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CANONICAL TORSOR BUNDLE OF PRESCRIBED RATIONAL FUNCTIONS ON COMPLEX CURVES

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Abstract. Prescribed rational functions constitute a subset of rational functions satisfying certain symmetry and analyticity conditions. We define and construct explicitly prescribed rational functions-valued bundle $\mathcal{W}_M$ over a smooth complex curve M. An intrinsic coordinate-independent formulation for such bundle is is given. The construction presented in this paper is useful for studies of the canonical cosimplicial cohomology of infinite-dimensional Lie algebras on smooth manifolds, as well as for purposed of conformal field theory, deformation theory, and the theory of foliations.

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1. Introduction

For purposes of continuous cohomology theory computation on complex manifolds [Fei, Wag] and their foliations [BS, Lo], it is important to be able to define and construct objects such as bundles on auxiliary manifolds [BS, PT, Wag]. One eager to introduce such bundles so that their sections would be canonical, i.e., defined over abstract disks independent of the choice of local coorinates over marked points on a complex manifold. In order to do this, one needs a precise description of transformation properties of sections under changes of coordinates. The construction of such bundles on a smooth complex curve M grounds on the notion of prescribed rational functions. Namely, functions with specified analytic behavior, expressed through matrix elements defined for infinite-dimensional Lie algebras, and satisfying certain symmetry properties.

The purpose of this paper is to construct, in an intrinsic coordinate-independent way, a canonical fiber bundle of rational functions with defined analytical properties (described in section 3) which we call prescribed rational functions. The construction involves torsors and twists of an infinite-dimensional Lie algebra $\mathcal{G}$ by the group of automorphisms of local coordinates indepdenent transformations of non-intersecting domains of a number of points on a smooth complex curve. Prescribed rational functions are obtained via non-degenerate bilinear forms on the algebraic completion of the space of $\mathcal{G}$-valued formal series. One of the essential ideas of this paper is to use the language of prescribed rational functions from the very beginning avoiding

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non-commutative ingredients as much as possible. Thus, analytical properties and convergence of such rational functions are assumed.

The idea of studies of certain bundles cohomology on a smooth manifold M and making connection to a cohomology of M has first appeared in [BS]. Let $Vect(M)$ be the Lie algebra of vector fields on M. Bott and Segal proved that the Gelfand-Fuks cohomology $H^*(Vect(M))$ [Fuks] is isomorphic to the singular cohomology $H^*(E)$ of the space E of continuous cross sections of a certain fiber bundle $\mathcal{E}$ over M. Authors of [PT, Sm] continued to use advanced topological methods of [BS] for more general cosimplicial spaces of maps.

Let us introduce the general notations used in this paper. We denote by boldface vectors of elements, e.g., $\mathbf{a}_n = (a_1, \ldots, a_n)$, and the same for all types of objects used in the text. We also express as $(\mathbf{a}_j)_n$ the j-th component of $\mathbf{a}_n$. The prime $'$ denotes the ordinary derivative. Let M be a smooth complex curve, $\mathcal{G}$ be an infinite-dimensional Lie algebra, $G_{\mathbf{z}_n}$ be the graded (with respect to a grading operator K_G) algebraic completion of the space of formal series individually in each of $\mathbf{z}_n$-variables. We denote $\mathbf{x}_n = (\mathbf{g}_n, \mathbf{z}_n \; \mathbf{dz}_n)$ for $\mathbf{g}_n$ of the n-th power $\mathbf{G}_n = G^n$ of $\mathcal{G}$-module G, and $G^*_{\mathbf{z}_n}$ be the dual to $G_{\mathbf{z}_n}$ with respect to non-degenerate bilinear form $(.,.)$. In case when formal variables $\mathbf{z}_n$ are associated to n points $\mathbf{p}_n$ on M, we denote $G_{\mathbf{z}_n}$ by $G_{\mathbf{p}_n}$, and when $\mathbf{z}_n$ are substituted by local coordinates $\mathbf{t}_{\mathbf{p}_n}$ in vicinities of $\mathbf{p}_n$, we replace $G_{\mathbf{z}_n}$ by $G_{\mathbf{t}_{\mathbf{p}_n}}$. For fixed $\theta \in G^*_{\mathbf{p}_n}$, and varying $\mathbf{x}_n \in \mathbf{G}_{\mathbf{z}_n}$ we consider a vector $\overline{F}(\mathbf{x}_n)$ of matrix elements of the form

$$F(\mathbf{x}_n) = (\theta, f(\mathbf{x}_n)) \in \mathbb{C}((z)), \tag{1.1}$$

where $F(\mathbf{x}_n)$ depends implicitly on $\mathbf{g}_n \in \mathbf{G}_n$. We may view the vector $\overline{F}(\mathbf{x}_n)$ of prescribed rational functions as a section of a fiber bundle over a collection of non-intersecting punctured discs $\mathbf{D}^\times_{\mathbf{x}_n} = \mathrm{Spec}_{\mathbf{x}_n} \, \mathbb{C}((\mathbf{z}_j)_n)$, $1 \leqslant j \leqslant n$, with an $\mathrm{End}\,(G_{\mathbf{z}_n})$-valued fiber $f(\mathbf{x}_n) \in G_{\mathbf{z}_n}$.

In this paper we explain how to construct the bundle mentioned above in the case when the space of prescribed rational functions carries an action of the group $\mathbf{Aut}_n \; \boldsymbol{\mathcal{O}}^{(1)}_n$ of local coordinates changes in vicinities of n points on M. This means that the action of the group $\mathbf{Aut}_n \; \boldsymbol{\mathcal{O}}^{(1)}_n = \mathrm{Aut}_1 \; \mathcal{O}^{(1)}_1 \times \ldots \times \mathrm{Aut}_n \; \mathcal{O}^{(1)}_n$ comes about by exponentiation of the action of corresponding Lie algebras $\left(Der_{0,j} \; \boldsymbol{\mathcal{O}}^{(1)}_j \right)_n$, $a \leqslant j \leqslant n$, via the action on $G_{\mathbf{p}_n}$. The construction of Aut $\mathcal{O}^{(1)}$ was a part of the formal geometry developed in [GK, GKF, BR] as a method of applying representation theory of infinite-dimensional Lie algebras to finite-dimensional geometry. The action of Aut $\mathcal{O}^{(1)}$ was also considered in [H2]. A vector $\overline{F}(\mathbf{x}_n)$ of matrix elements of $F(\mathbf{x}_n)$ gives rise to a section $\overline{\mathcal{F}}(\mathbf{p}_n)$ of the intrinsic bundle $\mathcal{W}_M|_{\mathbf{D}^\times_{\mathbf{p}_n}}$ with $\mathrm{End}\,(G_{\mathbf{p}_n})$-valued fibers.

The representation in term of formal series in $\mathbf{t}_{\mathbf{p}_n}$ allows us to find the precise transformation formula for all elements of $G_{\mathbf{p}_n}$ under the action of $\mathbf{Aut}_n \; \boldsymbol{\mathcal{O}}^{(1)}_n$. We then use this formula to give an intrinsic geometric meaning to sections $\overline{\mathcal{F}}(\mathbf{p}_n)$ of the fiber bundle in coordinate-free formulation. Namely, we attach to each $\mathcal{G}$, satisfying certain properties (c.f. definition 13 of an admissible Lie algebra) a fiber bundle $\mathcal{W}_M$ on an arbitrary smooth complex curve M. Such geometric realization of $\overline{\mathcal{F}}(\mathbf{p}_n)$ allows us to provide a global geometric meaning to the space of prescribed rational functions $F(\mathbf{x}_n)$ on arbitrary curves. Finally, we prove that the bundle $\mathcal{W}_M$ we constructed is canonical, i.e., its sections do not depend on a change $\mathbf{t}_{\mathbf{p}_n} \mapsto \widetilde{\mathbf{t}}_{\mathbf{p}_n}$ of coordinates around points $\mathbf{p}_n$ on M.

