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Formal series cohomology of complex curves

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FORMAL SERIES COHOMOLOGY OF COMPLEX CURVES

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Abstract. We introduce and study the local cohomology of a complex curve in terms of formal series with non-commutative coefficients. This includes a way to characterize a complex curve in terms of connections. Particular examples of a vertex algebra cohomologies on a complex curve are considered.

1. Introduction

The theory of cohomologies of complex curves is represented by a few approaches [2, 3, 6–9, 18]. In this paper we study the formal series cohomology with non-commutative coefficients defined for a complex curve in particular example of a grading-restricted vertex algebra [12]. Vertex algebras, generalizations of ordinary Lie algebras, constitute an algebraic language of conformal field theory. The geometric side of vertex algebra characters is in associating their formal parameters with local coordinates on a complex curve. Depending on geometry, one can obtain various consequences for a vertex algebra and its space of characters, and vice-versa, one can study geometrical property of a manifold by using algebraic nature of a vertex algebra attached. Let $\overline{W}$ be the algebraic complection of a vertex algebra V module W. We consider a cohomology of $\overline{W}$-valued rational functions with sets of formal parameters appropriately identified with local coordinates on sets of open domains on a complex curve $\mathcal{M}$. Manifolds of arbitrary dimensions will be considered elsewhere. It would be also important to establish relations to chiral de Rham complex theory on a smooth manifold introduced in [16]. In many cases it is useful to express cohomology in terms of connections. Connections numerously appear in conformal field theory [1, 5, 19].

In Section 2 we recall definition of $\mathcal{W}$-valued rational functions. In Section 3 we define the formal series cohomology of a complex curve associated to a quasi-conformal grading-restricted vertex algebra. The Section 4 discusses examples of vertex algebra cohomologies of a complex curve. In Appendixes we provide the material needed for construction of the vertex algebra cohomology of a complex curve. In Appendix 5 we recall the notion of a quasi-conformal grading-restricted vertex algebra and its modules. In Appendix 6 we describe the approach to cohomology in terms of connections. In Appendix 7 non-emptiness and canonicity of the construction is proved. In Appendix 8 we recall the notion of cohomology classes associated to vertex algebras and propose ways to characterize complex curves.

Key words and phrases. Complex curves; vertex operator algebras; cohomology; characteristic functions.

2. $\mathcal{W}$-valued rational functions

Recall the definition of shuffles. Let S_q be the permutation group. For $l \in \mathbb{N}$ and $1 \le s \le l-1$, let $J_{l;s}$ be the set of elements of S_l which preserve the order of the first s numbers and the order of the last $l-s$ numbers, that is,

$$J_{l,s} = \{\sigma \in S_l \mid \sigma(1) < \cdots < \sigma(s),\ \sigma(s+1) < \cdots < \sigma(l)\}.$$

The elements of $J_{l;s}$ are called shuffles, and we use the notation

$$J_{l;s}^{-1} = \{\sigma \mid \sigma \in J_{l;s}\}.$$

We define the configuration spaces:

$$F_n\mathbb{C} = \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid z_i \neq z_j, i \neq j\},$$

for $n \in \mathbb{Z}_+$. Let V be a grading-restricted vertex algebra (cf. Appendix 5), and W a a grading-restricted generalized V-module. By $\overline{W}$ we denote the algebraic completion of W,

$$\overline{W} = \prod_{n \in \mathbb{C}} W_{(n)} = (W')^*.$$

Definition 1. A $\overline{W}$-valued rational function in $(z_1, \dots, z_n)$ with the only possible poles at $z_i = z_j$, $i \neq j$, is a map

$$\begin{array}{rcl} f : F_n\mathbb{C} & \to & \overline{W}, \\ (z_1, \dots, z_n) & \mapsto & f(z_1, \dots, z_n), \end{array}$$

such that for any $w' \in W'$,

$$R_f(z_1, \dots, z_n) = R(\langle w', f(z_1, \dots, z_n)\rangle), \tag{2.1}$$

is a rational function in $(z_1, \dots, z_n)$ with the only possible poles at $z_i = z_j$, $i \neq j$. In this paper, such a map is called $\overline{W}$-valued rational function in $(z_1, \dots, z_n)$ with possible other poles. The space of $\overline{W}$-valued rational functions is denoted by $\overline{W}_{z_1,\dots,z_n}$. When we write $\Phi \in \overline{W}_{z_1,\dots,z_n}$ we will always assume $R_\Phi(z_1, \dots, z_n)$ (2.1).

For

$$R_{\Phi(v_1,\dots,v_n)}(z_1, \dots, z_n) = R(\langle w', \Phi(v_1, z_1; \dots; v_n, z_n)\rangle), \tag{2.2}$$

one defines an action of S_n on the space $\mathrm{Hom}(V^{\otimes n}, \overline{W}_{z_1,\dots,z_n})$ of linear maps from $V^{\otimes n}$ to $\overline{W}_{z_1,\dots,z_n}$ by

$$R_{\sigma(\Phi)(v_1,\dots,v_n)}(z_1, \dots, z_n) = R_{\Phi(v_{\sigma(1)},\dots,v_{\sigma(n)})}(z_{\sigma(1)}, \dots, z_{\sigma(n)}),$$

for $\sigma \in S_n$, and $v_1, \dots, v_n \in V$. We will use the notation $\sigma_{i_1,\dots,i_n} \in S_n$, to denote the the permutation given by $\sigma_{i_1,\dots,i_n}(j) = i_j$, for $j = 1, \dots, n$. In [12] it is proven that the subspace of $\mathrm{Hom}(V^{\otimes n}, \overline{W}_{z_1,\dots,z_n})$ consisting of linear maps having the $L(-1)$-derivative property, having the $L(0)$-conjugation property or being composable with m vertex operators is invariant under the action of S_n.

Let us introduce another definition:

Definition 2. We define the space $\mathcal{W}_{z_1,\dots,z_n}$ of $\overline{W}_{z_1,\dots,z_n}$-valued rational forms Φ with each vertex algebra element entry v_i, $1 \le i \le n$ of a quasi-conformal grading-restricted vertex algebra V tensored with power wt (v_i)-differential of corresponding formal parameter z_i, i.e.,

$$\Phi\left(dz_1^{\mathrm{wt}\ (v_1)} \otimes v_1, z_1; \dots; dz_n^{\mathrm{wt}\ (v_n)} \otimes v_n, z_n\right) \in \mathcal{W}_{z_1,\dots,z_n}. \tag{2.3}$$

We assume also that (2.3) satisfy $L_V(-1)$-derivative (2.5), $L_V(0)$-conjugation (2.10) properties, and the symmetry property with respect to action of the symmetric group S_n:

$$\sum_{\sigma\in J_{l;s}^{-1}} (-1)^{|\sigma|} R_{\Phi(v_{\sigma(1)},\dots,v_{\sigma(n)})}\left(z_{\sigma(l)},\dots,z_{\sigma(n)})\right) = 0. \tag{2.4}$$

In Section 3 we prove that (2.3) is invariant with respect to changes of formal parameters $(z_1,\dots,z_n)$.

2.1. **Properties of matrix elements for a grading-restricted vertex algebra.** Let V be a grading-restricted vertex algebra and W a grading-restricted generalized V-module. Let us recall some definitions and facts about matrix elements for a grading-restricted vertex algebra [12]. If a meromorphic function $f(z_1,\dots,z_n)$ on a domain in $\mathbb{C}^n$ is analytically extendable to a rational function in $z_1,\dots,z_n$, we denote this rational function by $R(f(z_1,\dots,z_n))$.

Definition 3. For $n \in \mathbb{Z}_+$, a linear map

$$\Phi(v_1, z_1; \dots; v_n, z_n) = V^{\otimes n} \to \mathcal{W}_{z_1,\dots,z_n},$$

with associated rational function (2.2), is said to have the $L(-1)$-derivative property if

$$R_{\partial_{z_i}\Phi(v_1,\dots,z_n)}(z_1,\dots,z_n) = R_{\Phi(v_1,\dots,L_V(-1)v_i,\dots,v_n)}(z_1,\dots,z_n), \tag{2.5}$$

for $i = 1,\dots,n$, $v_1,\dots,v_n \in V$, $w' \in W$, and

$$\sum_{i=1}^{n} \partial_{z_i} R_{\Phi(v_1,\dots,v_n)}(z_1,\dots,z_n) = R_{L_W(-1).\Phi(v_1,\dots,v_n)}(z_1,\dots,z_n), \tag{2.6}$$

with some action . of $L_W(-1)$ on $\Phi(v_1, z_1; \dots; v_n, z_n)$, and and $v_1,\dots,v_n \in V$.

Note that since $L_W(-1)$ is a weight-one operator on W, for any $z \in \mathbb{C}$, $e^{zL_W(-1)}$ is a well-defined linear operator on $\overline{W}$. In [12] we find the following

Proposition 1. *Let Φ be a linear map having the $L(-1)$-derivative property. Then for $v_1,\dots,v_n \in V$, $w' \in W'$, $(z_1,\dots,z_n) \in F_n\mathbb{C}$, $z \in \mathbb{C}$ such that $(z_1+z,\dots,z_n+z) \in F_n\mathbb{C}$,*

$$R_{e^{zL_W(-1)}\Phi(v_1,\dots,v_n)}(z_1,\dots,z_n) = R_{\Phi(v_1,\dots,v_n)}(z_1+z,\dots,z_n+z), \tag{2.7}$$

and for $v_1,\dots,v_n \in V$, $w' \in W'$, $(z_1,\dots,z_n) \in F_n\mathbb{C}$, $z \in \mathbb{C}$, and $1 \le i \le n$ such that

$$(z_1,\dots,z_{i-1}, z_i+z, z_{i+1},\dots,z_n) \in F_n\mathbb{C},$$

the power series expansion of

$$R_{\Phi(v_1,\dots,v_n)}(z_1,\dots,z_{i-1}, z_i+z, z_{i+1};\dots,z_n), \tag{2.8}$$

in z is equal to the power series

$$R_{\Phi(v_1,\ldots,v_{i-1},e^{zL(-1)}v_i,v_{i+1},\ldots,v_n)}(z_1,\ldots,z_n), \tag{2.9}$$

in z. In particular, the power series (2.9) in z is absolutely convergent to (2.8) in the disk $|z|<\min_{i\neq j}\{|z_i-z_j|\}$.

Finally, we have

Definition 4. A linear map

$$\Phi : V^{\otimes n}\to \mathcal{W}_{z_1,\ldots,z_n}$$

has the $L(0)$-conjugation property if for $v_1,\ldots,v_n\in V$, $w'\in W'$, $(z_1,\ldots,z_n)\in F_n\mathbb{C}$ and $z\in\mathbb{C}^\times$ so that $(zz_1,\ldots,zz_n)\in F_n\mathbb{C}$,

$$R_{z^{L_W(0)}\Phi(v_1,\ldots,v_n)}(z_1,\ldots,z_n)=R_{\Phi(z^{L(0)}v_1,\ldots,z^{L(0)}v_n)}(zz_1,\ldots,zz_n). \tag{2.10}$$

2.2. **Composability of $\overline{W}$ maps.** Let us recall the definition of maps composable with a number of vertex operators [12].

