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Associative algebra twisted bundles over compact topological spaces

Alexander Zuevsky

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ASSOCIATIVE ALGEBRA TWISTED BUNDLES OVER COMPACT TOPOLOGICAL SPACES

A. ZUEVSKY

Abstract. For the associative algebra $A(\mathfrak{g})$ of an infinite-dimensional Lie algebra $\mathfrak{g}$, we introduce twisted fiber bundles over arbitrary compact topological spaces. Fibers of such bundles are given by elements of algebraic completion of the space of formal series in complex parameters, sections are provided by rational functions with prescribed analytic properties. Homotopical invariance as well as covariance in terms of trivial bundles of twisted $A(\mathfrak{g})$-bundles is proven. Further applications of the paper's results useful for studies of the cohomology of infinite-dimensional Lie algebras on smooth manifolds, K-theory, as well as for purposes of conformal field theory, deformation theory, and the theory of foliations are mentioned.

1. Introduction

It is natural to consider bundles of modules related to associative algebras. Playing important roles in clarification of the cohomology theory on smooth manifolds, they are also important for elliptic and Witten genua [22, 21], the highest weight representations of Heisenberg and affine Kac-Moody algebras, and provide important examples for the construction of related associative algebras. In this paper we introduce twisted fiber bundles of modules of associative algebras of infinite-dimensional Lie algebras twisted in the form of group of automorphisms torsors originating from local geometry. Our original motivation for this work is to understand continuous cohomology [3, 4, 7, 8, 9, 17, 20] of non-commutatieve structrues over compact topological spaces. In particular [4], one hopes to relate cohomology of infinite-dimensional Lie algebras-valued series considered on complex manifolds to fiber bundles on auxiliary topological spaces [17]. We also plan to study applications of results of this paper for K-theories. Let $\mathfrak{g}$ be an infinite-dimensional Lie algebra [13]. Starting from algebraic completion $G_{\mathbf{z}}$ of the space of $\mathfrak{g}$-valued series in a few formal complex parameters, we introduce the category $\mathcal{O}_{A(\mathfrak{g})}$ of associative algebra modules for the associative algebra $A(\mathfrak{g})$ originating from $G_{\mathbf{z}}$ by means of factorization with respect to two natural multiplications [23]. Local parts of twisted bundles are constructed as principal bundles of products of $\mathrm{Aut}(\mathfrak{g})$ modules and spaces of all sets of local parameters of a X-covering. As in the untwisted case [6], this result is crucial in defining $A(\mathfrak{g})$ K-groups and studying the cohomology their properties.

Key words and phrases. Associative algebras, fiber bundles, rational functions with prescribed properties.

2. Prescribed rational functions

In this section the space of prescribed rational functions is defined as rational functions with certain analytical and symmetric properties [12]. Such rational functions depend implicitly on an infinite number of non-commutative parameters.

2.1. **Rational functions originating from matrix elements.** Let us introduce the general notations used in this paper. We denote by boldface vectors of elements, e.g., $\mathbf{a}_n = (a_1, \ldots, a_n)$, and the same for all types of objects used in the text. If n is omitted then $\mathbf{a}$ denotes any choice of $n \geqslant 0$. We also express as $(\mathbf{a}_j)_n$ the j-th component of $\mathbf{a}_n$. Let I be set of positive integers, and $X_{\boldsymbol{\alpha}} = \{X_\alpha, \alpha \in I\}$ be an open covering of a compact topological space X which gives a local trivialization of the $A(\mathfrak{g})$ fiber bundle. Let $\mathfrak{g}$, be an infinite-dimensional Lie algebra. Denote by G a $\mathfrak{g}$-module. Denote by $G_{\mathbf{z}_n}$ be the graded (with respect to a grading operator K_G) algebraic completion of the space of formal series individually in each of complex formal parameters $\mathbf{z}_n$, and satisfying certain properties described below. We denote $\mathbf{x}_n = (\mathbf{g}_n, \mathbf{z}_n)$ for $\mathbf{g}_n$ of the n-th power $\mathbf{G}_n = G^{\otimes n}$ of $\mathfrak{g}$-module G, and $G^*_{\mathbf{z}_n}$ be the dual to $G_{\mathbf{z}_n}$ with respect to non-degenerate bilinear form $(.,.)$. For fixed $\theta \in G^*_{\mathbf{z}_n}$, and varying $\mathbf{x}_n \in \mathbf{G}_{\mathbf{z}_n}$ we consider matrix elements $F(\mathbf{x}_n)$ of the form

$$F(\mathbf{x}_n) = (\theta, f(\mathbf{x}_n)) \in \mathbb{C}((z)), \tag{2.1}$$

where $F(\mathbf{x}_n)$ depends implicitly on $\mathbf{g}_n \in \mathbf{G}_n$. In this paper we consider meromorphic functions of several complex formal parameters defined on a compact topological space which are extendable to rational functions on larger domains on X. We denote such extensions by $R(f(\mathbf{z}_n))$.

Definition 1. Denote by $F_n\mathbb{C}$ the configuration space of $n \geqslant 1$ ordered coordinates in $\mathbb{C}^n$, $F_n\mathbb{C} = \{\mathbf{z}_n \in \mathbb{C}^n \mid z_i \neq z_j, i \neq j\}$.

In order to work with objects on X for a set of $\mathbf{G}_n$-elements $\mathbf{g}_n$ we consider converging rational functions $f(\mathbf{x}_n) \in G_{\mathbf{z}_n}$ of $\mathbf{z}_n \in F_n\mathbb{C}$.