2. Rational functions and differentials on abstract discs

In this section we describe (partially following [BZF]) the setup needed for formulation of further results. Let p be a point on a smooth complex curve M, and t_p be a local coordinate in a vicinity of p. We replace the field of Laurent series $\mathbb{C}((t_p))$ by any complete topological algebra non-canonically isomorphic to $\mathbb{C}((t_p))$.

2.1. Abstract discs.

We may consider the scheme underlying the $\mathbb{C}$-algebra $\mathbb{C}[[t_p]]$. Viewing $\mathbb{C}[[t_p]]$ as the ring of complex-valued functions on the affine scheme $D_{t_p} =$ Spec $\mathbb{C}[[t_p]]$, we call this scheme the standard disc D_{t_p}. As a topological space, D_p can be described by the origin (corresponding to the maximal ideal $t_p\, \mathbb{C}[[t_p]]$) and the generic point (the zero ideal). A morphism from D to an affine scheme $Z =$ Spec $\mathcal{R}$, where $\mathcal{R}$ is a $\mathbb{C}$-algebra, is a homomorphism of algebras $\mathcal{R} \to \mathbb{C}[[t_p]]$. If M is a curve, such a homomorphism may be constructed by realizing $\mathbb{C}[[t_p]]$ as a completion of $\mathcal{R}$. Geometrically, this is an identification of the disc D_p with the formal neighborhood of a point on the curve M.

Definition 1. An abstract disc is the affine scheme Spec $\mathcal{R}$, where $\mathcal{R}$ is a $\mathbb{C}$-algebra isomorphic to $\mathbb{C}[[t_p]]$. FOr the standard disc, the maximal ideal $t_p\mathbb{C}[[t_p]]$ has a preferred generator t_p.

For the abstract disk, there is no preferred generator in the maximal ideal of $\mathcal{R}$, and no preferred coordinate on an abstract disc. Denote by $\mathcal{O}_p$ the completion of the local ring of M. Then $\mathcal{O}_p$ is non-canonically isomorphic to $\mathcal{O} = \mathbb{C}[[t_p]]$. To specify such an isomorphism, or equivalently, an isomorphism between $D_p =$ Spec $\mathcal{O}_p$, and $D_{t_p} =$ Spec $\mathbb{C}[[t_p]]$, we need to choose a formal coordinate t_p at $p \in M$, i.e., a topological generator of the maximal ideal $\mathfrak{m}_p$ of $\mathcal{O}_p$. In general there is no preferred formal coordinate at $p \in M$, and D_p is an abstract disc.

2.2. Rational functions attached to discs.

Now we would like to attach rational functions to the standard $D_{t_p} =$ Spec $\mathbb{C}[[t_p]]$ and to any abstract discs D_p, where p is a point on M. We denote by $\mathcal{K}_x$ the field of fractions of the ring of integers $\mathbb{Z}$ is the rational field $\mathbb{Q}$, and the field of fractions of the polynomial ring $\mathcal{K}[\mathbf{t}_{\mathbf{p}_n}]$ over a field $\mathcal{K}$ is the field of rational functions $\mathcal{K}(\mathbf{t}_{\mathbf{p}_n}) = \big\{(R_1(\mathbf{t}_{\mathbf{p}_n}))/(R_2(\mathbf{t}_{\mathbf{p}_n})) : R_1, R_2 \in \mathcal{K}[\mathbf{t}_{\mathbf{p}_n}], g \neq 0\big\}$. The field of fractions of an integral domain $\mathcal{D}$ is the smallest field containing $\mathcal{D}$, since it is obtained from $\mathcal{D}$ by adding the least needed to make $\mathcal{D}$ a field, namely the possibility of dividing by any non-zero element. If we choose a coordinate t_p on D_p, then we obtain isomorphisms $\mathcal{O}_p = \mathbb{C}[[t_p]]$ and $\mathcal{K}_p = \mathbb{C}((t_p))$. We denote by D_p (resp., $D_p^\times$) the disc (resp., the punctured disc) at p, defined as Spec $\mathcal{O}_p^{(1)}$ (resp., Spec $\mathcal{K}_p$).

2.3. **Differentials.** In this subsection we recall basic definitions concerning differentials [Sch].

Definition 2. Let k be a rational number. A k-differential on a smooth curve is by definition a section of the k-th tensor power of the canonical line bundle Ω.

Choosing a local coordinate t_p we may trivialize $\Omega^{\otimes k}$ by the non-vanishing section $(dt_p)^{\otimes k}$. Any section of $\Omega^{\otimes k}$ may then be written as $f(t_p)(dt_p)^{\otimes k}$. If we choose another coordinate $\widetilde{t}_p = \rho(t_p)$, then the same section will be written as $g(\widetilde{t}_p)(d\widetilde{t}_p)^{\otimes k}$, where

$$f(t_p) = g(\rho(t_p))(\rho'(t_p))^{\otimes k}.$$

Now let us suppose that we have a section of $\Omega^{\otimes k}$ whose representation by a function does not depend on the choice of local coordinate, i.e., $g(\widetilde{t}_p) = f(\widetilde{t}_p)$, and $f(t_p) = f(\rho(t_p))(\rho'(t_p))^{\otimes k}$ for any change of variable $\rho(t_p)$. When we consider sections of $\Omega^{\otimes k}$ with values in a vector space that itself transforms non-trivially under changes of coordinates, canonical sections may exist.

Definition 3. We call $f(t_p)(dt_p)^{\otimes k}$ the canonical k-differential.

Let us denote by Ω_p the space of differentials on $D_p^\times$. In [BZF] we find the following lemma which we apply to G_{t_p}:

Lemma 1. *Given a linear map $\rho : \mathcal{K}_p \to \mathrm{End}\left(G_{t_p}\right)$, such that for any $\alpha \in G_{t_p}$ we have $\rho(\mathfrak{m}_p)^l \cdot \alpha = 0$, for large enough l, where $\mathfrak{m}_p$ is the maximal ideal of $\mathcal{O}_p$ at p. Then $\omega_p = \sum_{n\in\mathbb{Z}} \rho(t_p^n)\ t_p^{-n-1}\ dt_p$, is a canonical $\mathrm{End}\left(G_{t_p}\right)$-valued differential on $D_{t_p}^\times$ i.e., it is independent of the choice of coordinate t_p.*

3. Rational functions with prescribed analytic behavior

In this section the space of prescribed rational functions is defined as rational functions with certain analytical and symmetric properties [H2, H1]. Such rational functions depend implicitly on an infinite number of non-commutative parameters.

3.1. Rational functions originating from matrix elements.

Definition 4. Let M be a complex manifold. Denote by $\mathbf{p}_n$ be a set of n points on M. We denote by $\mathcal{U}_n$ a set of domains such that $\mathbf{p}_n \in \mathcal{U}_n$. Let $\mathbf{z}_n$ be n complex coordinates in $\mathcal{U}_n$ around origines $\mathbf{p}_n$. In this paper we consider meromorphic functions of several complex variables defined on sets of open domains of M with local coordinates $\mathbf{z}_n$ which are extendable to rational functions on larger domains on M. We denote such extensions by $R(f(\mathbf{z}_n))$.

Definition 5. Denote by $F_n\mathbb{C}$ the configuration space of $n \geqslant 1$ ordered coordinates in $\mathbb{C}^n$, $F_n\mathbb{C} = \{\mathbf{z}_n \in \mathbb{C}^n \mid z_i \neq z_j, i \neq j\}$.

In order to work with objects having coordinate invariant formulation, for a set of $\mathbf{G}_n$-elements $\mathbf{g}_n$ we consider converging rational functions $f(\mathbf{x}_n) \in G_{\mathbf{z}_n}$ of $\mathbf{z}_n \in F_n\mathbb{C}$.