Definition 5. For a V-module

$$W=\coprod_{n\in\mathbb{C}}W_{(n)},$$

and $m\in\mathbb{C}$, let

$$P_m:\overline{W}\to W_{(m)},$$

be the projection from $\overline{W}$ to $W_{(m)}$. Let

$$\Phi : V^{\otimes n}\to \mathcal{W}_{z_1,\ldots,z_n},$$

be a linear map. For $m\in\mathbb{N}$, Φ is called [12, 17] to be composable with m vertex operators if the following conditions are satisfied:

1) Let $l_1,\ldots,l_n\in\mathbb{Z}_+$ such that $l_1+\cdots+l_n=m+n$, $v_1,\ldots,v_{m+n}\in V$ and $w'\in W'$. Set

$$\Psi_i \quad=\quad R_{Y_W\ \ldots\ Y_W(v_{k_1},\ldots,v_{k_i})}(z_{k_1}-\zeta_i,\ldots,z_{k_i}-\zeta_i), \tag{2.11}$$

where

$$k_1=l_1+\cdots+l_{i-1}+1,\quad \ldots,\quad k_i=l_1+\cdots+l_{i-1}+l_i, \tag{2.12}$$

$$m_1=n_1+\cdots+n_{i-1}+1,\quad \ldots,\quad m_i=n_1+\cdots+n_{i-1}+n_i, \tag{2.13}$$

for $i=1,\ldots,n$. Then there exist positive integers $N^n_m(v_i,v_j)$ depending only on v_i and v_j for $i,j=1,\ldots,k$, $i\neq j$ such that the series

$$\mathcal{I}^n_m(\Phi)=\sum_{r_1,\ldots,r_n\in\mathbb{Z}}R_{\Phi(P_{r_1}\Psi_1,\ldots,P_{r_n}\Psi_n)}(\zeta_1,\ldots,\zeta_n), \tag{2.14}$$

is absolutely convergent when

$$|z_{l_1+\cdots+l_{i-1}+p}-\zeta_i|+|z_{l_1+\cdots+l_{j-1}+q}-\zeta_i|<|\zeta_i-\zeta_j|, \tag{2.15}$$

for $i,j=1,\ldots,k$, $i\neq j$ and for $p=1,\ldots,l_i$ and $q=1,\ldots,l_j$. The sum must be analytically extended to a rational function in $(z_1,\ldots,z_{m+n})$, independent of $(\zeta_1,\ldots,\zeta_n)$, with the only possible poles at $z_i = z_j$, of order less than or equal to $N^n_m(v_i, v_j)$, for $i, j = 1, \ldots, k$, $i \neq j$.

2) For $v_1, \ldots, v_{m+n} \in V$, there exist positive integers $N^n_m(v_i, v_j)$, depending only on v_i and v_j, for $i, j = 1, \ldots, k$, $i \neq j$, such that for $w' \in W'$, and

$$\mathcal{J}^n_m(\Phi) = \sum_{q \in \mathbb{C}} R_{Y_W \ldots Y_W P_q(\Phi)(v_1, \ldots, v_n)}(z_1, \ldots, z_n), \tag{2.16}$$

is absolutely convergent when

$$\begin{aligned} z_i \neq z_j, \quad i \neq j, \\ |z_i| > |z_k| > 0, \end{aligned} \tag{2.17}$$

for $i = 1, \ldots, m$, and $k = m+1, \ldots, m+n$, and the sum can be analytically extended to a rational function in $(z_1, \ldots, z_{m+n})$ with the only possible poles at $z_i = z_j$, of orders less than or equal to $N^n_m(v_i, v_j)$, for $i, j = 1, \ldots, k$, $i \neq j$,.

In [12], we the following useful proposition is proven:

Proposition 2. *Let $\Phi : V^{\otimes n} \to \overline{W}_{z_1, \ldots, z_n}$ be composable with m vertex operators. Then we have:*

1. *For $p \leq m$, Φ is composable with p vertex operators and for p, $q \in \mathbb{Z}_+$ such that $p + q \leq m$ and $l_1, \ldots, l_n \in \mathbb{Z}_+$ such that $l_1 + \cdots + l_n = p + n$,*
$$R_{\ \Phi(Y_V \ldots Y_V) \ldots (Y_V \ldots Y_V)(v_1, \ldots, v_{l_1}) \ \ldots \ (v_{l_{k_1}}, \ldots, v_{l_{k_n}})}(z_1, \ldots, z_{n+m-1}),$$
$$R_{Y_W \ldots Y_W \ \Phi(v_1, \ldots, v_{n+p})}(z_1, \ldots, z_{n+p}),$$
are composable with q vertex operators.
2. *For $p, q \in \mathbb{Z}_+$ such that $p+q \leq m$, $l_1, \ldots, l_n \in \mathbb{Z}_+$ such that $l_1 + \cdots + l_n = p+n$ and $k_1, \ldots, k_{p+n} \in \mathbb{Z}_+$ such that $n_1 + \cdots + n_{p+n} = q + p + n$, we have*
$$\begin{aligned} &R_{\ (\ \Phi(Y_V \ldots Y_V)\ )\ (Y_V \ldots Y_V)\ \left((v_1, \ldots, v_{l_1}) \ldots (v_{l_{k_1}}, \ldots, v_{l_{k_n}})\right)\ \left((v_1, \ldots, v_{m_{p_1}})\ \ldots\ (v_{n_{m_1}}, \ldots, v_{n_{m_{n+p}}})\right)} \\ &\qquad (z_1, \ldots, z_{n+p+m-1}) \\ &= R_{\ \Phi\left(\ (v_{l_1}, \ldots, v_{k_1}),\ \ldots,\ (v_{n_{m_1}}, \ldots, v_{n_{m_{n+p}}})\ \right)}\ (z_1, \ldots, z_{n+p+m-1}). \end{aligned}$$
3. *For $p, q \in \mathbb{Z}_+$ such that $p+q \leq m$ and $l_1, \ldots, l_n \in \mathbb{Z}_+$ such that $l_1 + \cdots + l_n = p + n$, we have*
$$\begin{aligned} &R_{(Y_W \ldots Y_W)\ (\Phi)((v_1, \ldots, v_q)(v_1, \ldots, v_{l_{k_1}}), \ldots, (v_{l_{k_1}}, \ldots, v_{l_{k_n}}))}(z, \ldots, z_{n+q-1}) \\ = \ &R_{((Y_W \ldots Y_W)\ \Phi)((v_1, \ldots, v_q)(v_1, \ldots, v_{l_{k_1}}), \ldots, (v_{l_{k_1}}, \ldots, v_{l_{k_n}}))}(z, \ldots, z_{n+q-1}). \end{aligned} \tag{2.18}$$
4. *For $p, q \in \mathbb{Z}_+$ such that $p + q \leq m$, we have*
$$\begin{aligned} &R_{Y_W \ldots Y_W (Y_W \ldots Y_W\ \Phi(v_1, \ldots v_{p+n+q}))}(z_1, \ldots z_{p+n+q}) \\ &\qquad = R_{Y_W \ldots Y_W\ \Phi(v_1, \ldots v_{n+p+q})}(z_1, \ldots z_{n+p+q}). \end{aligned} \tag{2.19}$$

In the construction of double complexes in Section 3.2 we would like to use linear maps from tensor powers of V to the space $\mathcal{W}_{z_1, \ldots, z_n}$ to define cochains in vertex algebra cohomology theory. For that purpose, in particular, to define the coboundary operator, we have to compose cochains with vertex operators. However, as mentioned in [12], the images of vertex operator maps in general do not belong to algebras or their modules. They belong to corresponding algebraic completions which constitute one of the most subtle features of the theory of vertex algebras. Because of this, we might not be able to compose vertex operators directly. In order to overcome this problem [14], we first write a series by projecting an element of the algebraic completion of an algebra or a module to its homogeneous components. Then we compose these homogeneous components with vertex operators, and take formal sums. If such formal sums are absolutely convergent, then these operators can be composed and can be used in constructions. Another question that appears is the question of associativity. Compositions of maps are usually associative. But for compositions of maps defined by sums of absolutely convergent series the existence of does not provide associativity in general. Nevertheless, the requirement of analyticity provides the associativity [12].

3. Cohomology associated to formal series

In this section we define the formal series cohomology for a grading-restricted vertex algebra cohomology on a complex curve. A consideration of cohomology of smooth manifolds of arbitrary dimension will be given elsewhere.

3.1. $C^n_m(V, \mathcal{W}, \mathbf{U}, \mathcal{M})$-spaces.

Let $\mathbf{U}$ be a family of intersecting open domains on $\mathcal{M}$. We assume that there exist homology embedding maps h_i, $i = 1, \ldots, k-1$, relating domains $U_1, \ldots, U_k \in \mathbf{U}$.

Definition 6. If α is a path between two points p_1 and p_2 on $\mathcal{M}$ and if U_1 and U_2 are domains centered at p_1 and p_2, then α defines a transport along $\mathcal{M}$ from a neighborhood of p_1 in U_1 to a neighborhood of p_2 in U_2, and hence a germ of a diffeomorphism

$$hol(\alpha) : (U_1, p_1) \hookrightarrow (U_2, p_2),$$

called the holonomy of the path α. Two homotopic paths always define the same holonomy.

Definition 7. If the above transport along α is defined in all of U_1 and embeds U_1 into U_2, this embedding $h : U_1 \hookrightarrow U_2$ will be denoted by $hol(\alpha) : U_1 \hookrightarrow U_2$. Embeddings of this form we call the holonomy embeddings. The composition of paths also induces an operation of composition on those holonomy embeddings.

We consider set of k points p_k on $\mathcal{M}$, $k \geq 0$, and a sequence of corresponding overlapping open domains U_k surrounding each p_k, such that there exist holonomy embeddings h_i as described above. We assume that each domain U_i is endowed with a local coordinate $c(p_i)$ centered at p_i. For association of formal parameters of mappings and vertex operators with points of $\mathcal{M}$ we will use in what follows their local coordinates $c(p_i)$ in domains U_i on $\mathcal{M}$.

For a set of n elements of a grading-restricted vertex algebra V

$$(v_1, \ldots, v_n), \tag{3.1}$$

we consider linear maps

$$\Phi : V^{\otimes n} \to \mathcal{W}_{z_1, \ldots, z_n} \tag{3.2}$$

(see Subsection 2 for the definition of a $\mathcal{W}_{z_1,\dots,z_n}$ space),

$$\Phi\left(dz_1^{\mathrm{wt}\ v_1}\otimes v_1, z_1; \dots; dz_n^{\mathrm{wt}\ v_n}\otimes v_n, z_n\right), \tag{3.3}$$

where we identify, as it is usual in the theory of characters for vertex operator algebras on curves [14, 19–21], n formal parameters $z_1, \dots, z_n$ of $\mathcal{W}_{z_1,\dots,z_n}$, with local coordinates $c_i(p_i)$ in vicinities of points p_i, $0 \le i \le n$, on $\mathcal{M}$. The construction of vertex algebra cohomology of a smooth complex curve $\mathcal{M}$ in terms of connections is parallel to ideas of [2]. Such a relation will be explained elsewhere. Note that similar to [1] (3.3) can be treated as $\left(\mathrm{Aut}_{p_1}\mathcal{O}^{(1)}\times\dots\times\mathrm{Aut}_{p_n}\mathcal{O}^{(1)}\right)$-torsor of the product of groups of coordinate transformations. In what follows, according to definitions of Section 2, when we write an element Φ of the space $\mathcal{W}_{z_1,\dots,z_n}$, we actually have in mind corresponding matrix element $\langle w', \Phi\rangle$ that absolutely converges (on certain domain) to a rational form-valued function $R_\Phi = R(\langle w', \Phi\rangle)$. Quite frequently we will write $\langle w', \Phi\rangle$ which would denote a rational $\mathcal{W}$-valued form. In notations, we would keep tensor products of vertex algebra elements with wt -powers of z-differentials when it is inevitable only.