Definition 2. For an arbitrary fixed $\theta \in G^*_{\mathbf{z}}$, we call a map linear in $\mathbf{g}_n$ and $\mathbf{z}_n$,

$$F : \mathbf{x}_n \mapsto R((\theta, f(\mathbf{x}_n)), \tag{2.2}$$

a rational function in $\mathbf{z}_n$ with the only possible poles at $z_i = z_j$, $i \neq j$. Abusing notations, we denote

$$F(\mathbf{x}_n) = R\left((\theta, f(\mathbf{x}_n))\right).$$

Definition 3. We define left action of the permutation group S_n on $F(\mathbf{z}_n)$ by

$$\sigma(F)(\mathbf{x}_n) = F\left(\mathbf{g}_n, \mathbf{z}_{\sigma(i)}\right).$$

2.2. **Conditions on $G_{\mathbf{z}}$.** For $G_{\mathbf{z}}$ we assume [12] that is $G_{\mathbf{z}} = \coprod_{\lambda \in \mathbb{C}} G_{\mathbf{z},\lambda}$, where $G_{\mathbf{z},\lambda} = \{w \in G_{\mathbf{z}} | K_0 w = \lambda w \ \ \lambda = \mathrm{wt}(w)\}$, such that $G_{\mathbf{z},\lambda} = 0$ when the real part of α is sufficiently negative, Moreover we require that $\dim G_{\mathbf{z},\lambda} < \infty$, i.e., it is finite, and for fixed λ, $G_{\mathbf{z},n+\lambda} = 0$, for all small enough integers n. In addition, assume that $G_{\mathbf{z}}$ equipped with a map $\omega_G : G_{\mathbf{z}} \to G[[\mathbf{z}, \mathbf{z}^{-1}]]$, $g \mapsto \omega_g(z) \equiv \sum_{l \in \mathbb{C}} g_l \ z^l$. In addition to that, For $g \in \mathfrak{g}$ and $g \in G$, $\omega_g(z)w$ contains only finitely many negative power terms, that is, $\omega_g(z)w \in G((z))$. We denote by $G'_{\mathbf{z}} = \coprod_{\lambda \in \mathbb{Z}} G^*_\lambda$ the dual to $G_{\mathbf{z}}$. Through matrix elements (2.1), locality and associativity conditions for g_1, $g_2 \in \mathfrak{g}$, $w \in G$, $\theta \in G'$, for $G_{\mathbf{z}_n}$ are assumed, i.e., the series $(\theta, \omega_{g_1}(z_1)\, \omega_{g_2}(z_2) w)$, $(\theta, \omega_{g_2}(z_2)\, \omega_{g_1}(z_1) w)$, $\left(\theta, \omega_{\omega_{g_1}(z_1-z_2) g_2}(z_2) w\right)$, are absolutely convergent in the regions $|z_1| > |z_2| > 0$, $|z_2| > |z_1| > 0$, $|z_2| > |z_1 - z_2| > 0$, respectively, to a common rational function in z_1 and z_2 with the only possible poles at $z_1 = 0 = z_2$ and $z_1 = z_2$.

Definition 4. Let $\mathfrak{G} \subset \mathrm{Aut}(\mathfrak{g})$ be a subgroup of $\mathrm{Aut}(\mathfrak{g})$. We say that $\mathfrak{G}$ acts on $G_{\mathbf{z}}$ as automorphisms if $g\, \omega_h(z)\, g^{-1} = \omega_{gh}(z)$, on $G_{\mathbf{z}}$ for all $g \in \mathfrak{G}$, $h \in \mathfrak{g}$.

2.3. **Conditions on rational functions.** Let $\mathbf{z}_n \in F_n\mathbb{C}$. Denote by T_G the translation operator [12]. We define now extra conditions on rational functions leading to the definition of restricted rational functions.

Definition 5. Denote by $(T_G)_i$ the operator acting on the i-th entry. We then define the action of partial derivatives on an element $F(\mathbf{x}_n)$

$$\begin{aligned} \partial_{z_i} F(\mathbf{x}_n) &= F((T_G)_i\, \mathbf{x}_n), \\ \sum_{i \geqslant 1} \partial_{z_i} F(\mathbf{x}_n) &= T_G F(\mathbf{x}_n), \end{aligned} \tag{2.3}$$

and call it T_G-derivative property.

Definition 6. For $z \in \mathbb{C}$, let

$$e^{zT_G} F(\mathbf{x}_n) = F(\mathbf{g}_n, \mathbf{z}_n + z). \tag{2.4}$$

Let $\mathrm{Ins}_i(A)$ denotes the operator of multiplication by $A \in \mathbb{C}$ at the i-th position. Then we define

$$F\left(\mathbf{g}_n, \mathrm{Ins}_i(z)\, \mathbf{z}_n\right) = F\left(\mathrm{Ins}_i(e^{zT_G})\, \mathbf{x}_n\right), \tag{2.5}$$

are equal as power series expansions in z, in particular, absolutely convergent on the open disk $|z| < \min_{i \neq j}\{|z_i - z_j|\}$.

Definition 7. A rational function has K_G-property if for $z \in \mathbb{C}^\times$ satisfies $(z\, \mathbf{z}_n) \in F_n\mathbb{C}$,

$$z^{K_G} F(\mathbf{x}_n) = F\left(z^{K_G} \mathbf{g}_n, z\, \mathbf{z}_n\right). \tag{2.6}$$

2.4. **Rational functions with prescribed analytical behavior.** In this subsection we give the definition of rational functions with prescribed analytical behavior on a domain of X. We denote by $P_k : G \to G_{(k)}$, $k \in \mathbb{C}$, the projection of G on $G_{(k)}$. For each element $g_i \in G$, and $x_i = (g_i, z)$, $z \in \mathbb{C}$ let us associate a formal series $\omega_{g_i}(z) = \sum_{k \in \mathbb{C}} g_{ik}\, z^k$, $i \in \mathbb{Z}$. Following [12], we formulate