Definition 6. For an arbitrary fixed $\theta \in G^*_{\mathbf{p}_n}$, we call a map linear in $\mathbf{g}_n$ and $\mathbf{z}_n$,

$$F : \mathbf{x}_n \mapsto R(\theta, f(\mathbf{x}_n)), \tag{3.1}$$

a rational function in $\mathbf{z}_n$ with the only possible poles at $z_i = z_j$, $i \neq j$. Abusing notations, we denote

$$F(\mathbf{x}_n) = R\left(\theta, f(\mathbf{x}_n)\right).$$

Definition 7. We define left action of the permutation group S_n on $F(\mathbf{z}_n)$ by

$$\sigma(F)(\mathbf{x}_n) = F\left(\mathbf{g}_n, \mathbf{z}_{\sigma(i)}\ \mathbf{dz}_{\sigma(i)}\right).$$

3.2. **Conditions on rational functions.** Let $\mathbf{z}_n \in F_n\mathbb{C}$. Denote by T_G the translation operator [H2]. We define now extra conditions on rational functions leading to the definition of restricted rational functions.

Definition 8. Denote by $(T_G)_i$ the operator acting on the i-th entry. We then define the action of partial derivatives on an element $F(\mathbf{x}_n)$

$$\begin{aligned} \partial_{z_i} F(\mathbf{x}_n) &= F((T_G)_i\ \mathbf{x}_n), \\ \sum_{i \geqslant 1} \partial_{z_i} F(\mathbf{x}_n) &= T_G F(\mathbf{x}_n), \end{aligned} \tag{3.2}$$

and call it T_G-derivative property.

Definition 9. For $z \in \mathbb{C}$, let

$$e^{zT_G} F(\mathbf{x}_n) = F(\mathbf{g}_n, (\mathbf{z}_n + z)\ \mathbf{dz}_n). \tag{3.3}$$

Let $\mathrm{Ins}_i(A)$ denotes the operator of multiplication by $A \in \mathbb{C}$ at the i-th position. Then we define

$$F\left(\mathbf{g}_n, \mathrm{Ins}_i(z)\ \mathbf{z}_n\ \mathbf{dz}_n\right) = F\left(\mathrm{Ins}_i(e^{zT_G})\ \mathbf{x}_n\right), \tag{3.4}$$

are equel as power series expansions in z, in particular, absolutely convergent on the open disk $|z| < \min_{i \neq j}\{|z_i - z_j|\}$.

Definition 10. A rational function has K_G-property if for $z \in \mathbb{C}^\times$ satisfies $(z\ \mathbf{z}_n) \in F_n\mathbb{C}$,

$$z^{K_G} F(\mathbf{x}_n) = F\left(z^{K_G}\mathbf{g}_n, z\ \mathbf{z}_n\ \mathbf{dz}_n\right). \tag{3.5}$$

3.3. **Rational functions with prescribed analytical behavior.** In this subsection we give the definition of rational functions with prescribed analytical behavior on a domain of complex manifold M of dimension n. We denote by $P_k : G \to G_{(k)}$, $k \in \mathbb{C}$, the projection of G on $G_{(k)}$. For each element $g_i \in G$, and $x_i = (g_i, z)$, $z \in \mathbb{C}$ let us associate a formal series $W_{g_i}(z) = W(x_i) = \sum_{k \in \mathbb{C}} g_i\ z^k\ dz$, $i \in \mathbb{Z}$. Following [H1], we formulate

Definition 11. We assume that there exist positive integers $\beta(g_{l',i}, g_{l",j})$ depending only on $g_{l',i}$, $g_{l'',j} \in G$ for i, $j = 1, \ldots, (l+k)n$, $k \geqslant 0$, $i \neq j$, $1 \leqslant l', l'' \leqslant n$. Let $\mathbf{l}_n$ be a partition of $(l+k)n = \sum_{i \geqslant 1} l_i$, and $k_i = l_1 + \cdots + l_{i-1}$. For $\zeta_i \in \mathbb{C}$, define $h_i = F(\mathbf{W}_{\mathbf{g}_{k_i+1_i}}(\mathbf{z}_{k_i+1_i} - \zeta_i))$, for $i = 1, \ldots, n$. We then call a rational function F satisfying properties (3.2)–(3.5), a rational function with prescribed analytical behavior, if under the following conditions on domains, $|z_{k_i+p}-\zeta_i|+|z_{k_j+q}-\zeta_j|<|\zeta_i-\zeta_j|$, for i, $j=1,\ldots,k$, $i\neq j$, and for $p=1,\ldots,l_i$, $q=1,\ldots,l_j$, the function $\sum\limits_{\mathbf{r}_n\in\mathbb{Z}^n} F(\mathbf{P}_{\mathbf{r_i}}\mathbf{h_i};(\zeta)_l)$, is absolutely convergent to an analytically extension in $\mathbf{z}_{l+k}$, independently of complex parameters $(\zeta)_l$, with the only possible poles on the diagonal of $\mathbf{z}_{l+k}$ of order less than or equal to $\beta(g_{l',i},g_{l'',j})$. In addition to that, for $\mathbf{g}_{l+k}\in G$, the series $\sum_{q\in\mathbb{C}} F(\mathbf{W}(\mathbf{g_k}, \mathbf{P}_q(\mathbf{W}(\mathbf{g}_{l+k},\mathbf{z}_k),\mathbf{z}_{k+\mathbf{l}}))$, is absolutely convergent when $z_i\neq z_j$, $i\neq j$ $|z_i|>|z_s|>0$, for $i=1,\ldots,k$ and $s=k+1,\ldots,l+k$ and the sum can be analytically extended to a rational function in $\mathbf{z}_{l+k}$ with the only possible poles at $z_i=z_j$ of orders less than or equal to $\beta(g_{l',i},g_{l'',j})$.

For $m\in\mathbb{N}$ and $1\leqslant p\leqslant m-1$, let $J_{m;p}$ be the set of elements of S_m which preserve the order of the first p numbers and the order of the last $m-p$ numbers, that is,

$$J_{m,p}=\{\sigma\in S_m \mid \sigma(1)<\cdots<\sigma(p),\ \sigma(p+1)<\cdots<\sigma(m)\}.$$

Let $J_{m;p}^{-1}=\{\sigma \mid \sigma\in J_{m;p}\}$. In addition to that, for some rational functions require the property:

$$\sum_{\sigma\in J_{n;p}^{-1}}(-1)^{|\sigma|}\sigma(F(\mathbf{g}_{\sigma(i)},\mathbf{z}_n))=0. \tag{3.6}$$

Let us also introduce the following vector containing rational functions with properties described above.

$$\overline{F}(\mathbf{x}_n)=\left[F\left(\mathbf{g}_n,\mathbf{z}_n\ \mathbf{dz}_{i(n)}\right)\right]. \tag{3.7}$$

where $i(j)$, $j=1,\ldots,n$, are cycling permutations of $(1,\ldots,n)$ starting with j.

Finally, we formulate

Definition 12. We define the space $\Theta\left(n,k,G_{\mathbf{z}_n},U\right)$ of prescribed rational functions as a space of vectors $\overline{F}_n(\mathbf{x}_n)$ of the form (3.7) of complex n-variable restricted rational functions with prescribed analytical behavior on a $F_n\mathbb{C}$-domain $U\subset M$, and satisfying T_G- and K_G-properties (3.2)–(3.5), definition (11), and (3.6).

4. The bundle $\mathcal{W}_M$ of prescribed rational functions

In this section we provide the construction of prescribed rational function bundle on M.

