ORIENTED STOCHASTIC DATA ENVELOPMENT MODELS: RANKING COMPARISON TO STOCHASTIC FRONTIER APPROACH

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Oriented stochastic data envelopment models: Ranking comparison to stochastic frontier approach*

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Abstract

Results of data envelopment analysis sensitively respond to stochastic noise in the data. In this paper, by introduction of output augmentation and input reduction I extend additive models for stochastic data envelopment analysis (SDEA), which were developed by Li (1998) to handle the noise in the data. Applying the linearization procedure by Li (1998) the linearized versions of models are derived. In the empirical part of this work, the efficiency scores of Indonesian rice farms are computed. The computed scores are compared to the stochastic frontier approach scores by Druska and Horrace (2004) and weak ranking consistency with results of stochastic frontier method is observed.

Abstrakt

Výsledky hodnocení efektivnosti získané analýzou obalu dat (DEA) jsou citlivé na přítomnost náhodného šumu v analyzovaných datech. V tomto článku odvodím orientované verze aditivních modelů prezentovaných v Li (1998), které berou v úvahu vliv náhodného šumu na efektivnost produkční jednotky. V části věnované aplikaci stochastických modelů analyzuji míru konzistence odhadů technické efektivnosti v závislosti na zvolené metodě. Skóre efektivnosti farem podle přístupu SDEA a DEA je porovnatelné s výsledky, které Druska and Horrace (2004) získal pomocí metody stochastické hranice produkční množiny.

Keywords: stochastic data envelopment analysis, linear programming, efficiency, rice farm
JEL classification: C14, C61, L23, Q12

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1 Introduction

Data envelopment analysis (DEA) involves an non–parametric principle for extracting information about observations of a population of production mixes, so called decision making units (DMUs), that are described by the same quantitative characteristics. The primary objective of this work is to extent the work of Huang and Li (2001) and Li (1998) on additive stochastic DEA models (SDEA) by derivation of SDEA models that allow for proportional input reduction and output augmentation – oriented SDEA models. The empirical part of this paper is motivated by Horrace and Schmidt's (1996) comparison of methods and by Mortimer's (2002) conclusion, that more comparative studies for the DEA and stochastic frontier approach are needed to evaluate the consistency of results with respect to method choice.

Data envelopment analysis, developed by Charnes, Cooper, and Rhodes (1978), involves an alternative approach to stochastic frontier analysis (SFA) that was developed at the same time by Aigner, Lovell, and Schmidt (1977), for efficiency evaluation of the decision process observations. The DEA approach is a nonparametric approach to production frontier estimation and requires specification of the production possibility set properties rather than the production function form that is required when the stochastic frontier approach is used. In contrast to parametric approaches for information extraction, the objective of the DEA is to identify the smallest set that satisfies production possibility properties.

The general model of production function is defined as: $y_j = f(x_j, \beta) + e_j$, where x_j represents inputs, β unknown parameters of production function $f(x_j, \beta)$ and y_j represents output of the DMU_j. The aggregate error term e_j is considered as extent of inefficiency in the DEA approach. In the SFA approach (e.g. Aigner, Lovell, and Schmidt (1977); Meeeusen and van den Broeck (1977)) the error component e_j is decomposed into a stochastic random component and a true technical efficiency component. Therefore, together with the extreme point nature of the DEA, the noise in data may lead to bias in the DEA technical efficiency measure. The dilemma of the efficiency evaluation approach depends on the trade off between the minimal specification of production function form that favors the DEA approach and the handling of stochastic error in measuring efficiency that favors the SFA approach. To compete with the SFA in error handling, the stochastic data envelopment analysis (SDEA) approach was developed by considering the used levels of inputs and outputs as random variables in the DEA model specification.

The theoretical part of this paper extends the work on derivation of almost 100% confidence SDEA models by Li (1998) and Huang and Li (2001) by specification of the performance improvement direction, so called model orientation. Further, assumptions to simplify the disturbance structure are taken and using linearization methods the linear deterministic equivalents of these models are derived. This is utilized in the application section where it allows for the use of the linear programming method to solve SDEA problems. These SDEA results are compared to SFA results, so the consistency of results across frontier estimation methods can be assessed.

The following literature review section presents details of the motivation for the SDEA. In the third and fourth section, notation and definitions used to construct SDEA models are presented. Subsequently, the derivation of Huang and Li's (2001) additive models is summarized and in the fifth section I introduce input reduction and output augmentation directions for efficiency measure definition. In the sixth and following sections, I derive oriented models and their linearized forms. The ninth section describes numerical methods used to solve derived linearized versions of the oriented SDEA models. In the tenth section, I evaluate the SDEA, DEA and SFA efficiency scores consistency assessing the results of the Indonesian rice farms efficiency evaluation, as in Horrace and Schmidt (1996). The comparison of methods reveals inconsistency between efficiency rankings acquired by the SFA approach and SDEA approach. All figures and tables that I reference to, are included in the appendix.

2 Literature review

As Charnes, Cooper, Lewin, and Seiford (1994) explain in their introduction, the story of data envelopment analysis began with Edwardo Rhodes's dissertation, which was the basis for the later published paper by Charnes, Cooper, and Rhodes (1978). In his dissertation, Rhodes used the production efficiency concept by Farrell (1957) to analyze the educational program for disadvantaged students in the USA. Rhodes compared the performance of students from schools participating and not participating in the program. Students' performance was recorded in terms of inputs and outputs, e.g. "increased self–esteem" (measured by psychological tests) as one of the outputs and "time spent by mother reading with child" as one of the inputs. The subsequent work on efficiency evaluation of multiple inputs and outputs technology led to Charnes, Cooper, and Rhodes's (1978) model (CCR model).

The introduced CCR model is suitable for analysis of the technological process under the constant returns to scale assumption. This fact is reflected in the shape of the production possibility frontier when the frontier is formed by a single half–ray and the DMU identified as efficient is an element of the production possibility frontier set up by this half–ray. To handle the variable returns to scale, introduced by Farrell and Fieldhouse (1962) in the SFA framework, the CCR model was reformulated by Banker, Charnes, and Cooper (1984) (BCC model). Since the production possibility frontier of the BCC model is a piecewise linear set, they defined weak efficiency (a weakly efficient DMU has nonzero slacks) and efficiency (an efficient DMU has zero slacks). To review the DEA models Table 1 summarizes a generalized versions of the aforementioned DEA models. The generalized versions of the DEA models collapse to the CCR model (constant returns to scale) for $\varphi = 0$ and for $\varphi = 1$ it matches the form of the BCC model (variable returns to scale).

As many applications suggest, the capability of handling multiple inputs–outputs and the fact that the specification of production function form is not required, make the DEA a powerful tool that is applied in various industries (e.g. in air transportation, Land, Lovell, and Thore (1993); fishing, Walden and Kirkley (2000); banking, Ševčovič, Halická, and Brunovský (2001); health care, Byrnes and Valdmanis (1989) where 123 US hospitals were covered; and in Halme and Korhonen (1998) dental care units were assessed) for technical efficiency evaluation. The expanding number of papers using the DEA approach helped to identify the limitations that an analyst should keep in mind when choosing whether or not to use the approach.

It is worth noting that the DEA approach performs very well when estimating the "relative" efficiency but it is not such a powerful technique when estimating "absolute" efficiency. In other words, the DEA reveals how well the considered DMU is doing compared to the DMU's peers but not compared to a "theoretical maximum". Figure 1 illustrates this situation as the difference between the true production frontier and the estimated production frontier. This difference results from the analyst's limitation in knowledge of the true production function.

A more remarkable limitation originates from the extreme point nature of the DEA approach which makes computed technical efficiency measure sensitive to changes in data. Therefore, noise (even symmetrical noise with zero mean) such as measurement error can cause significant problems. The literature on recent developments for noise incorporation in the DEA identifies three approaches: mixture of the DEA and SFA approaches, bootstrapping, and taking inputs and outputs as random variables.

Gstach (1998) proposes using the DEA technique to estimate a pseudo–production frontier (non–parametric production possibility set estimation) to select the efficient DMUs that identify the production possibility frontier. After this selection, he applies a maximum likelihood–technique to estimate the scalar value in production frontier form, by which this pseudo–frontier must be shifted downward to get the true production frontier (frontier location estimation), using the DEA–estimated efficiencies. Simar (2003) described the iterative bootstrapping method for improving the performance of the deterministic DEA frontier estimation. However, this bootstrapping approach is suitable only for cases where noise to signal ratio is low.

In this work, I focus on the approaches were the noise is introduced by considering DMUs as realizations of random variables. These theoretical attempts are based on Land, Lovell, and Thore's (1993) paper, where the authors use improved models to examine the efficiency of the same schooling program for disabled scholars as in Charnes, Cooper, and Rhodes (1978).Land, Lovell, and Thore (1993) offer the prospect of stochastic data envelopment analysis and constructed their own model (LLT model). The LLT model is derived as a chance constrained version of the BCC output oriented model in envelopment form. Further, they transform these chance constrained problems to their deterministic non–linear equivalents, which allow them to determine the efficient DMUs.

Olesen and Petersen (1995) present a different approach to incorporating the stochastic component into the DEA and their model (OP model) originates from the multiplier formulation of the BCC model. They assume that the inefficiency term of the considered DMU can be decomposed into true inefficiency and disturbance term as in the SFA approach. Further, Olesen (2002) compares the approaches of the models by Olesen and Petersen (1995) and Land, Lovell, and Thore (1993) and identifies weaknesses of both model types. The LLT model is criticized because it does not account for all the correlations that can occur in disturbances. Olesen (2002) criticizes the OP model because it ignores correlations between DMUs. A related weakness is the omission of the fact that a convex combination of two DMUs can have a lower variance than the DMUs considered solely. A straightforward remedy for the OP model is to take the union of confidence regions for any linear combination of the stochastic vectors themselves rather than using a piecewise linear envelopment of the confidence regions. Olesen (2002) implements this idea and derives the combined chance constrained model.

The approach that will be extended in this paper, originates from work by Huang and Li (2001), where inputs and outputs are introduced as random variables and the relation of stochastic efficiency dominance is defined. Huang and Li (2001) define the efficiency dominance of a DMU via joint probabilistic comparisons of inputs and outputs with other DMUs which are evaluated by solving a chance constrained programming problem. By utilizing the theory of chance constrained programming, deterministic equivalents are obtained for both situations of multivariate symmetric random disturbances and a single random factor in production relationships. Under the assumption of the single random factor, Huang and Li (2001) obtain linear deterministic equivalent to stochastic programming problems via linear programming theory. In this paper, I will propose the oriented form of the additive SDEA models derived by Huang and Li (2001). Further, by use of the reviewed linearization approach I linearize the proposed oriented SDEA models.

In the empirical part of this paper, I compare the results of the different methods to productivity evaluation as in Horrace and Schmidt (1996). This comparison is motivated by Mortimer's (2002) comparative study of recent literature that summarizes the results from SFA and DEA studies to identify the amount of correlation between scores in SFA and DEA comparative studies. Mortimer (2002) calls for more studies that will compare efficiency scores correlation across production efficiency approaches because the present comparative studies show either strong (e.g. Ferro–Luzzi, Ramirez, Flückiger, and Vassiliev (2003)) or very weak (e.g. Lan and Lin (2002), Wadud and White (2000)) correlation of obtained efficiency rankings.

The major problems associated with solving the DEA models are the analysis of a large set of DMUs and interpretation of the optimal solutions with zero elements. The analysis of a large data set leads to large size optimization problems that can be costly to solve. The solutions that contain many zero elements can make the results of the analysis questionable because the elements of optimal solutions are interpreted as shadow prices of inputs and outputs. Gonzales-Lima, Tapia, and Thrall (1996) present the primal–dual interior–points computational methods as the methods that significantly improve the reliability of the solution in comparison to simplex methods. The interior–points methods maximize the product of the positive components in the optimal solutions, so they identify optimal solution with the minimal number of zero components. Due to this property of the optimal solution it is easier to interpret the DEA models results. Therefore, as part of my theoretical work the interior point method solver is constructed.