Definition 8. We assume that there exist positive integers $\beta(g_{l',i}, g_{l'',j})$ depending only on $g_{l',i}$, $g_{l'',j} \in G$ for i, $j = 1, \ldots, (l+k)n$, $k \geqslant 0$, $i \neq j$, $1 \leqslant l', l'' \leqslant n$. Let $\mathbf{l}_n$ be a partition of $(l+k)n = \sum_{i \geqslant 1} l_i$, and $k_i = l_1 + \cdots + l_{i-1}$. For $\zeta_i \in \mathbb{C}$, define $h_i = F(\mathbf{W}_{\mathbf{g}_{k_i+1_i}}(\mathbf{z}_{k_i+1_i} - \zeta_i))$, for $i = 1, \ldots, n$. We then call a rational function F satisfying properties (2.3)–(2.6), a rational function with prescribed analytical behavior, if under the following conditions on domains, $|z_{k_i+p} - \zeta_i| + |z_{k_j+q} - \zeta_j| < |\zeta_i - \zeta_j|$, for i, $j =$

$1, \dots, k$, $i \neq j$, and for $p = 1, \dots, l_i$, $q = 1, \dots, l_j$, the function $\sum\limits_{\mathbf{r}_n \in \mathbb{Z}^n} F(\mathbf{P}_{\mathbf{r_i}} \mathbf{h_i}; (\zeta)_l)$, is absolutely convergent to an analytically extension in $\mathbf{z}_{l+k}$, independently of complex parameters $(\zeta)_l$, with the only possible poles on the diagonal of $\mathbf{z}_{l+k}$ of order less than or equal to $\beta(g_{l',i}, g_{l'',j})$. In addition to that, for $\mathbf{g}_{l+k} \in G$, the series $\sum_{q \in \mathbb{C}} F(\mathbf{W}(\mathbf{g_k}$, $\mathbf{P}_q(\mathbf{W}(\mathbf{g}_{l+k}, \mathbf{z}_k), \mathbf{z}_{k+1}))$, is absolutely convergent when $z_i \neq z_j$, $i \neq j$ $|z_i| > |z_s| > 0$, for $i = 1, \dots, k$ and $s = k+1, \dots, l+k$ and the sum can be analytically extended to a rational function in $\mathbf{z}_{l+k}$ with the only possible poles at $z_i = z_j$ of orders less than or equal to $\beta(g_{l',i}, g_{l'',j})$.

For $m \in \mathbb{N}$ and $1 \leqslant p \leqslant m-1$, let $J_{m;p}$ be the set of elements of S_m which preserve the order of the first p numbers and the order of the last $m-p$ numbers, that is,

$$J_{m,p} = \{\sigma \in S_m \mid \sigma(1) < \cdots < \sigma(p),\ \sigma(p+1) < \cdots < \sigma(m)\}.$$

Let $J_{m;p}^{-1} = \{\sigma \mid \sigma \in J_{m;p}\}$. In addition to that, for some rational functions require the property:

$$\sum_{\sigma \in J_{n;p}^{-1}} (-1)^{|\sigma|} \sigma(F(\mathbf{g}_{\sigma(i)}, \mathbf{z}_n)) = 0. \tag{2.7}$$

Then, we have

Definition 9. We define the space $\Theta\left(n, k, G_{\mathbf{z}_n}, U\right)$ of matrix elements $F(\mathbf{x}_n)$ of n formal complex parameters as the space of restricted rational functions with prescribed analytical behavior on a $F_n\mathbb{C}$-domain $U \subset X$, and satisfying T_G- and K_G-properties (2.3)–(2.6), definition (8), and (2.7).

3. Associative algebra $A(\mathfrak{g})$ of prescribed rational functions

In this section we define a twisted bundles corresponding to the associative algebra $A(\mathfrak{g})$, and describe their properties.

3.1. Associative algebra $A(\mathfrak{g})$ out of $\mathfrak{g}$.

In this subsection we recall [23, 5] a way how to derive an associative algebra $A(\mathfrak{g})$ out of an infinite-dimensional Lie algebra $\mathfrak{g}$.

Definition 10. For any homogeneous vectors $h, \widetilde{h} \in G$, one defines the multiplications

$$h *_\kappa \widetilde{h} = \mathrm{Res}_z \left((1+z)^{\mathrm{wt}(h)} \sum_{l \in \mathbb{C}} h_n z^{l-\kappa} \right) .\widetilde{h},$$

for $\kappa = 1$, 2, and extend it bilinearly it to $G \times G$.

Here, as usual, Res_z denotes the coefficient in front of z^{-1}.

Definition 11. For $h, \widetilde{h} \in G$, define $A(\mathfrak{g}) = G_{\mathbf{z}}/(\mathrm{span}(h *_2 \widetilde{h}))^{\theta}$.

For $\theta = 0$ we get back to $G_{\mathbf{z}}$ with associativity property described in subsection 2.2 expressed via matrix elements, while for and for $\theta = 1$ we obtain an associative algebra associated to $\mathfrak{g}$ with ordinary associativity. The following theorem is due to [23, §2] (also see [5]).

Theorem 1. *The bilinear operation $*_1$ makes $A(\mathfrak{g})$ into an associative algebra with the linear map $\phi : g \mapsto \exp\left(-z^2\partial_z\right)(-1)^{-z\partial_z} g$, inducing an anti-involution ν on $A(\mathfrak{g})$.*

In what follows, let us denote by $W \subset G_{\mathbf{z}}$ a subspace which is an $A(\mathfrak{g})$-module. For homogeneous $g \in G$ we set $o(g) = a_{\mathrm{wt}(g)-1}$ and extend linearly to all $g \in G$.

