INSTITUTE OF MATHEMATICS

THE CZECH ACADEMY OF SCIENCES

Cohomology of modular form connections on complex curves

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Preprint No. 38-2021

PRAHA 2021

COHOMOLOGY OF MODULAR FORM CONNECTIONS ON COMPLEX CURVES

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Abstract. We consider reduction cohomology of modular functions defined on complex curves via generalizations of holomorphic connections. The cohomology is explicitly found in terms of higher genus counterparts of elliptic functions as analytic continuations of solutions for functional equations. Examples of modular functions on various genera are provided.

1. Introduction

The natural problem of computation of continuous cohomologies for non-commutative structures on manifolds has proven to be a subject of great geometrical interest [BS, Kaw, PT, Fei, Fuks, Wag]. As it was demonstrated in [Fei, Wag], the ordinary Gelfand-Fuks cohomology of the Lie algebra of holomorphic vector fields on complex manifolds turns to be not the most effective and general one. For Riemann surfaces, and even for higher dimension complex manifolds, the classical cohomology of vector fields becomes trivial [Kaw]. The Lie algebra of holomorphic vector fields does not always work for cohomology. For example, it is zero for a compact Riemann surface of genus greater than one. In [Fei] Feigin obtained various results concerning (co)-homology of the Lie algebra cosimplicial objects of holomorphic vector fields $Lie(M)$. Inspite of results in previous approaches, it is desirable to find a way to enrich cohomological structures. This motivates constructions of more refined cohomology description for non-commutative algebraic structures. In [BS], it have been proven that the Gelfand-Fuks cohomology $H^*(Vect(M))$ of vector fields on a smooth compact manifold M is isomorphic to the singular cohomology of the space of continuous cross sections of a certain fibre bundle over M.

The main aim of this paper is to introduce and compute the reduction cohomology of modular functions on complex curves [FK, Bo, Gu, A]. Due to structure of modular forms [FMS, BKT, Fo] and reduction relations [Y, Zhu, MTZ, GT, TW] among them, one can form chain complexes of n-point modular forms that are fine enough to describe local geometry of complex curves. In contrast to more geometrical methods, e.g., of ordinary cosimplicial cohomology for Lie algebras [Fei, Wag], the reduction cohomology pays more attention to the analytical and modular structure of elements of chain complex spaces. Computational methods involving reduction formulas proved

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Key words and phrases. Cohomology; Complex curves; Modular functions; Elliptic functions; Quasi-Jacobi forms.

their effectiveness in conformal field theory, geometrical descriptions of intertwined modules for Lie algebras, and differential geometry of integrable models.

In section 2 we give the definition of the reduction cohomology as well as lemma relating it to the cohomology of generalized connections on M. The main proposition explicitly expressing the reduction cohomology in terms of spaces of generalized elliptic functions on M is proven. In Appendix 3 we provide examples of reduction formulas for various modular functions. Results of this paper are useful for cosimplisial cohomology theory of smooth manifolds, generalizations of the Bott-Segal theorem, and have their consequences in conformal field theory [Fei, Wag], deformation theory, non-commutative geometry, modular forms, and the theory of foliations.

2. The chain complex and cohomology

2.1. **Chain complex spaces of n-variable modular forms.** In this section we introduce the chain complex spaces for modular functions on complex curves [EZ, Zag, Zhu, Miy, Miy1, MTZ, GT, TW]. Mark n points $\mathbf{p}_n = (p_1, \ldots, p_n)$ on a compact complex curve M of genus g. Denote by $\mathbf{z}_n = (z_1, \ldots, z_n)$ local coordinates around $\mathbf{p}_n \in M$. On genus g complex curves an n-point modular function $\mathcal{Z}(\mathbf{z}_n, \mu)$ has certain specific form depending on g, M (cf. [Y]) and kind of modular form. In addition to that, it depends on a set of moduli parameters $\mu \in \mathcal{M}$ where we denote by $\mathcal{M}$ a subset of the moduli space of genus g complex curve M.

Definition 1. On a complex curve M of genus g, we consider the spaces of n-point modular forms with moduli parameters μ.

$$C^n(\mu) = \{\mathcal{Z}(\mathbf{z}_n, \mu), n \geqslant 0\}, \tag{2.1}$$

that possess reduction formulas.

The co-boundary operator $\delta^n(\mathbf{z}_{n+1})$ on $C^n(\mu)$-space is defined according to the reduction formulas for μ-modular functions (cf. particular examples in Appendix 3, [Fo, Zhu, MTZ, GT, TW]).

Definition 2. For $n \geqslant 0$, and any $z_{n+1} \in \mathbb{C}$, define

$$\delta^n : C^n(\mu) \quad \to \quad C^{n+1}(\mu), \tag{2.2}$$

given by

$$\delta^n(\mathbf{z}_{n+1})\ \mathcal{Z}(\mathbf{z}_n, \mu) = \sum_{l=1}^{l(g)} \sum_{k=0}^{n} \sum_{m \geqslant 0} f_{k,l,m}(\mathbf{z}_{n+1}, l, \mu)\ T_{l,k,m}(\mu).\mathcal{Z}_n(\mathbf{z}_n, \mu), \tag{2.3}$$

where $l(g) \geqslant 0$ is a constant depending on g, and the meaning of indexes $1 \leqslant k \leqslant n$, $1 \leqslant l \leqslant l(g)$, $m \geqslant 0$ explained below.

For each particular genus $g \geqslant 0$ of M and type of modular form defined by the moduli parameter μ, known operator-valued functions $f_{k,l,m}(\mathbf{z}_{n+1}, \mu)T_{k,l,m}(\mu)$. change the k-argument of $\mathcal{Z}(\mathbf{z}_n, \mu)$ by changing μ. The reduction formulas have the form:

$$\mathcal{Z}(\mathbf{z}_{n+1}, \mu) = \delta^n(\mathbf{z}_{n+1}).\mathcal{Z}(\mathbf{z}_n, \mu). \tag{2.4}$$

For $n \geqslant 0$, let us denote by $\mathfrak{Z}_n$ the domain of all $\mathbf{z}_n \in \mathbb{C}^n$, such that the chain condition

$$\delta^{n+1}(\mathbf{z}_{n+1})\ \delta(\mathbf{z}_n).\mathcal{Z}\left(\mathbf{z}_n, \mu\right) = 0, \tag{2.5}$$

for the coboundary operators (2.3) for spaces $C^n(\mu)$ is satisfied. Explicitly, the chain condition (2.5) leads to an infinite $n \geqslant 0$ set of equations involving functions $f_{k,l,m}\left(\mathbf{z}_{n+1}, \mu\right)$ and $\mathcal{Z}\left(\mathbf{z}_n, \mu\right)$:

$$\sum_{\substack{l'=1 \\ l=1}}^{l(g)} \sum_{\substack{k' \\ k=1}}^{\substack{n+1 \\ n}} \sum_{\substack{m' \\ m\geqslant 0}} f_{k',l',m'}\left(\mathbf{z}_{n+2}, \mu\right) f_{k,m,l}\left(\mathbf{z}_{n+1}, \mu\right) T_{k',l',m'}(\mu) T_{k,l,m}(\mu).\mathcal{Z}\left(\mathbf{z}_n, \mu\right) = 0. \tag{2.6}$$

Definition 3. The spaces with conditions (2.6) constitute a chain complex

$$0 \longrightarrow C^0 \xrightarrow{\delta^0} C^1 \xrightarrow{\delta^1} \ldots \xrightarrow{\delta^{n-2}} C^{n-1} \xrightarrow{\delta^{n-1}} C^n \longrightarrow \ldots. \tag{2.7}$$

For $n \geqslant 1$, we call corresponding cohomology

$$H^n(\mu) = \mathrm{Ker}\ \delta^n(\mathbf{z}_{n+1}) / \mathrm{Im}\ \delta^{n-1}(\mathbf{z}_n), \tag{2.8}$$

the n-th reduction cohomology of μ-modular forms on a complex curve M.