3 Notation

In this section, the notation used to construct the oriented stochastic DEA models is introduced. Additional notation will be introduced in the following section to describe the considered error structure. In contrast to the deterministic approach to envelopment analysis, where DMUs are observations of decision realization, the DMUs in the stochastic approach are characterized by random variables and the technology realizations are observations of these random variables. The notation in this paper coincides with the notation usually found in data envelopment analysis literature (e.g. Charnes, Cooper, Lewin, and Seiford (1994),Cooper, Huang, Lelas, Li, and Olesen (1998), and Huang and Li (2001)).[1] The task is to analyze the set of DMU_j, where $1 \leq j \leq n$. Each of the DMUs is described by a random vector $\tilde{x}_j$, $\tilde{x}_j = (\tilde{x}_{1j}, \ldots, \tilde{x}_{mj})^T$ of m input amounts (random variables) that are used to produce s outputs in amounts described by random vector $\tilde{y}_j$, $\tilde{y}_j = (\tilde{y}_{1j}, \ldots, \tilde{y}_{sj})^T$. These vectors are aggregated to matrices of random vectors of inputs and outputs, so the following matrix notation will be used:

[1]In the following text the random variables are denoted by ˜ and means of these variables are denoted by an upper bar.

matrix of inputs random vectors	$\tilde{X} = (\tilde{x}_1, \ldots, \tilde{x}_n)$
i^{th} row of "input" matrix $\tilde{X}$	${}_i\tilde{x} = (\tilde{x}_{i1}, \ldots, \tilde{x}_{in}),\ i = 1, \ldots, m$
$m \times n$ matrix of expected inputs	$\bar{X} = (\bar{x}_1, \ldots, \bar{x}_n)$
i^{th} row of expected "input" matrix $\bar{X}$	${}_i\bar{x} = (\bar{x}_{i1}, \ldots, \bar{x}_{in}),\ i = 1, \ldots, m$
matrix of outputs random vectors	$\tilde{Y} = (\tilde{y}_1, \ldots, \tilde{y}_n)$
r^{th} row of "output" matrix $\tilde{Y}$	${}_r\tilde{x} = (\tilde{y}_{r1}, \ldots, \tilde{y}_{rn}),\ r = 1, \ldots, s$
$s \times n$ matrix of expected outputs	$\bar{Y} = (\bar{y}_1, \ldots, \bar{y}_n)$
r^{th} row of expected "output" matrix $\bar{Y}$	${}_r\bar{y} = (\bar{y}_{r1}, \ldots, \bar{y}_{rn}),\ r = 1, \ldots, s.$

4 Stochastic efficiency dominance

In this section, the efficiency dominance relation and derivation of additive almost 100% chance constrained models by Huang and Li (2001) is reviewed. These theorems and definitions form the basis for derivation of the oriented SDEA derived in the following sections.

Definition 1. General stochastic production possibility set $T \subset \mathbb{R}_+^{m+s}$ is defined as: $T = \{(\tilde{x}, \tilde{y}) \mid$ outputs $\tilde{y}$ can be produced using inputs $\tilde{x}\}$.[2]

This definition of the stochastic production possibility set relates to random vectors that characterize DMUs and it means that all DMUs are required to be an element of the stochastic production possibility set but not all observations of DMUs are required to be in the stochastic production possibility set. As mentioned in the literature review, the function form is not known, therefore the estimate of the production possibility set is identified by the properties that the production possibility set should fulfill.

Almost 100% confidence production possibility set T constructed from the set of $\text{DMU}_j,\ j = 1, \ldots, n$ should fulfill the following properties:

Property 1. Convexity: If $(\tilde{x}_j, \tilde{y}_j) \in T$, $j = 1, \ldots, n$ and $\lambda \in \mathbb{R}_+^n$, $\Rightarrow (\tilde{X}\lambda, \tilde{Y}\lambda) \in T$.

[2]Here, $\mathbb{R}_+$ means set of positive real numbers and $\mathbf{1}$ is column vector of ones.

Property 2. Inefficiency property: If $(\bar{x}, \bar{y}) \in T$ and $x \geq \bar{x}$, then $(x, \bar{y}) \in T$. If $(\bar{x}, \bar{y}) \in T$ and $y \leq \bar{y}$ then $(\bar{x}, y) \in T$.

Property 3. Minimum extrapolation: T is the intersection of all sets satisfying convexity and inefficiency property and subject to each of the observed random vectors $(\tilde{x}_j, \tilde{y}_j) \in T, j = 1, \ldots, n$.

From the first two properties follows that less output can be produced with the same amount of inputs. This reflects the situation when some portion of inputs is wasted in the production process. The parametric production possibility set T_φ; $T_\varphi = \{(\tilde{x}, \tilde{y}) \mid \tilde{x} \geq \tilde{X}\lambda,\ \tilde{y} \leq \tilde{Y}\lambda,\ \varphi(\mathbf{1}^T\lambda) = \varphi,\ \lambda \geq 0\}$, where $\varphi \in \{0, 1\}$, satisfies all aforementioned properties. T_0 is the stochastic generalization of the production possibility set under the assumption of the constant returns to scale production function as used by Charnes, Cooper, and Rhodes (1978) in the derivation of the CCR model. Similarly, the stochastic generalization of the production possibility set T_1 will be used to derive models with variable returns to scale as in a case of the BCC model by Banker, Charnes, and Cooper (1984).

The concept of efficiency in the DEA (based on the following relative efficiency definition) is used to define the α–stochastic efficiency dominance.

Definition 2. Relative Efficiency: A DMU is to be identified as efficient on the basis of available evidence if and only if the performances of other DMUs does not show that some of its inputs or outputs can be improved without worsening some of its other inputs or outputs.

The efficient point of the production possibility set is identified if there is no other production point that produces more output without consuming more input, or consumes less input without producing less output. This leads to the following efficiency domination definition of the production possibility set element:

Definition 3. Efficiency dominance relation: The point (x, y) is not dominated in the sense of efficiency if $\nexists\ (x^*, y^*)$ in the production possibility set such that $x^* \leq x$

or $y^* \geq y$ with at least one strict inequality for input or output components.

This definition demonstrates the efficiency concept of the DEA and is used to derive the deterministic models with no possibility of a violation of the production possibility set properties or efficiency dominance. In the deterministic environment, the non–dominated DMUs are elements of the production possibility set frontier. Figure 1 illustrates this situation where the set of DMUs is divided into efficient (DMU1, DMU2 and DMU3) and inefficient DMUs (DMU4 and DMU5). The efficient DMUs – points that dominate in efficiency the other elements of the production possibility set – are used to identify the production possibility frontier.

In the stochastic framework, where efficiency dominance can be violated due to random errors, the efficiency dominance violations are allowed with the probability α, $0 \leq \alpha \leq 1$. In chance constrained programming methodology the term $1 - \alpha$ is interpreted as the modeler's confidence level and α is interpreted as the modeler's risk (the extent of conditions violations). In the almost 100% confidence approach, the production possibility constraints are almost certainly not violated and the efficiency dominance can be violated with probability α. For the case of the almost 100% confidence chance constrained approach, Li (1998) and Huang and Li (2001) define the α–stochastically efficiency of point as:

Definition 4. α–stochastic efficiency of point in set T_φ: $(\tilde{x}^*, \tilde{y}^*) \in T_\varphi$ is called α–stochastically efficient point associated with $T_\varphi \Leftrightarrow$ if the analyst is confident that $(\tilde{x}^*, \tilde{y}^*)$ is efficient with probability $1 - \alpha$ in the set T_φ.

Definition 4 means that point $(\tilde{x}^*, \tilde{y}^*)$, considered as α–stochastically efficient may be dominated (in the sense of efficiency dominance) by any other point in T_φ with a probability less or equal to α. For the DMU_j associated with this point this definition is used to evaluate the α–stochastic efficiency of DMU_j.

This definition and the aforementioned properties of the set T_φ straightforwardly imply that for the efficient DMU_j and for any $\lambda_j \in \mathbb{R}^n_+$ such that $\varphi(\mathbf{1}^T\lambda_j) = \varphi, \lambda \geq 0$

the expression $Prob(\tilde{X}\lambda_j \leq \tilde{x}_j, \tilde{Y}\lambda_j \geq \tilde{y}_j) \leq \alpha$ holds with at least one strict inequality in input–output constraints.

To illustrate the DEA and almost 100% confidence SDEA approach, Figure 1 illustrates the relation of the deterministic frontier to the possible true production possibility frontier. The solid piecewise linear line is the possible true production possibility frontier and the dashed line is the DEA estimate of this production possibility frontier. In Figure 2 the expected values of DMUs (same values as the observations in Figure 1) are pictured and the set of α–efficiency dominant elements is presented as a grey shaded area. A comparison of Figures 1 and 2 shows that for the almost 100% confidence SDEA approach, the deterministic production possibility set frontier is a subset of the stochastic possibility set frontier. Due to this fact more DMUs can be identified as efficiency dominant in the stochastic framework than in the deterministic.

4.1 Stochastic model

In this subsection, the derivation of the almost 100% confidence chance constrained problem is reviewed. The reviewed stochastic model for assessing efficiency of DMU$_j$ is the equivalent to the additive DEA model and serves as the basis for the further theoretical development of SDEA models. In the following subsection, specific assumptions about the error structure in the data are made and the stochastic model is transformed into its deterministic equivalent.

Now, from the set properties for the virtual peers $(\tilde{X}\lambda, \tilde{Y}\lambda)$ that are used for evaluation of efficiency of DMU$_j$ follows that

$$\{\tilde{X}\lambda \leq \tilde{x}_j, \tilde{Y}\lambda \geq \tilde{y}_j\} \quad \subset \quad \{\mathbf{1}^T(\tilde{X}\lambda - \tilde{x}_j) + \mathbf{1}^T(\tilde{y}_j - \tilde{Y}\lambda) < 0\} \qquad (1)$$

and using the probability properties the following inequality is derived:[3]

$$Prob(\tilde{X}\lambda \leq \tilde{x}_j, \tilde{Y}\lambda \geq \tilde{y}_j) \leq Prob(\mathbf{1}^T(\tilde{X}\lambda - \tilde{x}_j) + \mathbf{1}^T(\tilde{y}_j - \tilde{Y}\lambda) < 0).$$

Therefore, for $\lambda \in \mathbb{R}^n_+$ such that $\varphi(\mathbf{1}^T\lambda) = \varphi$ and $\lambda \geq 0$ the condition

$$Prob(\mathbf{1}^T(\tilde{X}\lambda - \tilde{x}_j) + \mathbf{1}^T(\tilde{y}_j - \tilde{Y}\lambda) < 0) \leq \alpha$$

is a necessary condition for the DMU$_j$ to be α–stochastically efficient. Using the necessary condition for α–stochastic efficiency of the DMU$_j$, the following almost 100% confidence chance constrained problem (in matrix notation) for the technical efficiency evaluation of the DMU$_j$, $j = 1, \ldots, n$ is constructed (Cooper, Huang, Lelas, Li, and Olesen (1998), Li (1998) and Huang and Li (2001))

$$\begin{aligned} \max_{\lambda_j} \quad & Prob(\mathbf{1}^T(\tilde{X}\lambda_j - \tilde{x}_j) + \mathbf{1}^T(\tilde{y}_j - \tilde{Y}\lambda_j) < 0) - \alpha \\ s.t. \quad & Prob({}_i\tilde{x}\lambda_j < \tilde{x}_{ij}) \geq 1 - \epsilon, \qquad i = 1, \ldots, m; \\ & Prob({}_r\tilde{y}\lambda_j > \tilde{y}_{rj}) \geq 1 - \epsilon, \qquad r = 1, \ldots, s; \\ & \varphi(\mathbf{1}^T\lambda_j) = \varphi, \\ & \lambda_j \geq 0, \end{aligned} \tag{2}$$

where ϵ is a non–Archimedean infinitesimal quantity.[4] The optimal solution of problem 2 is related to the stochastic efficiency of the DMU$_j$ by following two theorems which are direct corollaries of Theorem 3 by Cooper, Huang, Lelas, Li, and Olesen (1998):[5]

[3] The inequality type change is due to the additional restriction that $\{\tilde{X}\lambda \leq \tilde{x}_j, \tilde{Y}\lambda \geq \tilde{y}_j\}$ holds with at least one strict inequality. The accuracy of this simplification is closely discussed in Ruszczynski and Shapiro (2003).

[4] This means that ϵ is a very small positive number such that $\sum_{i=1}^{n} \epsilon < 1$ no matter how large is n. According to the chapter "Computational Aspects of DEA" in Charnes, Cooper, Lewin, and Seiford (1994), $\epsilon < \min_{j=1,\ldots,n} 1/(\sum_{i=1}^{m} x_{ij})$ is selected in the calculations of these models.

[5] See Theorem 3 and its proof in Cooper, Huang, Lelas, Li, and Olesen (1998).