Definition 12. We now define the space of lowest weight vectors of G:

$$L(W) = \{g, \in G, w \in W | g_{\mathrm{wt}(h)+m} w = 0, h \in W, m \geqslant 0\}.$$

Remark 1. For $W = \bigoplus_{\lambda\in\mathbb{C}} W_\lambda$, $L(W) = \bigoplus_{\lambda\in\mathbb{C}} L(W)_\lambda$ is naturally graded, and each homogeneous subspace $L(W)_\lambda = L(W) \cap W_\lambda$ is finite dimensional.

It is easy to see the following

Lemma 1. *For W, $\widetilde{W}$ be two $A(\mathfrak{g})$-modules, and for $\varphi : W \to \widetilde{W}$ an $A(\mathfrak{g})$-module homomorphism, one has $\varphi(L(W)) \subset L(\widetilde{W})$. In particular, if φ is an isomorphism then $\varphi(L(W)) = L(\widetilde{W})$.*

3.2. **Category $\mathcal{O}_{A(\mathfrak{g})}$ of $A(\mathfrak{g})$-modules.** Let W_z be an $A(\mathfrak{g})$-module and we denote the dual space of W with respect to the form $(.,.)$ by W'. The following lemma is obvious [6]:

Lemma 2. *W' is an $A(\mathfrak{g})$-module such that $(a\, m', m) = (m', \nu(a)\, m)$, for $a \in A(\mathfrak{g})$, $m' \in W'$, and $m \in W$.*

Definition 13. A form $(.,.)$ defined on an $A(\mathfrak{g})$-module W_z is called invariant if $(a\, w_1, w_2) = (w_1, \nu(a)\, w_2)$ for $w_i \in W_z$ and $a \in A(\mathfrak{g})$.

We also need to define the category $\mathcal{O}_{A(\mathfrak{g})}$ of $A(\mathfrak{g})$-modules.

Definition 14. An $A(\mathfrak{g})$-module W is in $\mathcal{O}_{A(\mathfrak{g})}$ if there exist $\boldsymbol{\lambda}_s \in \mathbb{C}^n$, such that $W = \bigoplus_{\substack{i=1,\\ n\geqslant 0}}^{s} W_{\lambda_i+n}$, is a direct sum of finite dimensional $A(\mathfrak{g})$-modules and $\mathrm{Hom}_{A(\mathfrak{g})}(W_\lambda, W_\mu) = 0$, if $\mu \neq \lambda$.

Theorem 2. *Let $W_0 \neq 0$. Then the linear map $o : W_{\mathbf{z}} \to \mathrm{End}(L(W_{\mathbf{z}}))$, $g \mapsto o(g)|L(W_{\mathbf{z}})$, induces a homomorphism from W_z to $\mathrm{End}(L(W_{\mathbf{z}}))$, and $L(W_{\mathbf{z}})$ is a left $A(\mathfrak{g})$-module. For all $\lambda \in \mathbb{C}$, $L(W_{\mathbf{z}})_\lambda$ is an finite-dimensional $A(\mathfrak{g})$-module. $\mathcal{O}_{A(\mathfrak{g})}$ is invariant with respect to definition of L.*

Note that for $\lambda \neq \mu$, $\mathrm{Hom}_{A(\mathfrak{g})}(L(W_{\mathbf{z}})_\lambda, L(W_{\mathbf{z}})_\mu) = 0$. Thus $L(W_{\mathbf{z}})$ is an element of $\mathcal{O}_{A(\mathfrak{g})}$.

4. Twisted $A(\mathfrak{g})$-bundles

As it was shown in [6], it turns out that we can introduce corresponding bundles for a large class of associative algebras. In this section we define the main objects of this paper, associative algebra twisted $A(\mathfrak{g})$-bundles.

4.1. **Torsors and twists under groups of automorphisms.** We now explain how to collect elements of the space $\Theta\left(n, k, W_{\mathbf{z},\lambda}, X_\alpha\right)$ of prescribed rational functions into sections of a twisted $A(\mathfrak{g})$-bundle on X. Let $\mathcal{H}$ be a subgroup of the group $\mathbf{Aut_z}\, \boldsymbol{\mathcal{O}}_X$ of independent formal parameters $\mathbf{z}$ automorphisms on X. We recall here the notion of a torsor with respect to a group.

Definition 15. Let $\mathfrak{H}$ be a group, and $\mathcal{X}$ a non-empty set. Then $\mathcal{X}$ is called a $\mathfrak{H}$-torsor if it is equipped with a simply transitive right action of $\mathfrak{H}$, i.e., given ξ, $\widetilde{\xi} \in \mathcal{X}$, there exists a unique $h \in \mathfrak{H}$ such that $\xi \cdot h = \widetilde{\xi}$, where for h, $\widetilde{h} \in \mathfrak{H}$ the right action is given by $\xi \cdot (h \cdot \widetilde{h}) = (\xi \cdot h) \cdot \widetilde{h}$. The choice of any $\xi \in \mathcal{X}$ allows us to identify $\mathcal{X}$ with $\mathfrak{H}$ by sending $\xi \cdot h$ to h.

Using similar results for W_z of [2], one shows that certain subspaces $W_z \subset G_{\mathbf{z}}$ form $\mathcal{H}$-modules. Applying the definition of a group twist to the group $\mathcal{H}$ and its module $W_{\mathbf{z}}$ we obtain

Definition 16. Given a $\mathcal{H}$-module $W_{\mathbf{z}}$ and a $\mathcal{H}$-torsor X_α, one defines the X_α-twist of $W_{\mathbf{z}}$ as the set

$$\mathcal{E}_{X_\alpha} = W_{\mathbf{z}} \underset{\mathcal{H}}{\times} X_\alpha = W_{\mathbf{z}} \times X_\alpha / \left\{ (w, a \cdot \xi) \sim (aw, \xi) \right\},$$

for $\xi \in X_\alpha$, $a \in \mathcal{H}$, and $w \in W_{\mathbf{z}}$.