Later in this section we prove, that for arbitrary $v_i \in V$, $1 \le i \le n$, points p_i with local coordinates $c_i(p_i)$ on $U_i \in \mathbf{U}$ of $\mathcal{M}$, an element (3.3) as well as the vertex operators

$$\omega_W\left(dc_i(p_i)^{\mathrm{wt}\ (v_i)}\otimes v_i, c_i(p_1)\right) = Y\left(d(c_i(p_i))^{\mathrm{wt}\ (v_i)}\otimes v_i, c_i(p_i)\right), \tag{3.4}$$

are invariant with respect to the action of the group $\left(\mathrm{Aut}_{p_1}\mathcal{O}^{(1)}\times\dots\times\mathrm{Aut}_{p_n}\mathcal{O}^{(1)}\right)$. In (3.4) we mean the ordinary vertex operator (as defined in Appendix 5) not affecting the tensor product with corresponding differential. We assume that the maps (3.2) are composable (according to Definition (5) of Subsection 2.2), with k vertex operators $\omega_W(v_i, c_i(p_i))$, $1 \le i \le k$ for any choice of k vertex algebra elements from (3.1), and corresponding formal parameters associated with local coordinates on k domains U_i of $\mathbf{U}$, $i = 1, \dots, k$ for p_i.

The composability of a map Φ with a number of vertex operators consists of two conditions on Φ. The first requires the existence of positive integers $N_m^n(v_i, v_j)$ depending just on v_i, v_j, and the second restricts orders of poles of corresponding sums (2.14) and (2.16). Taking into account these conditions, we will see that the construction of the space (3.6) does depend on the choice of vertex algebra elements (3.1). In this subsection we construct the spaces for a double complex defined for a complex curve $\mathcal{M}$, and associated to a grading-restricted vertex algebra.

In order to define vertex algebra cohomology of $\mathcal{M}$, mappings Φ are supposed to be composable with a number of vertex operators with a number of vertex algebra elements, and formal parameters identified with local coordinates of points $p_1, \dots, p_k$ on each of k elements U_j, $1 \le j \le k$. The above setup is considered for a set of vertex algebra elements, which could be varied accordingly. We first introduce

Definition 8. Let $p_1, \dots, p_n$ be points taken on domains $U_j \in \mathbf{U}$, $j \ge 1$. Assuming $k \ge 0$, $n \ge 0$, we denote by $C^n(V, \mathcal{W}, \mathcal{M})(U_j)$, $0 \le j \le k$, the space of all linear maps (3.2)

$$\Phi : V^{\otimes n} \to \mathcal{W}_{c_1(p_1),\dots,c_n(p_n)}, \tag{3.5}$$

composable with a k of vertex operators (3.4) with formal parameters identified with local coordinates $c_j(p_j)$ functions around points p_j on each of domains U_j, $1 \leq j \leq k$.

The set of vertex algebra elements (3.1) plays the role of parameters in our further construction of the vertex algebra cohomology on a smooth curve $\mathcal{M}$. A holonomy embedding maps a domain of $\mathbf{U}$ and a coordinate chart into a domain and coordinate chart on another domain of $\mathbf{U}$. Let us now introduce the following spaces:

Definition 9. For $n \geq 0$, and $m \geq 0$, we define the space

$$C_m^n(V, \mathcal{W}, \mathbf{U}, \mathcal{M}) = \bigcap_{\substack{U_1 \overset{h_1}{\hookrightarrow} \ldots \overset{h_{m-1}}{\hookrightarrow} U_m \\ 1 \leq j \leq m}} C^n(V, \mathcal{W}, \mathcal{M})(U_j), \tag{3.6}$$

where the intersection ranges over all possible m-tuples of holonomy embeddings h_i, $i \in \{1, \ldots, m-1\}$, between domains

$$U_1 \overset{h_1}{\hookrightarrow} \ldots \overset{h_{m-1}}{\hookrightarrow} U_m,$$

of $\mathbf{U}$ on $\mathcal{M}$ (for $m = 0$ there is no a sequence of embeddings above).

Remark 1. Since m can be sent to infinity, (3.6) still works for infinite sequences of overlapping open domains on non-compact $\mathcal{M}$.

First, we have the following

Lemma 1. (3.6) *is non-empty.*

The main statement of this section is contained in the following

Proposition 3. *For a quasi-conformal grading-restricted vertex algebra V and its module W, the construction* (3.6) *is canonical, i.e., does not depend on the choice of local coordinates $c_i(p_i)$, $i = 1, \ldots, m$, on $U_i \in \mathbf{U}$.*

The proofs of Lemma 1 and Proposition 3 are contained in Appendix 7.

Remark 2. The condition of quasi-conformality is necessary in the proof of invariance of elements of the space $\mathcal{W}_{z_1, \ldots, z_n}$ with respect to a vertex algebraic representation (cf. Appendix 5) of the group $\left(\mathrm{Aut}_{p_1} \mathcal{O}^{(1)} \times \ldots \times \mathrm{Aut}_{p_n} \mathcal{O}^{(1)}\right)$. In what follows, when it concerns the spaces (3.6) we will always assume the quasi-conformality of V. Generalizations of Lemma 1, and Proposition 3 proofs for the case of a arbitrary n-dimensional smooth manifold will be given elsewhere.

Let W be a grading-restricted V module. Since for $n = 0$, Φ does not include variables, and due to Definition 5 of the composability, we can put:

$$C_m^0(V, \mathcal{W}, \mathcal{M}) = W, \tag{3.7}$$

for $m \geq 0$. Nevertheless, according to our Definition 3.6, mappings that belong to (3.7) are assumed to be composable with a number of vertex operators with depending on local coordinates of m points p_i, $i = 1, \ldots, m$ on m open domains of $\mathbf{U}$. Since V, $\mathcal{W}$ and $\mathcal{M}$ are fixed in our construction we will omit them in what follows. We then have

Lemma 2.

$$C^n_m(\mathbf{U}) \subset C^n_{m-1}(\mathbf{U}). \tag{3.8}$$

The proof of this Lemma is contained in Appendix 7.

3.2. **Coboundary operators.** For the double complex spaces (3.6), the coboundary operator δ^n_m is defined as the form of a multi-point vertex algebra connection (cf. Definition 6.1 in Appendix 6, cf. [12]):

$$R_{\delta^n_m \Phi(v_1,\ldots,v_n)}(p_1,\ldots,p_n) = G(p_1,\ldots,p_{n+1}), \tag{3.9}$$

where

$$\begin{aligned} G(p_1,\ldots,p_{n+1}) &= R_{\sum_{i=1}^n (-1)^i \; \Phi(\omega_V(v_i,v_{i+1}))}(p_i - p_{i+1}) \\ &+ R_{\;\omega_W \; \Phi(v_1,\ldots,v_n)}(p_1,\ldots,p_n) \\ &+ R_{\;(-1)^{n+1}\omega_W \; \Phi(v_{n+1},v_1,\ldots,v_n)}(p_{n+1},p_1,\ldots,p_n). \end{aligned}$$

Note that it is assumed that the coboundary operator does not affect $dc_i(p_i)^{\mathrm{wt}\;(v_i)}$-tensor multipliers in Φ. Inspecting construction of the double complex spaces (3.6) we see that the action (3.9) of the δ^n_m on an element of $C^n_m(\mathbf{U})$ provides a coupling (in terms of $\mathcal{W}_{z_1,\ldots,z_n}$-valued rational functions) of vertex operators taken at some of the points p_j, $j=1,\ldots,m$ with local coordinates $c_j(z_{p_j})$ at the vicinities of p_i in $U_j \in \mathbf{U}$ with elements Φ of $C^n_{m-1}(\mathbf{U})$ taken at remaining points among p_i, $1 \le i \le n$. Then we have

Proposition 4. *The operator* (3.9) *provides the chain-cochain complex*

$$\delta^n_m : C^n_m(\mathbf{U}) \to C^{n+1}_{m-1}(\mathbf{U}), \tag{3.10}$$

$$\delta^{n+1}_{m-1} \circ \delta^n_m = 0, \tag{3.11}$$

$$0 \longrightarrow C^0_m(\mathbf{U}) \xrightarrow{\delta^0_m} C^1_{m-1}(\mathbf{U}) \xrightarrow{\delta^1_{m-1}} \cdots \xrightarrow{\delta^{m-1}_1} C^m_0(\mathbf{U}) \longrightarrow 0. \tag{3.12}$$

The proof of this proposition follows from the construction (3.9) of the coboundary operator, and from Proposition 2.

3.3. **Cohomology.** Recall definitions of vertex algebra connections and their forms given in Appendix 6. In this subsection we define the formal series cohomology for $\mathbf{U}$, as well as the formal series cohomology of whole complex curve $\mathcal{M}$ associated to a grading-restricted vertex algebra V.

Definition 10. We define the n-th cohomology $H^n_m(\mathbf{U})$ of $\mathbf{U}$ with coefficients in $\mathcal{W}_{z_1,\ldots,z_n}$ (containing maps composable with m vertex operators defined on domains of $\mathbf{U}$) to be the factor space of closed multi-point connections by the space of connection forms:

$$H^n_k(\mathbf{U}) = \mathcal{C}on^n_{k;\,cl}(\mathbf{U})/G^{n-1}_{k+1}(\mathbf{U}). \tag{3.13}$$

Note that due to (3.9), (4.18), and Definitions 6.1 and 6.2 (cf. Section 3.2), it is easy to see that (3.13) is equivalent to the standard cohomology definition

$$H^n_k(\mathbf{U}) = \ker \delta^n_k|_{\mathbf{U}}/\mathrm{im}\; \delta^{n-1}_{k+1}|_{\mathbf{U}}. \tag{3.14}$$

According to the construction of (3.6), the cohomology (3.14) has a local manner, i.e., it depends on $\mathbf{U}$. In the next section we provide applications and examples of (3.14). Up to now, the geometrical picture used to define the spaces $C^n_m(\mathbf{U})$ was local. Now let us formulate

Definition 11. Let $\mathcal{U} = \bigcup\limits_{\mathcal{M}} \mathbf{U}$ be the covering of whole $\mathcal{M}$ performed by sets of open domains $\mathbf{U}$. Define corresponding chain complex

$$\left(C^n_{\mathbf{m}} \left(\mathcal{U}, \mathcal{M} \right), \widetilde{\delta}^n_{\mathbf{m}} \right), \tag{3.15}$$

with

$$C^n_{\mathbf{m}} \left(\mathcal{U}, \mathcal{M} \right) = \bigcup_{\mathbf{U} \in \mathcal{M}} C^n_{m_{\mathbf{U}}}(\mathbf{U}),$$

and

$$\widetilde{\delta}^n_{\mathbf{m}} = (\delta^n_{m_{\mathbf{U}}})|_{\mathcal{U}},$$

where $\delta^n_{m_{\mathbf{U}}}$ acts separately on each element $\mathbf{U}$ of $\mathcal{U}$, and each individual $m_{\mathbf{U}}$ is chosen suitable to individual $\mathbf{U}$.