Theorem 1. *Let the DMU_j be α–stochastically efficient. The optimal value of the objective function in the chance constrained programming problem 2 is less than or equal to zero.*

Theorem 2. *If the optimal value objective functional of problem 2 is greater than zero, then DMU_j is not α–stochastically efficient.*

Theorem 2 implies that if the maximum value of the chance functional $Prob(\mathbf{1}^T(\tilde{X}\lambda_j - \tilde{x}_j) + \mathbf{1}^T(\tilde{y}_j - \tilde{Y}\lambda_j) < 0)$ exceeds α, then the considered DMU_j is not α–stochastically efficient. The value of the chance functional of the additive SDEA model represented by problem 2 can be used as the simplest efficiency measure when interpreted as the sum of input excess and output slack. In the section on derivation of the oriented SDEA models, I introduce measures based on possible proportional input reduction or output augmentation.

4.2 Error structure

In this subsection, the error structure that allows the transformation of the model from a chance constrained problem to a linear deterministic equivalent is introduced and the linearization approach by Cooper, Huang, Lelas, Li, and Olesen (1998) is summarized. The following structure of m inputs and s outputs of the DMU_j, for $j = 1, \ldots, n$ with noise driven by normally distributed shocks is considered

$$\begin{aligned} \tilde{x}_{ij} &= \bar{x}_{ij} + a_{ij}\zeta_{ij} \qquad i = 1, \ldots, m; \\ \tilde{y}_{ij} &= \bar{y}_{ij} + b_{ij}\xi_{rj}, \qquad r = 1, \ldots, s; \end{aligned} \tag{3}$$

where it is assumed $E(\zeta_{ij}) = E(\xi_{rj}) = 0$, $j = 1, \ldots, n$ and the following variance–covariance structure of errors for all DMUs is assumed:[6]

$$
\begin{aligned}
Var(\zeta_{ij}) = Var(\xi_{rj}) = \sigma_\varepsilon^2 \quad & 1 \le i \le m;\ 1 \le r \le s;\ 1 \le j \le n; \\
Cov(\zeta_{ij}, \zeta_{kl}) = 0 \quad & 1 \le i, k \le m;\ 1 \le j, l \le n; \\
Cov(\xi_{rj}, \xi_{kl}) = 0 \quad & 1 \le r, k \le s;\ 1 \le j, l \le n; \\
Cov(\xi_{rj}, \zeta_{il}) = 0 \quad & 1 \le r \le s;\ 1 \le i \le m;\ , 1 \le j, l \le n.
\end{aligned}
$$

Under this error structure follows that inputs and outputs are normally distributed with $E(\tilde{x}_{ij}) = \bar{x}_{ij}$, $E(\tilde{y}_{rj}) = \bar{y}_{rj}$ and variance $Var(\tilde{x}_{ij}) = (a_{ij}\sigma_\varepsilon)^2$, $Var(\tilde{y}_{rj}) = (b_{rj}\sigma_\varepsilon)^2$.

When assessing the production processes it is also reasonable to consider the case of log–normally distributed variables. In the case of log–normality of inputs and outputs with disturbances driven by normal random variables, the following structure of inputs and outputs can be considered:

$$
\begin{aligned}
\tilde{x}_{ij}^{log} &= exp(\bar{x}_{ij} + a_{ij}\zeta_{ij}) \qquad i = 1, \ldots, m; \\
\tilde{y}_{ij}^{log} &= exp(\bar{y}_{ij} + b_{ij}\xi_{rj}), \qquad r = 1, \ldots, s.
\end{aligned} \tag{4}
$$

The log–normal input–output structure can be transformed to normal input–output structure by taking logs, therefore in the following text I assume only the input–output structure with normally distributed input and output variables.

Additionally, when assuming $\varepsilon = \xi_{ij} = \xi_{kl} = \zeta_{rj} = \zeta_{il}$, for $1 \le r \le s$; $1 \le i \le m$; $1 \le j, l \le n$ then the assumed error structure collapses to a single factor symmetric error structure where ε follows normal distribution with $E(\varepsilon) = 0, Var(\varepsilon) = \sigma_\varepsilon^2$. To simplify this notation, the vectors

[6] For linearization procedure the standard normal distribution $N(0, 1)$ can be assumed. The scaling of the measurement units is used when numerical problems with tiny diagonals of the input–output variance matrices occurs, therefore the more general assumption of $N(0, \sigma_\varepsilon^2)$ is used. This simplifying assumption also reduces the number of parameters to be estimated for efficiency evaluation to $2n(m + s)$. Without simplifying assumption $[n^2(m + s)^2 + 3n(m + s)]/2$ parameters are needed to be estimated.

$$a_j = (a_{1j}, \ldots, a_{mj})^T, \quad b_j = (b_{1j}, \ldots, b_{sj})^T, \quad j = 1, \ldots, n;$$
$${}_ia = (a_{i1}, \ldots, a_{in}), \qquad {}_rb = (b_{r1}, \ldots, b_{rn}), \quad i = 1, \ldots, m, \; r = 1, \ldots, s;$$

are introduced and these vectors are aggregated to construct the following matrices of input and output variations $A_{m\times n} = (a_1, \ldots, a_n), B_{s\times n} = (b_1, \ldots, b_n)$. Using the properties of normal distribution it is derived that ${}_i\tilde{x}\lambda_j - \tilde{x}_{ij}$ is distributed according to $N({}_i\bar{x}\lambda_j - \bar{x}_{ij}; ({}_ia\lambda_j - a_{ij})^2\sigma_\varepsilon^2)$ and $({}_r\tilde{y}\lambda_j - \tilde{y}_{rj})$ is normally distributed according to $N({}_r\bar{y}\lambda_j - \bar{y}_{rj}; (b_{rj} - {}_rb\lambda_j)^2\sigma_\varepsilon^2)$. Applying the inverse cumulative distribution function $\Phi^{-1}(\alpha)$, the constraints and objective function in the almost 100% confidence chance constrained problem 2 can be rewritten as in Cooper, Huang, Lelas, Li, and Olesen (1998) or Huang and Li (2001) and the following deterministic equivalent of problem 2 is derived:

$$\min_{\lambda_j \in \mathbb{R}_+^{m+s}} \; \mathbf{1}^T(\bar{X}\lambda_j - \bar{x}_j) + \mathbf{1}^T(\bar{y}_j - \bar{Y}\lambda_j) + \mid \mathbf{1}^T(A\lambda_j - a_j) + \mathbf{1}^T(b_j - B\lambda_j) \mid \sigma_\varepsilon \Phi^{-1}(\alpha) \quad (5)$$

$$\begin{aligned} s.t. \quad & {}_i\bar{x}\lambda_j \le \bar{x}_{ij} + \mid {}_ia\lambda_j - a_{ij} \mid \sigma_\varepsilon\Phi^{-1}(\epsilon), \quad i = 1, \ldots, m, \\ & \bar{y}_{rj} \le {}_r\bar{y}\lambda_j + \mid b_{rj} - {}_rb\lambda_j \mid \sigma_\varepsilon\Phi^{-1}(\epsilon), \quad r = 1, \ldots, s, \\ & \varphi(\mathbf{1}^T\lambda_j) = \varphi, \\ & \lambda_j \ge 0. \end{aligned}$$

Applying the linearization procedure, new variables q_{1r}, q_{2r}, h_{1i}, h_{2i} and the cumulative term $\epsilon(\sum_{r=1}^{s}(q_{1r} + q_{2r}) + \sum_{i=1}^{m}(h_{1i} + h_{2i}))$ introduced into the objective function allows for the decomposition of the absolute value terms and to linearize the constraints in problem 5.[7] Moreover, this modification does not affect the optimal solutions of problem 5 and this problem is equivalent to the following

[7] For simplicity of notation, in the following text the index j is omitted in the terms q_{1r}, q_{2r}, h_{1i}, h_{2i} that are used to replace the absolute value term.

problem with linear constraints:

$$\min_{\lambda_j, q_{kr}, h_{ki}} \mathbf{1}^T(\bar{X}\lambda_j - \bar{x}_j) + \mathbf{1}^T(\bar{y}_j - \bar{Y}\lambda_j) + \tag{6}$$

$$+ \mid \mathbf{1}^T(A\lambda_j - a_j) + \mathbf{1}^T(b_j - B\lambda_j) \mid \sigma_\varepsilon \Phi^{-1}(\alpha) + \epsilon(\sum_{r=1}^{s}(q_{1r} + q_{2r}) + \sum_{i=1}^{m}(h_{1i} + h_{2i}))$$

$$\begin{aligned} s.t. \quad & {}_i\bar{x}\lambda_j \leq \bar{x}_{ij} + (h_{1i} + h_{2i})\sigma_\varepsilon \Phi^{-1}(\epsilon), & \\ & {}_i a\lambda_j - a_{ij} = h_{1i} - h_{2i}, & i = 1, \ldots, m, \\ & \bar{y}_{rj} \leq {}_r\bar{y}\lambda_j + (q_{1r} + q_{2r})\sigma_\varepsilon \Phi^{-1}(\epsilon), & \\ & b_{rj} - {}_r b\lambda_j = q_{1r} - q_{2r}, & r = 1, \ldots, s, \\ & \varphi(\mathbf{1}^T\lambda_j) = \varphi, & \\ & \lambda_j \geq 0, q_{kr} \geq 0, h_{ki} \geq 0, & k = 1, 2. \end{aligned}$$

In the following step, the absolute value from the objective function is removed. The inverse of cumulative distribution function $\Phi(\alpha)$ takes a positive or negative values; to account for this factor let's define δ such that

$$\delta = \begin{cases} -1 & \text{if } \alpha < 0.5; \\ 0 & \text{if } \alpha = 0.5; \\ 1 & \text{if } \alpha > 0.5. \end{cases}$$

The absolute value term in the objective function is the sum of the absolute value terms in the constraints of problem 6; therefore, the decomposition that was used in these constraints is just substituted in the objective function. Thus as in used literature (e.g. Li (1998) and Huang and Li (2001)), the absolute value terms are eliminated from the objective function and the following problem with a linear objective function is obtained:

$$\min_{\lambda_j, q_{kr}, h_{ki}} \mathbf{1}^T(\bar{X}\lambda_j - \bar{x}_j) + \mathbf{1}^T(\bar{y}_j - \bar{Y}\lambda_j) + \tag{7}$$

$$+\delta(\mathbf{1}^T(A\lambda_j - a_j) + \mathbf{1}^T(b_j - B\lambda_j))\sigma_\varepsilon\Phi^{-1}(\alpha) + \epsilon(\sum_{r=1}^{s}(q_{1r} + q_{2r}) + \sum_{i=1}^{m}(h_{1i} + h_{2i}))$$

$$\begin{aligned} s.t.\quad & {}_i\bar{x}\lambda_j \leq \bar{x}_{ij} + (h_{1i} + h_{2i})\sigma_\varepsilon\Phi^{-1}(\epsilon), & \\ & {}_i a\lambda_j - a_{ij} = h_{1i} - h_{2i}, & i = 1, \ldots, m, \\ & \bar{y}_{rj} \leq {}_r\bar{y}\lambda_j + (q_{1r} + q_{2r})\sigma_\varepsilon\Phi^{-1}(\epsilon), & \\ & b_{rj} - {}_r b\lambda_j = q_{1r} - q_{2r}, & r = 1, \ldots, s, \\ & \varphi(\mathbf{1}^T\lambda_j) = \varphi, & \\ & \lambda_j \geq 0, q_{kr} \geq 0, h_{ki} \geq 0, & k = 1, 2. \end{aligned}$$

Problem 7 is known as the envelopment formulation of the DEA model, because the optimal solution identifies the projected point on to the envelopment surface for DMU_j. Using Li's (1998) definition of the dual problem, the dual problem 8 to primal problem 7 is restated as:

$$\max_{\mu,\nu,\eta,\omega,\psi_j} \mu^T\bar{y}_j - \nu^T\bar{x}_j - \eta^T b_j - \omega^T a_j - \varphi\psi_j \tag{8}$$

$$\begin{aligned} s.t.\quad & \mu^T\bar{y}_l - \nu^T\bar{x}_l - \eta^T b_l - \omega^T a_l - \varphi\psi_j \leq 0,\ l = 1, \ldots, n; \\ & -\sigma_\varepsilon\Phi^{-1}(\varepsilon)\mu + \eta \geq -\sigma_\varepsilon(\Phi^{-1}(\varepsilon) + \varepsilon)\mathbf{1} - \delta\sigma_\varepsilon\Phi^{-1}(\alpha)\mathbf{1}, \\ & -\sigma_\varepsilon\Phi^{-1}(\varepsilon)\mu - \eta \geq -\sigma_\varepsilon(\Phi^{-1}(\varepsilon) + \varepsilon)\mathbf{1} + \delta\sigma_\varepsilon\Phi^{-1}(\alpha)\mathbf{1}, \\ & -\sigma_\varepsilon\Phi^{-1}(\varepsilon)\nu - \omega \geq -\sigma_\varepsilon(\Phi^{-1}(\varepsilon) + \varepsilon)\mathbf{1} - \delta\sigma_\varepsilon\Phi^{-1}(\alpha)\mathbf{1}, \\ & -\sigma_\varepsilon\Phi^{-1}(\varepsilon)\nu + \omega \geq -\sigma_\varepsilon(\Phi^{-1}(\varepsilon) + \varepsilon)\mathbf{1} + \delta\sigma_\varepsilon\Phi^{-1}(\alpha)\mathbf{1}, \\ & \mu \geq \mathbf{1} \\ & \nu \geq \mathbf{1}, \\ & \eta, \omega, \psi_j \text{ unconstrained.} \end{aligned}$$