Remark 1. Note that the reduction cohomology can be defined as soon as for a type of modular functions there exist reduction formulas (2.4).

Operators $T_{k,l,m}(\mu)$, $0 \leqslant l \leqslant l(g)$, $m \geqslant 0$, $1 \leqslant k \leqslant n$, form a set of generators of an infinite-dimensional continual Lie algebra $\mathfrak{g}(\mu)$ endowed with a natural grading indexed l, m. Indeed, we set the space of functions $\mathcal{Z}(\mathbf{z}_n, \mu)$ as the base algebra [Sav, SV1, SV2, V] for the continual Lie algebra $\mathfrak{g}(\mu)$, and the generators as

$$X_{k,l,m}\left(\mathcal{Z}\left(\mathbf{z}_n, \mu\right)\right) \quad = \quad T_{k,l,m}(\mu).\mathcal{Z}\left(\mathbf{z}_n, \mu\right), \tag{2.9}$$

for $0 \leqslant l \leqslant l(g)$, $m \geqslant 0$, $1 \leqslant k \leqslant n$. Then the commutation relations for non-commutative operators $T_{k,l,m}$. $1 \leqslant k \leqslant n$ inside $\mathcal{Z}(\mathbf{z}_n, \mu)$ represent the commutation relations of the continual Lie algebra $\mathfrak{g}(\mu)$. Jacobi identities for $\mathfrak{g}(\mu)$ follow from Jacobi identities of the Lie algebra of operators $T_{k,l,m}$.

2.2. **Geometrical meaning of reduction formulas and conditions** (2.6)**.** In this section we show that the reduction formulas have the form of multipoint connections generalizing ordinary holomorphic connections on complex curves [Gu]. Let us define the notion of a multipoint connection which will be useful for identifying reduction cohomology in section 2.3. Motivated by the definition of a holomorphic connection for a holomorphic bundle [Gu] over a smooth complex curve M, we introduce the definition of a multiple point connection over M.

Definition 4. Let $\mathcal{V}$ be a holomorphic vector bundle on M, and $M_0 \subset M$ be its subdomain. Denote by $\mathcal{SV}$ the space of sections of $\mathcal{V}$. A multi-point connection $\mathcal{G}$ on $\mathcal{V}$ is a $\mathbb{C}$-multi-linear map

$$\mathcal{G} : M^n \to \mathbb{C},$$

such that for any holomorphic function f, and two sections $\phi(p)$ and $\psi(p')$ of $\mathcal{V}$ at points p and p' on $M_0 \subset M$ correspondingly, we have

$$\sum_{q,q' \in M_0 \subset M} \mathcal{G}\left(f(\psi(q)).\phi(q')\right) = f(\psi(p'))\ \mathcal{G}\left(\phi(p)\right) + f(\phi(p))\ \mathcal{G}\left(\psi(p')\right), \tag{2.10}$$

where the summation on left hand side is performed over locuses of points q, q' on M_0. We denote by $\mathcal{C}on_n$ the space of n-point connections defined over M.

Geometrically, for a vector bundle $\mathcal{V}$ defined over M, a multi-point connection (2.10) relates two sections ϕ and ψ at points p and p' with a number of sections on $M_0 \subset M$.

Definition 5. We call

$$G(\phi,\psi) = f(\phi(p))\, \mathcal{G}\left(\psi(p')\right) + f(\psi(p'))\, \mathcal{G}\left(\phi(p)\right) - \sum_{q,q' \in M_0 \subset \mathcal{X}} \mathcal{G}\left(f(\psi(q')).\phi(q)\right), \quad (2.11)$$

the form of a n-point connection $\mathcal{G}$. The space of n-point connection forms will be denoted by G^n.

Here we prove the following

Lemma 1. *n-point modular functions of the space $\{\mathcal{Z}\left(\mathbf{z}_n,\mu\right), n \geqslant 0\}$ form n-point connections. For $n \geqslant 0$, the reduction cohomology of a compact complex curve of genus g is*

$$H^n(\mu) = \mathcal{C}on^n / G^{n-1}. \quad (2.12)$$

Proof. For non-vanishing $f(\phi(p))$ let us write set

$$\begin{aligned}
\mathcal{G} &= \mathcal{Z}\left(\mathbf{z}_n,\mu\right), \quad (2.13)\\
\psi(p') &= \left(\mathbf{z}_{n+1},\mu\right),\\
\phi(p) &= \left(\mathbf{z_n},\mu\right),\\
\mathcal{G}\left(f(\psi(q)).\phi(q')\right) &= T^{(g)}_{k,l,m}(\mu).\mathcal{Z}\left(\mathbf{z}_n,\mu\right),\\
-\frac{f(\psi(p'))}{f(\phi(p))}\,\mathcal{G}\left(\phi(p)\right) &= \sum_{l=1}^{l(g)} f_{0,l,m}\left(\mathbf{z}_{n+1},\mu\right)\ T_{0,l,m}.\mathcal{Z}\left(\mathbf{z}_n,\mu\right),\\
\frac{1}{f(\phi(p))}\sum_{\substack{q_n, q'_n \in \\ \mathcal{X}_0 \subset M}} \mathcal{G}\left(f(\psi(q)).\phi(q')\right) &= \sum_{k=1}^{n}\sum_{m \geqslant 0} f_{k,l,m}\left(\mathbf{z}_{n+1},\mu\right) T_{k,l,m}(\mu).\mathcal{Z}\left(\mathbf{z}_n,\mu\right).
\end{aligned}$$

Thus, the formula (2.13) gives (2.4). □

The geometrical meaning of (2.6) consists in the following. Due to modular properties of n-point functions $\mathcal{Z}(\mathbf{z}_n,\mu)$, (2.6) is also interpreted as relations among modular forms. The condition (2.4) defines a complex variety in $\mathbf{z}_n \in \mathbb{C}^n$. As most identities (e.g., trisecant identity [Fa, Mu] and triple product identity [MTZ]) for n-point functions (2.6) has its algebraic-geometrical meaning. The condition (2.6) relates finite series of modular functions on M with rational function coefficients (at genus $g = 0$) [Zhu], or (deformed) elliptic functions (at genus $g = 1$) [Zhu, MTZ], or generalizations of classical elliptic functions (at genus $g \geqslant 2$) [GT, TW].

