilde{v}_{k+n-r+1}, z_{k+n-r+1}) \\
&\qquad\qquad \Psi(\widetilde{v}_{k+1}, z_{k+1}; \ldots; \widetilde{v}_{k+n-r}, z_{k+n-r}), \zeta_2) \rangle
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{l \in \mathbb{Z}} \epsilon^l \langle w',\ Y^W_{VW}(\delta^k_m \Phi(\widetilde{v}_1, z_1; \ldots; \widetilde{v}_k, z_k), \zeta_1) u \rangle \\
&\qquad\qquad\qquad \langle w', Y^W_{VW}(\Psi(\widetilde{v}_{k+1}, z_{k+1}; \ldots; \widetilde{v}_{k+n-r}, z_{k+n-r}), \zeta_2)\overline{u}\rangle \\
&\quad + (-1)^k \sum_{l \in \mathbb{Z}} \epsilon^l \langle w', Y^W_{VW}(\Phi\,(\widetilde{v}_1, z_1; \ldots; \widetilde{v}_k, z_k), \zeta_1)\, u \rangle \\
&\qquad\qquad \langle w', Y^W_{VW}(\delta^{n-r}_{m'-t} \Psi(\widetilde{v}_{k+1}, z_{k+1}; \ldots; \widetilde{v}_{k+n-r}, z_{k+n-r}), \zeta_2)\overline{u}\rangle
\end{aligned}
$$

$$
\begin{aligned}
&= \langle w', \delta^k_m \Phi(\widetilde{v}_1, z_1; \ldots; \widetilde{v}_k, z_k) \cdot \langle w', \Psi(\widetilde{v}_{k+1}, z_{k+1}; \ldots; \widetilde{v}_{k+n-r}, z_{k+n-r}) \rangle \\
&\quad + (-1)^k \langle w', \Phi\,(\widetilde{v}_1, z_1; \ldots; \widetilde{v}_k, z_k) \cdot_\epsilon \delta^{n-r}_{m'-t} \Psi(\widetilde{v}_{k+1}, z_{k+1}; \ldots; \widetilde{v}_{k+n-r}, z_{k+n-r}) \rangle,
\end{aligned}
$$

since,

$$
\begin{aligned}
&\sum_{l \in \mathbb{Z}} \epsilon^l \langle w', (-1)^{k+1} Y^W_{VW}(\omega_W\,(\widetilde{v}_{k+1}, z_{k+1})\ \ \Phi(\widetilde{v}_1, z_1; \ldots; \widetilde{v}_k, z_k), \zeta_1) u \rangle \\
&\qquad\qquad \langle w', Y^W_{VW}(\Psi(\widetilde{v}_{k+2}, z_{k+2}; \ldots; \widetilde{v}_{k+n-r}, z_{k+n-r}), \zeta_2)\overline{u}\rangle
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{l \in \mathbb{Z}} \epsilon^l \langle w', (-1)^{k+1} e^{\zeta_1 L_W(-1)} Y_W(u, -\zeta_1)\ \omega_W\,(\widetilde{v}_{k+1}, z_{k+1})\ \ \Phi(\widetilde{v}_1, z_1; \ldots; \widetilde{v}_k, z_k)) \rangle \\
&\qquad\qquad \langle w', Y^W_{VW}(\Psi(\widetilde{v}_{k+2}, z_{k+2}; \ldots; \widetilde{v}_{k+n-r}, z_{k+n-r}), \zeta_2)\overline{u}\rangle
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{l \in \mathbb{Z}} \epsilon^l \langle w', (-1)^{k+1} e^{\zeta_1 L_W(-1)} \omega_W\,(\widetilde{v}_{k+1}, z_{k+1})\, Y_W(u, -\zeta_1)\ \ \Phi(\widetilde{v}_1, z_1; \ldots; \widetilde{v}_k, z_k)) \rangle \\
&\qquad\qquad \langle w', Y^W_{VW}(\Psi(\widetilde{v}_{k+2}, z_{k+2}; \ldots; \widetilde{v}_{k+n-r}, z_{k+n-r}), \zeta_2)\overline{u}\rangle
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{v \in V} \langle w', (-1)^{k+1}\ \omega_W\,(\widetilde{v}_{k+1}, z_{k+1} + \zeta_1)\ \ e^{\zeta_1 L_W(-1)} Y_W(u, -\zeta_1)\ \ \Phi(\widetilde{v}_1, z_1; \ldots; \widetilde{v}_k, z_k)) \rangle \\
&\qquad\qquad \langle w', Y^W_{VW}(\Psi(\widetilde{v}_{k+2}, z_{k+2}; \ldots; \widetilde{v}_{k+n-r}, z_{k+n-r}), \zeta_2)\overline{u}\rangle
\end{aligned}
$$