4.1. **Admissible Lie algebras.** Let us assume that for an infinite-dimensional Lie algebra $\mathcal{G}$, the grading of G_{t_p} is bounded from below by some subspace index $\kappa=\in\mathbb{C}$, i.e., $G_{t_p}=\oplus_{i\geqslant \mathrm{Re}(\kappa)}G_{t_p,i}$, with finite dimensional $\dim G_{t_p,i}<\infty$ grading subspaces $G_{t_p,i}$. Then we have a filtration $G_{t_{p_j},\leqslant m}=\bigoplus\limits_{i\geqslant\mathrm{Re}(\kappa)}^{m} G_{t_{p_j},i}$, of $G_{t_{p_j}}$ by finite-dimensional $\mathrm{Aut}_{p_j}\ \mathcal{O}_j^{(1)}$-submodules, $1\leqslant j\leqslant n$. In addition to that, we assume that: $\mathcal{G}$ carries an action of $\mathbf{Der}_n\ \boldsymbol{\mathcal{O}}_n^{(1)}$; the element $\left(-\partial_{t_p}\right)$ acts as the translation operator on G_{t_p} semi-simply with integral eigenvalues; the Lie subalgebra $\left(\mathbf{Der}_+\right)_n\ \boldsymbol{\mathcal{O}}_n^{(1)}$ acts locally nilpotently; and the operator $\left(-t_p\ \partial_{t_p}\right)$ provides a $\mathbb{C}$-grading. Finally, we assume also that the action of the Lie algebras $\mathbf{Der}_n\ \boldsymbol{\mathcal{O}}_n^{(1)}$ on $G_{t_{\mathbf{p}_n}}$ can be exponentiated to an action of the group $\mathbf{Aut}_n\ \boldsymbol{\mathcal{O}}_n^{(1)}$.

Definition 13. We call $\mathcal{G}$ subject to the assumptions of this subsection an admissible Lie algebra.

4.2. **Torsors and twists under groups of automorphisms.** We now explain how to collect elements of the space $\Theta\left(n, k, G_{\mathbf{z}_n}, U\right)$ of prescribed rational functions into an intrinsic object on a collection of abstract discs. We consider a configuration of n-points $\mathbf{p}_n$ on a complex curve M lying in non-intersecting local disks, and we assume that at each point of $\mathbf{p}_n$ a coordinate changes independently of changing of coordinates on other disks. Therefore, the general element of the group of independent automorphisms of coordinates of n points on M $\mathbf{Aut}_n\ \boldsymbol{\mathcal{O}}^{(1)}_{\mathbf{p}_n}$ has the form $\mathbf{t}_{\mathbf{p}_n} \mapsto \left(\boldsymbol{\rho}_j(\mathbf{t}_{\mathbf{p}_j})\right)_n$, $1 \leqslant j \leqslant n$. We recall here the notion of a torsor we use to prove the independence of the choice of coordinates for prescribed rational functions.

Definition 14. Let $\mathfrak{G}$ be a group, and X a non-empty set. Then X is called a $\mathfrak{G}$-torsor if it is equipped with a simply transitive right action of $\mathfrak{G}$, i.e., given ξ, $\widetilde{\xi} \in X$, there exists a unique $h \in \mathfrak{G}$ such that $\xi \cdot h = \widetilde{\xi}$, where for h, $\widetilde{h} \in \mathfrak{G}$ the right action is given by $\xi \cdot (h \cdot \widetilde{h}) = (\xi \cdot h) \cdot \widetilde{h}$. The choice of any $\xi \in X$ allows us to identify X with $\mathfrak{G}$ by sending $\xi \cdot h$ to h.

Applying the definition of a group twist to the group Aut $\mathcal{O}^{(1)}$ and its module G_z we obtain

Definition 15. Given a Aut $\mathcal{O}^{(1)}$-module G_z and a Aut $\mathcal{O}^{(1)}$-torsor X, one defines the X-twist of G_z as the set

$$\mathcal{V}_X = G_z \underset{\text{Aut}\ \mathcal{O}^{(1)}}{\times} X = G_z \times X / \left\{(g, a \cdot \xi) \sim (ag, \xi)\right\}.$$

for $\xi \in X$, $a \in \text{Aut}\ \mathcal{O}^{(1)}$, and $g \in G_z$.

Given $\xi \in X$, we may identify G_z with $\mathcal{V}_X$, by $g \mapsto (\xi, g)$. This identification depends on the choice of ξ. Since Aut $\mathcal{O}^{(1)}$ acts on G_z by linear operators, the vector space structure induced by the above identification does not depend on the choice of ξ, and $\mathcal{V}_X$ is canonically a vector space. If one thinks of X as a principal Aut $\mathcal{O}^{(1)}$-bundle over a point, then $\mathcal{V}_X$ is simply the associated vector bundle corresponding to G_z. Any structure on G_z (such as a bilinear form or multiplicative structure) that is preserved by Aut $\mathcal{O}^{(1)}$ will be inherited by $\mathcal{V}_X$.

Now we wish to attach to any disc a certain twist $\mathcal{V}_{t_p}$ of G_{t_p}, so that G_{t_p} is attached to the standard disc, and for any coordinate t_p on D_p we have an isomorphism

$$i_{t_p,p} : G_{t_p} \xrightarrow{\sim} \mathcal{V}_{t_p}. \tag{4.1}$$

We then associate to elements of G_{t_p} sections of some bundles on $D^\times_{t_p}$. The system of isomorphisms $i_{t_p,p}$ should satisfy certain compatibility condition. Namely, if t_p and $\widetilde{t}_p$ are two coordinates on D_p such that $\widetilde{t}_p = \rho(t_p)$, then we obtain an automorphism $(i^{-1}_{\widetilde{t}_p,p} \circ i_{t_p,p})$ of G_{t_p}. The condition is that the assignment $\rho(z) \mapsto i^{-1}_{\widetilde{t}_p,p} \circ i_{t_p,p}$, defines a representation on G_{t_p} of the group Aut $\mathcal{O}^{(1)}$ of changes of coordinates. If this condition is satisfied, then $\mathcal{V}_{t_p}$ is canonically identified with the twist of G_{t_p} by the Aut $\mathcal{O}^{(1)}$-torsor of formal coordinates at p.

In the next subsection we will show, given the space $\Theta\left(n,k,G_{\mathbf{z}_n},U\right)$ of prescribed rational functions associated to an admissible Lie algebra, one can attach to it a vector bundle $\mathcal{W}_M$ on any smooth curve M. I.e., the elements of $\Theta\left(n,k,G_{\mathbf{z}_n},U\right)$ give rise to a collection of coordinate-independent sections $\mathcal{F}(\mathbf{p}_n)$ of the dual bundle $\mathcal{W}^*_M$ in the neighborhoods of a collection of points $\mathbf{p}_n \in M$.

The construction is based on the principal bundle for the group $\mathbf{Aut}_n\ \mathcal{O}^{(1)}_n$, which naturally exists on an arbitrary smooth curve and on any collection $\mathbf{D}_{\mathbf{p}_n}$ of non-intersecting discs. We denote by $Aut_{\mathbf{p}_n}$ the set of all coordinates $\mathbf{t}_{\mathbf{p}_n}$ on disks $\mathbf{D}_{\mathbf{p}_n}$, centered at points $\mathbf{p}_n$. It comes equipped with a natural right action of the group of automorphisms $\mathbf{Aut}_n\ \mathcal{O}^{(1)}_n$. If $t_{p_i} \in Aut_{p_i}$, and $\rho(z_i) \in \mathrm{Aut}_i\ \mathcal{O}^{(1)}_i$, then $\rho_i(t_{p_i}) \in Aut_{p_i}$. Furthermore, as it was shown in [BZF] that $(\rho_i * \mu_i)(t_{p_i}) = \mu_i(\rho_i(t_{p_i}))$, for $1 \leqslant i \leqslant n$, it defines a right simply transitive action of $\mathrm{Aut}_i\ \mathcal{O}^{(1)}_i$ on Aut_{p_i}. Next we ave

Lemma 2. *The group $\mathbf{Aut}_n\ \mathcal{O}^{(1)}_n$ acts naturally on $Aut_{\mathbf{p}_n}$, and is a $\mathbf{Aut}_n\ \mathcal{O}^{(1)}_n$-torsor.*

Thus, we can define the following twist.