2.3. **Cohomology.** In this section we compute the reduction cohomology defined by (2.7)–(2.8). The main result of this paper is the following

Proposition 1. *The n-th reduction cohomology of the spaces $C^n(\mu)$ (2.1) of modular forms $\mathcal{Z}(\mathbf{z}_n, \mu)$ is the space of recursively generated (by reduction formulas (2.4)) functions with $z_i \notin \mathfrak{Z}_i$, for $1 \leqslant i \leqslant n$, satisfying the condition*

$$\sum_{l=1}^{l(g)} \sum_{k=1}^{n} \sum_{m \geqslant 0} f_{k,l,m}\left(\mathbf{z}_{n+1}, \mu\right) \; T_{l,k,m}.\mathcal{Z}\left(\mathbf{z}_n, \mu\right) = 0. \tag{2.14}$$

Remark 2. The first cohomology is given by the space of transversal (i.e., with vanishing sum over q, q') one-point connections $\mathcal{Z}\left(x_1, \mu\right)$ provided by coefficients in terms of series of special functions. The second cohomology is given by a space of generalized higher genus complex kernels corresponding to M.

Proof. By definition (2.8), the n-th reduction cohomology is defined by the subspace of $C^n(\mu)$ of functions $\mathcal{Z}\left(\mathbf{z}_n, \mu\right)$ satisfying (2.14) modulo the subspace of $C^n(\mu)$ n-point modular functions $\mathcal{Z}\left(\mathbf{z}'_n, \mu\right)$ resulting from:

$$\mathcal{Z}\left(\mathbf{z}'_n, \mu\right) \quad = \quad \sum_{l=1}^{l(g)} \sum_{k=1}^{n-1} \sum_{m \geqslant 0} f_{k,l,m}\left(\mathbf{z}'_n, \mu\right) \; T_{k,l,m}.\, \mathcal{Z}\left(\mathbf{z}'_{n-1}, \mu\right). \tag{2.15}$$

We assume that, subject to other fixed μ-parameters, n-point modular functions are completely determined by all choices $\mathbf{z}_n \in \mathbb{C}^n$. Thus, the reduction cohomology can be treated as depending on set of $\mathbf{z}_n$ only with appropriate action of endomorphisms generated by z_{n+1}. Consider a non-vanishing solution $\mathcal{Z}\left(\mathbf{z}_n, \mu\right)$ to (2.14) for some $\mathbf{z}_n$. Let us use the reduction formulas (2.4) recursively for each z_i, $1 \leqslant i \leqslant n$ of $\mathbf{z}_n$ in order to express $\mathcal{Z}\left(\mathbf{z}_n, \mu\right)$ in terms of nul-point modular form $\mathcal{Z}\left(\mu\right)$, i.e., we obtain

$$\mathcal{Z}\left(\mathbf{z}_n, \mu\right) = \mathcal{D}(\mathbf{z}_n, \mu)\mathcal{Z}\left(\mu\right), \tag{2.16}$$

as in [MTZ]. It is clear that $z_i \notin \mathfrak{Z}_i$ for $1 \leqslant i \leqslant n$, i.e., at each stage of the recursion procedure towards (2.16), otherwise $\mathcal{Z}\left(\mathbf{z}_n, \mu\right)$ would be zero. Thus, $\mathcal{Z}\left(\mathbf{z}_n, \mu\right)$ is explicitly known and is repsented as a series of auxiliary functions $\mathcal{D}(\mathbf{z}_n)$ depending on moduli space parameters μ. Consider now $\mathcal{Z}\left(\mathbf{z}'_n\right)$ given by (2.15). It is either vanishes when $z_{n-i} \in \mathfrak{Z}_{n-i}$, $2 \leqslant i \leqslant n$, or given by (2.16) with $\mathbf{z}'_n$ arguments. The general idea of deriving reduction formulas is to consider the double integration of $\mathcal{Z}\left(\mathbf{z}_n\right)$ along small circles around two auxiliary variables with the action of reproduction kernels inserted. Then, these procedure leads to recursion formulas relating $\mathcal{Z}(\mathbf{z}_{n+1}, \mu)$ and $\mathcal{Z}(\mathbf{z}_n, \mu)$ with functional coefficients depending on the nature of corresponding modular functions, and M. In [Y, MTZ] formulas to n-point modular functions in various specific examples were explicitely and recursively obtained. In terms of z_{n+1}, we are able to transfer in (2.14) the action of $T_{k,l,m}$-operators into an analytical continuation of $\mathcal{Z}\left(\mathbf{z}_n, \mu\right)$ multi-valued holomorphic functions to domains $D_n \subset M$ with $z_i \neq z_j$ for $i \neq j$. Namely, in (2.14), the operators $T_{k,l,m}$ shift the formal parameters $\mathbf{z}_n$ by z_{n+1}, i.e., $\mathbf{z}'_n = \mathbf{z}_n + z_{n+1}$. Thus, the n-th reduction cohomology is given by the space of analytical continuations of n-point modular functions $\mathcal{Z}\left(\mathbf{z}_n, \mu\right)$ with $\mathbf{z}_{n-1} \notin \mathfrak{Z}_{n-1}$ that are solutions of (2.14). □

Acknowledgments

The author would like to thank H. V. Lê and A. Lytchak for related discussions. Research of the author was supported by the GACR project 18-00496S and RVO: 67985840.

3. Appendix: Examples

The reduction cohomology depends on the kind of modular forms (via moduli parameters which we denote μ) and genus of M. Modular functions we consider in this section satisfy certain modular properties with respect to corresponding groups [Zhu, MTZ, GT, TW]. As it was shown in [Miy, KMI, KMII], existence of reduction formulas is related in some sense to modularity.

3.1. **Rational case.** In (cf., e.g., [Zhu]) we find for the rational case n-point functions, the reduction formulas

$$\mathcal{Z}(\mathbf{z}_{n+1}, \mu) = \sum_{k=0}^{n} \sum_{m \geqslant 0} f_{k,m}(z_{n+1}, z_k)\, T_{k,m}.\, \mathcal{Z}(\mathbf{z}_n, \mu), \tag{3.1}$$

where $f_{k,m}(z, w)$ is a rational function defined by

$$f_{n,m}(z, w) = \frac{z^{-n}}{m!} \left(\frac{d}{dw}\right)^m \frac{w^n}{z - w},$$

$$\iota_{z,w} f_{n,m}(z, w) = \sum_{j \in \mathbb{N}} \binom{n+j}{m} z^{-n-j-1} w^{n+j-1}.$$

Let us take z_{n+1} as the variable of expansion. Then the n-th reduction cohomology $H^n(\mu)$ is given by the space of rational functions recursively generated by (2.4) with $\mathbf{z}_n \notin \mathfrak{Z}_n$, satisfying (2.14) with rational function coefficients $f_{k,m}(z_{n+1}, z_k)$, and modulo the space of n-point functions obtained by the recursion procedure, not given by $\delta^{n-1}\mathcal{Z}(\mathbf{z}_{n-1}, \mu)$. It is possible to rewrite (2.14), in the form

$$\left(\partial_{z_{n+1}} + \sum_{k=1}^{n} \widetilde{f}^{(0)}_{k,m}(z_{n+1}, z_k)\right)\ \mathcal{Z}(\mathbf{z}_n + (z_{n+1})_k, \mu) = 0, \tag{3.2}$$

which is an equation for an analytical continuation of $\mathcal{Z}(\mathbf{z}_n + (z_{n+1})_k, \mu)$ with different functions $\widetilde{f}_{k,m}$. Using the reduction formulas (2.4) we obtain

$$\mathcal{Z}(\mathbf{z}_n + (z_{n+1})_k, \mu) = \mathcal{D}(\mathbf{z}_{n+1}, \mu),$$

where $\mathcal{D}(\mathbf{z}_{n+1}, \mu)$ is given by the series of rational-valued functions in $\mathbf{z}_{n+1} \notin \mathfrak{Z}_n$ resulting from the recursive procedure starting from n-point function to the partition function. Thus, in this example, the n-th cohomology is the space of analytic extensions of rational function solutions to the equation (2.14) with rational function coefficients.