Definition 12. We define the cohomology of $\mathcal{U}$ as cohomology

$$H^n_{\mathbf{m}}(\mathcal{U}) = \ker \widetilde{\delta}^n_{\mathbf{m}} / \mathrm{im}\ \widetilde{\delta}^{n-1}_{\mathbf{m+1}},$$

of the complex (3.15).

Under a refinement $\mathcal{V}$ of $\mathcal{U}$ by subdomains $\overline{\mathbf{U}} = (\overline{U}_1, \ldots, \overline{U}_m)$ $\overline{U}_i \subset U_i$ such that new $\mathcal{V}$ minimally covers the whole complex curve $\mathcal{M}$., we obtain a map suitable for the construction of a cohomology of whole $\mathcal{M}$. $\overline{U}_i \subset U_i$ such that new $\mathcal{V}$ minimally covers the whole complex curve $\mathcal{M}$. Let

$$H^n_{\mathbf{m}}(\mathcal{U}) \to H^n_{\mathbf{m}}(\mathcal{V}),$$

be the map of cohomologies under a refinement. Then we define

$$H^n(\mathcal{M}) = \varinjlim_{\mathcal{U}} H^n_{\mathbf{m}}(\mathcal{U}).$$

by the direct limit of this system. In what follows we consider local cohomologies associated to a grading-restricted vertex algebra.

4. Examples

In this section we consider examples of lower formal series cohomologies and characterization of $\mathbf{U}$.

4.1. **Fixed point double compexes.**

For our further purposes we have to define spaces suitable for the definition of a double complex with a fixed point. Such double complex will be needed for the construction of first vertex algebra cohomologies, in particular, for $H^1_m(\mathbf{U})$, $m \geq 0$ (see Section 3.2).

Definition 13. Let us fix a point p_r and a domain $U_r \in \mathbf{U}$, $r \geq 1$. Assuming $k \geq 0$, $n \geq 0$, consider the space $C^n(p_r)(U_r)$, of linear mappings

$$\Phi : V^{\otimes n} \to \mathcal{W}_{c_1(p_1),\ldots,c_n(p_n)}, \tag{4.1}$$

composable with k vertex operators with formal parameters identified with local coordinates $\{c_1(p_1), \ldots, c_r(p_r)|_{p_r}, \ldots, c_n(p_k)\}$, on each of k domains of $\mathbf{U}$.

The holonomy embeddings h_j provide a map of local coordinate functions

$$h_j : c_j(p_j) \to c_{j+1}(p_{j+1}),$$

and we have a sequence of mappings

$$\mathbf{h} = p_1 \stackrel{h_1}{\to} \ldots \stackrel{h_{r-1}}{\to} p_r \stackrel{h_r}{\to} \ldots \stackrel{h_{m-1}}{\to} p_m. \tag{4.2}$$

Let us now introduce the following spaces:

Definition 14. For $n \geq 0$, and $m \geq 0$, consider the space

$$C^n_m(p_r; \mathbf{U}) = \bigcap_{\mathbf{h},\ j \in \{1,\ldots,m\}} C^n(p_r)(U_j), \tag{4.3}$$

where the intersection is taken over all possible $m-1$-sequences (4.2) of holonomy mappings h_i, $i \in \{1, \ldots, m-1\}$ among points on domains of $\mathbf{U}$ with the fixed point p_r.

Then we have the following

Lemma 3. *The double complex $(C^n_k(p_r; \mathbf{U}), \delta^n_k|_{p_r})$ is a subcomplex of double chain-cochain complex $(C^n_k(\mathbf{U}), \delta^n_k)$.*

Proof. We assume that in the construction of (3.6), the points $\{p_j\}$, $1 \leq r-1$, $r+1 \leq k$ in sequences (4.2) of holonomy mappings can be shifted all over corresponding domains $\{U_j\} \in \mathbf{U}$. In Definition (3.6) of $C^n_k(\mathbf{U},)$, the points $\{p_j\}$, $1 \leq r-1$, $r+1 \leq k$ exhaust domains $\{U_j\}$ of $\mathbf{U}$. Thus,

$$C^n_k(p_r; \mathbf{U}) \subset C^n_k(\mathbf{U}).$$

It is clear that the operator $\delta^n_k|_{p_r}$ is a reduction of δ^n_k, and satisfies the chain-cochain property as in Proposition (4). □

4.2. **Fixed point cohomology.** Here we formulate

Definition 15. Let U_r, $r \geq 1$, be a domain of $\mathbf{U}$, and $p_r \in U_r$ be a fixed point. We define the fixed point cohomology as

$$H^n_k(\mathbf{U}) = \mathcal{C}on^n_{p;\ k;\ cl}|_{\mathbf{U}} / G^{n-1}_{p;\ k+1}|_{\mathbf{U}}, \tag{4.4}$$

which is equivalent to

$$H^n_k\,(p_r; \mathbf{U}) = \mathrm{Ker}\ \delta^n_k|_{p_r,\mathbf{U}} / \mathrm{Im}\ \delta^n_k|_{p_r,\mathbf{U}}.$$

From Lemma 3 it follows

Lemma 4. *The cohomology $H^n_m\,(p; \mathbf{U})$ is given by*

$$H^n_m\,(\mathbf{U}) = \bigcup_{p'_r \in U_r} H^n_m\,(p_r; \mathbf{U})\,.$$

4.3. **Computation of the cohomologies $H^1_m(\mathbf{U})$.** In [13], lower cohomologies for a grading-restricted vertex algebra were computed. In this paper we determine the first grading-restricted vertex algebra cohomologies $H^1_m(\mathbf{U})$ of $\mathbf{U}$ on a complex curve $\mathcal{M}$. Let us first consider one-variable reduction of multi-point connections (which is called in the derivation [12]). We introduce the following definition of the derivation applicable to maps from V to $\mathcal{W}$.

Definition 16. Let V be a grading-restricted vertex algebra and W a V-module. A grading-preserving linear map

$$g : V \to \mathcal{W},$$

is called a derivation if

$$\begin{aligned} g\left(\omega_V(u,z)v,0\right) &= e^{zL_W(-1)}\omega_W(v,-z)\ g(u,0) + \omega_W(u,z)\ g(v,0) \\ &= \omega^W_{WV}(g(u,0),z)v + \omega_W(u,z)\ g(v,0), \end{aligned}$$

for u, $v \in V$, where $\omega^W_{WV}(v,z)$ is the intertwiner-valued vertex operator in accordance with notaions of (3.4). We use Der $(V,\mathcal{W})$ to denote the space of all such derivations. It is clear that

$$g(v,0) = \mathcal{G}(v,0).$$

As we see from the definition of the derivation over V, it depends on one element of V only. The space of one V-element two point holomorphic connections reduces to the space of derivations over $\mathcal{W}$ [12]. In [13] it is proven the following

Lemma 5. *Let $g(v,0) : V \to \mathcal{W}$ be a derivation. Then $g(\mathbf{1}_V,0) = 0$.*

We will need another statement proven in [13]

Lemma 6. *Let*

$$\Phi : V \to \mathcal{W}_z,$$

be an element of $C^1_m(\mathbf{U})$ satisfying

$$\delta^1_m \Phi = 0.$$

Then $\Phi(v,0)$ is a grading-preserving linear map from V to $\mathcal{W}$, i.e.,

$$z^{L(0)}\Phi(v,0) = \Phi(z^{L(0)}v,0) = z^n\Phi(v,0).$$

In [13], the first cohomologies $H^1_M(V,W)$ of a grading-restricted vertex algebra were related to the space of derivations $\mathrm{Der}(V,W)$. We find the following

Proposition 5. *Let V be a grading-restricted vertex algebra and W a V-module. Then $H^1_m(V,W)$ is linearly isomorphic to the space* Der (V,W) *of derivations from V to W for any $m \in \mathbb{Z}_+$.*

In the case of a complex curve we have the following identifications in (6.4)

$$\begin{aligned} \mathcal{G}(\phi(p)) &= \mathcal{G}(v,c(p)) = \Phi(v,c(p)), \\ f(\psi(p)) &= \omega(v,c(p)), \\ \phi(p) &= (u,p), \\ f(\psi(p')).\phi(p) &= \omega(v,c(p')-c(p))u, \end{aligned} \tag{4.5}$$

and a multi-point holomorphic connection $\mathcal{G}$ on $\mathbf{U}$, is a $\mathbb{C}$-linear map

$$\mathcal{G} : V^{\otimes^n} \to \mathcal{W}_{z_1,\ldots,z_n}.$$

Thus, the multi-point holomorphic connection has the form

$$\sum_{q,q'\in M} \Phi\left(\omega_V(v_q, c(q)-c(q'))u, q\right) = \omega_W(u, c(p'))\ \Phi\left(v, c(p)\right) + \omega_W(v, c(p))\ \Phi\left(u, c(p')\right). \tag{4.6}$$

Remark 3. Due to Proposition 3, the definition of the multi-point holomorphic connection on (4.6) does not depend on the choice of coordinates on $\mathbf{U}$.

The meaning of the name of a transversal holmophic connection (6.5) is clear when we consider elements of the space $\mathcal{W}_{z_1,\ldots,z_n}$ for $\mathcal{M}$,

$$G(p,p') = \omega_W(u, c(p'))\ \mathcal{G}(v, c(p)) + \omega_W(u, c(p'))\ \mathcal{G}(u, c(p')) = 0,$$

with formal parameters associated to local coordinates $c(p)$. This type of connections will appear in considerations of the second vertex algebra cohomology $H^2_{ex}(\mathbf{U})$ in Subsection 3.2. In what follows, to shortcut notations, we will denote by p the origin of a local coordinate $c(p)$ at p, i.e., $c(p)|_p = 0$. Let us introduce another

Definition 17. A one fixed-point p' holomorphic connection for the space (4.3) is defined by

$$\sum_{q,q'\in M} \Phi\left(\omega_V(v_q, c(q)-c(q'))u, q\right) = \omega_W(u, p')\ \Phi\left(v, c(p)\right) + \omega_W(v, c(p))\ \Phi\left(u, p'\right). \tag{4.7}$$

In particular, for the space $C^1_m(p_r;\mathbf{U})$ we obtain

$$\Phi\left(\omega_V(v, p'-c(p))u, c(p)\right) = \omega_W(u, p')\ \Phi\left(v, c(p)\right) + \omega_W(v, c(p))\ \Phi\left(u, p'\right). \tag{4.8}$$