Definition 16. We can introduce the $\mathbf{Aut}_n\ \mathcal{O}^{(1)}_n$-twist of $G_{\mathbf{p}_n}$

$$\mathcal{V}_{\mathbf{p}_n} = G_{\mathbf{p}_n} \underset{\mathbf{Aut}_n\ \mathcal{O}^{(1)}_n}{\times} Aut_{\mathbf{p}_n}.$$

The original definition similar to (16) was given in [BD, Wi]. For each set of formal coordinates $\mathbf{t}_{\mathbf{p}_n}$ at $\mathbf{p}_n$, and $\mathbf{g}_n \in \mathbf{G}_n$, any element of the twist $\mathcal{V}_{\mathbf{p}_n}$ may be written uniquely as a pair $(\mathbf{g}_n, \mathbf{t}_{\mathbf{p}_n})$.

4.3. **Definition of prescribed rational functions bundle.** Now let us formulate the definition of fiber bundle associated through vectors of elements $F(\mathbf{x}_n)$ with $\mathbf{x}_n = (\mathbf{g}_n, \mathbf{t}_{\mathbf{p}_n})$ to the space $\Theta\left(n,k,G_{\mathbf{t}_{\mathbf{p}_n}}\mathbf{D}_{\mathbf{t}_{\mathbf{p}_n}}\right)$ of prescribed rational functions on any set of standard disks $\mathbf{D}_{\mathbf{t}_{\mathbf{p}_n}}$ around points $\mathbf{p}_n$ with local coordinates $\mathbf{t}_{\mathbf{p}_n}$. For that purpose we involve the notion of a principal bundle for the group $\mathbf{Aut}_{\mathbf{p}_n}\ \mathcal{O}^{(1)}_n$ which naturally exists on an arbitrary smooth curve and on any disc.

For the fiber space provided by vectors of elements $f(\mathbf{x}_n) \in G_{\mathbf{t}_{\mathbf{p}_n}}$, using the property of prescribed rational functions we form a principal $\mathbf{Aut}_n\ \mathcal{O}_n$-bundle, which is a fiber bundle $\mathcal{W}_M|_{\mathbf{D}_{\mathbf{t}_{\mathbf{p}_n}}}$ defined by trivializations

$$i_{\mathbf{t}_{\mathbf{p}_n}} : \overline{F}(\mathbf{x}_n) = [(\theta, f(\mathbf{x}_n))] \to \mathbf{D}_{\mathbf{t}_{\mathbf{p}_n}},$$

together with a continuous right action

$$\overline{F}(\mathbf{x}_n) \times \mathbf{Aut}_n\ \mathcal{O}^{(1)}_n \to \overline{F}(\mathbf{x}_n),$$

such that $\mathbf{Aut}_n\ \mathcal{O}^{(1)}_n$ preserves $\overline{F}(\mathbf{x}_n)$, i.e., ζ, $\zeta.a$ are sections of $\mathcal{W}_M|_{\mathbf{D}_{\mathbf{t}_{\mathbf{p}_n}}}$ for all $a \in \mathbf{Aut}_n\ \mathcal{O}^{(1)}_n$, and acts freely and transitively, i.e., the map $a \mapsto \zeta.a$ is a homeomorphism. Thus, we have [BZF]

Lemma 3. *For $1 \leqslant i \leqslant n$, the projection $Aut_{\mathbf{p}_n} \to M$ is a principal $\mathbf{Aut}_{\mathbf{p}_n}\ \mathcal{O}_n$-bundle. The fiber of this bundle at points $\mathbf{p}_n$ is the $\mathbf{Aut}_n\ \mathcal{O}^{(1)}_n$-torsor $Aut_{\mathbf{p}_n}$.*

As we observed above, we have a representation of the product of the group of autormorphisms $\mathbf{Aut}_n\, \mathcal{O}_n^{(1)}$ on $G_{\mathbf{t}_{\mathbf{p}_n}}$. Then we obtain

Definition 17. Let $(Aut_M)_n$ be the set of n-tuples of local coordinates $Aut_{\mathbf{p}_n}$ all over the complex curve M. Given a finite-dimensional $\mathbf{Aut}_n\, \mathcal{O}_n^{(1)}$-module $G_{i,\mathbf{p}_n}$, let

$$\mathcal{W}_M|_{\mathbf{D}_{\mathbf{t}_{\mathbf{p}_n}}} = G_{i,\mathbf{t}_{\mathbf{p}_n}} \underset{\mathbf{Aut}_n\, \mathcal{O}_n^{(1)}}{\times} (Aut_M)_n,$$

be the fiber bundle associated to $G_{i,\mathbf{t}_{\mathbf{p}_n}}$ and $(Aut_M)_n$. Thus, $\mathcal{W}_M|_{\mathbf{D}_{\mathbf{t}_{\mathbf{p}_n}}}$ is a bundle of finite rank over $M|_{\mathbf{D}_{\mathbf{t}_{\mathbf{p}_n}}}$ whose fiber at a collection of points $\mathbf{p}_n \in M$ is the vector $[f(\mathbf{x}_n)]$.

In a vicinity of every point of $\mathbf{p}_n$ on M we can choose disks $\mathbf{D}_{\mathbf{p}_n}$ such that the bundle $\mathcal{W}_M$ over $\mathbf{D}_{\mathbf{p}_n}$ is $\mathbf{D}_{\mathbf{p}_n} \times \mathcal{F}(\mathbf{x}_n)$, where $\mathcal{F}(\mathbf{x}_n)$ is a section of $\mathcal{W}_M$. The fiber bundle $\mathcal{W}_M$ with fiber $[f(\mathbf{p}_n)]$ is a map $\mathcal{W}_M : \mathbb{C}^n \to M$ where $\mathbb{C}^n$ is the total space of $\mathcal{W}_M$ and M is its base space. For every set of points $\mathbf{p}_n \in M$ with local disks $\mathbf{D}_{\mathbf{t}_{\mathbf{p}_n}}$ $i^{-1}_{\mathbf{t}_{\mathbf{p}_n}}$ is homeomorphic to $\mathbf{D}_{\mathbf{t}_{\mathbf{p}_n}} \times \mathbb{C}^n$. Namely, we have for $[f(\mathbf{p}_n)] : i^{-1}_{\mathbf{t}_{\mathbf{p}_n}} \to \mathbf{D}_{\mathbf{t}_{\mathbf{p}_n}} \times \mathbb{C}^n$, that $\mathcal{P} \circ [f(\mathbf{p}_n)] = i_{\mathbf{t}_{\mathbf{p}_n}}|_{i^{-1}_{\mathbf{t}_{\mathbf{p}_n}}} \left(\mathbf{D}_{\mathbf{t}_{\mathbf{p}_n}}\right))$, where $\mathcal{P}$ is the projection map on $\mathbf{D}_{\mathbf{t}_{\mathbf{p}_n}}$. Now we are able to formulate the definition of a prescribed rational functions bundle over a complex curve M.

Definition 18. For an $\mathbf{Aut}_n\, \mathcal{O}_n^{(1)}$-module $G_{\mathbf{t}_{\mathbf{p}_n}}$ which has a filtration by finite-dimensional submodules $G_{\mathbf{t}_{\mathbf{p}_n},i}$, $i \geqslant 0$, we consider the directed inductive limit $\mathcal{W}_M$ of a system of finite rank bundles $\mathcal{W}^i_M$ on M defined by embeddings $\mathcal{W}^i_M \to \mathcal{W}^j_M$, for $i \leqslant j$, i.e., $\mathcal{W}_M$ it as a fiber bundle of infinite rank over M. Similarly, the dual bundle $\mathcal{W}^*_M$ is inverse system of bundles $(\mathcal{W}^i_M)^*$, $i \geqslant 0$, and surjections $(\mathcal{W}^j_M)^* \to (\mathcal{W}^i_M)^*$, for $i \leqslant j$, a projective limit of bundles of finite rank.

In the next subsection we will identify sections $\mathcal{F}(\mathbf{p}_n)$ of $\mathcal{W}_M|_{\mathbf{D}^\times_{\mathbf{p}_n}}$ with an operation which takes vector of elements of $F(\mathbf{x}_n) \in \Theta\left(n, k, G_{\mathbf{x}_n}, D^\times_{\mathbf{t}_{\mathbf{p}_n}}\right)$, and assigns to it a dual element.