For the DMU_j represented by point $(\tilde{x}_j, \tilde{y}_j)$, the following stochastic hyperplane

$Prob(c^T\tilde{x}_j + d^T\tilde{y}_j + f_j \leq 0) = 1-\epsilon$ is the supporting hyperplane for T_φ at $(\tilde{x}_j, \tilde{y}_j)$ if and only if

$$c^T\tilde{x}_j + d^T\tilde{y}_j + f_j + \Phi^{-1}(\epsilon)\sigma_\varepsilon \mid c^T a_j + d^T b_j \mid = 0 \tag{9}$$

$$\text{and for } \forall\, (\tilde{x}, \tilde{y}) \in T_\varphi : \; c^T\tilde{x} + d^T\tilde{y} + f_j + \Phi^{-1}(\epsilon)\sigma_\varepsilon \mid c^T a_j + d^T b_j \mid \geq 0. \tag{10}$$

The dual problem 8 is known as the multiplier problem because the optimal solutions $(\mu_j^*, \nu_j^*, \eta_j^*, \omega_j^*, \psi_j^*)$, for $j = 1, \ldots, n$, set up the supporting hyperplanes that are used to construction the production possibility frontier. If there is an unique optimal solution $(\mu_j^*, \nu_j^*, \eta_j^*, \omega_j^*, \psi_j^*)$ to problem 8 that satisfies

$$\mu_j^{*T}(b_j - b_k) + \nu_j^{*T}(a_j - a_k) - \Phi^{-1}(\epsilon)\sigma_\varepsilon(\mid \mu_j^{*T} b_j - \nu_j^{*T} a_j \mid - \mid \mu_j^{*T} b_k - \nu_j^{*T} b_k \mid) \geq 0,$$

for $k = 1, \ldots, n$, then the optimal solution $(\mu_j^*, \nu_j^*, \eta_j^*, \omega_j^*, \psi_j^*)$ identifies the following stochastic hyperplane $Prob(\mu_j^{*T}\tilde{y}_j - \nu_j^{*T}\tilde{x}_j + f_j^* \leq 0) = 1-\epsilon$, where $f_j^* = -\eta_j^{*T} b_j - \omega_j^{*T} a_j - \varphi\psi_j^* + \Phi^{-1}(\epsilon)\sigma_\varepsilon \mid \mu_j^{*T} b_j - \nu_j^{*T} a_j \mid$. This almost 100% confidence hyperplane is the supporting hyperplane to T_φ at the DMU_j. Further, in the section on returns to scale, the sign of f_j is related to the returns to scale type and these relations are summarized in Table 2. In a case without a unique optimal solution to problem 8, the supporting hyperplane for T_φ at $(\tilde{x}_j, \tilde{y}_j)$ is not uniquely identified.

5 Efficiency measure

In this section, by introducing the input reducing and output augmenting direction for projection into the data envelopment I derive the extension to the reviewed additive models. As explained in the previous section, the optimal solution to the envelopment problem 7 for the DMU$_j$ identifies the point $(\hat{x}_j, \hat{y}_j) = (\bar{X}\lambda_j^*, \bar{Y}\lambda_j^*)$ and the optimal solution of the multipliers problem 8 identifies the supporting hyperplane assigned to the DMU_j. Therefore, the simplest inefficiency measure can be defined by the distance measure of a discrepancy between the projected and expected point as: $|(\hat{x}_j, \hat{y}_j) - (\bar{x}_j, \bar{y}_j)|$. This discrepancy measure expresses the difference between the efficient frontier represented by the projected point $(\hat{x}_j, \hat{y}_j)$ and the present position of the DMU_j. Starting from $(\bar{x}_j, \bar{y}_j)$, various projection paths on the corresponding part of the envelopment surface can be followed as is illustrated by Figure 3. Figure 3 illustrates directions of inputs reduction and augmentation in outputs. I will use these two directions to derive the input and output oriented efficiency measures that are used to state the oriented SDEA models.

First, for inputs of the DMU$_j$ let's denote $e_{ij} \in \mathbb{R}_+$, $e_{ij} = \bar{x}_{ij} - {}_i\bar{x}\lambda_j$, $i = 1, \ldots, m$ and define the column vector of inputs excess $e_j \in \mathbb{R}^m_+$, $e_j = (e_{1j}, \ldots, e_{mj})^T$. If the following inequality $Prob({}_i\tilde{x}\lambda_j < \tilde{x}_{ij}) > 1 - \epsilon$ holds there must exist $e_{ij} > 0$, $i \in \{1, \ldots, m\}$ such that $Prob(e_{ij} \leq \tilde{x}_{ij} - {}_i\tilde{x}\lambda_j) = 1 - \epsilon$. Therefore, for inputs of the DMU_j, by following the path $-e_j$ the inputs can be decreased and the projected point is moved towards the production possibility frontier. This projection direction is given in Figure 3 as the input reduction direction and the point DMU5i is the input oriented projection of the DMU#5.

Similarly, the DEA output oriented model is derived using the column vector of output slacks $s_j \in \mathbb{R}^s_+, s_j = (s_{1j}, \ldots, s_{sj})^T$, $s_{rj} = {}_r\bar{y}\lambda_j - \bar{y}_{rj}$, $r = 1, \ldots, s$. For $r \in \{1, \ldots, s\}$ such that $Prob({}_r\tilde{y}\lambda_j > \tilde{y}_{rj}) > 1 - \epsilon$ exists $s_{rj} > 0$ for which the following equality holds: $Prob({}_r\tilde{y}\lambda_j - \tilde{y}_{rj} \geq s_{rj}) = 1 - \epsilon$. The path s_j projects the DMU$_j$ on to the production possibility frontier in an outputs augmenting direction and the projected point is shown in Figure 3 as the DMU5o.

Next, to determine the maximal scale effects in inputs reduction or outputs augmentation, the projection paths s_j, e_j are decomposed to a proportional increase (decrease) of output (input) and residual as follows: $s_j = \rho_j\bar{y}_j + \delta^j_s$, $e_j = \gamma_j\bar{x}_j + \delta^j_e$, where a proportional increase of outputs ρ_j and proportional decrease of inputs γ_j

for $j = 1, \ldots, n$ are defined as

$$\rho_j = \min_{r=1,\ldots,s} \frac{\hat{y}_{rj} - \bar{y}_{rj}}{\bar{y}_{rj}} \geq 0,$$
$$\gamma_j = \min_{i=1,\ldots,m} \frac{\bar{x}_{ij} - \hat{x}_{ij}}{\bar{x}_{ij}} \geq 0,$$

and $\delta_e^j \geq 0, \delta_s^j \geq 0,\ j = 1, \ldots, n$.[8]

Next as in Ali and Seiford (1993), the new variables for the output oriented model are defined as $\phi_j = 1 + \rho_j$ and for the input oriented model $\theta_j = 1 - \gamma_j$. From the construction of the scaling parameters, the θ_j satisfies $0 < \theta_j \leq 1$ and for ϕ_j in the output problem we have $\phi_j \geq 1$. The maximal output scale effect is identified by optimal value ϕ_j^* and the maximal input reduction is identified by the optimal value of θ_j^*.

For the identification of possible proportional scaling of inputs or outputs and efficiency evaluation of the DMU_j, two stage models are constructed. In the first model stage, the maximal ϕ_j or minimal θ_j is found to identify the maximal equi–proportional effect. In the second stage of modelling, the identified scale effect is utilized to evaluate the efficiency of the DMU_j with optimally reduced levels of inputs (augmented levels of outputs, in case of the output oriented model). These two stage models are summarized in Table 3. The optimal solution to the first stage for the DMU_j is denoted as $\hat{\theta}_j$ and in the case of the output oriented model $\hat{\phi}_j$. The second stage of almost 100% confidence problem is constructed by replacing $\bar{x}_j$ (in output oriented model: $\bar{y}_j$) with $\hat{\theta}_j \bar{x}_j$ (respectively for input model with: $\hat{\phi}_j \bar{y}_j$) in constraints and objective function of problem 2 as presented in Table 3.

When the two stage models are used, the inefficiency of the DMU_j can be evaluated by use of values of $\hat{\phi}_j^{-1}$ or $\hat{\theta}_j$. The major drawback of use of $\hat{\phi}_j^{-1}$ and $\hat{\theta}_j$ as inefficiency measures of the DMU_j is that these measures do not uniquely identify efficient points. This shortage is present because for $\hat{\phi}_j = 1$ ($\hat{\theta}_j = 1$) the

[8] Note that at least one component of each δ is zero because of the projection on to the production possibility frontier.

DMU_j is the boundary point of T_φ but the positive non–proportional slacks can be present. The elements of production possibility set with $\hat{\phi}_j = 1$ ($\hat{\theta}_j = 1$) and positive non–proportional slacks are usually referred to as weakly efficient points. Due to the aforementioned shortage, the identification of efficiency of the DMU_j has to be done in two stages. Therefore, the DMU_j is identified as efficient if the proportional scaling parameter equality $\hat{\phi}_j = 1$ ($\hat{\theta}_j = 1$) holds and the second stage model identify the DMU_j as α–stochastically efficient. The additional condition on slacks is referred to as the sum of slacks and for α–stochastic efficiency it is required that it holds with probability $1 - \alpha$.

6 Oriented SDEA models

In both stages the objective function optimization is subject to the same constraints, the only difference being the objective function, therefore the two stage oriented SDEA models can be merged into a one–stage model. To merge these stages in one optimization problem, the non–Archimedean ϵ is used as a weight for the second stage objective function. The choice of non–Archimedean ϵ as the weight guarantees that proportional movement towards the frontier pre–empts the additive slacks optimization.

Output oriented model The one stage model for evaluation of efficiency of the DMU_j is derived from the two stages optimization model presented in Table 3 and can be stated as:

$$\begin{aligned}
\max_{\lambda_j,\phi_j} \quad & \phi_j + \epsilon(Prob(\mathbf{1}^T(\tilde{X}\lambda_j - \tilde{x}_j) + \mathbf{1}^T(\phi_j\tilde{y}_j - \tilde{Y}\lambda_j) < 0) - \alpha) && (11)\\
s.t. \quad & Prob({}_i\tilde{x}\lambda_j < \tilde{x}_{ij}) \geq 1 - \epsilon, && i = 1, \ldots, m;\\
& Prob({}_r\tilde{y}\lambda_j > \phi_j\tilde{y}_{rj}) \geq 1 - \epsilon, && r = 1, \ldots, s;\\
& \varphi(\mathbf{1}^T\lambda_j) = \varphi;\\
& \lambda_j \geq 0.
\end{aligned}$$

After the same linearization procedure that was applied to problem 2 and reviewed in the fourth section of this paper, the following linear model is derived:

$$\begin{aligned}
\max_{\lambda_j,q_{kr},h_{ki},\phi_j} \quad & \phi_j - \epsilon[\mathbf{1}^T(\bar{X}\lambda_j - \bar{x}_j) + \mathbf{1}^T(\phi_j\bar{y}_j - \bar{Y}\lambda_j) + && (12)\\
& +\delta(\mathbf{1}^T(A\lambda_j - a_j) + \mathbf{1}^T(\phi_j b_j - B\lambda_j))\sigma_\varepsilon\Phi^{-1}(\alpha)] + \epsilon(\sum_{r=1}^{s}(q_{1r} + q_{2r}) + \sum_{i=1}^{m}(h_{1i} + h_{2i}))\\
s.t. \quad & {}_i\bar{x}\lambda_j \leq \bar{x}_{ij} + (h_{1i} + h_{2i})\sigma_\varepsilon\Phi^{-1}(\epsilon),\\
& {}_ia\lambda_j - a_{ij} = h_{1i} - h_{2i}, && i = 1, \ldots, m,\\
& \phi_j\bar{y}_{rj} \leq {}_r\bar{y}\lambda_j + (q_{1r} + q_{2r})\sigma_\varepsilon\Phi^{-1}(\epsilon),\\
& \phi_j b_{rj} - {}_rb\lambda_j = q_{1r} - q_{2r}, && r = 1, \ldots, s,\\
& \varphi(\mathbf{1}^T\lambda_j) = \varphi,\\
& \lambda_j \geq 0, q_{kr} \geq 0, h_{ki} \geq 0, && k = 1, 2.
\end{aligned}$$