3.2. **Modular and elliptic functions.** For a variable x, set $D_x = \frac{1}{2\pi i}\partial_x$, and $q_x = e^{2\pi i x}$. Define for $m \in \mathbb{N} = \{\ell \in \mathbb{Z} : \ell > 0\}$, the elliptic Weierstrass functions

$$P_1(w,\tau) = -\sum_{n\in\mathbb{Z}\backslash\{0\}} \frac{q_w^n}{1-q^n} - \frac{1}{2}, \quad (3.3)$$

$$P_{m+1}(w,\tau) = \frac{(-1)^m}{m!} D_w^m \left(P_1(w,\tau)\right) = \frac{(-1)^{m+1}}{m!} \sum_{n\in\mathbb{Z}\backslash\{0\}} \frac{n^m q_w^n}{1-q^n}. \quad (3.4)$$

Next, we have

Definition 6. The modular Eisenstein series $E_k(\tau)$, defined by $E_k = 0$ for k for odd and $k \geqslant 2$ even

$$E_k(\tau) = -\frac{B_k}{k!} + \frac{2}{(k-1)!}\sum_{n\geqslant 1} \frac{n^{k-1}q^n}{1-q^n},$$

where B_k is the k-th Bernoulli number defined by

$$(e^z-1)^{-1} = \sum_{k\geqslant 0} \frac{B_k}{k!} z^{k-1}.$$

It is convenient to define $E_0 = -1$. E_k is a modular form for $k > 2$ and a quasi-modular form for $k = 2$. Therefore,

$$E_k(\gamma\tau) = (c\tau+d)^k E_k(\tau) - \delta_{k,2}\frac{c(c\tau+d)}{2\pi i}.$$

Definition 7. For w, $z \in \mathbb{C}$, and $\tau \in \mathbb{H}$ let us define

$$\widetilde{P}_1(w,z,\tau) = -\sum_{n\in\mathbb{Z}} \frac{q_w^n}{1-q_z q^n}.$$

We also have

Definition 8.

$$\widetilde{P}_{m+1}(w,z,\tau) = \frac{(-1)^m}{m!} D_w^m \left(\widetilde{P}_1(w,z,\tau)\right) = \frac{(-1)^{m+1}}{m!} \sum_{n\in\mathbb{Z}} \frac{n^m q_w^n}{1-q_z q^n}. \quad (3.5)$$

It is thus useful to give

Definition 9. For $m \in \mathbb{N}_0$, let

$$P_{m+1,\lambda}(w,\tau) = \frac{(-1)^{m+1}}{m!} \sum_{n\in\mathbb{Z}\backslash\{-\lambda\}} \frac{n^m q_w^n}{1-q^{n+\lambda}}. \quad (3.6)$$

On notes that

$$P_{1,\lambda}(w,\tau) = q_w^{-\lambda}(P_1(w,\tau) + 1/2),$$

with

$$P_{m+1,\lambda}(w,\tau) = \frac{(-1)^m}{m!} D_w^m \left(P_{1,\lambda}(w,\tau)\right).$$

We also consider the expansion

$$P_{1,\lambda}(w,\tau) = \frac{1}{2\pi i w} - \sum_{k\geqslant 1} E_{k,\lambda}(\tau)(2\pi i w)^{k-1},$$

where we find [Zag]

$$E_{k,\lambda}(\tau) = \sum_{j=0}^{k} \frac{\lambda^j}{j!} E_{k-j}(\tau). \tag{3.7}$$

Definition 10. We define another generating set $\widetilde{E}_k(z,\tau)$ for $k \geqslant 1$ together with $E_2(\tau)$ given by [Ob]

$$\widetilde{P}_1(w,z,\tau) = \frac{1}{2\pi i w} - \sum_{k\geqslant 1} \widetilde{E}_k(z,\tau)(2\pi i w)^{k-1}, \tag{3.8}$$

where we find that for $k \geqslant 1$,

$$\widetilde{E}_k(z,\tau) = -\,\delta_{k,1}\frac{q_z}{q_z-1} - \frac{B_k}{k!} + \frac{1}{(k-1)!}\sum_{m,n\geqslant 1}\left(n^{k-1}q_z^m + (-1)^k n^{k-1} q_z^{-m}\right) q^{mn}, \tag{3.9}$$

and $\widetilde{E}_0(z,\tau) = -1$.

3.3. **Elliptic case.** Let $q = e^{2\pi i \tau}$, $q_i = e^{z_i}$, where τ is the torus modular parameter. Then the genus one Zhu recursion formula is given by the following [Zhu]

$$\mathcal{Z}(\mathbf{z}_{n+1},\mu,\tau) = \mathcal{Z}\left(\mathbf{z}_n,\mu_0,\tau\right) + \sum_{k=1}^{n}\sum_{m\geqslant 0} P_{m+1}(z_{n+1}-z_k,\tau)\ \mathcal{Z}(\mathbf{z}_n,\mu_{k,m},\tau). \tag{3.10}$$

Here $P_m(z,\tau)$ denote higher Weierstrass functions defined by

$$P_m(z,\tau) = \frac{(-1)^m}{(m-1)!}\sum_{n\in\mathbb{Z}_{\neq 0}} \frac{n^{m-1}q_z^n}{1-q^n}.$$

3.4. **Case of deformed elliptic functions.** Let $w_{n+1} \in \mathbb{R}$ and define $\phi \in U(1)$ by

$$\phi = \exp(2\pi i\ w_{n+1}). \tag{3.11}$$

For some $\theta \in U(1)$, we obtain the following generalization of Zhu's Proposition 4.3.2 [Zhu] for the n-point function [MTZ]:

Theorem 1. *Let θ and ϕ be as as above. Then for any $\mathbf{z}_n \in C^n$ we have*

$$\begin{aligned}\mathcal{Z}\left(\mathbf{x_{n+1}},\mu,\tau\right) &= \delta_{\theta,1}\delta_{\phi,1}\mathcal{Z}\left(\mathbf{x_n},\mu_0,\tau\right)\\ &+ \sum_{\substack{k=1\\ m\geqslant 0}}^{n} p(n,k)\ P_{m+1}\left[\begin{array}{c}\theta\\ \phi\end{array}\right](z_{n+1}-z_k,\tau)\ \mathcal{Z}(\mathbf{z}_n;\mu_{k,m},\tau).\end{aligned} \tag{3.12}$$

The deformed Weierstrass function is defined as follows [MTZ]. Let $(\theta,\phi) \in U(1)\times U(1)$ denote a pair of modulus one complex parameters with $\phi = \exp(2\pi i\lambda)$ for $0 \leqslant \lambda < 1$. For $z \in \mathbb{C}$ and $\tau \in \mathbb{H}$ we define deformed Weierstrass functions for $k \geqslant 1$,

$$P_k\left[\begin{array}{c}\theta\\ \phi\end{array}\right](z,\tau) = \frac{(-1)^k}{(k-1)!}{\sum_{n\in\mathbb{Z}+\lambda}}' \frac{n^{k-1}q_z^n}{1-\theta^{-1}q^n},$$

for $q = q_{2\pi i\tau}$ where $\sum'$ means we omit $n = 0$ if $(\theta,\phi) = (1,1)$.