We denote the space of such connections with a fixed point p as $\mathcal{C}on_{p'}(m;\mathbf{U})$. Above, we have introduced the notion (Definition 15) of a fixed-point cohomology $H^n_m\,(p;\mathbf{U})$. In particular, for $n=1$,

$$H^1_m\,(p_r;\mathbf{U}) = \mathrm{Ker}\ \delta^1_m|_{p_r,\mathbf{U}}/\mathrm{Im}\ \delta^0_{m+1}|_{p_r,\mathbf{U}},$$

for a point $p_r \in U_r$ in the set of domains $\mathbf{U}$. The result of this subsection is in the following

Proposition 6. *The vertex algebra first cohomologies $H^1_m(\mathbf{U})$, $m \geq 0$, of $\mathbf{U}$ on a complex curve $\mathcal{M}$ are isomorphic to the space $\mathcal{C}on_{p_r}(m,\mathbf{U})$, for all $p_r \in U_r$, $1 \leq r \leq m$, of holomorphic fixed point two point connections with mappings composable with m vertex operators on domains of $\mathbf{U}$.*

Proof. Let us fix a point p_r with the local coordinate $c_r(p_r)$ on the domain U_r with origin at p_r, i.e., $c_r(p_r)|_{p_r} = 0$. According to Proposition 5 (cf. (1.1) in [13]), the cohomologies $H^1_m(V,W)$ of V are given by the space of derivations. In terms of Definition 30, it coincides with the space of fixed point holomorphic connections, i.e., $\mathrm{Der(V,W)} = \mathcal{C}on_{p_r}(V,W)$. Note that, for any

$$\Phi(v, c_r(p_r)|_{p_r}) \in C^1_m(p_r;\mathbf{U}),$$

such that

$$R_{\delta^1_m \Phi}(v, p_r) = \langle w', G_2\,(p_r, p_2)\rangle = 0,$$

i.e., results in an element of the space $\mathcal{C}on_{p_r}(\mathbf{U})$ of one fixed point p_r holomorphic connections. In addition, by direct computation for any $\Phi' \in C^0_m(p_r; \mathbf{U})$, we find

$$\mathrm{Im}\;\delta^0_{m+1}\Phi' = \{0\}.$$

Conversely, for any element $g(v, 0)$ of $\mathcal{C}on_{p_r}(\mathbf{U})$, and $v \in V$, let us consider

$$\Phi_g = g(\omega_V(v, z)\mathbf{1}_V, p_r) = \omega^W_{WV}(g(v, p_r), z)\mathbf{1}_V, \tag{4.9}$$

where we have used Lemma 5. We had to express (4.9) in terms of intertwining operator in order to show that (4.9) is indeed composable with m vertex operators and belong to the space $C^1_m(p_r; \mathbf{U})$ with a fixed point p_r. As it follows from [4], the map from V to $\mathcal{W}_z$ given by

$$v \mapsto \omega^W_{WV}(\Phi_g(v, p_r), z_1)\mathbf{1}_V,$$

is composable with m vertex operators for any $m \in \mathbb{N}$. Thus, $\Phi_g \in C^1_m(\mathbf{U})$ for any $m \in \mathbb{N}$. For v_1, $v_2 \in V$, and $w' \in W'$, by using (5.8), we find by direct computation that

$$\begin{aligned} &R_{\delta^1_m \Phi_g}(v_1, p_1; v_2, p_2) \\ &= -R_{\omega^W_{WV}\; g\; \omega_V(v_1, v_2)}(p_r, p_1, p_2) + R_{\omega_W\; \omega^W_{WV}\; g(v_2, v_1)}(p_2, p_r, p_1). \end{aligned}$$

By using Theorem 5.6.2 in [4], we see that (4.10) vanishes. Therefore we obtain a linear map

$$g(v, p_r) \mapsto \Phi_g,$$

from the space

$$\mathcal{C}on_{p_r}(\mathbf{U}) = \mathrm{Der}\;(\mathbf{U}) \to H^1_m(\mathbf{U}) = C^1_m(p_r; \mathbf{U}).$$

Thus, we find, that

$$H^1_m(p_r; \mathbf{U}) = \mathcal{C}on_{p_r}(\mathbf{U}). \tag{4.10}$$

By shifting $p_r \in U_r$ all along $c_r(p_r)$ we exhaust all points of U_r, we obtain connections of $\mathcal{C}on_{U_r}(\mathbf{U})$ on the whole U_r. By using Lemma 3, we extend (4.10) to we obtain the statement of Proposition:

$$H^1_m(\mathbf{U}) = \bigcup_{p_r \in U_r} \mathcal{C}on_{p_r}(\mathbf{U}).$$

$\square$

4.4. **Classes associated with the first cohomologies $H^1_m(\mathbf{U})$.** For the first cohomology $H^1_m(\mathbf{U})$, we have the following corollary from Proposition 6:

Corollary 1. *The $H^1_m(\mathbf{U})$ cohomological class of the grading-restricted vertex algebra cohomology of $\mathbf{U}$ is given by*

$$\left[\delta^1_m \Phi\right], \tag{4.11}$$

for $\Phi \in C^1_m(\mathbf{U})$. It vanishes if and only if Φ is given by a two point holomorphic connection.

Proof. $\left[\delta^1_m \Phi\right]$ for $\Phi \in C^1_m(\mathbf{U})$. It is easy to see that it remains cohomologically invariant under a substitution $\Phi \mapsto \Phi + \Phi_0$, due to properties of (4.11). The second statement of the proposition follows from the proof of Proposition 6. □

Since $\Phi \in C^1_m(\mathbf{U})$ and $\delta^1_m \Phi \in C^2_{m-1}(\mathbf{U})$, we obtain the characteristic two-form for $\mathbf{U}$

$$\mathcal{R}(c(p), c(p')) = R_{\delta^1_m \Phi(v,v')}(p, p')\,. \tag{4.12}$$

Here $\delta^1_m \Phi$ represents a cohomological class of $H^1_m(\mathbf{U})$. Instead, for a local characterization of $\mathcal{M}$, we may choose elements

$$\Phi_g = g(v, 0) \in \mathcal{W},$$

which do not depend on z, and, hence, matrix elements become computable. For non-vanishing invariants (4.11) (i.e., which are not two point connections $G(\Phi)$) we obtain the non-vanishing form

$$\begin{aligned}\mathcal{R}(c(p)) = \quad & R_{\ \delta^1_m \Phi_g(v)}(p) \\ & = \big(R_{\omega_W\ g(u,v)}(p, 0) + R_{\ e^{zL_W(-1)}\omega_W\ g(v,u)}(-p, 0) - R_{g(\omega_V(u,v))}(p, 0)\big).\end{aligned} \tag{4.13}$$

The form of dependence of Φ or $g(v, z)$ on $v \in V$ determines the result of computation of the matrix element in (4.13). In order to compute (4.13) we use the properties of the grading-restricted vertex algebra V, in particular, one expands $\omega(v, c(p))$ as in (5.1), and acts on $g(v, 0)$.

4.5. C^2_{ex}**-complex.** In this subsection we consider a particular example of the double complex (3.6), which takes into account C^2_0- conditions on elements of the space $C^2_0(\mathbf{U})$. In addition to the double complex $(C^n_m(\mathbf{U}), \delta^n_m)$ provided by (3.6) and (3.9), there exists an C^2_0- short double complex which we call transversal connection complex. We have

Lemma 7. *For $n = 2$, there exists a subspace $C^0_{ex}(\mathbf{U})$*

$$C^2_m(\mathbf{U}) \subset C^0_{ex}(\mathbf{U}) \subset C^2_0(\mathbf{U}),$$

for all $m \geq 1$, with the action of coboundary operator δ^2_m defined.

Proof. Let us consider the space $C^2_0(\mathbf{U})$. The space $C^2_0(\mathbf{U})$ contains elements of $\mathcal{W}_{p_1,p_2}$ so that the action of δ^2_0 is zero. Nevertheless, as for $\mathcal{J}^n_m(\Phi)$ in (2.16), Definition 5, let us consider sum of projections

$$P_r : \mathcal{W}_{z_i,z_j} \to W_r,$$

for $r \in \mathbb{C}$, and $(i, j) = (1, 2), (2, 3)$, so that the condition (2.16) is satisfied for some connections similar to the action (2.16) of δ^2_0. Separating the first two and the second two summands in (3.9), we find that for a subspace of $C^2_0(\mathbf{U})$, which we denote as $C^2_{ex}(\mathbf{U})$, consisting of three-point connections Φ such that for v_1, v_2, $v_3 \in V$, and arbitrary $\zeta \in \mathbb{C}$, the following forms of connections

$$
\begin{aligned}
& G_1(c_1(p_1), c_2(p_2), c_3(p_3)) \\
&= \sum_{r \in \mathbb{C}} \big(R_{\;\omega_W P_r(\Phi)(v_1, v_2, v_3)}\, (p_1, p_2 - \zeta, p_3 - \zeta) \\
&\quad + R_{\Phi\;\, P_r(\omega_V; \omega_V)(v_1, v_2, v_3)}\, (p_1, p_2 - \zeta, p_3 - \zeta) \big),
\end{aligned} \tag{4.14}
$$

$$
\begin{aligned}
& G_2(c_1(p_1), c_2(p_2), c_3(p_3)) \\
&= \sum_{r \in \mathbb{C}} \big(R_{\Phi(P_r(\omega_V\;\, \omega_V)(v_1, v_2, v_3)}(p_1 - \zeta, p_2 - \zeta, p_3) \\
&\quad + R_{\omega_V\;\, P_r(\Phi)(v_3, v_1, v_2)}\, (p_3, p_1 - \zeta; p_2 - \zeta) \big),
\end{aligned} \tag{4.15}
$$

are absolutely convergent in the regions $|c_1(p_1) - \zeta| > |c_2(p_2) - \zeta|$, $|c_2(p_2) - \zeta| > 0$, $|\zeta - c_3(p_3)| > |c_1(p_1) - \zeta|$, $|c_2(p_2) - \zeta| > 0$, where c_i, $1 \le i \le 3$ are coordinate functions, respectively, and can be analytically extended to rational form-valued functions in $c_1(p_1)$ and $c_2(p_2)$ with the only possible poles at $c_1(p_1)$, $c_2(p_2) = 0$, and $c_1(p_1) = c_2(p_2)$. Note that (4.14) and (4.15) constitute the first two and the last two terms of (3.9) correspondingly. According to Proposition 2 (cf. Appendix 2.2), $C^2_m(\mathbf{U})$ is a subspace of $C^2_{ex}(\mathbf{U})$, for $m \ge 0$, and $\Phi \in C^2_m(\mathbf{U})$ are composable with m vertex operators. Note that (4.14) and (4.15) represent sums of forms $G_{tr}(p, p')$ of transversal connections (6.6) (cf. Section 4). $\square$

Then we have

Definition 18. The coboundary operator

$$
\delta^2_{ex} : C^2_{ex}(\mathbf{U}) \to C^3_0(\mathbf{U}), \tag{4.16}
$$

is defined by three point connection of the form

$$
R_{\delta^2_{ex}\Phi(v_1, v_2, v_3)}(p_1, p_2, p_3) = G_{ex}(p_1, p_2, p_3), \tag{4.17}
$$

$$
\begin{aligned}
G_{ex}(p_1, p_2, p_3) \;\;=\;\; & R_{\omega_W\;\, \Phi(v_1, v_2, v_3)}\, (p_1, p_2, p_3) - R_{\Phi(\omega_V\;\, \omega_V)(v_1, v_2, v_3)}(p_1, p_2, p_3) \\
& + R_{\Phi(\omega_V\;\, \omega_V)(v_1, v_2, v_3)}(p_1, p_2, p_3) + R_{\omega_W\;\, \Phi(v_3, v_1, v_2)}(p_3, p_1, p_2),
\end{aligned} \tag{4.18}
$$

for $w' \in W'$, $\Phi \in C^2_{ex}(\mathbf{U})$, $v_1, v_2, v_3 \in V$ and $(c_1(p_1), c_2(p_2), c_3(p_3)) \in F_3\mathbb{C}$.