$$= \sum_{v\in V} \sum_{u\in V_l} \sum_{l\in\mathbb{Z}} \epsilon^l \sum_{u\in V_l} \langle v', (-1)^{k+1}\ \omega_W\left(\widetilde{v}_{k+1}, z_{k+1}+\zeta_1\right) w\rangle$$
$$\langle w', e^{\zeta_1 L_W(-1)} Y_W(u, -\zeta_1)\ \ \Phi(\widetilde{v}_1, z_1; \ldots; \widetilde{v}_k, z_k)\rangle$$
$$\langle w', Y^W_{VW}(\Psi(\widetilde{v}_{k+2}, z_{k+2}; \ldots; \widetilde{v}_{k+n-r}, z_{k+n-r}), \zeta_2)\overline{u}\rangle$$
$$= \sum_{l\in\mathbb{Z}} \epsilon^l \langle w', e^{\zeta_1 L_W(-1)} Y_W(u, -\zeta_1)\ \ \Phi(\widetilde{v}_1, z_1; \ldots; \widetilde{v}_k, z_k)\rangle$$
$$\sum_{v\in V} \langle v', (-1)^{k+1}\ \omega_W\left(\widetilde{v}_{k+1}, z_{k+1}+\zeta_1\right) w\rangle$$
$$\langle w', Y^W_{VW}(\Psi(\widetilde{v}_{k+2}, z_{k+2}; \ldots;$$
$$\widetilde{v}_{k+n-r}, z_{k+n-r}), \zeta_2)\overline{u}\rangle$$
$$= \sum_{l\in\mathbb{Z}} \epsilon^l \langle w', Y^W_{VW}(\Phi(\widetilde{v}_1, z_1; \ldots; \widetilde{v}_k, z_k), \zeta_1) u\ \rangle$$
$$\langle w', (-1)^{k+1}\ \omega_W\left(\widetilde{v}_{k+1}, z_{k+1}+\zeta_1\right)$$
$$Y^W_{VW}(\Psi(\widetilde{v}_{k+2}, z_{k+2}; \ldots; \widetilde{v}_{k+n-r}, z_{k+n-r}), \zeta_2)\overline{u}\rangle$$
$$= \sum_{l\in\mathbb{Z}} \epsilon^l \langle w', Y^W_{VW}(\Phi(\widetilde{v}_1, z_1; \ldots; \widetilde{v}_k, z_k), \zeta_1) u\ \rangle$$
$$\langle w', (-1)^{k+1}\ \omega_W\left(\widetilde{v}_{k+1}, z_{k+1}+\zeta_1\right)$$
$$e^{\zeta_2 L_W(-1)} Y_W(\overline{u}, -\zeta_2)\ \Psi(\widetilde{v}_{k+2}, z_{k+2}; \ldots; \widetilde{v}_{k+n-r}, z_{k+n-r})\rangle$$
$$= \sum_{l\in\mathbb{Z}} \epsilon^l \langle w', Y^W_{VW}(\Phi(\widetilde{v}_1, z_1; \ldots; \widetilde{v}_k, z_k), \zeta_1) u\ \rangle$$
$$\langle w', (-1)^{k+1}\ \ e^{\zeta_2 L_W(-1)}\ Y_W(\overline{u}, -\zeta_2)\ \omega_W\left(\widetilde{v}_{k+1}, z_{k+1}+\zeta_1-\zeta_2\right)$$
$$\Psi(\widetilde{v}_{k+2}, z_{k+2}; \ldots; \widetilde{v}_{k+n-r}, z_{k+n-r})\rangle$$
$$= \sum_{l\in\mathbb{Z}} \epsilon^l \langle w', Y^W_{VW}(\Phi(\widetilde{v}_1, z_1; \ldots; \widetilde{v}_k, z_k), \zeta_1) u\rangle$$
$$\langle w', Y^W_{VW}(\omega_W(\widetilde{v}_{k+1}, z_{k+1})\ \Psi(\widetilde{v}_{k+2}, z_{k+2}; \ldots; \widetilde{v}_{k+n-r}, z_{k+n-r}), \zeta_2)\overline{u}\rangle,$$

due to locality (9.6) of vertex opertors, and arbitrarness of $\widetilde{v}_{k+1} \in V$ and z_{k+1}, we can always put

$$\omega_W\left(\widetilde{v}_{k+1}, z_{k+1}+\zeta_1-\zeta_2\right) = \omega_W(\widetilde{v}_{k+2}, z_{k+2}),$$

for $\widetilde{v}_{k+1} = \widetilde{v}_{k+2}$, $z_{k+2} = z_{k+1} + \zeta_2 - \zeta_1$. The statement of the proposition for δ^2_{ex} (6.7) can be checked accordingly. □

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