3.5. **Reduction formulas for Jacobi n-point functions.** In this subsection we recall the reduction formulas derived in [MTZ, BKT]. For $\alpha \in \mathbb{C}$, we now provide the following reduction formula for formal Jacobi n-point functions.

Proposition 2. *Let $\mathbf{z}_{n+1} \in \mathbb{C}^{n+1}$, $\alpha \in \mathbb{C}$. For $\alpha z \notin \mathbb{Z}\tau + \mathbb{Z}$, we have*

$$\mathcal{Z}\left(\mathbf{z}_{n+1}, \mu, \tau\right) = \sum_{k=1}^{n} \sum_{m \geqslant 0} \widetilde{P}_{m+1}\left(\frac{z_{n+1} - z_k}{2\pi i}, \alpha z, \tau\right) \mathcal{Z}\left(\mathbf{z}_n, \mu_{k,m}, \tau\right). \tag{3.13}$$

Recall the definition of $\widetilde{P}$.

Proposition 3. *For $\alpha z = \lambda\tau + \mu \in \mathbb{Z}\tau + \mathbb{Z}$, we have*

$$\begin{aligned}&\mathcal{Z}\left(\mathbf{z}_{n+1}, \mu, \tau\right) \\ &\quad = e^{-z_{n+1}\lambda} \mathcal{Z}\left(\mathbf{z}_n, \mu_{0,\lambda}, \tau\right) + \sum_{k=1}^{n} \sum_{m \geqslant 0} P_{m+1,\lambda}\left(\frac{z_{n+1} - z_k}{2\pi i}, \tau\right) \mathcal{Z}\left(\mathbf{z}_n, \mu_{k,m}, \tau\right),\end{aligned} \tag{3.14}$$

with $P_{m+1,\lambda}(w, \tau)$ defined in (3.6).

Next we provide the reduction formula for Jacobi n-point functions.

Proposition 4. *For $l \geqslant 1$ and $\alpha z \notin \mathbb{Z}\tau + \mathbb{Z}$, we have*

$$\begin{aligned}&\mathcal{Z}\left(\mathbf{z}_{n+1}), \mu_{1,-l}, \tau\right) \\ &\quad = \sum_{m \geqslant 0} (-1)^{m+1} \binom{m+l-1}{m} \widetilde{G}_{m+l}(\alpha z, \tau) \mathcal{Z}\left(\mathbf{z}_n; \mu_{1,m}\tau\right) \\ &\quad + \sum_{k=2}^{n} \sum_{m \geqslant 0} (-1)^{l+1} \binom{m+l-1}{m} \widetilde{P}_{m+l}\left(\frac{z_1 - z_k}{2\pi i}, \alpha z, \tau\right) \mathcal{Z}\left(\mathbf{z}_n, \mu_{k,m}, \tau\right).\end{aligned} \tag{3.15}$$

Propositions 3 and 4 imply the next result [BKT]:

Proposition 5. *For $l \geqslant 1$ and $\alpha z = \lambda\tau + \mu \in \mathbb{Z}\tau + \mathbb{Z}$, we have*

$$\begin{aligned}&\mathcal{Z}\left(\mathbf{z}_{n+1}, \mu_{1,-l}); B\right) \\ &\quad = (-1)^{l+1} \frac{\lambda^{l-1}}{(l-1)!} \mathcal{Z}\left(\mathbf{z}_{n+1}, \mu_{0,-1}, \tau\right) \\ &\quad + \sum_{m \geqslant 0} (-1)^{m+1} \binom{m+l-1}{m} E_{m+l,\lambda}(\tau)\, \mathcal{Z}\left(\mathbf{z}_n, \mu_{1,m}, \tau\right) \\ &\quad + \sum_{k=2}^{n} \sum_{m \geqslant 0} (-1)^{l+1} \binom{m+l-1}{m} P_{m+l,\lambda}\left(\frac{x_1 - x_k}{2\pi i}, \tau\right) \mathcal{Z}\left(\mathbf{z}_n, \mu_{k,m}, \tau\right),\end{aligned}$$

for $E_{k,\lambda}$ given in (3.7).

3.6. **Multiparameter Jacobi forms.** For multiparameter Jacobi forms [EZ, Zag, KMI, KMII, BKT], the reduction formulas are found using an analysis that is similar to that in [Zhu, MTZ]. The following two lemmas reduce any n-point multiparameter Jacobi function to a linear combination of $(n-1)$-point Jacobi functions with modular coefficients.

Lemma 2. *For each $1 \leqslant j \leqslant m$ we have*

$$\begin{aligned} &\mathcal{Z}(\mathbf{z}_{n+1}, \mu, \tau) \\ &= \delta_{\mathbf{z}_n \cdot (\alpha)_n, \mathbb{Z}}\ \mathcal{Z}(\mathbf{z}_n, (\alpha)_n, \mu(m)) \\ &+ \sum_{s=1}^{n} \sum_{k \geqslant 0} \tilde{P}_{k+1}(z_s - z, \mathbf{z}_n \cdot (\alpha_n), \tau)\ \mathcal{Z}(\mathbf{z}_n, \mu_{s,k}\tau), \end{aligned} \tag{3.16}$$

where $\delta_{\mathbf{z}\cdot(\mu)_n,\mathbb{Z}}$ is 1 if $\mathbf{z}_n \cdot (\mu)_n \in \mathbb{Z}$ and is 0 otherwise.

Lemma 3. *Let the assumptions be the same as in the previous lemma. Then for $p \geqslant 1$,*

$$\begin{aligned} &\mathcal{Z}(\mathbf{z}_{n+1}, \mu_{1,-p}, \tau) \\ &= \delta_{\mathbf{z}_n \cdot (\alpha)_n, \mathbb{Z}}\ \delta_{p,1}\ \mathcal{Z}(\mathbf{z}_n, \mu_0, \tau) \\ &+ (-1)^{p+1} \sum_{k \geqslant 0} \binom{k+p-1}{p-1} \tilde{E}_{k+p}(\tau, \mathbf{z}_n \cdot (\alpha)_n)\ \mathcal{Z}(\mathbf{z}_n, \mu_{1,k}, \tau) \\ &+ (-1)^{p+1} \sum_{s=2}^{n} \sum_{k \geqslant 0} \binom{k+p-1}{p-1} \tilde{P}_{k+p}(z_s - z_1, \tau, \mathbf{z}_n \cdot (\alpha)_n)\ \mathcal{Z}(\mathbf{z}_n \mu_{s,k}, \tau). \end{aligned}$$

Remark 3. The difference of a minus sign between these equations and those found in [MTZ] can be attributed to the minus sign difference in our definitions of the functions $P_k \left[{\zeta \atop 1} \right] (w, \tau)$ and the action of $\mathrm{SL}_2(\mathbb{Z})$.