Now we wish to attach to any X_α a twist $\mathcal{E}_{X_\alpha}$ of $W_{\mathbf{z}}$. We have an isomorphism $i_{\mathbf{z},X_\alpha} : W_{\mathbf{z}} \xrightarrow{\sim} \mathcal{E}_{X_\alpha}$. The system of isomorphisms $i_{\mathbf{z},\mathbf{X}_\alpha}$ should satisfy certain compatibility conditions. Namely, an automorphism $(i^{-1}_{\mathbf{z},X_\alpha} \circ i_{\mathbf{z},X_\beta})$ of $W_{\mathbf{z}}$ should define a representation on $W_{\mathbf{z}}$ of the group $\mathcal{H}$. Then $\mathcal{E}_{X_\alpha}$ is canonically identified with the twist of $W_{\mathbf{z}}$ by the $\mathcal{H}$-torsor of X_α. The elements of $\Theta\left(n, k, W_{\mathbf{z}}, X_\alpha\right)$ give rise to a collection of sections $F(\mathbf{x})$. The construction of local parts of a twisted $A(\mathfrak{g})$-bundle is grounded on the notion of a principal bundle for the group $\mathcal{H}$ naturally existing on X. Let Aut_{X_α} be the space of all sets of local parameters on X_α. Next we ave (c.f. [2])

Lemma 3. *The group $\mathcal{H}$ acts naturally on Aut_{X_α} which is a $\mathcal{H}$-torsor.*

Thus, we can define the following twist.

Definition 17. We introduce the $\mathcal{H}$-twist of $W_{\mathbf{z}}$ $\mathcal{E}_{\mathbf{z}} = W_{\mathbf{z}} \underset{\mathcal{H}}{\times} Aut_{X_\alpha}$. The original definition similar to (17) was given in [1, 21].

4.2. **Definition of the local part of prescribed rational functions bundle $\mathcal{E}(W_{\mathbf{z},\lambda})$.** We now fix an infinite-dimensional Lie algebra $\mathfrak{g}$ satisfying requirements of subsection 2.2. Suppose that a $\mathbb{C}$-grading is generated by K_0 on $W_{\mathbf{z}}$. Let $\mathfrak{G} \subset Aut(\mathfrak{g})$ be a $W_{\mathbf{z}}$-grading preserving subgroup of $Aut(\mathfrak{g})$. Denote by $\mathcal{O}_{\mathfrak{g},A(\mathfrak{g})}$ a subcategory of $\mathcal{O}_{A(\mathfrak{g})}$ consisting of $A(\mathfrak{g})$-modules $W_{\mathbf{z}}$ such that $\mathfrak{G}$ acts on $W_{\mathbf{z}}$ as automorphisms. By using the ideas of [2], we formulate here the definition of the local part $\mathcal{E}(W_{\mathbf{z},\lambda})$ of the fiber bundle associated to $\mathfrak{g}$ through matrix elements $F(\mathbf{x})$ with $\mathbf{x} = (\mathbf{g}, \mathbf{z})$, to the space $\Theta\left(n, k, W_{\mathbf{z},\lambda}\right)$ for all n, $k \geqslant 0$, of prescribed rational functions on a finite part $\{X_\alpha, \alpha \in I_0\}$ of a covering $\{X_\alpha\}$ of X. For the fiber space provided by elements $f(\mathbf{x}) \in W_{\mathbf{z}}$, using the property of prescribed rational functions we form a principal

$\mathcal{H}$-bundle, which is a fiber bundle $\mathcal{E}(W_{\mathbf{z},\lambda})$ defined by trivializations $i_{\mathbf{z}} : F(\mathbf{x}) = (\theta, f(\mathbf{x})) \to X_a$, together with a continuous right action $F(\mathbf{x}) \times \mathcal{H} \to F(\mathbf{x})$, such that $\mathcal{H}$ preserves $F(\mathbf{x})$, i.e., ζ, $\zeta.a$ are sections of $\mathcal{E}(n, k, W_{\mathbf{z},\lambda})$ for all $a \in \mathcal{H}$, and acts freely and transitively, i.e., the map $a \mapsto \zeta.a$ is a homeomorphism. Thus, we have [2]

Lemma 4. *The projection $Aut_{X_\alpha} \to X$ is a principal $\mathcal{H}$-bundle. The fiber of this bundle over X_α is the $\mathcal{H}$-torsor Aut_{X_α}.*

Then we obtain

Definition 18. Given a finite-dimensional $\mathcal{H}$-module $W_{i_{\mathbf{z}_n},\lambda}$, let

$$\mathcal{E}(W_{\mathbf{z},\lambda}) = \bigoplus_{n,k \geqslant 0} W_{i_{\mathbf{z}_n},\lambda} \underset{\mathcal{H}}{\times} Aut_{X_a},$$

be the fiber bundle associated to $W_{i_{\mathbf{z}},\lambda}$, Aut_{X_a}, and with sections provided by elements of $\Theta(n, k, W_{\mathbf{z}_n}, X_\alpha)$, for n, $k \geqslant 0$.

On X we can choose $\{X_\alpha\}$ such that the bundle $\mathcal{E}(W_{\mathbf{z},\lambda})$ over X_α is $X_\alpha \times F(\mathbf{x})$. The fiber bundle $\mathcal{E}(W_{\mathbf{z},\lambda})$ with fiber $f(\mathbf{x}_n)$ is a map $\mathcal{E}(W_{\mathbf{z},\lambda}) : \mathbb{C}^n \to X$ where $\mathbb{C}^n$ is the total space of $\mathcal{E}(W_{\mathbf{z},\lambda})$ and X is its base space. For every X_α of X $i^{-1}_{\mathbf{z}_n}$ is homeomorphic to $X_a \times \mathbb{C}^n$. Namely, we have for $f(\mathbf{x}_n) : i^{-1}_{\mathbf{z}_n} \to X_\alpha \times \mathbb{C}^n$, that $\mathcal{P} \circ f(\mathbf{x}_n) = i_{\mathbf{z}_n} \circ i^{-1}_{\mathbf{z}_n}(X_\alpha)$, where $\mathcal{P}$ is the projection map on X_α.