4.4. **Explicit construction of canonical intrinsic setup for $\mathcal{W}_M$.** In order to be able to define a section $\overline{\mathcal{F}}(\mathbf{p}_n)$ defined on abstract disks $D^\times_{\mathbf{p}_n}$ in the coordinate independent description of the bundle $\mathcal{W}_M$, we have to associate in some way $\overline{\mathcal{F}}(\mathbf{p}_n)$ to $\overline{F}(\mathbf{x}_n)$. Now let us give

Definition 19. For each triple $\mathbf{p}_n$, $\mathbf{g}_n \in \mathbf{G}_n$, and fixed $\theta \in G^*_{\mathbf{p}_n}$, we define a intrinsic $\mathbb{C}^n$-valued meromorphic section $\overline{\mathcal{F}}(\mathbf{p}_n)$ of the bundle $\mathcal{W}^*_{\mathbf{D}_{\mathbf{p}_n}}$ on the punctured discs $\mathbf{D}^\times_{\mathbf{p_n}}$ by an operation

$$(\theta,\ \mathbf{g}_n,\ \mathbf{p}_n) \mapsto \left(\theta, \mathcal{F}_{i_{\mathbf{p}_n}}\right), \tag{4.2}$$

assigning to a vector $\overline{\mathcal{F}}(\mathbf{p}_n)$ of $\mathcal{W}_{\mathbf{D}_{\mathbf{p}_n}}$ an element of $\mathcal{K}_{\mathbf{p}_n}$ (i.e., functions on $\mathbf{D}^\times_{\mathbf{p}_n}$), defined by the $\mathcal{W}^*_{\mathbf{D}_{\mathbf{p}_n}}$-fiber $\mathcal{F}_{i_{t_{\mathbf{p}_n}}} \in G_{\mathbf{p}_n}$.

We now formulate the main statement of this paper which is contained in the following proposition for prescribed rational functions bundle $\mathcal{W}_M$.

Proposition 1. *A $\mathbb{C}^n$-valued canonical (i.e., independent of the choice of coordinates $\mathbf{t}_{\mathbf{p}_n}$ on $\mathbf{D}^\times_{\mathbf{p}_n}$) section $\overline{\mathcal{F}}(\mathbf{p}_n)$ of the bundle $\mathcal{W}^*_M|_{\mathbf{D}^\times_{\mathbf{p}_n}}$ on of the $G_{\mathbf{p}_n}$-valued fibers $\mathcal{F}_{i_{\mathbf{t}_{\mathbf{p}_n}}}$ defined by (4.1) on $\mathbf{D}^\times_{\mathbf{p}_n}$ dual to $\mathcal{W}_M|_{D^\times_{\mathbf{p}_n}}$ is given by the formula*

$$\overline{\mathcal{F}}(\mathbf{p}_n) = \left[\left(\mathbf{y}_n, \mathcal{F}_{i_{\mathbf{t}_{\mathbf{p}_n}}}(\mathbf{h}_n)\right)\right] = [(\theta, f(\mathbf{x}_n))] = \overline{F}(\mathbf{x}_n), \tag{4.3}$$

for $\mathbf{y}_n = (\theta, \mathbf{t}_{\mathbf{p}_n})$, $f(\mathbf{x}_n) \in G_{\mathbf{t}_{\mathbf{p}_n}}$, where $\mathbf{t}_{\mathbf{p}_n}$ are coordinates on $\mathbf{D}^\times_{\mathbf{p}_n}$, and $\mathbf{h}_n \in \mathbf{G}_n$.

Proof. Now let us proceed with the explicit construction of $\mathcal{F}_{i_{\mathbf{p}_n}}$. By choosing coordinates $\mathbf{t}_{\mathbf{p}_n}$ on a collection of discs $\mathbf{D}^\times_{\mathbf{p}_n}$, we obtain a trivialization

$$i_{\mathbf{t}_{\mathbf{p}_n}} : \overline{F}\left(G[[\mathbf{t}_{\mathbf{p}_n}]]\right) \overset{\sim}{\to} \Gamma\left(\mathcal{W}_M|_{\mathbf{D}^\times_{\mathbf{t}_{\mathbf{p}_n}}}\right),$$

of the bundle $\mathcal{W}_{\mathbf{D}^\times_{\mathbf{p}_n}}$ which we call the $\mathbf{t}_{\mathbf{p}_n}$-trivialization. We also obtain trivializations of the fiber $G_{\mathbf{p}_n} \overset{\sim}{\to} \gamma\left(\mathcal{W}_M|_{\mathbf{D}^\times_{\mathbf{t}_{p_n}}}\right)$, and its dual $G^*_{\mathbf{p}_n} \overset{\sim}{\to} \gamma\left(\mathcal{W}^*_M|_{\mathbf{D}^\times_{\mathbf{t}_{p_n}}}\right)$. Let us denote by $(\mathbf{g}_n, \mathbf{t}_{\mathbf{p}_n})$ the image of $\mathbf{g}_n \in \mathbf{G}_n$ in $\mathcal{W}_M|_{\mathbf{D}^\times_{\mathbf{t}_{p_n}}}$ and by $(\theta, \mathbf{t}_{\mathbf{p}_n})$ the image of $\theta \in G^*_{\mathbf{p}_n}$ in $\mathcal{W}^*_M|_{\mathbf{D}^\times_{\mathbf{t}_{p_n}}}$ under $\mathbf{t}_{\mathbf{p}_n}$-trivialization. In order to define the required section $\overline{\mathcal{F}}(\mathbf{p}_n)$ with respect to these trivializations through its matrix elements we need to attach an element of $(\mathbb{C}(\mathbf{t}_{p_n}))^n$ to each triple $(\mathbf{g}_n, \mathbf{t}_{\mathbf{p}_n}) \in \mathcal{W}_M|_{\mathbf{D}^\times_{\mathbf{t}_{p_n}}}$, $(\theta, \mathbf{t}_{\mathbf{p}_n}) \in \mathcal{W}^*_M|_{\mathbf{D}^\times_{\mathbf{t}_{p_n}}}$, and a section $i_{\mathbf{t}_{\mathbf{p}_n}}(\mathbf{h}_n)$ of $\mathcal{W}|_{\mathbf{D}^\times_{\mathbf{t}_{\mathbf{p}_n}}}$ for $\mathbf{h}_n \in F\left(G_{\mathbf{t}_{\mathbf{p}_n}}\right)$. The operation we define above is $\mathbb{C}^n$-linear in $\mathbf{g_n}$, and $\theta \in G^*_{\mathbf{p}_n}$ and $\mathbb{C}[[(\mathbf{t}_{p_i})_n]]$-linear in $F(\mathbf{x}_n)$. It is sufficient to assign a function to the triples $\mathbf{x}_n$, $\theta \in G_{\mathbf{p}_n}$, $\mathbf{h}_n \in \mathbf{G}_n$ in the $\mathbf{t}_{\mathbf{p}_n}$-trivialization. Thus, we identify a $\mathbb{C}^n$-valued section $\overline{\overline{\mathcal{F}}}(\mathbf{p}_n)$ of $\mathcal{W}^*_{\mathbf{D}^\times_{\mathbf{p}_n}}$, with the section $\overline{F}(\mathbf{x}_n)$ of $\mathcal{W}_{\mathbf{D}^\times_{\mathbf{p}_n}}$ by means of formula (4.3).