3.7. **Genus two counterparts of Weierstrass functions.** In this subsection we recall the definition of genus two Weierstrass functions [GT]. For m, $n \geqslant 1$, we first define a number of infinite matrices and row and column vectors:

$$\begin{aligned} \Gamma(m,n) &= \delta_{m,-n+2p-2}, \\ \Delta(m,n) &= \delta_{m,n+2p-2}. \end{aligned} \tag{3.17}$$

We also define the projection matrix

$$\Pi = \Gamma^2 = \begin{bmatrix} \mathbb{1}_{2p-1} & 0 \\ 0 & \ddots \end{bmatrix}, \tag{3.18}$$

where Id_{2p-3} denotes the $2p-3$ dimensional identity matrix and $\mathrm{Id}_{-1} = 0$. Let Λ_a for $a \in \{1, 2\}$ be the matrix with components

$$\Lambda_a(m, n; \tau_a, \epsilon) = \epsilon^{(m+n)/2} (-1)^{n+1} \binom{m+n-1}{n} E_{m+n}(\tau_a). \tag{3.19}$$

Note that

$$\Lambda_a = S A_a S^{-1}, \tag{3.20}$$

for A_a given by

$$A_a = A_a(k,l,\tau_a,\epsilon) = \frac{(-1)^{k+1}\epsilon^{(k+l)/2}}{\sqrt{kl}} \frac{(k+l-1)!}{(k-1)!(l-1)!} E_{k+l}(\tau_a).$$

introduce the infinite dimensional matrices for S a diagonal matrix with components

$$S(m,n) = \sqrt{m}\delta_{mn}. \tag{3.21}$$

Let $\mathbb{R}(x)$ for x on the torus be the row vector with components

$$\mathbb{R}(x;m) = \epsilon^{\frac{m}{2}} P_{m+1}(x,\tau_a), \tag{3.22}$$

for $a \in \{1,2\}$. Let $\mathbb{X}_a$ be the column vector with components

$$\begin{aligned}
\mathbb{X}_1(m) &= \mathbb{X}_1\left(m; z_{n+1}, \mathbf{z}_n; \mu\right) \\
&= \epsilon^{-m/2} \sum_{u\in V} \mathcal{Z}\left(\mathbf{z}_k, \mu_{k,m}, \tau_1\right)\ \mathcal{Z}\left(\mathbf{x}_{k+1,n}, \mu', \tau_2\right), \\
\mathbb{X}_2(m) &= \mathbb{X}_2\left(m; z_{n+1}, \mathbf{z}_n; \mu\right) \\
&= \epsilon^{-m/2} \sum_{u\in V} \mathcal{Z}\left(\mathbf{x}_k, \mu, \tau_1\right) \mathcal{Z}\left(\mathbf{x}_{n-k}, \mu_{n-k,m}, \tau_2\right).
\end{aligned} \tag{3.23}$$

Introduce also $\mathbb{Q}(p;x)$ an infinite row vector defined by

$$\mathbb{Q}(p;x) = \mathbb{R}(x)\Delta\left(\mathbb{1} - \widetilde{\Lambda}_{\overline{a}}\widetilde{\Lambda}_a\right)^{-1}, \tag{3.24}$$

for x on the torus. Notice that

$$\widetilde{\Lambda}_a = \Lambda_a \Delta.$$

One introduces

$$\mathbb{P}_{j+1}(x) = \frac{(-1)^j}{j!}\mathbb{P}_1(x),$$

and $j \geqslant 0$, is the column with components

$$\mathbb{P}_{j+1}(x;m) = \epsilon^{\frac{m}{2}} \binom{m+j-1}{j} \left(P_{j+m}(x,\tau_a) - \delta_{j0}E_m(\tau_a)\right). \tag{3.25}$$

Definition 11. One defines

$$\mathcal{P}_1(p;x,y) = \mathcal{P}_1(p;x,y;\tau_1,\tau_2,\epsilon),$$

for $p \geqslant 1$ by

$$\begin{aligned}
\mathcal{P}_1(p;x,y) &= P_1(x-y,\tau_a) - P_1(x,\tau_a) \\
&- \mathbb{Q}(p;x)\widetilde{\Lambda}_{\overline{a}}\,\mathbb{P}_1(y) - (1-\delta_{p1})\left(\mathbb{Q}(p;x)\Lambda_{\overline{a}}\right)(2p-2),
\end{aligned}$$

for x, y on the torus, and

$$\begin{aligned}
\mathcal{P}_1(p;x,y) &= (-1)^{p+1}\Big[\mathbb{Q}(p;x)\mathbb{P}_1(y) + (1-\delta_{p1})\epsilon^{p-1}P_{2p-1}(x) \\
&+ (1-\delta_{p1})\left(\mathbb{Q}(p;x)\widetilde{\Lambda}_{\overline{a}}\Lambda_a\right)(2p-2)\Big],
\end{aligned}$$

for x and y on two torai.

For $j > 0$, define

$$\mathcal{P}_{j+1}(p;x,y) = \frac{1}{j!}\partial_y^j\left(\mathcal{P}_1(p;x,y)\right),$$
$$\mathcal{P}_{j+1}(p;x,y) = \delta_{a,\bar{a}}P_{j+1}(x-y) + (-1)^{j+1}.\mathbb{Q}(p;x)\left(\widetilde{\Lambda}_{\overline{a}}\right)^{\delta_{a,\bar{a}}}\ \mathbb{P}_{j+1}(y). \quad (3.26)$$

Definition 12. One calls $\mathcal{P}_{j+1}(p;x,y)$ the genus two generalized Weierstrass functions.

3.8. **Genus two case.** In this subsection we recall [GT] the construction and reduction formulas for modular funcitons defined on genus two complex curve. In particular, we use the geometric construction developed in [Y].

Definition 13. For a complex parameter $\epsilon = z_1 z_2$, the null-point modular form is defined on a genus two complex curve by

$$\mathcal{Z}\left(\mu\right) = \sum_{r\geqslant 0}\epsilon^r \mathcal{Z}(z_1,\mu_1\tau_1)\ \mathcal{Z}(z_2,\mu_2,\tau_2), \quad (3.27)$$

where parameters μ_1 and μ_2 are related.

We then recall [GT] the formal genus two reduction formulas for n-point modular functions.

Definition 14. Let x_{n+1}, $\mathbf{y}_k$ and $\mathbf{y}'_l$ be inserted on two torai. We consider the genus two n-point modular function

$$\mathcal{Z}\left(z_{n+1},\mathbf{z}_k;\mathbf{z}'_l,\mu)\right) = \sum_{r\geqslant 0}\epsilon^r\mathcal{Z}\left(z_{n+1},\mathbf{x}_k,\mu_1,\tau_1\right)\ \mathcal{Z}\left(\mathbf{x}'_l,\mu_2,\tau_2\right), \quad (3.28)$$

where the sum as in (3.27).