4.3. **Definitions of a twisted $A(\mathfrak{g})$-bundle.** In this subsection we formulate (generalizing examples of other cases considered in [6]) the definition of a twisted fiber bundle associated to $A(\mathfrak{g})$-module $W_{\mathbf{z}} \in \mathcal{O}_{\mathfrak{G},A(\mathfrak{g})}$. We obtain

Definition 19. A twisted $A(\mathfrak{g})$-bundle $\mathcal{E}$ over X with fiber $W_{\mathbf{z}}$ and $\Theta\,(n, k, W_{\mathbf{z}}, X)$, n, $k \geqslant 0$-valued sections is a direct sum of vector bundles $\mathcal{E} = \bigoplus_{\lambda \in \mathbb{C}} \mathcal{E}(W_{\mathbf{z},\lambda})$, such that all transition functions are $A(\mathfrak{g})$-module isomorphisms, and a family of continuous isomorphisms $H_\alpha = \{H_{\alpha,\lambda}, \lambda \in \mathbb{C}\}$, of fiber bundles

$$H_{\alpha,\lambda} : \mathcal{E}(W_{\mathbf{z},\lambda})|_{X_\alpha} \to W_{i_{\mathbf{z}},\lambda} \underset{\mathfrak{G}}{\times} Aut_{X_\alpha},$$

such that for transition functions $g_{\alpha\beta,\lambda} = H_{\alpha,\lambda} *_2 H^{-1}_{\beta,\lambda}$, for all $\lambda \in \mathbb{C}$, then

$$g_{\alpha\beta}(x) = (g_{\alpha\beta,\lambda}(\xi)) : W_{\mathbf{z}} \to W_{\mathbf{z}},$$

is an $A(\mathfrak{g})$-module isomorphism for any $\xi \in (X_\alpha \bigcap X_\beta)$, where the transition functions $g_{\alpha,\beta}(x)$ are $G_{\mathbf{z}}$-valued.

Note that definitions of direct sum of bundles, sub-bundles and quotient bundles appear accordingly. We are also able to define graded twisted $A(\mathfrak{g})$-bundles. For that purpose, replace $W_{\mathbf{z},\lambda} \to L(W_{\mathbf{z},\lambda})$, $\mathcal{E}(W_{\mathbf{z},\lambda}) \to \mathcal{E}^{\mathrm{gr}}(W_{\mathbf{z},\lambda})$, $g_{\alpha\beta,\lambda}$ by $\left(g^{\mathrm{gr}}_{\alpha\beta,\lambda}\right) = \left(H^{\mathrm{gr}}_{\alpha,\lambda}\right) *_2 \left(H^{\mathrm{gr}}_{\beta,\lambda}\right)^{-1}$, and introduce $(H^{\mathrm{gr}}_\alpha) = \left(H_\alpha|_{(\mathcal{E}^{\mathrm{gr}}(W_{\mathbf{z},\lambda}))}\right)$, for all $\lambda \in \mathbb{C}$. Then using Theorem 2, we obtain

Lemma 5. *The graded transition functions $(g^{\mathrm{gr}}_{\alpha\beta})(\xi)$ provide an $A(\mathfrak{g})$-module isomorphism $\left(g^{\mathrm{gr}}_{\alpha\beta,\lambda}\right)(x) : L(W_{\mathbf{z}_n}) \to L(W_{\mathbf{z}_n})$, for any $\lambda \in \mathbb{C}$ and $x \in X_\alpha \bigcap X_\beta$. By Lemma 1, $\mathcal{E}^{\mathrm{gr}}$ is a twisted $A(\mathfrak{g})$-bundle over X.*

Let $A(\mathfrak{g})$ and $\widetilde{A}(\widetilde{\mathfrak{g}})$ be two associative algebras with anti-involutions ν_A and $\nu_{\widetilde{A}}$ respectively. Then, similar to [6], one has

Lemma 6. *$A(\mathfrak{g}) \otimes_{\mathbb{C}} \widetilde{A}(\widetilde{\mathfrak{g}})$ is an associative algebra with anti-involution $\nu_A \otimes \nu_{\widetilde{A}}$.*

5. Properties of twisted $A(\mathfrak{g})$-bundles

In this section we reveal properties of twisted $A(\mathfrak{g})$-bundles. In particular we show that twisted $A(\mathfrak{g})$-bundle behaves well under standard operations, and are invariant under homotopy transformations of X. Note that in the simplest case of $A(\mathfrak{g}) = \mathbb{C}$ the twisted $A(\mathfrak{g})$-bundle is a classical complex vector bundle over X. Let $\mathcal{E}$, $\widetilde{\mathcal{E}}$ be $A(\mathfrak{g})$ and $A(\widetilde{\mathfrak{g}})$-bundles over X. Then we have

Lemma 7. *Then $\mathcal{E} \otimes \widetilde{\mathcal{E}}$ is a $A(\mathfrak{g}) \otimes_{\mathbb{C}} A(\widetilde{\mathfrak{g}})$-bundle over X. In particular, If $A(\mathfrak{g}) = \mathbb{C}$ then $\mathcal{E} \otimes \widetilde{\mathcal{E}}$ is again a $\widetilde{A}$-bundle over X.*

Let $\mathcal{E}$ be an $A(\mathfrak{g})$-bundle over X. Introduce $\mathcal{E}' = \oplus_{\lambda \in \mathbb{C}} (\mathcal{E}(W_{\mathbf{z},\lambda}))^*$. Then, due to this definition and properties of the non-degenerate bilinear form $(.,.)$, we obtain