Let $\widetilde{\mathbf{t}}_{\mathbf{p}_n} = \left(\rho_j(\mathbf{t}_{\mathbf{p}_j})\right)_n$ be another coordinate. Then, using the above arguments, we construct analogously a section $\overline{\widetilde{\overline{\mathcal{F}}}}(\mathbf{p}_n)$ by the formula

$$\overline{\widetilde{\overline{\mathcal{F}}}}(\mathbf{p}_n) = \left[\left(\widetilde{\mathbf{y}}_n, \widetilde{\mathcal{F}}_{i_{\widetilde{\mathbf{t}}_{\mathbf{p}_n}}}(\widetilde{\mathbf{h}})\right)\right] = \left[\left(\widetilde{\theta}, f(\widetilde{\mathbf{x}}_n)\right)\right] = \overline{F}(\widetilde{\mathbf{x}}_n),$$

for $\widetilde{\mathbf{y}}_n = (\widetilde{\theta}, \widetilde{\mathbf{t}}_{\mathbf{p}_n})$. Since $\left(i^{-1}_{\widetilde{\mathbf{t}}_{\mathbf{p}_n}} \circ i_{\mathbf{t}_{\mathbf{p}_n}}\right)$ is an automorphism of $G_{\mathbf{p}_n}$, we represent a change of variables $\widetilde{t}_{p_j} = \rho_j(z_j)$ in terms of composition of trivializations

$$\rho_j(z_j) \mapsto i^{-1}_{\widetilde{t}_{j,p}} \circ i_{t_{j,p}}, \tag{4.4}$$

and, therefore, relate $\mathcal{F}_{i_{\widetilde{\mathbf{t}}_{\mathbf{p}_n}}}(\widetilde{\mathbf{h}}_n)$ with $\mathcal{F}_{i_{\mathbf{t}_{\mathbf{p}_n}}}(\mathbf{h}_n)$. Since (4.4) defines a representation on G of the group $\mathrm{Aut}_j\ \mathcal{O}^{(1)}_j$ of changes of coordinates, then then $G_{\mathbf{p}_n}$ is canonically identified with the twist of G by the $\mathbf{Aut}_n\ \boldsymbol{\mathcal{O}}^{(1)}_n$-torsor of formal coordinate at p_j. Using definitions 14 one sees that prescribed rational functions of the space $\Theta\left(n, k, G_{\mathbf{t}_{\mathbf{p}_n}}, U\right)$ can be treated as $\mathbf{Aut}_n\ \boldsymbol{\mathcal{O}}^{(1)}_n$-torsor of the product of groups of a coordinate transformation, namely, that $\mathbf{x}_n = \left(R(\rho_n)^{-1}.(\mathbf{g}_n), \widetilde{\mathbf{t}}_{\mathbf{p}_n}\right)$, $\mathbf{y}_n = \left(\theta.R(\rho_n), \widetilde{\mathbf{t}}_{\mathbf{p}_n}\right)$. Thus, we relate the l.h.s and r.h.s. of (4.3).

To finish proof of proposition, it remains to show that $\overline{F}(\mathbf{x}_n)$ is invariant with respect to changes of coordinates. We have the following

Lemma 4. *For generic elements of the space of prescribed rational functions* $F(\mathbf{x}_n) \in \Theta(n, k, G_{\mathbf{z}_n}, U)$ *for an admissible Lie algebra,* $\overline{F}(\mathbf{x}_n)$ *are canonical, i.e., independent on changes*

$$\mathbf{z}_{n+k} \mapsto \widetilde{\mathbf{z}}_{n+k} = \left(\boldsymbol{\rho}_i(\mathbf{z}_i)\right)_{n+k}, \quad 1 \leqslant i \leqslant n+k, \tag{4.5}$$

as local coordinates of $\mathbf{z}_n$ *and* $\widetilde{\mathbf{z}}_k$*, at points* $\mathbf{p}_n$ *and* $\widetilde{\mathbf{p}}_k$*.*

Remark 1. A generalization Lemma 4 for the case of a arbitrary smooth manifold will be given in [Z].

Indeed, consider the vector

$$\overline{F}(\widetilde{\mathbf{x}}_n) = \left[F\left(\mathbf{g}_n, \widetilde{\mathbf{z}}_n\ \mathbf{d}\widetilde{\mathbf{z}}_{i(n)}\right)\right]. \tag{4.6}$$

Note that $d\widetilde{z}_j = \sum\limits_{i=1}^{n} dz_i\ \partial_{z_i}\rho_j,\ \partial_{z_i}\rho_j = \frac{\partial \rho_j}{\partial z_i}$. By the definition of the action of $\mathbf{Aut}_n\ \boldsymbol{\mathcal{O}}_n^{(1)}$, when rewriting $d\widetilde{\mathbf{z}}_i$, we have

$$\begin{aligned}
\overline{F}(\widetilde{\mathbf{x}}_n) &= \overline{\mathcal{F}}(\mathbf{g}_n, \widetilde{\mathbf{z}}_n\ \mathbf{d}\widetilde{\mathbf{z}}_n) \\
&= \mathrm{R}(\boldsymbol{\rho}_n)\ \left[F\left(\mathbf{g}_n, \mathbf{z}_n\ \mathbf{d}\widetilde{\mathbf{z}}_{i(n)}\right)\right] \\
&= \mathrm{R}(\boldsymbol{\rho}_n)\ \left[F\left(\mathbf{g}_n, \mathbf{z}_n\ \sum_{j=1}^{n} \partial_j \rho_{i(n)}\ dz_j\right)\right].
\end{aligned}$$

By using (3.3) and linearity of the mapping F, we obtain from the last equation

$$\overline{F}(\widetilde{\mathbf{x}}_n) = \overline{F}(\mathbf{g}_n, \widetilde{\mathbf{z}}_n\ \mathbf{d}\widetilde{z}_n) = \left[F\left(\mathbf{g}_n, \mathbf{z}_n\ \mathbf{d}\mathbf{z}_{i(n)}\right)\right], \tag{4.7}$$

with

$$\mathrm{R}(\boldsymbol{\rho}_n) = \left[\widehat{\partial}_J \rho_{i(I)}\right] = \begin{bmatrix} \widehat{\partial}_J \rho_{i_1(I)} \\ \widehat{\partial}_J \rho_{i_2(I)} \\ \cdots \\ \widehat{\partial}_J \rho_{i_n(I)} \end{bmatrix}. \tag{4.8}$$

The index operator J takes the value of index z_j of arguments in the vector (4.7), while the index operator I takes values of index of differentials dz_i in each entry of the vector $\overline{F}$ (4.6). Thus, the index operator $i(I) = (i_I, \ldots, i_n(I))$ is given by consequent cycling permutations of I. Taking into account the property (3.3), we define the operator

$$\widehat{\partial}_J \rho_a = \exp\left(-\sum_{\mathbf{r}_n,\ \sum\limits_{i=1}^{n} r_i \geqslant 1} r_J\ \beta^{(a)}_{\mathbf{r}_n}\ \zeta_1^{r_1}\ \ldots\ \zeta_J^{r_J} \ldots \zeta_n^{r_n}\ \partial_{z_J}\right), \tag{4.9}$$

which contain index operators J as index of a dummy variable ζ_J turning into z_j, $j = 1, \ldots, n$. (4.9) acts on each argument of maps F in the vector $\overline{F}$ (4.6). Due to properties of the Lie algebra $\mathcal{G}$ required by the definition 13, the action of operators $R(\boldsymbol{\rho}_n)$ on $\mathbf{g}_n \in G$ results in a sum of finitely many terms. In [BZF], it is proven

Lemma 5. *The mappings*

$$\boldsymbol{\rho}_n(\mathbf{z}_j) \mapsto R\left(\boldsymbol{\rho}_n\right),$$

for $1 \leqslant j \leqslant n$*, define a representation of* $\mathbf{Aut}_n\ \boldsymbol{\mathcal{O}}_n$ *on* $\mathbf{G}_n$ *by*

$$\mathrm{R}\left(\rho \circ \widetilde{\rho}\right) = \mathrm{R}\left(\rho\right)\ \mathrm{R}\left(\widetilde{\rho}\right),$$

for ρ*,* $\widetilde{\rho} \in \mathbf{Aut}_n\ \boldsymbol{\mathcal{O}}_n$*.*

Using Lemma 5, we then conclude that the vector $\overline{F}$ (4.6) is invariant, i.e.,

$$\overline{F}(\widetilde{\mathbf{x}}_n) = \overline{F}\left(\mathbf{g}_n, \widetilde{\mathbf{z}}_n\ \mathbf{d}\widetilde{\mathbf{z}}_n\right) = \overline{F}\left(\mathbf{g}_n, \mathbf{z}_n\ \mathbf{dz}_n\right) = \overline{F}(\mathbf{x}_n).$$