Input oriented model Similarly, as for the output oriented model, the almost 100% confidence chance constrained input oriented model for efficiency evaluation of the DMU_j is derived as:

$$\begin{aligned} \min_{\lambda_j,\theta_j} \quad & \theta_j - \epsilon(Prob(\mathbf{1}^T(\tilde{X}\lambda_j - \theta_j\tilde{x}_j) + \mathbf{1}^T(\tilde{y}_j - \tilde{Y}\lambda_j) < 0) - \alpha) && (13) \\ s.t. \quad & Prob({}_i\tilde{x}\lambda_j < \theta_j\tilde{x}_{ij}) \geq 1-\epsilon, && i = 1,\dots,m; \\ & Prob({}_r\tilde{y}\lambda_j > \tilde{y}_{rj}) \geq 1-\epsilon, && r = 1,\dots,s; \\ & \varphi(\mathbf{1}^T\lambda_j) = \varphi; \\ & \lambda_j \geq 0. \end{aligned}$$

Finally, the linearized form of the almost 100% confidence chance constrained input oriented model is stated as:

$$\begin{aligned} \min_{\lambda_j,q_{kr},h_{ki},\theta_j} \quad & \theta_j + \epsilon[\mathbf{1}^T(\bar{X}\lambda_j - \theta_j\bar{x}_j) + \mathbf{1}^T(\bar{y}_j - \bar{Y}\lambda_j) + && (14) \\ & +\delta(\mathbf{1}^T(A\lambda_j - \theta_j a_j) + \mathbf{1}^T(b_j - B\lambda_j))\sigma_\varepsilon\Phi^{-1}(\alpha)] + \epsilon(\sum_{r=1}^{s}(q_{1r}+q_{2r}) + \sum_{i=1}^{m}(h_{1i}+h_{2i})) \\ s.t. \quad & {}_i\bar{x}\lambda_j \leq \theta_j\bar{x}_{ij} + (h_{1i}+h_{2i})\sigma_\varepsilon\Phi^{-1}(\epsilon), \\ & {}_ia\lambda_j - \theta_j a_{ij} = h_{1i} - h_{2i}, && i = 1,\dots,m, \\ & \bar{y}_j\lambda_j \leq {}_r\bar{y} + (q_{1r}+q_{2r})\sigma_\varepsilon\Phi^{-1}(\epsilon), \\ & b_{rj} - {}_rb\lambda_j = q_{1r} - q_{2r}, && r = 1,\dots,s, \\ & \varphi(\mathbf{1}^T\lambda_j) = \varphi, \\ & \lambda_j \geq 0, q_{kr} \geq 0, h_{ki} \geq 0, && k = 1,2. \end{aligned}$$

Furthermore, the optimal solution $(\lambda_j^*, \mathbf{q}_{1j}^*, \mathbf{q}_{2j}^*, \mathbf{h}_{1j}^*, \mathbf{h}_{2j}^*, \phi_j^*)$ of output oriented problem (12) (alternatively the optimal solution $(\lambda_j^*, \mathbf{q}_{1j}^*, \mathbf{q}_{2j}^*, \mathbf{h}_{1j}^*, \mathbf{h}_{2j}^*, \theta_j^*)$ of input oriented problem (14)) is used to evaluate the technical efficiency of the DMU_j. The DMU_j is α–stochastic efficient, when the following two conditions are satisfied:

1. $\phi_j^* = 1$ $(\theta_j^* = 1)$;
2. $\mathbf{1}^T(\bar{X}\lambda_j^* - \bar{x}_j) + \mathbf{1}^T(\phi_j^*\bar{y}_j - \bar{Y}\lambda_j^*) + |\mathbf{1}^T(A\lambda_j^* - a_j) + \mathbf{1}^T(\phi_j^* b_j - B\lambda_j^*)|\sigma_\varepsilon\Phi^{-1}(\alpha) \geq 0$ $(\mathbf{1}^T(\bar{X}\lambda_j^* - \theta_j^*\bar{x}_j) + \mathbf{1}^T(\bar{y}_j - \bar{Y}\lambda_j^*) + |\mathbf{1}^T(A\lambda_j^* - \theta_j^* a_j) + \mathbf{1}^T(b_j - B\lambda_j^*)|\sigma_\varepsilon\Phi^{-1}(\alpha) \geq 0)$.

As mentioned in the section on efficiency measure introduction, a class of weakly efficient DMUs can be defined. The analyzed DMU$_j$ is identified as weakly efficient when the optimal solution of the associated problem satisfies $\phi_j^* = 1$ or $\theta_j^* = 1$.

7 Introducing returns to scale

As mentioned in the second section, the CCR model was designed to analyze the technology with property of constant returns to scale. Later, the BCC model and its variations were developed by Banker, Charnes, and Cooper (1984) to analyze the production function with variable returns to scale. Here, I follow this concept to introduce the variable returns to scale into the stochastic framework. The following definition uses the expected values to define types of returns to scale:

Definition 5. Returns to scale. Let the DMU$_j$ be stochastically efficient and the point $Z_\delta = ((1+\delta)\bar{x}_j, (1+\delta)\bar{y}_j)$ is a point in δ–neighborhood of $(\bar{x}_j, \bar{y}_j)$:

- The Non–Decreasing returns to scale are present $\Leftrightarrow \exists\ \delta^* > 0$ such that $Z_\delta \in T_\varphi$ for $\delta^* > \delta \geq 0$ and $Z_\delta \notin T_\varphi$ for $-\delta^* < \delta < 0$
- The Constant returns to scale are present $\Leftrightarrow \exists\ \delta^* > 0$ such that $Z_\delta \in T_\varphi$ for $\mid \delta \mid < \delta^*$
- The Non–Increasing returns to scale are present $\Leftrightarrow \exists\ \delta^* > 0$ such that $Z_\delta \notin T_\varphi$ for $\delta^* > \delta \geq 0$ and $Z_\delta \in T_\varphi$ for $-\delta^* < \delta < 0$.

The differences in types of returns to scale are reflected by different shapes of the production possibility set frontier that is set up by the intersection of supporting hyperplanes identified by optimal solutions of multiplier formulation of the DEA models. In the case of constant returns to scale (the CCR model by Charnes, Cooper, and Rhodes (1978)) the envelopment surface consists of a single half line that passes through the origin as shown in Figure 4. In the case of variable returns to scale, the production frontier is a piecewise linear set. Therefore, Figure 4 also shows the production possibility frontier of the model with the variable returns to scale that is referred to as the BCC model (Banker, Charnes, and Cooper (1984)) and in Figure 5 the BCC frontier is related to the frontier under the assumption of increasing returns to scale. These frontiers of production possibility set under various types of returns to scale are parameterized via the selection of φ and constraint type associated with the φ as follows:

$$\varphi = \begin{cases} 0 & \text{Constant returns to scale (CCR model)} \\ 1 & \text{Variable returns to scale (BCC model).} \end{cases}$$

Since the α–stochastically efficient point $(\tilde{x}_j, \tilde{y}_j)$ satisfies condition 9, for the point $Z_\delta = ((1+\delta)\bar{x}_j, (1+\delta)\bar{y}_j)$ can be derived

$$\begin{aligned} c^T(1+\delta)\tilde{x}_j + d^T(1+\delta)\tilde{y}_j + f_j + (1+\delta)\Phi^{-1}(\epsilon)\sigma_\varepsilon \mid c^T a_j + d^T b_j \mid \ &= \\ = (1+\delta)(c^T\tilde{x}_j + d^T\tilde{y}_j + f_j + \Phi^{-1}(\epsilon)\sigma_\varepsilon \mid c^T a_j + d^T b_j \mid) - \delta f_j = -\delta f_j & \end{aligned} \tag{15}$$

and the point $Z_\delta \in T_\varphi$ if and only if $-\delta f_j \geq 0$. Using definition 5, the relations between the type of the returns to scale and the sign of f_j is revealed and these relations are summarized in Table 2 together with choice of constrain on intensity variable vector λ_j.

8 Summary of SDEA models

In the previous sections, the oriented SDEA models were derived and these models are summarize in Table 4. It should be stressed that even the models using the same efficiency dominance definition but with different orientation choice result in different efficiency scores. Therefore, the choice of the efficiency dominance type, returns to scale and projection path to the envelopment surface (the set of dominating points in the production possibility set) are crucial for the efficiency analysis and the choice should reflect the aims of this study.

The returns to scale choice affects the shape of the production possibility set envelopment. The restrictions on returns to scale are related to four types of the envelopment surface shape through the geometry of the production possibility set and these restrictions are interpreted as the restriction on intensity variable λ in the envelopment problem or a restriction on supporting hyperplanes in the multiplier problem.

The evaluation of the efficiency score is based on distance measurement between the point that represents DMU and the associated point on the envelopment surface. This distance measure used in additive models is the most simple efficiency measure. A more sophisticated efficiency measure is created using the measure of maximal proportional inputs reduction (output augmentation) while keeping the levels of outputs (inputs) fixed. This proportional input (output) scaling approach is interpreted as the selection of a projection path towards the envelopment surface and results in the creation of oriented SDEA models.

The use of Non–Archimedean infinitesimal ϵ is closely related to the unit invariance property of the objective function values of the derived models because the result of multiplication by ϵ is not unit dependent. The use of unit invariant models also delivers the possibility of units of measurement change to avoid numerical problems (e.g., tiny diagonal matrices) when the SDEA models are solved.

Table 4 compares the derived SDEA with the most popular DEA models that appear in the present studies on efficiency evaluation. The additional SDEA models can be derived as extensions of models covered in this paper using the extensions procedures for the DEA models.

9 Method for SDEA model solving

To solve the linear optimization problems associated with the derived SDEA models the variant of the interior point method (IPM) is used because it is less computationally costly than the simplex methods when large sized problems are solved. For the purpose of the IPM employment the linearized problems 12 and 14 can be easily transformed to the standard linear programming form:[9]

$$\begin{array}{llll} \text{Primal:} & \min_{\mathbf{x}} \mathbf{c}^T \mathbf{x} & \text{Dual:} & \max_{\mathbf{y},\mathbf{z}} \mathbf{b}^T \mathbf{y} \\ & \text{s.t. } \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq 0 & & \text{s.t. } \mathbf{A}^T\mathbf{y} + \mathbf{z} = \mathbf{c}, \mathbf{z} \geq 0. \end{array} \tag{16}$$

Using the complementarity constraint $\mathbf{z}^T\mathbf{x} = 0$ (equivalent to duality gap condition $\mathbf{c}^T\mathbf{x} - \mathbf{b}^T\mathbf{y} = 0$) together with the feasibility constraints the following optimality condition for problem 16 is stated as

$$\begin{pmatrix} \mathbf{A}\mathbf{x} - \mathbf{b} \\ \mathbf{A}^T\mathbf{y} + \mathbf{z} - \mathbf{c} \\ \mathbf{z}^T\mathbf{x} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ 0 \end{pmatrix}, \tag{17}$$

where $\mathbf{z}, \mathbf{x} \geq 0$. To solve problem 17, I use Mehrotra's predictor–corrector algorithm that belongs to the class of the central path following IPM algorithms.[10] This primal–dual algorithm uses the combination of Newton's direction (duality gap reduction direction) and centering direction to solve the sequence of problems that comes from problem 17, where the complementarity constraint is modified to $\mathbf{x}_k^T\mathbf{z}_k = \mu_k$ and sequence $\{\mu_k\}$ converges to 0 for $k \to \infty$. So, the IPM algorithm generates an infinite sequence of points that converges to an optimal solution and the iteration process stops when the iterations are sufficiently close to the optimal

[9] In the case of linearized stochastic problems, vectors $\mathbf{x}$, $\mathbf{c}$, $\mathbf{z} \in \mathbb{R}^{n+3(m+s)+1}$; vectors $\mathbf{y}$, $\mathbf{b} \in \mathbb{R}^{2(m+s)+1}$ and matrix $\mathbf{A} \in \mathbb{R}^{(2(m+s)+1)\times(n+3(m+s)+1)}$.