First, one defines the functions $\mathcal{Z}_{n,a}$ for $a \in \{1,2\}$, via elliptic quasi-modular forms

$$\mathcal{Z}_{n,1}\left(\mathbf{z}_{n+1};\mu\right) = \sum_{r\geqslant 0}\epsilon^r\mathcal{Z}(\mathbf{z}_{n+1},\mathbf{z}_k\mu_0,\tau)\ \mathcal{Z}_{n-k}\left(\mathbf{x}_{k+1,n},\mu',\tau_2\right),$$

$$\mathcal{Z}_{n,2}\left(\mathbf{z}_{n+1};\mu\right) = \sum_{r\geqslant 0}\epsilon^r\mathcal{Z}_k\left(\mathbf{x}_k,\mu',\tau_1\right)\ \mathcal{Z}\left(z_{n+1}),\mathbf{z}_{k+1,n}\right),$$
$$\mathcal{Z}_{n,3}\left(\mathbf{z}_{n+1};\mu\right) = \mathbb{X}_1^{\Pi},$$

of (3.23). We also define

Definition 15. Let $f_a^{(2)}(p;z_{n+1})$, for $p \geqslant 1$, and $a = 1, 2$ be given by

$$f_a^{(2)}(p;z_{n+1}) = 1^{\delta_{ba}} + (-1)^{p\delta_{b\overline{a}}}\epsilon^{1/2}\left(\mathbb{Q}(p;z_{n+1})\left(\widetilde{\Lambda}_{\overline{a}}\right)^{\delta_{ba}}\right)(1), \quad (3.29)$$

for $z_{n+1} \in \widehat{\Sigma}_b^{(1)}$. Let $f_3^{(2)}(p;z_{n+1})$, for $z_{n+1} \in \Sigma_a^{(1)}$ be an infinite row vector given by

$$f_3^{(2)}(p;z_{n+1}) = \left(\mathbb{R}(z_{n+1}) + \mathbb{Q}(p;z_{n+1})\left(\widetilde{\Lambda}_{\overline{a}}\Lambda_a + \Lambda_{\overline{a}}\Gamma\right)\right)\Pi. \quad (3.30)$$

In [GT] it is proven that the genus two $n = k + l$-point function inserted at x_{n-k}, $\mathbf{y}_k$ on two torai has the following reduction formula

$$\begin{aligned}\mathcal{Z}(\mathbf{x}_{n+1}, \mu) &= \sum_{l=1}^{3} f_l(p; z_{n+1})\ \mathcal{Z}_{n,l}\left(\mathbf{z}_{n+1}; \mu\right), \\ &= \sum_{i=1}^{n} \sum_{j \geqslant 0} \mathcal{P}_{j+1}(p; z_{n+1}, z_i)\ \mathcal{Z}\left(\mathbf{z}_n; \mu_{i,j}\right),\end{aligned} \tag{3.31}$$

where p is some parameter. with $\mathcal{P}_{j+1}(p; x, y)$ of (3.26).

3.9. **Genus g generalizations of elliptic functions.** For purposes of the formula (3.54) we recall here certain definitions [TW]. Define a column vector

$$X = (X_a(m)),$$

indexed by $m \geqslant 0$ and $a \in \mathcal{I}$ with components

$$X_a(m) = \rho_a^{-\frac{m}{2}} \sum_{\boldsymbol{\mu_{a,m}}} \mathcal{Z}(\ldots; w_a, \mu_{a,m}; \ldots), \tag{3.32}$$

and a row vector

$$p(x) = (p_a(x, m)),$$

for $m \geqslant 0, a \in \mathcal{I}$ with components

$$p_a(x, m) = \rho_a^{\frac{m}{2}}\, \partial^{(0,m)} \psi_p^{(0)}(x, w_a). \tag{3.33}$$

Introduce the column vector

$$G = (G_a(m)),$$

for $m \geqslant 0, a \in \mathcal{I}$, given by

$$G = \sum_{k=1}^{n} \sum_{j \geqslant 0} \partial_k^{(j)}\ q(y_k)\ \mathcal{Z}(\mathbf{z}_n, \mu_{k,j}),$$

where $q(y) = (q_a(y; m))$, for $m \geqslant 0$, $a \in \mathcal{I}$, is a column vector with components

$$q_a(y; m) = (-1)^p \rho_a^{\frac{m+1}{2}}\, \partial^{(m,0)} \psi_p^{(0)}(w_{-a}, y), \tag{3.34}$$

and

$$R = (R_{ab}(m, n)),$$

for m, $n \geqslant 0$ and a, $b \in \mathcal{I}$ is a doubly indexed matrix with components

$$R_{ab}(m, n) = \begin{cases} (-1)^p \rho_a^{\frac{m+1}{2}} \rho_b^{\frac{n}{2}}\, \partial^{(m,n)} \psi_p^{(0)}(w_{-a}, w_b), & a \neq -b, \\ (-1)^p \rho_a^{\frac{m+n+1}{2}}\, \mathcal{E}_m^n(w_{-a}), & a = -b, \end{cases} \tag{3.35}$$

where

$$\mathcal{E}_m^n(y) = \sum_{\ell=0}^{2p-2} \partial^{(m)} f_\ell(y)\ \partial^{(n)} y^\ell, \tag{3.36}$$

$$\psi_p^{(0)}(x,y) = \frac{1}{x-y} + \sum_{\ell=0}^{2p-2} f_\ell(x) y^\ell, \tag{3.37}$$

for any Laurent series $f_\ell(x)$ for $\ell = 0, \ldots, 2p-2$. Define the doubly indexed matrix $\Delta = (\Delta_{ab}(m,n))$ by

$$\Delta_{ab}(m,n) = \delta_{m,n+2p-1}\delta_{ab}. \tag{3.38}$$

Denote by

$$\widetilde{R} = R\Delta,$$

and the formal inverse $(I-\widetilde{R})^{-1}$ is given by

$$\left(I-\widetilde{R}\right)^{-1} = \sum_{k\geqslant 0} \widetilde{R}^{\,k}. \tag{3.39}$$

Define $\chi(x) = (\chi_a(x;\ell))$ and

$$o(\boldsymbol{y}_k, \mu_0) = (o_a(\boldsymbol{y}_k;\mu_0,\ell)),$$

are finite row and column vectors indexed by $a \in \mathcal{I}$, $0 \leqslant \ell \leqslant 2p-2$ with

$$\chi_a(x;\ell) = \rho_a^{-\frac{\ell}{2}}(p(x) + \widetilde{p}(x)(I-\widetilde{R})^{-1}R)_a(\ell), \tag{3.40}$$

$$o_a(\ell) = o_a(\boldsymbol{y}_k, \mu_0, \ell) = \rho_a^{\frac{\ell}{2}} X_a(\ell), \tag{3.41}$$

and where

$$\widetilde{p}(x) = p(x)\Delta.$$

$\psi_p(x,y)$ is defined by

$$\psi_p(x,y) = \psi_p^{(0)}(x,y) + \widetilde{p}(x)(I-\widetilde{R})^{-1}q(y). \tag{3.42}$$

For each $a \in \mathcal{I}_+$ we define a vector

$$\theta_a(x) = (\theta_a(x;\ell)),$$

indexed by $0 \leqslant \ell \leqslant 2p-2$ with components

$$\theta_a(x;\ell) = \chi_a(x;\ell) + (-1)^p \rho_a^{p-1-\ell}\chi_{-a}(x;2p-2-\ell). \tag{3.43}$$

Now define the following vectors of formal differential forms

$$P(x) = p(x)\, dx^p,$$
$$Q(y) = q(y)\, dy^{1-p}, \tag{3.44}$$

with

$$\widetilde{P}(x) = P(x)\Delta.$$

Then with

$$\Psi_p(x,y) = \psi_p(x,y)\, dx^p\, dy^{1-p}, \tag{3.45}$$

we have

$$\Psi_p(x,y) = \Psi_p^{(0)}(x,y) + \widetilde{P}(x)(I-\widetilde{R})^{-1}Q(y). \tag{3.46}$$

Defining

$$\Theta_a(x;\ell) = \theta_a(x;\ell)\, dx^p, \tag{3.47}$$

and
$$O_a(\boldsymbol{y}_k, \mu_0, \ell) = o_a(\boldsymbol{y}_k, \mu_0, \ell)\, \boldsymbol{dy}_k^{\beta}, \tag{3.48}$$
for some parameter β.