Lemma 8. *The dual bundle $\mathcal{E}'$ is also a twisted $A(\mathfrak{g})$-bundle.*

Definition 20. Let $\mathcal{E}$, $\widetilde{\mathcal{E}}$ be two twisted $A(\mathfrak{g})$-bundles on X. A map $\eta : \mathcal{E} \to \widetilde{\mathcal{E}}$, is called a twisted $A(\mathfrak{g})$-bundle morphism if there exist a family of continuous morphisms of fiber bundles

$$\eta_\lambda : \mathcal{E}(W_{\mathbf{z},\lambda}) \to \widetilde{\mathcal{E}}(W_{\mathbf{z},\lambda}),$$

such that with $\eta = (\eta_\lambda)$, for all $\lambda \in \mathbb{C}$, and $\eta_\lambda : \mathcal{E} \to \widetilde{\mathcal{E}}$, is an $A(\mathfrak{g})$-module morphism for any $\xi \in X$.

It follows from [6] that the following lemma is true for $A(\mathfrak{g})$.

Lemma 9. *Let $\mathcal{E}$ be a twisted $A(\mathfrak{g})$-bundle on X. Then $\mathcal{E} \bigoplus \mathcal{E}'$ is a twisted $A(\mathfrak{g})$-bundle, with nondegenerate symmetric invariant bilinear form*

$$\left(g^*_{\alpha\beta}(\xi)\theta, g_{\alpha\beta}(\xi)u\right) = (\theta, u)\,, \tag{5.1}$$

which is an invariant of $\mathcal{E}$ (i.e., does not depend on $g_{\alpha\beta}$ for all α, $\beta \in I$, $\xi \in X_\alpha \cap X_\beta$, $\xi \in G_{\mathbf{z}_n}$, $\theta \in G'_{\mathbf{z}_n}$) induced from the natural bilinear from on $G_{\mathbf{z}} \oplus G'_{\mathbf{z}}$.

Proof. For ordinary bundles this lemma was proven in [6]. Here we have to check (5.1) in the twisted case. Namely, for particular $\lambda \in \mathbb{C}$, using the definition of $g_{\alpha\beta}$ and multiplication $*_2$, consider $\left(g^*_{\alpha\beta,\lambda}(\xi)\theta, g_{\alpha\beta,\lambda}(\xi)u\right) = \left((H_{\alpha,\lambda} *_2 H^{-1}_{\beta,\lambda})^*\theta, (H_{\alpha,\lambda} *_2 H^{-1}_{\beta,\lambda})u\right)$. One sees that it equals to (θ, u). Thus, the form $(g^*_{\alpha\beta}(\xi)\theta, g_{\alpha\beta}(\xi)u)$ does not depend on any α, $\beta \in I$. □

It is useful to introduce the following

Definition 21. A twisted $A(\mathfrak{g})$-bundle $\mathcal{E}$ is called trivial if there exists an $A(\mathfrak{g})$-bundle isomorphism $\varphi : \mathcal{E} \to W \times X$, here $W \times X$ is the natural $A(\mathfrak{g})$-bundle on X with W as fibers.

Finally, we provide proofs for generalizations of two propositions given in [6] for the case of twisted $A(\mathfrak{g})$-bundles.

Proposition 1. *For any twisted $A(\mathfrak{g})$-bundle $\mathcal{E}$, there exists a twisted $A(\mathfrak{g})$-bundle $\widetilde{\mathcal{E}}$ such that $\mathcal{E} \bigoplus \widetilde{\mathcal{E}}$ is geometrically covariant for $\mathcal{E}$, i.e., $\mathcal{E} \bigoplus \widetilde{\mathcal{E}}$ is a trivial $A(\mathfrak{g})$-bundle.*

Proof. Due to the properties of non-degenerate bilinear form, it is naturaly to use it to characterize a twisted $A(\mathfrak{g})$-bundle $\mathcal{E}$. By Lemma 9 we can assume that there is a nondegenerate invariant symmetric bilinear form on $\mathcal{E}$ induced from the form on $G_{\mathbf{z}}$. We are able to choose a finite covering $\{X_\alpha, \alpha \in I_0\}$ in the definition 19. For $\alpha \in I_0$, let us set $H_\alpha = (h_\alpha(\epsilon), \pi(\epsilon))$, where $\pi(\epsilon) : \mathcal{E} \to X$ is the natural projection map and $h_\alpha(\epsilon) \in W$. Let $(\chi_\alpha h_\alpha)$ be representatives of an element W of $\mathcal{O}_{\mathfrak{G}, A(\mathfrak{g})}$ over X_α, and χ_a such that $\sum\limits_{\alpha \in I_0} \chi_a(\xi)\chi_\alpha(\xi) = Id_W$ is the identity operator for $\xi \in X$ on the covering $\{X_\alpha, \alpha \in I_0\}$. Define an $A(\mathfrak{g})$-bundle injective and bilinear form preserving homomorphism $\psi : \mathcal{E} \to W^{\oplus k} \times X$, $\psi(e) = ((\chi_\alpha(\xi) \; h_\alpha(\epsilon)), \xi)$, for $\alpha \in I_0$, where $\xi = \pi(\epsilon)$, and k is the number of independent domains in the covering $\{X_\alpha\}$. The ψ sends $\mathcal{E}$ to the trivial $A(\mathfrak{g})$-bundle $W^{\oplus k} \times X$. We are able to extend non-degenerate bilinear form $(.,.)$ to $W^{\oplus k} \times X$. By Lemma 9, the transition functions preserve the bilinear form on W, thus for any ϵ, $\widetilde{\epsilon} \in \mathcal{E}_x$ one finds

$$\begin{aligned}
(\psi(\epsilon), \psi(\widetilde{\epsilon})) &= \sum_{\alpha \in I_0} ((\chi_\alpha(\xi) \; h_\alpha(\epsilon)), (\chi_\alpha(\xi) \; h_\alpha(\widetilde{\epsilon}))) \\
&= \sum_{\alpha \in I_0} \chi_\alpha^2(\xi) \, (g_{\alpha\beta}(\xi) \; h_\beta(\epsilon), g_{\alpha\beta}(\xi) \; h_\beta(\widetilde{\epsilon})) \\
&= (\epsilon, \widetilde{\epsilon}).
\end{aligned}$$