[10] The solver for the stated oriented SDEA models is constructed using the procedures package known as PCx linear solver obtained from Optimization Technology Center at Argonne National Laboratory and Northwestern University.

solution or the limit for the number of iterations is reached. The advantage of the primal–dual version of the interior point method is that the primal and dual problem 16 are solved simultaneously.

Further, the IPM solutions satisfy the strong complementarity slackness condition (SCSC). The SCSC solution is the solution with the maximal product of the positive components of the optimal solution and therefore it is the optimal solutions with a minimal number of zero components. The SCSC property of optimal solutions helps to eliminate interpretation problems when the optimal solution to the DEA model are rendered as the shadow prices of inputs and outputs.[11]

10 Indonesian rice farms efficiency

To demonstrate the use of the oriented SDEA models, the results from the proposed SDEA models are compared to the DEA and SFA results. This comparison is motivated by Horrace and Schmidt's (1996) work, where parametric methods for efficiency estimation are compared using data on Indonesian rice farms. To compare with results presented in Druska and Horrace's (2004) methodological work on spatial effects in the SFA framework, I use the same data set to compute the SDEA and DEA scores.

Indonesia is the biggest rice importer in Asia at the same time almost 70% of the country's 213 million people are farmers, hence the identification of the linkages between different factors and rice yield in the West Java area is the subject of many studies on farming efficiency (e.g. Wadud (2002) and Daryanto, Battese, and Fleming (2002)). For research purposes, the Indonesian Ministry of Agriculture surveyed rice farms over six growing periods (3 wet and 3 dry periods) in six villages in the area of the Cimanuk River basin in West Java. The data set from this survey is filtered for outliers that reported yields over the maximum hectare yields

[11]For more details on the use of interior point methods solutions of the DEA related problems see Brázdik (2001).

reached in laboratory conditions. After this correction, the panel used for analysis is balanced and describes the production mixes of 160 rice farms with average yield of 3265.20 kg/ha that resemble the observed average yields in this area.

For the purpose of comparison with the SFA results, I use the same inputs and outputs to specify the inputs–output production mixes of the surveyed rice farms as were used in the SFA study by Druska and Horrace (2004). The considered inputs include total area of rice cultivation in hectares (Size), seed in kilograms (Seed), urea in kilograms (Urea), phosphate in kilograms (Phosphate) and total labor (Labor). As the measure of output the total output of rough rice in kilograms (Gross yield) is used and the summary statistics for the used inputs and output are presented in Table 5. All of the production factors exhibit very high variation and presence of noise that influence efficiency evaluation is expected. The presence of noise provides rationale for use of the SDEA approach.

To calculate the DEA efficiency scores, the output oriented DEA model presented in Table 1 is used. The α–stochastic efficiency of farms is evaluated by use of the linearized output oriented SDEA model described by problem 12. Moreover, I also compute the time average DEA efficiency scores and the DEA scores calculated using the mean values of farms' production mixes. The average DEA score for a rice farm is calculated by averaging the farm's efficiency scores when the data set is considered as a sample of 960 individual observations. The DEA–mean score is calculated using a sample with 160 observations, where each farm is characterized by mean values of its production mix characteristics.

For all data envelopment models, I consider the cases of normal (denoted by subscript N or $Norm$) and log–normal (denoted by subscript LN or $LogN$) distribution of the farms' inputs and outputs. Under the assumption of log–normal distribution, inputs and output are transformed by taking logs, therefore the efficiency scores are no more scale of operations invariant. The DEA and SDEA efficiency scores are calculated under assumption of constant returns to scale (choice $\varphi = 0$

and denoted by CCR) and variable returns to scale ($\varphi = 1$, BCC). The efficiency scores estimated by almost 100% chance constrained SDEA models are reported for $\alpha = 0.05$ as a level of modeler's risk because calculations shows that for higher levels the SDEA method suffers from a loss of discriminatory power and too many DMUs are evaluated as efficient.

The descriptive statistics of the computed DEA, SDEA and SFA efficiency scores are summarized in Table 6 and compared to Druska and Horrace's (2004) SFA scores FE and $FEsp$ that are estimated by the fixed effect method and fixed effect method with correction for spatially corrected errors, respectively. Table 6 reports higher mean values of efficiency scores for data envelopment approaches than for SFA scores. These SDEA and DEA results suggest that Indonesian rice farms are operating closer to the production frontier than in the SFA studies. Wadud (2002) observes a similar pattern for Bangladesh rice farms efficiency scores and he reports 0.80 as the mean score for the SFA and 0.86 and 0.91 for the CCR and BCC data envelopment models, respectively. From this comparison, I deduce that on average the considered Indonesian rice farms were operating at lower efficiency levels than rice farms in Bangladesh. As Table 6 reports, scores calculated by data envelopment approaches show a variance twice as high as scores calculated by the SFA. This is contrary to results by Wadud (2002), Ferro–Luzzi, Ramirez, Flückiger, and Vassiliev (2003) and Jaforullah and Premachandra (2003) that report comparable variance for SFA and DEA efficiency scores.

Further, to highlight differences in efficiency scores among the used approaches, Table 7 compares efficiency scores for group of chosen DMUs. These DMUs were chosen according to the SFA efficiency scores estimates by Druska and Horrace (2004) to represent farms with the highest, median and the lowest technical efficiency scores. Due to the differences in nature of the compared methods differences in efficiency scores estimates are expected. However, the differences in efficiency rankings presented in Table 8 indicate inconsistency of efficiency evaluation across the assessed methods.

The nature of the SFA approach allows only one DMU to achieve a score of 1 while the data envelopment approaches assign efficiency score 1 to all DMUs on the production possibility frontier. Therefore, the peak at 1 with height proportional to the numbers of DMUs identified as efficient occurs in distribution of efficiency scores calculated by use of the data envelopment approaches. Keeping this fact in mind, the shapes of efficiency score distributions displayed in Figure 6, Figure 7 and Figure 8 can be compared. Examination of these figures reveals that the shape of the SFA efficiency score distribution function is matched at best by the distribution function estimate for the DEA average efficiency score under assumption of linearly distributed production characteristics for constant (CCRnorm) and variable (BCCnorm) returns to scale specification.

Due to the aforementioned differences in nature of efficiency scores, the results' consistency among the used approaches should be assessed through correlation of efficiency rankings rather than an efficiency scores. For ranking correlation evaluation, Spearman's (1904) correlation coefficient is used because its important feature is lower sensitivity to extreme values when compared to the standard correlation coefficient. Further, by evaluating the significance of calculated rankings correlations the hypothesis that considered rankings are not correlated is tested. Table 9 presents correlation coefficients for rankings generated using DEA on mean values, oriented SDEA and SFA efficiency scores. In Table 10, correlation coefficients for DEA on mean values, the oriented SDEA, and SFA efficiency rankings are summarized.

When the rankings correlation coefficients presented in Table 9 and Table 10 are assessed, I conclude that higher level of rankings consistency is observed between SFA efficiency rankings and data envelope analysis rankings than between SFA and SDEA rankings. The highest DEA–mean ranking correlation coefficients values are 0.7205 and 0.5531 and the values 0.8539, 0.8214 for average DEA scores are substantially higher than the highest values 0.2534, 0.2448 of the SFA–SDEA correlation coefficients. The presented SFA and DEA rankings correlation results correspond to findings in recent studies on the SFA and DEA ranking consistency. Wadud (2002) reports the highest correlation coefficients values ranging from 0.61 to 0.83, Jaforullah and Premachandra (2003) report 0.74 and Ferro–Luzzi, Ramirez, Flückiger, and Vassiliev (2003) report significant correlation coefficients between SFA and DEA ranking in range from 0.594 to 0.677.

The purpose of this work was to improve the stochastic non–parametric approach for efficiency evaluation by introducing frontier projection direction. Therefore, the improvement in consistency of the SFA and SDEA results is expected. Contrary to this expectation, more consistency (in terms of significance of correlation coefficients and their absolute values) is found between the SFA and DEA (SFA–average DEA in range 0.1130, 0.8539, SFA–DEA mean in −0.0231, 0.5016) rankings than between the SFA–SDEA rankings (from −0.0835 to 0.2534). The observed low consistency of SFA–SDEA rankings may be a consequence of the high variance of the rice production characteristics that affects the accuracy of efficiency dominating set approximation. This conclusion originates from comparison of the DEA on mean values and SDEA efficiency rankings, where rankings correlations are insignificant or low and simultaneously the SDEA approach is derived from DEA on mean values approach by including correction for variance in data. Therefore, high values of the ranking correlation between SDEA and DEA–mean rankings are expected to be achieved when considered DMUs are characterized by random variables with low variances.

11 Conclusion

In the theoretical part of this work, I reviewed the technique used to derive linear deterministic equivalents to Huang and Li's (2001) SDEA models and this technique was used to develop the oriented stochastic DEA models and to describe their properties. Using the techniques of stochastic problems linearization the proposed oriented SDEA models were linearized, so the solver based on the interior point method for linear problems can be used to solve linear programming problems associated with the models. The created solver for problems associated with the SDEA and DEA models implements the primal–dual interior point method algorithm.

The empirical part of this work that was motivated by Horrace and Schmidt's (1996) comparison of SFA methods and presents results of the technical efficiency evaluation of Indonesian rice farms by the SDEA and DEA models. Further, efficiency rankings were constructed and compared with the SFA rankings constructed by Druska and Horrace (2004). While I was able to reject the hypothesis that the DEA, SDEA and SFA rankings are independent in the majority of the considered cases the consistency of results from the SFA and oriented SDEA models is questionable due to the low values of ranking correlation coefficients. Assessing the results of the DEA on the mean values approach, I conclude that in this data set the low rankings consistency originate from high variance present in the data. In spite of the low consistency of the SFA–SDEA approach the findings on the SFA–DEA rankings correlation are consistent with the recent studies on the SFA and DEA comparisons, e.g. Wadud and White (2000) and Jaforullah and Premachandra (2003) that report considerable consistency of efficiency rankings.