3.10. **Genus g Schottky case.** In this subsection we recall [TW, T2] the construction and reduction relations for n-point modular functions defined on a genus g Riemann surface M formed in the Schottky parameterization. All expressions here are functions of formal variables $w_{\pm a}$, $\rho_a \in \mathbb{C}$. Then we recall the genus g reduction formula with universal coefficients that have a geometrical meaning and are meromorphic on M. These coefficients are generalizations of the elliptic Weierstrass functions [L]. For a $2g$ local coordinates
$$\boldsymbol{w}_{2g} = (w_{-1}, w_1; \ldots; w_{-g}, w_g),$$
of $2g$ points $(p_{-1}, p_1; \ldots; p_{-g}, p_g)$ on the Riemann sphere, consider the genus zero $2g$-point function
$$\begin{aligned}\mathcal{Z}(\boldsymbol{w}_{2g}, \mu) =& \mathcal{Z}(w_{-1}, w_1; \ldots; w_{-g}, w_g, \mu)\\ =& \prod_{a \in \mathcal{I}_+} \rho_a^{\beta_a} \mathcal{Z}(w_{-1}, w_1; \ldots; w_{-g}, w_g, \mu),\end{aligned}$$
where $\mathcal{I}_+ = \{1, 2, \ldots, g\}$, and β_a are certain parameters related to μ. Let us denote
$$\begin{aligned}\boldsymbol{z}_+ &= (z_1, \ldots, z_g),\\ \boldsymbol{z}_- &= (z_{-1}, \ldots, z_{-g}).\end{aligned}$$
Let w_a for $a \in \mathcal{I}$ be $2g$ formal variables. One identify them with the canonical Schottky parameters (for details of the Schottky construction, see [TW, T2]). One can define the genus g null-point modular function as
$$\mathcal{Z} = (\boldsymbol{w}_{2g}, \boldsymbol{\rho}_{2g}, \mu) = \sum_{\boldsymbol{z}_+} \mathcal{Z}(\boldsymbol{z}_{2g}, \boldsymbol{w}_{2g}, \mu), \tag{3.49}$$
for
$$(\boldsymbol{w}_{2g}, \boldsymbol{\rho}_{2g}) = (w_{\pm 1}, \rho_1; \ldots; w_{\pm g}, \rho_g).$$
Now we recall the formal reduction formulas for all genus g Schottky n-point functions. One defines the genus g formal n-point modular function for $\boldsymbol{y}_n$ by
$$\mathcal{Z}(\boldsymbol{y}_n, \mu) = \mathcal{Z}(\boldsymbol{y}_n; \boldsymbol{w}_{2g}, \boldsymbol{\rho}_{2g}, \mu) = \sum_{\boldsymbol{z}_+} \mathcal{Z}(\boldsymbol{y}_n; \boldsymbol{w}_{2g}, \mu), \tag{3.50}$$
where
$$\mathcal{Z}(\boldsymbol{y}_n; \boldsymbol{w}_{2g}, \mu) = \mathcal{Z}(\boldsymbol{y}_n; \boldsymbol{w}_{-1,g}, \mu).$$
$$\mathcal{Z}(\boldsymbol{y}_n, \mu) = \sum_{\mathbf{z}_+ \in \boldsymbol{\alpha}_g} \mathcal{Z}(\boldsymbol{y}_n; \boldsymbol{w}_{2g}, \mu), \tag{3.51}$$
where here the sum is over a basis $\boldsymbol{\alpha}$. It follows that
$$\mathcal{Z}(\boldsymbol{y}_n, \mu) = \sum_{\boldsymbol{\alpha}_g \in \boldsymbol{A}} \mathcal{Z}^{(g)}_{\boldsymbol{\alpha}_g}(\boldsymbol{y}_n, \mu), \tag{3.52}$$

where the sum ranges over $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_g) \in \boldsymbol{A}$, for $\boldsymbol{A} = A^{\otimes g}$. Finally, one defines corresponding formal n-point correlation differential forms

$$Z(\boldsymbol{y}_n, \mu) = \mathcal{Z}(\boldsymbol{y}_n, \mu)\ \boldsymbol{dy}_n^{\boldsymbol{\beta}},$$
$$Z_{\boldsymbol{\alpha}_g}(\boldsymbol{y}_n, \mu) = \mathcal{Z}_{\boldsymbol{\alpha}}(\boldsymbol{y}_n, \mu)\ \boldsymbol{dy}_n^{\boldsymbol{\beta}}, \tag{3.53}$$

where

$$\boldsymbol{dy}_n^{\boldsymbol{\beta}} = \prod_{k=1}^{n} dy_k^{\beta_k}.$$

In [TW] they prove that the genus g $(n+1)$-point formal modular differential $Z(x; \boldsymbol{y}, \mu)$, for x_{n+1}, and point p_0, with the coordinate y_{n+1}, and $\mathbf{p}_n$ with coordinates $\mathbf{y}_n$ satisfies the recursive identity for $\mathbf{z}_n = (\boldsymbol{y})$

$$\begin{aligned} Z\left(x_{n+1}, \mathbf{z}_n, \mu\right) &= \sum_{a=1}^{g} \Theta_a(y_{n+1})\ O_a^{W_{\boldsymbol{\alpha}}}\left(z_{n+1}; \mathbf{z}_n\right) \\ &= \sum_{k=1}^{n} \sum_{j \geqslant 0} \partial^{(0,j)}\ \Psi_p(y_{n+1}, y_k)\ \mathcal{Z}\left(\mathbf{x_n}, \mu_{k,j}\right)\ dy_k^j \end{aligned}$$

Here $\partial^{(0,j)}$ is given by

$$\partial^{(i,j)} f(x, y) = \partial_x^{(i)} \partial_y^{(j)} f(x, y),$$

for a function $f(x, y)$, and $\partial^{(0,j)}$ denotes partial derivatives with respect to x and y_j. The forms $\Psi_p(y_{n+1}, y_k)\ dy_k^j$ given by (3.45), $\Theta_a(x)$ is of (3.47), and $O_a^{W_{\boldsymbol{\alpha}}}(z_{n+1}, \mathbf{z}_n, \mu)$ of (3.48).

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