Thus the homomorphism ψ preserves the bilinear form and the restriction of the bilinear form to $\psi(\mathcal{E})$ is nondegenerate. the form $(\psi(\epsilon), \psi(\widetilde{\epsilon}))$ does not depend on the choice of Let us take $\widetilde{\mathcal{E}} = \psi(\mathcal{E})^\dagger$ with respect to the bilinear form. According to Lemma 8, $\widetilde{\mathcal{E}}$ is an $A(\mathfrak{g})$-bundle on X, $(W^{\oplus k} \times X) = \widetilde{\mathcal{E}} \oplus \psi(\mathcal{E}) \cong \widetilde{\mathcal{E}} \oplus \mathcal{E}$. and such $A(\mathfrak{g}) \oplus A(\mathfrak{g})$-bundle is trivial is a geometrical covariant. □

A twisted $A(\mathfrak{g})$ exhibits the following homotopy-stability property:

Proposition 2. *The construction of a twisted $A(\mathfrak{g})$-bundle $\mathcal{E}$ is homotopy-invariant. I.e., let $\widetilde{X}$ be a compact Hausdorff space, $\tau_t : \widetilde{X} \to X$, for $0 \leqslant t \leqslant 1$, a homotopy and $\mathcal{E}$ a twisted $A(\mathfrak{g})$-bundle over X. Then $\tau_0^*(\mathcal{E}) \simeq \tau_1^*(\mathcal{E})$.*

Proof. Denote by $\mathcal{I}$ the unit interval and let $\tau : \widetilde{X} \times \mathcal{I} \to X$, be the homotopy, so that $\tau(\widetilde{\xi}, t) = \tau_t(\widetilde{\xi})$, and let $\pi : \widetilde{X} \times \mathcal{I} \to \widetilde{X}$ denote the projection onto the first factor. For a collection of $\xi_i \in \widetilde{X}$, $k \geqslant 1$, and an element $w_i \in W$, let us choose a finite open covering $\{\widetilde{X}_{\xi_i}\}_{i=1}^k$ of $\widetilde{X}$ so that $\tau^*(\mathcal{E}) = w_i \times \xi_i$ is trivial over each $\widetilde{X}_{\xi_i} \times \mathcal{I}$. For each $\xi \in \widetilde{X}$ we can find open neighborhood $U_{\xi,\mathbf{k}}$ in $\widetilde{X}$, and a partition $\{t_i, 0 = t_0 < t_1 \cdots < t_k = 1\}$ of $[0,1]$ such that the bundle is trivial over each $U_{\xi,i} \times [t_{i-1}, t_i]$. Set $U = U_{\xi,\mathbf{k}} = \bigcap\limits_{i=1}^{k} U_{\xi,i}$. Then the twisted bundle $\tau^*(\mathcal{E})$ is trivial over $\widetilde{X}_\xi \times \mathcal{I}$. Indeed, by choosing appropriate elements of Aut_{X_ξ} we could find for $t_{i-1,i} = [t_{i-1}, t_i]$, isomorphisms of trivializations such that

$$h_i : \tau^*(\mathcal{E})|_{\widetilde{X}_\xi \times t_{i-1,i}} \to W \times \widetilde{X}_\xi \times t_{i-1,i},$$

for $1 \leqslant i \leqslant k$. For $u \in U$, we take the $A(\mathfrak{g})$-bundle isomorphisms

$$h'_i(u,t_i) = (h_{i-1} \circ h_i^{-1})(u,t_i) \circ h_i(u,t_{i-1}) : \\ \tau^*(\mathcal{E})|_{U_\xi \times [t_i, t_{i+1}]} \to W \times U_\xi \times [t_i, t_{i+1}],$$

then $h_i = h'_{i+1}$ on $U_\xi \times \{t_i\}$, thus they define a trivialization on $\widetilde{X}_\xi \times [t_{i-1}, t_{i+1}]$, and thus $\tau^*(\mathcal{E})$ is trivial over $\widetilde{X}_x \times \mathcal{I}$. Let χ_i be a partition of unity of $\widetilde{X}$ with support of χ_i contained in U_{ξ_i}. For $i \geqslant 0$, let $q_j = \sum_{i=1}^{j} \chi_i$. In particular, $q_0 = 0$ and $q_n = 1$. Consider the subspace of $U \times \mathcal{I}$ consisting of points of the form $(\xi, q_i(\xi))$, and let $\pi_i : \mathcal{E}_i \to W_i$ be the restriction of the bundle $\mathcal{E}$ over W_i. Since $\mathcal{E}$ is trivial on $U_{x_i} \times \mathcal{I}$, the projection homeomorphism $W_i \to W_{i-1}$ induces homomorphisms $\varepsilon_i : \mathcal{E}_i \to \mathcal{E}_{i-1}$, which is identity outside $\pi_i(U_{x_i})$, and which takes each fiber of $\mathcal{E}_i$ isomorphically onto the corresponding fiber of $\mathcal{E}_{i-1}$. The composition $\varepsilon = \prod_{i=1}^{k} \circ \varepsilon_i$ is then an isomorphism from $\mathcal{E}|_{U \times \{1\}}$ to $\mathcal{E}|_{U \times \{0\}}$. □

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Institute of Mathematics, Czech Academy of Sciences, Zitna 25, Prague, Czech Republic

E-mail address: zuevsky@yahoo.com