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A Figures and Tables

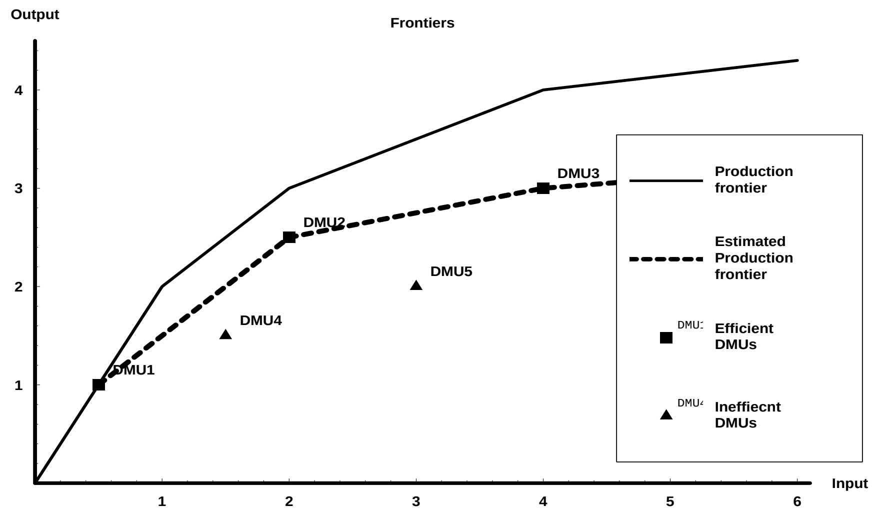

Figure 1: DEA estimate of production possibility frontier

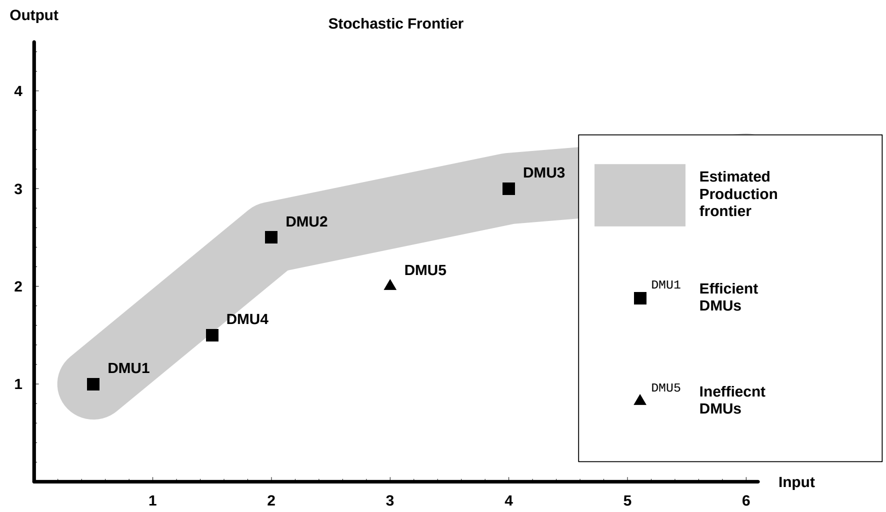

Figure 2: Set of α–stochastic dominant points

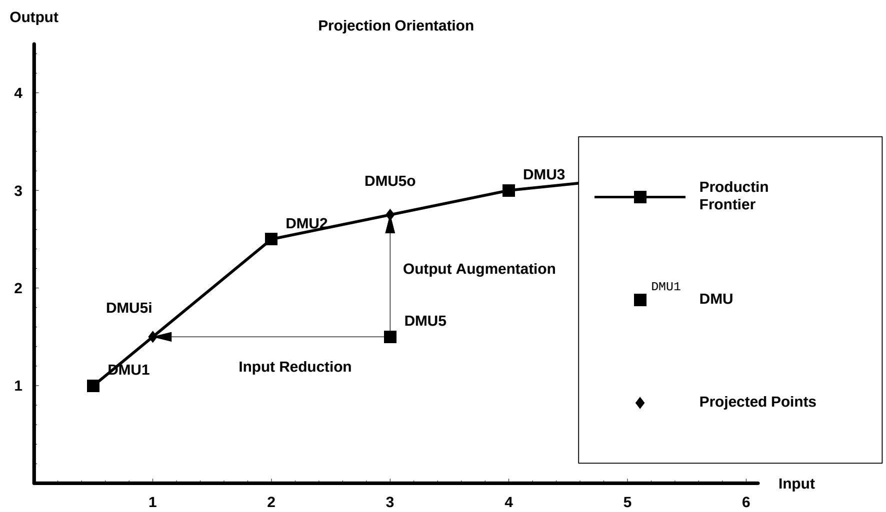

Figure 3: Projection on the production possibility frontier

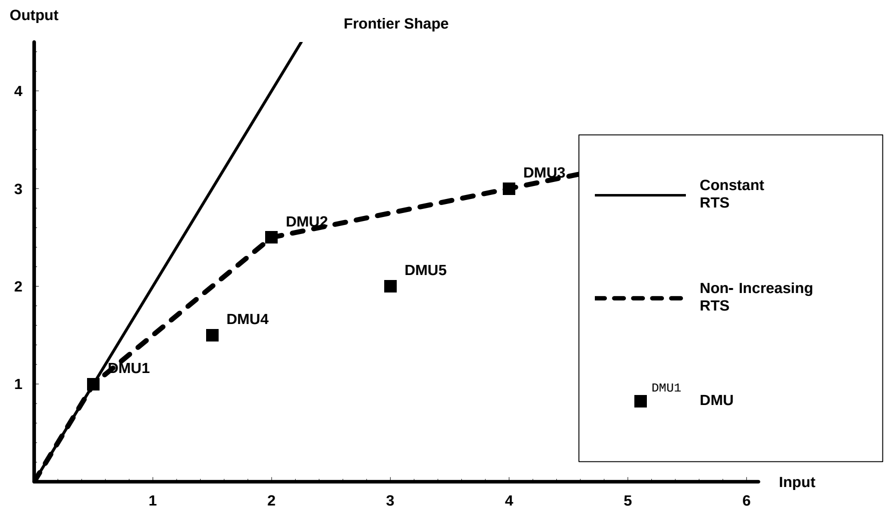

Figure 4: Returns to scale – Constant, Non–Increasing

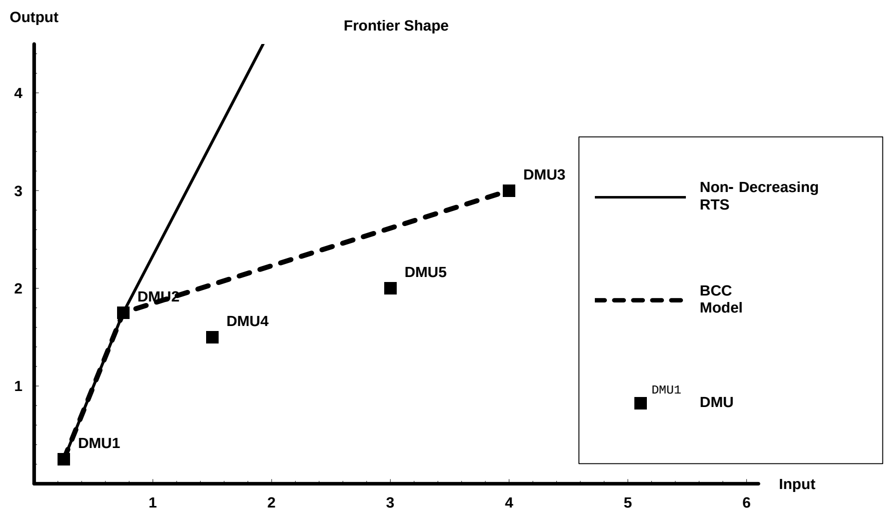

Figure 5: Returns to scale – Non–Decreasing, BCC model

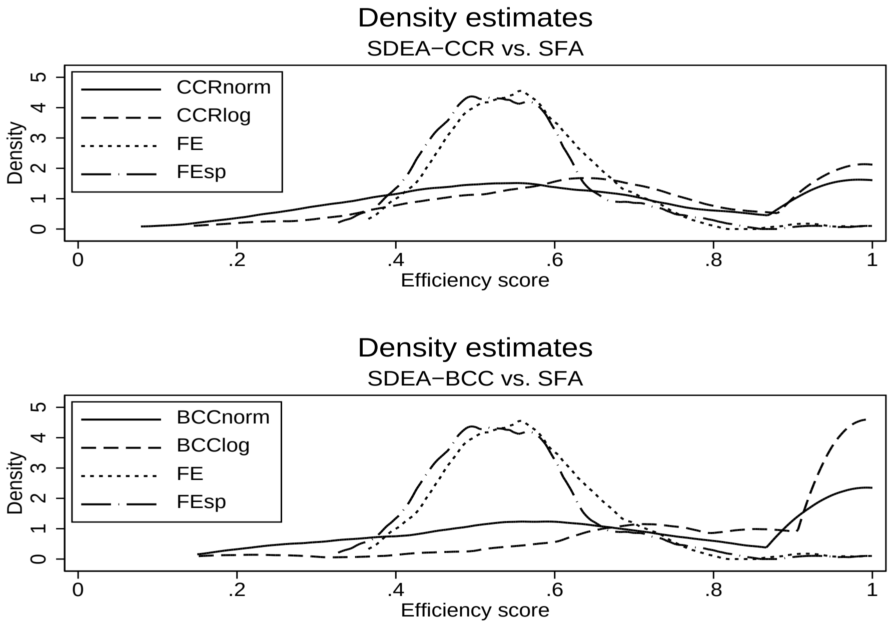

Figure 6: Kernel density estimates SDEA vs. SFA

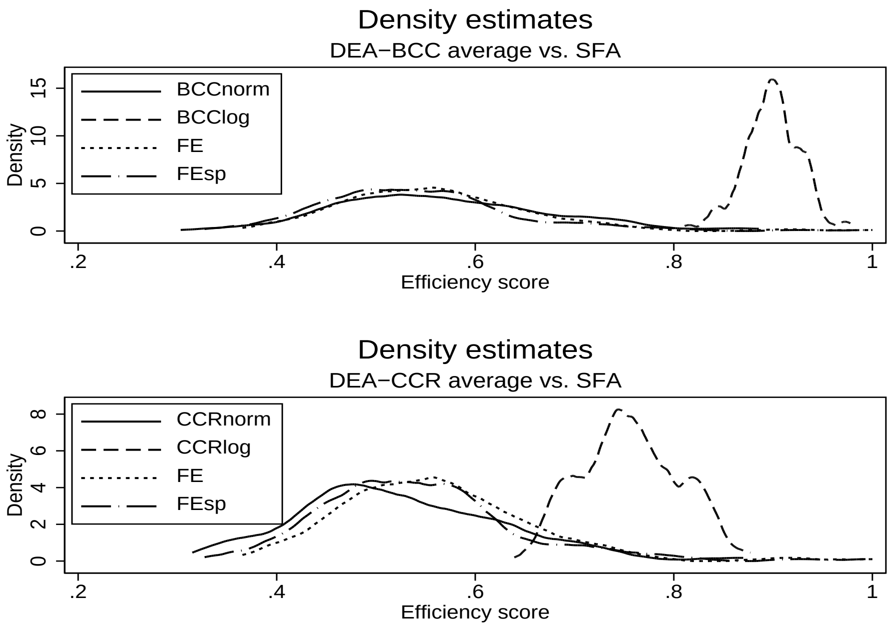

Figure 7: Kernel density estimates DEA average vs. SFA

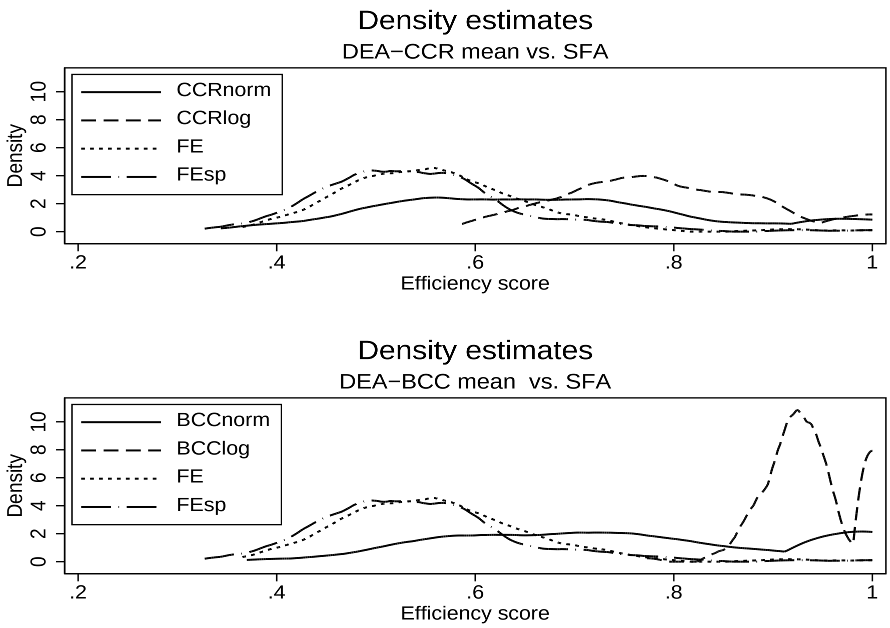

Figure 8: Kernel density estimates DEA–mean vs. SFA

Output oriented model		
$\max_{\lambda_j,\phi_j}$	$\phi_j + \epsilon(\mathbf{1}^T(X\lambda_j - x_j) + \mathbf{1}^T(\phi_j y_j - Y\lambda_j))$	
$s.t.$	${}_ix\lambda_j < x_{ij},$	$i = 1, \ldots, m;$
	${}_ry\lambda_j > \phi_j y_{rj},$	$r = 1, \ldots, s;$
	$\varphi(\mathbf{1}^T\lambda_j) = \varphi;$	
	$\lambda_j \geq 0$	

Input oriented model		
$\min_{\lambda_j,\theta_j}$	$\theta_j - \epsilon(\mathbf{1}^T(X\lambda_j - \theta_j x_j) + \mathbf{1}^T(y_j - Y\lambda_j))$	
$s.t.$	${}_ix\lambda_j < \theta_j x_{ij}$	$i = 1, \ldots, m;$
	${}_ry\lambda_j > y_{rj}$	$r = 1, \ldots, s;$
	$\varphi(\mathbf{1}^T\lambda_j) = \varphi;$	
	$\lambda_j \geq 0$	

Table 1: Generalized versions of input and output oriented DEA models

Model (Orientation)	Returns to scale	Constraint	Hyperplane(s)
CCR model			
(Input, Output)	Constant	None, $\varphi = 0$	Passes trough origin
BCC model			
(Input, Output)	Variable	$\mathbf{1}^T\lambda_j = 1$	Not constrained
SDEA models			
(Input)	Non–Decreasing	$\mathbf{1}^T\lambda_j \geq 1$	$f_j^* \geq 0$
(Input)	Non–Increasing	$\mathbf{1}^T\lambda_j \leq 1$	$f_j^* \leq 0$
(Input)	Constant	None	$f_j^* = 0$
(Output)	Non–Decreasing	$\mathbf{1}^T\lambda_j \geq 1$	$f_j^* \leq 0$
(Output)	Non–Increasing	$\mathbf{1}^T\lambda_j \leq 1$	$f_j^* \geq 0$
(Output)	Constant	None	$f_j^* = 0$