A particular consideration as in Proposition 4 results in

Proposition 7. *The operator* (4.16) *provides the chain-cochain complex*

$$
\delta^2_{ex} \circ \delta^1_2 = 0,
$$

$$
0 \longrightarrow C^0_3(\mathbf{U}) \xrightarrow{\delta^0_3} C^1_2(\mathbf{U}) \xrightarrow{\delta^1_2} C^2_{ex}(\mathbf{U}) \xrightarrow{\delta^2_{ex}} C^3_0(\mathbf{U}) \longrightarrow 0, \tag{4.19}
$$

on the spaces (3.6).

4.6. **Classes associated with C^2_{ex}- cohomology.** In this subsection we consider the cohomology $H^2_{ex}(\mathbf{U})$ associated to the short complex (4.19), and corresponding cohomological class. Let us first mention the geometrical meaning of the square-zero extension (V_W, γ, α) of V by W. Let us consider u, v belong to the square-zero ideal of a grading-restricted vertex algebra V, then

$$\omega_V(u, c(p))v = 0.$$

Then, geometrically it means that corresponding vertex algebra holomorphic connections are transversal (cf. Definition 32):

$$G_{tr}(p, p') = \omega_W\left(v, c(p')\right) \Phi\left(u, c(p)\right) + \omega_W\left(u, c(p)\right)\ \Phi\left(\psi, c(p')\right) = 0. \tag{4.20}$$

Note that, for a square-zero ideal, the full form of holomorphic connection has a reduced form (4.20). In our setup the holomorphic connection plays the following role: if it has does not have an full closed form (4.6), then the cohomology class it non-trivial. In [13] we find the proof of the following algebraic result for the second cohomology of a grading-restricted vertex algebra V, $H^2_{ex}(V, W)$ of V with coefficients in W. It follows from that Proposition, that the difference between two square-zero extensions are controled by the vertex operator map for the square-zero extension defined for $Z = V \bigoplus W$.

Proposition 8. *Let V be a grading-restricted vertex algebra and W a V-module. Then the set of the equivalence classes of square-zero extensions of V by W corresponds bijectively to $H^2_{ex}(V, W)$.*

Now we formulate the following corollary from Proposition 2.

Corollary 2. *Let V be a grading-restricted vertex algebra and W a V-module. The classes of square-zero extensions of V by W are isomorphic to classes of cohomological invariants Φ (5.19) of $H^2_{ex}(\mathbf{U})$.*

Proof. As in the proof of Proposition 2 we check that Φ (5.19) satisfies the $L(-1)$-derivative and $L(0)$- conjugation properties. Since Z is a grading-restricted vertex algebra, by using the associativity property for vertex operators (5.18), we see that the conditions (4.14) and (4.15) for forms G_i, $i = 1$, 2, in the proof of Lemma 7 of the space $C^2_{ex}(\mathbf{U})$ for Φ are satisfied, and $\Phi \in C^2_{ex}(\mathbf{U})$. Using again corresponding associativity properties for vertex operators in Z, we find that Φ is closed (according to our Definition 33), i.e., $\delta^2_{ex}\Phi = 0$. Thus, we see that, for a representative of the class of square-zero extension (Z, Y_Z, p_1, i_2) corresponds by the formula (5.19) for ω_Z to an element of $H^2_{ex}(\mathbf{U})$, $[\Phi] = \Phi + \eta$, where η be an element $\delta^1_2 C^1_2(\mathbf{U})$. It is easy to see that, according to properties of the above construction Φ is invariant with respect to a substitution $\Phi \mapsto \Phi + \mu$, for $\mu \in C^2_{ex}(\mathbf{U})$. Thus, Φ (5.19) belongs to the cohomology class $H^2_{ex}(\mathbf{U})$.

Let us prove the inverse statement. For an element $\Phi \in C^2_{ex}(\mathbf{U})$ which is a representative of $H^2_{ex}(\mathbf{U})$, according to Definition 5 of composibility, it follows that for any v_1, $v_2 \in V$, there exists $N^2_0(v_1, 0)$ such that for $w' \in W'$,

$$G_2(c_1(p_1), c_2(p_2)) = R_{\Phi(v_1, v_2)}(z_1, z_2),$$

is a rational $\mathcal{W}_{z_1,z_2}$-valued form with the only possible pole at $z_1 = z_2$ of order less than or equal to $N_0^2(v_1, v_2)$. For v_1, $v_2 \in V$, let us define $\omega_\Psi(v_1, \zeta)v_2 \in \mathcal{W}((\zeta))$ such that

$$R_{\omega_\Psi(v_1,v_2)}(\zeta)|_{\zeta=z} = R_{\Phi(v_1,v_2)}(z, z_2),$$

for $z \in \mathbb{C}^\times$. For v_1, $v_2 \in V$, we can define $Y_Z(v_1, \zeta)v_2$ using (5.18). Thus, we obtain a vertex operator map Y_Z, and Z is endowed with the structure of a grading-restricted vertex algebra. Finally, we have

Corollary 3. *Two elements of* $\ker \delta^2_{ex}$ *differ by an element* $\delta^1_2 C^1_2(\mathbf{U})$ *if and only if the corresponding square-zero extensions of* V *by* W *are equivalent.*

$\square$

Recall definitions of the forms G_1 (4.14) and G_2 (4.15) from Section 3.2. We define the following characteristic functions as triple integrals associated to the these forms:

$$F(c(p), c(p'), c(p'')) = \int_{(q_1,q_1',q_1'')}^{(q_2,q_2',q_2'')} G_i\left(v, c(p); v', c(p'); v'', c(p'')\right), \qquad (4.21)$$

with $i = 1$, 2. By assumption containing in Subsection 4.5, the forms (4.14) and (4.15) have nice convergence properties. Moreover, they contain only parts of the connection (functions do not vanish), and can be used in order to describe $\mathcal{M}$. For the invariant related to the second cohomology $H^2_{ex}(\mathbf{U})$, we obtain for (5.19)

$$F(c(p), c(p'), c(p'')) = R_{\Phi(v,v',v'')}\left(p, p', p''\right). \qquad (4.22)$$

In addition to (4.22), one uses the particular form of G_i, $i = 1$, 2

$$G_1(p_1, p_2, p_3) \quad = \quad R_{\omega_\Psi\ \omega_V\ .\omega_V(v_1,v_2,v_3)}(p_1, p_2, p_3) + R_{\omega_W\ \omega_\Psi\ \omega_V(v_1,v_2,v_3)}(p_1, p_2, p_3),$$

$$\begin{aligned} G_2(p_1, p_2, p_3) \quad = \quad & R_{\omega_\Psi(\omega_V\omega_V)(v_1,v_2,v_3)}(p_1 - p_2, p_2, p_3) \\ & + R_{\omega^W_{WV}(\omega_\Psi\omega_V)(v_1,v_2,v_3)}(p_1 - p_2, p_2, p_3), \end{aligned}$$

(4.14) and (4.15) in (4.21) (cf. Subsection 4.5).

$\Phi \in \mathcal{W}_{z_1,\ldots,z_n}$ is associated to $\mathcal{R}$ which is supposed to be a rational form with poles at $z_i = z_j$, $i \neq j$ only. Thus, the general principle is the following. By associating z_i to $c_i(p)$ on $\mathcal{M}$ and computing (8.1), we study its analytic behavior. If (8.1) has poles then they could be related to singular points of $\mathcal{M}$. Next, for (2.14), (2.16), (4.14), and (4.15), for $z_i = c_i(p_i)$, we determine the domains of convergence. When such a domain is limited to one point, then $\mathcal{M}$ might have a singular point. Finally, consider $\delta^0_1\Phi$, for $\Phi \in C^0_1(\mathbf{U})$, and identify z to $c(w)$, where $c(w)$ is a local coordinate on $\mathcal{M}$. Thus, in case of singular point we have different values of, e.g., integrals (8.2) in these directions.

4.7. **A relation to ordinary setup for Čech-de Rham cohomology.** Recall the setup for the Čech-de Rham cohomology given in Appendix 6. In particular, we have the following

Lemma 8. *The construction of the double complex* $\left(C^{k,l}, \delta\right)$, (6.7), (6.8) *follows from the construction of the double complex* $(C^n_m(\mathbf{U}), \delta^n_m)$ *of* (3.12). *Thus, the Čech-de Rham cohomology of a smooth complex curve* $\mathcal{M}$ *results from grading-restricted vertex algebra* V *cohomology of* $\mathcal{M}$.

Proof. One constructs the space of differential forms of degree k by elements Φ of $C^n_m(\mathbf{U})$

$$R_{\Phi\left(dc_1(p_1)^{\mathrm{wt}(v_1)}\otimes v_1,\ldots,dc_n(p_n)^{\mathrm{wt}(v_n)}v_n\right)}(p_1,\ldots,p_n), \tag{4.23}$$

such that $n=k$ the total degree

$$\sum_{i=1}^{n} \mathrm{wt}\ (v_i) = l,$$

for $v_i \in V$. The condition of composability of Φ with m vertex operators allows us make the association of the differential form $\varpi(h_1,\ldots,h_n)$ with (4.23) $(h_1^*,\ldots,h_k^*)$ with $(v_i,\ldots,v_k)$, and to represent a sequence of holomorphic embeddings $h_1,\ldots,h_p$ for $U_0,\ldots,U_p$ in (6.7) by vertex operators ω_W, i.e,

$$(h(h_1^*)\ldots h(h_n^*))\,(z_1,\ldots,z_n)) = \omega_W\,(v_1, c_1(p_1))\ldots\omega_W\,(v_l, c_n(p_n))\,.$$

Then, by using Definitions of coboundary operator (3.9), we see that the definition of the coboundary operator for the Čech-de Rham cohomology. □

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5. Appendix: Grading-restricted vertex algebras and their modules

In this section, following [12] we recall basic properties of grading-restricted vertex algebras and their grading-restricted generalized modules, useful for our purposes in later sections. We work over the base field $\mathbb{C}$ of complex numbers.

Definition 19. A vertex algebra $(V, Y_V, \mathbf{1}_V)$, (cf. [15]), consists of a $\mathbb{Z}$-graded complex vector space

$$V = \coprod_{n\in\mathbb{Z}} V_{(n)}, \quad \dim V_{(n)} < \infty,$$

for each $n \in \mathbb{Z}$, and linear map

$$Y_V : V \to \mathrm{End}\ (V)[[z, z^{-1}]],$$

for a formal parameter z and a distinguished vector $\mathbf{1}_V \in V$. The evaluation of Y_V on $v \in V$ is the vertex operator

$$Y_V(v) \equiv Y_V(v,z) = \sum_{n\in\mathbb{Z}} v(n) z^{-n-1}, \tag{5.1}$$

with components $(Y_V(v))_n = v(n) \in \mathrm{End}\,(V)$, where $Y_V(v,z)\mathbf{1}_V = v + O(z)$.