The definition 11 of prescribed rational functions $F(\mathbf{x}_n) \in \Theta\left(n, k, G_{\mathbf{z}_n}, U\right)$ consists of two conditions on F. The first requires the existence of positive integers $\beta^n_m(v_i, v_j)$ depending on v_i, v_j only, and the second restricts orders of poles of corresponding sums. The insertions of Lie algebra k elements $(\mathbf{g}_k, \mathbf{t}_{\mathbf{p}_k}\ \mathbf{dt}_{\mathbf{p}_k})$ which are present in definition 11 of prescribed rational functions keep functions F invariant with respect to coordinate changes (4.5). Thus, the construction of spaces 12 is invariant under the action of the group $\mathbf{Aut}_{n+k}\ \boldsymbol{\mathcal{O}}^{(1)}_{n+k}$. □

4.5. **The bundle dual to $\mathcal{W}_M$.** It is still possible to define a fiber bundle in the dual formulation when the conditions on grading subspaces of $G_{\mathbf{p}_n}$ are missing. The advantage of the dual (defined with respect to an appropriate form) fiber bundle $\mathcal{W}^\dagger_M$ is that to define it we do not need to assume that the $\mathbb{C}$-grading on $G_{\mathbf{p}_n}$ is bounded from below or that the graded components are finite-dimensional. Nevertheless, we have to assume that a Lie algebra $\mathcal{G}$ satisfies remaining conditions of subsection 4.1. Introduce the canonical residue map

$$\mathrm{Res}_{\ \mathbf{t}_{\mathbf{p}_n}} : \mathbf{t}_{\mathbf{p}_n} \longrightarrow \mathbb{C}^n, \tag{4.10}$$

with separate residues for each variable. Since the space of differentials $\mathbf{\Omega}_{\mathbf{p}_n}$ on the punctured discs $\mathbf{D}^\times_{\mathbf{p}_n}$ is already represented in the definition 1.1 of $F(\mathbf{p}_n)$, then the the map (4.10) gives rise to a pairing

$$\gamma\left(\mathcal{W}^\dagger_M|_{\mathbf{D}^\times_{\mathbf{p}_n}}\right) \times \gamma\left(\mathcal{W}_M|_{\mathbf{D}^\times_{\mathbf{p}_n}}\right) \to \mathbb{C}^n,$$
$$\eta, \mu \mapsto \mathrm{Res}_{\ \mathbf{t}_{\mathbf{p}_n}}\left(\eta, \mu\right),$$

for $\eta \in \gamma\left(\mathcal{W}^\dagger|_{D^\times_{\mathbf{p}_n}}\right)$ and $\mu \in \gamma\left(\mathcal{W}|_{D^\times_{\mathbf{p}_n}}\right)$ belong to corresponding space of fibers. Using this pairing, we obtain for each fiber μ of $\mathcal{W}_M|_{\mathbf{D}^\times_{\mathbf{p}_n}}$, a linear operator on $\mathcal{W}_M|_{\mathbf{D}^\times_{\mathbf{p}_n}}$ given by $\mathrm{Res}_{\ \mathbf{t}_{\mathbf{p}_n}}\left(\mathcal{W}^\dagger_M, \mu\right)$. Thus, we obtain a well-defined linear map

$$\mathcal{W}^\dagger_{\mathbf{D}\times_{\mathbf{p}_n}} : \gamma\left(\mathcal{W}_M|_{\mathbf{D}\times_{\mathbf{p}_n}}\right) \to \mathrm{End}\left(\gamma\left(\mathcal{W}_M|_{\mathbf{D}\times_{\mathbf{p}_n}}\right)\right). \tag{4.11}$$

For formal coordinates $\mathbf{t}_{\mathbf{p}_n}$ on $\mathbf{D}^\times_{\mathbf{p}_n}$, a fiber $\mu = f(\mathbf{x})$ of $W_M|_{\mathbf{D}^\times_{\mathbf{p}_n}}$ with $g_{\mathbf{p}_n} \in G_{\mathbf{p}_n}$ with respect to the $\mathbf{t}_{\mathbf{p}_n}$-trivialization, the map (4.11) is just $\mathbf{x}_n$. The geometrical information contained in $\mathcal{W}^\dagger_M|_{\mathbf{D}^\times_{\mathbf{p}_n}}$ is equivalent to that of $\mathcal{W}_M|_{\mathbf{D}^\times_{\mathbf{p}_n}}$.

Proposition 1 provides us with a way how to, starting from an admissible infinite-dimensional Lie algebra construct explicitly a fiber bundle $\mathcal{W}_M$ over a smooth complex curve M, with canonical sections $\mathcal{F}(\mathbf{p}_n)$ of $\mathcal{W}_M|_{\mathbf{D}_{\mathbf{p}_n}}$ and fibers with values in

$\mathrm{End}\,(G_{\mathbf{p}_n})$ for any collection of non-intersecting disks $\mathbf{D}_{\mathbf{p}_n}$ on M. Due to the assumptions in the definition of an admissible Lie algebra, the filtration $G_{\mathbf{t}_{\mathbf{p}_n},\leqslant m} = \bigoplus_{n=K}^{m} G_{\mathbf{t}_{\mathbf{p}_n},n}$, is preserved by $\mathbf{Aut}_n\ \mathcal{O}_n^{(1)}$. Then it is possible to prove that the exact sequences of $\mathbf{Aut}_n\ \mathcal{O}_n^{(1)}$-modules

$$0 \to G_{\mathbf{t}_{\mathbf{p}_n},\leqslant(m-1)} \to G_{\mathbf{t}_{\mathbf{p}_n},\leqslant m} \to G_{\mathbf{t}_{\mathbf{p}_n},m} \to 0,$$

gives rise to an exact sequence of vector bundles

$$0 \to \mathcal{W}_M^{\leqslant(m-1)} \to \mathcal{W}_M^{\leqslant m} \to \mathcal{W}_M^{m} \to 0.$$

5. Applications

In this section we list multiple applications of the notion of the bundle of prescribed rational functions on complex manifolds [Fei, Wag, DiMaSe, TUY] in deformation theory [Ma, BG, HinSch], and algebraic topology of foliations [Bott].

The fiber bundle associated to spaces of prescribed rational functions on domains of arbitrary complex manifolds can be used in construction of generalizations of the Bott–Segal theorem [BS]. As it was demonstrated in [Wag], the ordinary cohomology of vector fields on complex manifolds turns to be not the most effective and general one. In order to avoid trivialization and reveal a richer cohomological structure of complex manifolds cohomology, one has to treat [Fei] holomorphic vector fields as a sheaf rather than taking global sections. In analogy with the construction of [BS], the cohomology of a foliation over a smooth complex curve M can be expressed in terms of cohomology of a canonical complex (C_M^n, δ_m^n) for an auxillarly bundle $\mathcal{W}_M$ of prescribed rational functions with intrinsic action of δ_m^n. Constructions presented in this paper are also useful for purposes of cosimplitial cohomology [Wag] and computations in the deformation theory of complex manifolds [Ma, Fei, HinSch, GerSch, Kod].

Prescribed rational functions approach is applicable to studies of cohomology and characteristic classes of foliations of complex manifolds. In [Lo] a smooth structure on the leaf space $M/\mathfrak{F}$ of a foliation $\mathfrak{F}$ of codimension k on a smooth manifold M that allows to apply to $M/\mathfrak{F}$ the same techniques as to smooth manifolds. In [Lo] characteristic classes for a foliation as elements of the cohomology of certain bundles over the leaf space $M/\mathfrak{F}$ are defined. It would be interesting to develop also intrinsic and purely coordinate independent theory of a smooth manifold foliation cohomology involving bundles of prescribed rational functions.

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