Table 2: Returns to scale

Output oriented model

First stage	Second stage
$\max_{\lambda_j,\phi_j} \phi_j$	$\max_{\lambda_j} Prob(\mathbf{1}^T(\tilde{X}\lambda_j - \tilde{x}_j) + \mathbf{1}^T(\hat{\phi}_j\tilde{y}_j - \tilde{Y}\lambda_j)) - \alpha$
s.t. $Prob({}_i\tilde{x}\lambda_j < \tilde{x}_{ij}) \geq 1 - \epsilon$	s.t. $Prob({}_i\tilde{x}\lambda_j < \tilde{x}_{ij}) \geq 1 - \epsilon$
$Prob({}_r\tilde{y}\lambda_j > \phi\tilde{y}_{rj}) \geq 1 - \epsilon$	$Prob({}_r\tilde{y}\lambda_j > \hat{\phi}_j\tilde{y}_{rj}) \geq 1 - \epsilon$
$\varphi(\mathbf{1}^T\lambda_j) = \varphi$	$\varphi(\mathbf{1}^T\lambda_j) = \varphi$
$\lambda_j \geq 0$	$\lambda_j \geq 0$
	$i = 1, \ldots, m;\ r = 1, \ldots, s.$

Input oriented model

First stage	Second stage
$\min_{\lambda_j,\theta_j} \theta_j$	$\max_{\lambda_j} Prob(\mathbf{1}^T(\tilde{X}\lambda_j - \hat{\theta}_j\tilde{x}_j) + \mathbf{1}^T(\tilde{y}_j - \tilde{Y}\lambda_j)) - \alpha$
s.t. $Prob({}_i\tilde{x}\lambda_j < \theta_j\tilde{x}_{ij}) \geq 1 - \epsilon$	s.t. $Prob({}_i\tilde{x}\lambda_j < \hat{\theta}_j\tilde{x}_{ij}) \geq 1 - \epsilon$
$Prob({}_r\tilde{y}\lambda_j > \tilde{y}_{rj}) \geq 1 - \epsilon$	$Prob({}_r\tilde{y}\lambda_j > \tilde{y}_{rj}) \geq 1 - \epsilon$
$\varphi(\mathbf{1}^T\lambda_j) = \varphi$	$\varphi(\mathbf{1}^T\lambda_j) = \varphi$
$\lambda_j \geq 0$	$\lambda_j \geq 0$
	$i = 1, \ldots, m;\ r = 1, \ldots, s.$

Table 3: Two stages of oriented almost 100% confidence chance constrained models

Model (Orientation)	Returns to Scale	Envelopment Type	Range	Units Invariant	Involves Non–Archimedean
Additive	Variable	Piecewise linear	objective value≤ 0	No	No
	Constant	Piecewise linear		No	No
Almost 100% confidence	Constant	St. Hyperplane	objective value$\leq \sigma_\varepsilon \Phi^{-1}(\epsilon)$	No	Yes
additive model; Problem (7)	Variable	St. Hyperplanes	$\mid \mathbf{1}^T(A\lambda_j - a_j) + \mathbf{1}^T(b_j - B\lambda_j) \mid$	No	Yes
BCC model (input)	Variable	Piecewise linear	$0 < \theta \leq 1$	Yes	Yes
BCC model (output)	Variable	Piecewise linear	$1 \leq \phi$	Yes	Yes
CCR model (input)	Constant	Piecewise linear	$0 < \theta \leq 1$	Yes	Yes
CCR model (output)	Constant	Piecewise linear	$1 \leq \phi$	Yes	Yes
Almost 100% confidence					
oriented models,	Variable	St. Hyperplanes	$0 < \theta \leq 1,\ 1 \leq \phi$	Yes	Yes
Problems (14),(12)	Constant	St. Hyperplane	$0 < \theta \leq 1,\ 1 \leq \phi$	Yes	Yes
(input, output)					

Table 4: Comparison of models

Data summary statistics

Variable	Obs.	Mean	Std. Dev.	Minimum	Maximum
Size	960	0.4398	0.5607	0.0140	5.3220
Seed	960	18.4708	46.6819	1.0000	1250.0000
Urea	960	96.5250	130.3932	1.0000	1250.0000
Phosphate	960	33.8072	48.3489	0.0000	700.0000
Labor	960	394.2240	496.0169	17.0000	4774.0000
Gross yield	960	1413.9340	1966.0950	42.0000	20960.0000

Table 5: Indonesian rice farm summary statistics

Efficiency scores summary statistics

Model	Obs	Mean	Std. Dev.	Minimum	Maximum
DEA					
BCC_{Norm}	960	0.5672	0.2044	0.1912	1
CCR_{Norm}	960	0.5256	0.1943	0.1775	1
BCC_{LogN}	960	0.8987	0.0565	0.6484	1
CCR_{LogN}	960	0.7561	0.0817	0.5143	1
DEA–mean					
BCC_{Norm}	160	0.7641	0.1723	0.3698	1
CCR_{Norm}	160	0.6721	0.1616	0.3436	1
BCC_{LogN}	160	0.9360	0.0427	0.7730	1
CCR_{LogN}	160	0.7918	0.1026	0.5867	1
SDEA					
BCC_{Norm}	160	0.7343	0.2614	0.1500	1
CCR_{Norm}	160	0.6594	0.2569	0.0791	1
BCC_{LogN}	160	0.8714	0.1867	0.1519	1
CCR_{LogN}	160	0.7260	0.2331	0.1456	1
SFA					
FE	160	0.5613	0.0992	0.3655	1
$FEspatial$	160	0.5435	0.1023	0.3274	1

Table 6: Efficiency scores summary statistics

Efficiency scores

Score	Farm	SFA		SDEA				DEA average efficiency score				DEA–mean			
		FE	FE_{sp}	CCR_N	BCC_N	CCR_{LN}	BCC_{LN}	CCR_N	BCC_N	CCR_{LN}	BCC_{LN}	CCR_N	BCC_N	CCR_{LN}	BCC_{LN}
High	164	1.0000	1.0000	0.6660	0.7109	0.6442	0.6808	0.8635	0.8613	0.7782	0.9690	1.0000	1.0000	0.7362	1.0000
	118	0.9323	0.9269	0.6875	1.0000	1.0000	1.0000	0.8699	0.8754	0.7926	0.9778	1.0000	1.0000	0.7853	1.0000
	152	0.8993	0.8152	0.4109	0.6398	0.2872	0.2940	0.7922	0.8269	0.8595	0.9707	1.0000	1.0000	1.0000	1.0000
	153	0.7717	0.7487	0.7604	0.7899	0.9128	1.0000	0.6589	0.6710	0.7734	0.9347	0.8717	0.8768	0.7528	0.9459
Medium	40	0.5535	0.5824	0.9622	1.0000	1.0000	1.0000	0.5969	0.6298	0.7348	0.9118	0.8476	0.8590	0.6776	0.9787
	101	0.5518	0.5282	0.5967	0.6117	0.8212	1.0000	0.5117	0.5252	0.6864	0.9028	0.6680	0.7005	0.6893	0.9311
	80	0.5518	0.5166	0.2974	0.3012	0.5673	0.7255	0.5528	0.6064	0.7741	0.8842	0.5723	0.6305	0.8240	0.9205
	149	0.5495	0.5173	1.0000	1.0000	1.0000	1.0000	0.4588	0.5494	0.8046	0.8789	0.5981	1.0000	0.8589	0.8544
Low	86	0.3980	0.3907	1.0000	1.0000	0.5822	1.0000	0.3351	0.3527	0.7280	0.8381	0.3859	0.4478	0.7608	0.8452
	143	0.3837	0.3596	0.4127	0.4960	1.0000	1.0000	0.3150	0.3539	0.7438	0.8202	0.4933	0.5247	0.7591	0.8722
	117	0.3790	0.3713	1.0000	1.0000	1.0000	1.0000	0.3944	0.4998	0.6907	0.8109	0.5387	0.8970	0.8572	0.8800
	45	0.3655	0.3274	0.4770	0.6235	0.5744	0.7485	0.3814	0.5945	0.8252	0.8474	0.4896	1.0000	0.8862	1.0000

Note: Farm identification number is from original sample.

Table 7: Comparison of technical efficiency scores

Efficiency rankings

Score	Farm	SFA		SDEA				DEA average efficiency score				DEA–mean			
		FE	FE_{sp}	CCR_N	BCC_N	CCR_{LN}	BCC_{LN}	CCR_N	BCC_N	CCR_{LN}	BCC_{LN}	CCR_N	BCC_N	CCR_{LN}	BCC_{LN}
High	164	1	1	71	84	96	138	2	3	54	3	1	1	111	1
	118	2	2	67	1	1	1	1	2	39	1	1	1	81	1
	152	3	3	131	96	155	157	3	4	3	2	1	1	1	1
	153	4	7	56	74	54	1	19	27	60	17	23	48	97	61
Medium	40	79	48	42	1	1	1	41	44	109	51	25	51	140	34
	101	80	82	88	103	61	1	82	100	144	70	76	96	134	81
	80	81	91	148	148	120	123	56	54	59	115	111	116	56	102
	149	82	89	1	1	1	1	117	82	33	125	103	1	44	157
Low	86	157	154	1	1	114	1	158	159	114	156	157	157	91	158
	143	158	158	130	126	1	1	160	158	96	159	142	149	93	152
	117	159	157	1	1	1	1	145	115	142	160	125	45	46	147
	45	160	160	115	99	116	121	148	60	18	153	144	1	28	1

Note: Farm identification number is from original sample.

Table 8: Comparison of technical efficiency rankings

Efficiency rankings correlations

	SDEA				DEA average efficiency score				SFA	
	CCR_N	BCC_N	CCR_{LN}	BCC_{LN}	CCR_N	BCC_N	CCR_{LN}	BCC_{LN}	FE	FE_{sp}
SDEA										
CCR_N	1.0000									
BCC_N	0.8573***	1.0000								
CCR_{LN}	0.4954***	0.4320***	1.0000							
BCC_{LN}	0.3936	0.4615	0.6203***	1.0000						
DEA av.										
CCR_N	0.2863***	0.2835***	-0.0491	0.0012	1.0000					
BCC_N	0.2825***	0.3269***	-0.0334	0.0299	0.8547***	1.0000				
CCR_{LN}	0.0835*	0.0599*	0.0862*	-0.0449	0.3054***	0.5137***	1.0000			
BCC_{LN}	0.2321***	0.2774***	-0.0814**	-0.0153	0.8458***	0.8074***	0.2338***	1.0000		
SFA										
FE	0.2534***	0.2448***	-0.0224	-0.0292	0.8214***	0.7127***	0.2949***	0.8539***	1.0000	
FE_{sp}	0.2115***	0.2399***	-0.0835***	-0.0762**	0.7988***	0.6297***	0.1130***	0.8263***	0.8983***	1.0000

Note: ***,** and * coefficient significance at 1%,5% and 10% level.

Table 9: Spearman ranking correlation coefficients and significance levels

Efficiency rankings correlations

	SDEA				DEA–mean				SFA	
	CCR_N	BCC_N	CCR_{LN}	BCC_{LN}	CCR_N	BCC_N	CCR_{LN}	BCC_{LN}	FE	FE_{sp}
SDEA										
CCR_N	1.0000									
BCC_N	0.8573***	1.0000								
CCR_{LN}	0.4954***	0.4320***	1.0000							
BCC_{LN}	0.3936	0.4615	0.6203***	1.0000						
DEA mean										
CCR_N	0.4481***	0.4143***	0.1158***	0.0582*	1.0000					
BCC_N	0.4671***	0.5016***	0.1525***	0.1115***	0.6491***	1.0000				
CCR_{LN}	0.0316	-0.0231	0.1448***	-0.0266	-0.0226	0.3010***	1.0000			
BCC_{LN}	0.2930***	0.3156***	0.0151	0.0621*	0.5627***	0.7695***	0.2460***	1.0000		
SFA										
FE	0.2534***	0.2448***	-0.0224	-0.0292	0.7205***	0.5531***	0.0461	0.5438***	1.0000	
FE_{sp}	0.2115***	0.2399	-0.0835***	-0.0762**	0.7130***	0.4461***	-0.2274***	0.4114***	0.8983***	1.0000

Note: ***,** and * coefficient significance at 1%,5% and 10% level.

Table 10: Spearman ranking correlation coefficients and significance levels

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