Definition 20. A grading-restricted vertex algebra satisfies the following conditions:

(1) Grading-restriction condition: $V_{(n)}$ is finite dimensional for all $n \in \mathbb{Z}$, and $V_{(n)} = 0$ for $n \ll 0$;

(2) Lower-truncation condition: For u, $v \in V$, $Y_V(u,z)v$ contains only finitely many negative power terms, that is,

$$Y_V(u,z)v \in V((z)),$$

(the space of formal Laurent series in z with coefficients in V);

(3) Identity property: Let Id_V be the identity operator on V. Then

$$Y_V(\mathbf{1}_V, z) = \mathrm{Id}_V;$$

(4) Creation property: For $u \in V$,

$$Y_V(u,z)\mathbf{1}_V \in V[[z]],$$

and

$$\lim_{z\to 0} Y_V(u,z)\mathbf{1}_V = u;$$

(5) Duality: For $u_1, u_2, v \in V$,

$$v' \in V' = \coprod_{n\in\mathbb{Z}} V^*_{(n)},$$

where $V^*_{(n)}$ denotes the dual vector space to $V_{(n)}$ and $\langle\, .\,,.\rangle$ the evaluation pairing $V' \otimes V \to \mathbb{C}$, the series

$$R_{Y_V Y_V(u_1,u_2)}(z_1,z_2),\ R_{Y_V Y_V(u_2,u_1)}(z_2,z_1),\ R_{Y_V Y_V(u_1 u_2)}(z_1-z_2,z_2), \tag{5.2}$$

are absolutely convergent in the regions

$$|z_1| > |z_2| > 0,\ |z_2| > |z_1| > 0,\ |z_2| > |z_1 - z_2| > 0,$$

respectively, to a common rational function in z_1 and z_2 with the only possible poles at $z_1 = 0 = z_2$ and $z_1 = z_2$;

(6) $L_V(0)$-bracket formula: Let $L_V(0) : V \to V$, be defined by

$$L_V(0)v = nv, \qquad n = \mathrm{wt}\,(v),$$

for $v \in V_{(n)}$. Then

$$[L_V(0), Y_V(v,z)] = Y_V(L_V(0)v, z) + z\frac{d}{dz} Y_V(v,z),$$

for $v \in V$.

(7) $L_V(-1)$-derivative property: Let

$$L_V(-1): V \to V,$$

be the operator given by

$$L_V(-1)v = \mathrm{Res}_z z^{-2} Y_V(v,z)\mathbf{1}_V = Y_{(-2)}(v)\mathbf{1}_V,$$

for $v \in V$. Then for $v \in V$,

$$\frac{d}{dz} Y_V(u,z) = Y_V(L_V(-1)u,z) = [L_V(-1), Y_V(u,z)]. \tag{5.3}$$

In addition to that, we recall here the following definition (cf. [1]):

Definition 21. A grading-restricted vertex algebra V is called conformal of central charge $c \in \mathbb{C}$, if there exists a non-zero conformal vector (Virasoro vector) $\omega \in V_{(2)}$ such that the corresponding vertex operator

$$Y_V(\omega,z) = \sum_{n\in\mathbb{Z}} L_V(n) z^{-n-2},$$

is determined by modes of Virasoro algebra $L_V(n): V \to V$ satisfying

$$[L_V(m), L_V(n)] = (m-n)L(m+n) + \frac{c}{12}(m^3-m)\delta_{m+b,0}\,\mathrm{Id_V}.$$

Definition 22. A vector A which belongs to a module W of a quasi-conformal grading-restricted vertex algebra V is called primary of conformal dimension $\Delta(A) \in \mathbb{Z}_+$ if

$$\begin{aligned} L_W(k)A &= 0, \ k > 0, \\ L_W(0)A &= \Delta(A)A. \end{aligned}$$

Definition 23. A grading-restricted generalized V-module is a vector space W equipped with a vertex operator map

$$\begin{aligned} Y_W : V \otimes W &\to W[[z,z^{-1}]], \\ u \otimes w &\mapsto Y_W(u,w) \equiv Y_W(u,z)w = \sum_{n\in\mathbb{Z}} (Y_W)_n(u,w) z^{-n-1}, \end{aligned}$$

and linear operators $L_W(0)$ and $L_W(-1)$ on W satisfying the following conditions:

(1) Grading-restriction condition: The vector space W is $\mathbb{C}$-graded, that is,

$$W = \coprod_{\alpha\in\mathbb{C}} W_{(\alpha)},$$

such that $W_{(\alpha)} = 0$ when the real part of α is sufficiently negative;

(2) Lower-truncation condition: For $u \in V$ and $w \in W$, $Y_W(u,z)w$ contains only finitely many negative power terms, that is, $Y_W(u,z)w \in W((z))$;

(3) Identity property: Let Id_W be the identity operator on W. Then

$$Y_W(\mathbf{1}_V, z) = \mathrm{Id}_W;$$

(4) Duality: For $u_1, u_2 \in V$, $w \in W$,
$$w' \in W' = \coprod_{n\in\mathbb{Z}} W^*_{(n)},$$
W' denotes the dual V-module to W and $\langle .,. \rangle$ their evaluation pairing, the series $R_{Y_W Y_W(u_1 u_2)}(z_1, z_2)$, $R_{Y_W Y_W(u_2,u_1)}(z_2, z_1)$, $R_{Y_W Y_W(u_1,u_2)}(z_1 - z_2, z_2)$, are absolutely convergent in the regions
$$|z_1| > |z_2| > 0,\ |z_2| > |z_1| > 0,\ |z_2| > |z_1 - z_2| > 0,$$
respectively, to a common rational function in z_1 and z_2 with the only possible poles at $z_1 = 0 = z_2$ and $z_1 = z_2$.

(5) $L_W(0)$-bracket formula: For $v \in V$,
$$[L_W(0), Y_W(v,z)] = Y_W(L_V(0)v, z) + z\frac{d}{dz}Y_W(v,z);$$

(6) $L_W(0)$-grading property: For $w \in W_{(\alpha)}$, there exists $N \in \mathbb{Z}_+$ such that
$$(L_W(0) - \alpha)^N w = 0; \tag{5.4}$$

(7) $L_W(-1)$-derivative property: For $v \in V$,
$$\frac{d}{dz}Y_W(u,z) = Y_W(L_V(-1)u, z) = [L_W(-1), Y_W(u,z)]. \tag{5.5}$$

The translation property of vertex operators
$$Y_W(u,z) = e^{-z'L_W(-1)} Y_W(u, z+z') e^{z'L_W(-1)}, \tag{5.6}$$
for $z' \in \mathbb{C}$, follows from from (5.5). For $v \in V$, and $w \in W$, the intertwining operator
$$\begin{aligned} &Y^W_{WV} : V \to W, \\ &v \mapsto Y^W_{WV}(w,z)v, \end{aligned} \tag{5.7}$$
is defined by
$$Y^W_{WV}(w,z)v = e^{zL_W(-1)} Y_W(v,-z)w. \tag{5.8}$$
We will also use the following property of intertwining operators (5.7) [13]. For a function $f(u)$, $u \in V$,
$$f\left(Y_V(u,z)\mathbf{1}_V\right) = Y^W_{WV}(f(u), z)\mathbf{1}_V.$$

Let us recall some further facts from [1] relating generators of Virasoro algebra with the group of automorphisms in complex dimension one. Let us represent an element of Aut $\mathcal{O}^{(1)}$ by the map
$$z \mapsto \rho = \rho(z), \tag{5.9}$$
given by the power series
$$\rho(z) = \sum_{k\geq 1} a_k z^k, \tag{5.10}$$
$\rho(z)$ can be represented in an exponential form
$$\widetilde{\rho}(z) = \exp\left(\sum_{k>-1} \beta_k\ z^{k+1}\partial_z\right) (\beta_0)^{z\partial_z}\ .z, \tag{5.11}$$

where we express $\beta_k \in \mathbb{C}$, $k \geq 0$, through combinations of a_k, $k \geq 1$. A representation of Virasoro algebra modes in terms of differential operators is given by [15]

$$L_W(m) \mapsto -\zeta^{m+1}\partial_\zeta, \tag{5.12}$$

for $m \in \mathbb{Z}$. By expanding (5.11) and comparing to (5.10) we obtain a system of equations which, can be solved recursively for all β_k. In [1], $v \in V$, they derive the formula

$$[L_W(n), Y_W(v,z)] = \sum_{m \geq -1} \frac{1}{(m+1)!} \left(\partial_z^{m+1} z^{m+1}\right) \; Y_W(L_V(m)v, z), \tag{5.13}$$

of a Virasoro generator commutation with a vertex operator. Given a vector field

$$\beta(z)\partial_z = \sum_{n \geq -1} \beta_n z^{n+1}\partial_z, \tag{5.14}$$

which belongs to local Lie algebra of Aut $\mathcal{O}^{(1)}$, one introduces the operator

$$\beta = -\sum_{n \geq -1} \beta_n L_W(n).$$

We conlclude from (5.14) with the following

Lemma 9.

$$[\beta, Y_W(v,z)] = \sum_{m \geq -1} \frac{1}{(m+1)!} \left(\partial_z^{m+1} \beta(z)\right) \; Y_W(L_V(m)v, z). \tag{5.15}$$

The formula (5.15) is used in [1] (Chapter 6) in order to prove invariance of vertex operators multiplied by conformal weight differentials in case of primary states, and in generic case.

Let us give some further definition:

Definition 24. A grading-restricted vertex algebra V-module W is called quasi-conformal if it carries an action of local Lie algebra of Aut $\mathcal{O}$ such that commutation formula (5.15) holds for any $v \in V$, the element $L_W(-1) = -\partial_z$, as the translation operator T,

$$L_W(0) = -z\partial_z,$$

acts semi-simply with integral eigenvalues, and the Lie subalgebra of the positive part of local Lie algebra of Aut $\mathcal{O}^{(n)}$ acts locally nilpotently.

Recall [1] the exponential form $\widetilde{\rho}(\zeta)$ (5.11) of the coordinate transformation (5.9) $\rho(z) \in$ Aut $\mathcal{O}^{(1)}$. A quasi-conformal vertex algebra possesses the formula (5.15), thus, it is possible by using the identification (5.12), to introduce the linear operator representing $\widetilde{\rho}(\zeta)$ (5.11) on $\mathcal{W}_{z_1,\ldots,z_n}$,

$$P\left(\widetilde{\rho}(\zeta)\right) = \exp\left(\sum_{m>0} (m+1)\, \beta_m \, L_V(m)\right) \beta_0^{L_W(0)}, \tag{5.16}$$

(note that we have a different normalization in it). In [1] (Chapter 6) it was shown that the action of an operator similar to (5.16) on a vertex algebra element $v \in V_n$ contains finitely meny terms, and subspaces

$$V_{\leq m} = \bigoplus_{n \geq K}^{m} V_n,$$

are stable under all operators $P(\widetilde{\rho})$, $\widetilde{\rho} \in \mathrm{Aut}\ \mathcal{O}^{(1)}$. In [1] they proved the following

Lemma 10. *The assignment*

$$\widetilde{\rho} \mapsto P(\widetilde{\rho}),$$

defines a representation of Aut $\mathcal{O}^{(1)}$ *on* V*,*

$$P(\widetilde{\rho}_1 * \widetilde{\rho}_2) = P(\widetilde{\rho}_1)\ P(\widetilde{\rho}_2),$$

which is the inductive limit of the representations $V_{\leq m}$*,* $m \geq K$*.*

Similarly, (5.16) provides a representation operator on $\mathcal{W}_{z_1,\dots,z_n}$.

5.1. **Square-zero extensions of V.** Let us first recall some definitions [13] concerning the notion of square-zero extension of V by its module W.

Definition 25. Let V be a grading-restricted vertex algebra. A square-zero ideal of V is an ideal W of V such that for any u, $v \in W$,

$$Y_V(u, x)v = 0.$$

Definition 26. Let V be a grading-restricted vertex algebra and W a $\mathbb{Z}$-graded V-module. A square-zero extension (V_W, γ, α) of V by W is a grading-restricted vertex algebra V_W together with a surjective homomorphism

$$\gamma : V_W \to V,$$

of grading-restricted vertex algebras such that $\ker \gamma$ is a square-zero ideal of V_W (and therefore a V-module) and an injective homomorphism α of V-modules from W to V_W such that

$$\alpha(W) = \ker \gamma.$$

Definition 27. Two square-zero extensions $(V_{W,1}, \gamma_1, \alpha_1)$ and $(V_{W,2}, \gamma_2, \alpha_2)$ of V by W are equivalent if there exists an isomorphism of grading-restricted vertex algebras $h : V_{W,1} \to V_{W,2}$ such that the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & W & \xrightarrow[\alpha_1]{} & V_{W,1} & \xrightarrow[\gamma_1]{} & V & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \mathrm{Id}_W} & & \Big\downarrow{\scriptstyle h} & & \Big\downarrow{\scriptstyle \mathrm{Id}_V} & & \\
0 & \longrightarrow & W & \xrightarrow[\alpha_2]{} & V_{W,2} & \xrightarrow[\gamma_2]{} & V & \longrightarrow & 0,
\end{array}$$

is commutative.

Let (V_W, γ, α) be a square-zero extension of V by W. It is possible to construct a realization of the square-zero extension of V by W on $Z = V \bigoplus W$. Then there exists an injective linear map $\Gamma : V \to V_W$, such that the linear map

$$h : Z \to V_W,$$

given by

$$h(v, w) = \Gamma(v) + \alpha(w),$$

is a linear isomorphism. By definition, the restriction of h to W is the isomorphism α from W to $\ker \gamma$. Then the grading-restricted vertex algebra structure and the V-module structure on V_W give a grading-restricted vertex algebra structure and a V-module structure on Z such that the embedding $i_2 : W \to Z$ and the projection $p_1 : Z \to V$, are homomorphisms of grading-restricted vertex algebras. In addition to that, $\ker p_1$ is a square-zero ideal of Z, i_2 is an injective homomorphism such that $i_2(W) = \ker p_1$ and the diagram

$$\begin{tikzcd}
0 \arrow[r] & W \arrow[r, "i_2"] \arrow[d, "\mathrm{Id}_W"'] & Z \arrow[r, "p_1"] \arrow[d, "h"'] & V \arrow[r] \arrow[d, "\mathrm{Id}_V"] & 0 \\
0 \arrow[r] & W \arrow[r, "\alpha"'] & V_W \arrow[r, "\gamma"'] & V \arrow[r] & 0
\end{tikzcd} \tag{5.17}$$

of V-modules is commutative. Thus, one obtains a square-zero extension (Z, p_1, i_2) equivalent to (V_W, γ, α). It is enough then to consider square-zero extensions of V by W of the particular form (Z, p_1, i_2). The difference between two such square-zero extensions consists in the vertex operator maps. Such square-zero extensions will be denoted by (Z, Y_Z, p_1, i_2).

The explicit definition for Z-vertex operator was introduced in [13]. We denote by (Z, Y_Z, p_1, i_2) a suitable square-zero extension of V by W. Then there exists

$$\omega_\Psi\,(u, z)\, v \;\in \mathcal{W}((z)),$$

for u, $v \in V$ such that

$$\begin{aligned}
\omega_Z((v_1, 0), z)(v_2, 0) &= (\omega_V(v_1, z)v_2, \omega_\Psi(v_1, z)v_2), \\
\omega_Z((v_1, 0), z)(0, w) &= (0, \omega_V(v_1, z)w_2), \\
\omega_Z((0, w_1), z)(v_2, 0) &= (0, \omega^W_{WV}(w, z)v_2), \\
\omega_Z((0, w_1), z)(0, w_2) &= 0,
\end{aligned}$$

for v_1, $v_2 \in V$ and w_1, $w_2 \in W$. Thus, one has

$$\begin{aligned}
&\omega_Z((v_1, w_1), z)(v_2, w_2) \\
&= \left(\omega_V(v_1, z)v_2, \omega_W(v_1, z)\; w_2 + \omega^W_{WV}(w_1, z)\; v_2 + \omega_\Psi(v_1, z)\; v_2\right),
\end{aligned} \tag{5.18}$$

for $v_1, v_2 \in V$ and $w_1, w_2 \in W$. The vacuum of Z is given by $(\mathbf{1}_V, 0)$, and

$$\omega_\Psi(v, z)\mathbf{1}_V = 0,$$

and the dual space Z' for Z is identified with

$$Z' = V' \oplus W'.$$

By Definition 20 of a grading-restricted vertex algebra, for v, $v' \in V$, vertex operators $\omega_\Psi(v,z)$ and $\omega_V(v',z')$ in extension (V_W, γ, α), satisfy the associativity property, i.e., their matrix elements of (5.2) converge (under appropriate conditions for local coordinates of points) to the same $\mathcal{W}_{z_1,z_2}$-valued rational function. Thus, for $v_1, v_2 \in V$, and $(z_1,z_2) \in F_2\mathbb{C}$, we introduce a linear map

$$\Phi : V \otimes V \to \mathcal{W}_{z_1,z_2},$$

such that

$$\begin{aligned} R_{\Phi(v_1,v_2)}(z_1,z_2) &= R_{\omega_\Psi \omega_V(v_1,v_2)}(z_1,z_2) \\ &= R_{\omega_\Psi \omega_V(v_2,v_1)}(z_2,z_1) \\ &= R_{\omega^W_{WV}(\omega_\Psi)(v_1,v_2)}(z_1-z_2,z_2). \end{aligned} \tag{5.19}$$

6. Appendix: Cohomology in terms of connections

6.1. **Multi-point holomorphic connections.** In this subsection, motivated by the definition of the holomorphic connection for a vertex algebra bundle (cf. Section 6, [1]) over a smooth complex variety, we introduce the definition of the multiple point holomorphic connection over a smooth complex curve $\mathcal{M}$.

Definition 28. Let $\mathcal{V}$ be a holomorphic vector bundle over $\mathcal{M}$, and $\mathcal{M}_0$ its submanifold. A holomorphic multi-point connection $\mathcal{G}$ on $\mathcal{V}$ is a $\mathbb{C}$-multi-linear map

$$\mathcal{G} : \mathcal{E} \to \mathcal{E} \otimes \Omega,$$

such that for any holomorphic function f, and two sections $\phi(p)$ and $\psi(p')$ at points p and p' on $\mathcal{M}_0$ correspondingly, we have

$$\sum_{q,q' \mathcal{M}_0 \subset \mathcal{M}} \mathcal{G}\left(f(\psi(q)).\phi(q')\right) = f(\psi(p'))\, \mathcal{G}\left(\phi(p)\right) + f(\phi(p))\, \mathcal{G}\left(\psi(p')\right), \tag{6.1}$$

where the summation on left hand side is performed over a locus of points q, q' on $\mathcal{M}_0$. We denote by $\mathcal{C}on_{\mathcal{M}_0}(\mathcal{S})$ the space of such connections defined over a smooth complex curve $\mathcal{M}$. We will call $\mathcal{G}$ satisfying (6.1), a closed connection, and denote the space of such connections by $\mathcal{C}on^n_{\mathcal{M}_0;cl}$.

Geometrically, for a vector bundle $\mathcal{V}$ defined over a complex variety $\mathcal{M}$, a multi-point holomorphic connection (6.1) relates two sections ϕ and ψ of E at points p and p' with a number of sections at a subvariety $\mathcal{M}_0$ of $\mathcal{M}$.

Definition 29. We call

$$G(\phi,\psi) = f(\phi(p))\, \mathcal{G}\left(\psi(p')\right) + f(\psi(p'))\, \mathcal{G}\left(\phi(p)\right) - \sum_{q,q' \mathcal{M}_0 \subset \mathcal{M}} \mathcal{G}\left(f(\psi(q')).\phi(q)\right), \tag{6.2}$$

the form of a holomorphic connection $\mathcal{G}$. The space of form for n-point holomorphic connection forms will be denoted by $G^n(p,p',q,q')$.

Definition 30. A fixed point holomorphic connection on E is defined by the condition

$$\sum_{p_0;\, q,q' \in \mathcal{M}_0 \subset \mathcal{M}} \mathcal{G}\left(f(\psi(q')).\phi(q)\right) = f(\psi(p_0'))\, \mathcal{G}\left(\phi(p)\right) + f(\phi(p))\, \mathcal{G}\left(\psi(p_0)\right), \tag{6.3}$$

where a point p_0 is fixed on $\mathcal{M}_0$.

Definition 31. A holomorphic connection defined for a vector bundle $\mathcal{V}$ over a smooth complex variety $\mathcal{M}$ (the two point case of the multi-point holomorphic connection (6.1)) is called a two point connection when for any holomorphic function f,

$$\mathcal{G}\left(f(\psi(p')) \,.\phi(p)\right) = f(\psi(p'))\, \mathcal{G}(\phi(p)) + f(\phi(p))\, \mathcal{G}(\psi(p')), \tag{6.4}$$

for two sections $\psi(p')$ and $\phi(p)$ of E. We denote the space of such connections as $\mathcal{C}on^2_{p,p_0;\,\mathcal{M}_0}$.

Let us formulate another definition which we use in the next section:

Definition 32. We call a multi-point holomorphic connection $\mathcal{G}$ the transversal connection, i.e., when it satisfies

$$f(\psi(p'))\, \mathcal{G}(\phi(p)) + f(\phi(p))\, \mathcal{G}(\psi(p')) = 0. \tag{6.5}$$

We call

$$G_{tr}(p,p') = (\psi(p'))\, \mathcal{G}(\phi(p)) + f(\phi(p))\, \mathcal{G}(\psi(p')), \tag{6.6}$$

the form of a transversal connection. The space of such connections is denoted by G^2_{tr}.

In various situations it is sometimes effective to use an interpretation of cohomology in terms of connections. In particular in our supporting example of vertex algebra cohomology of codimension one foliations. It is convenient to introduce multi-point connections over a graded space and to express coboundary operators and cohomology in terms of connections:

$$\delta^n\phi \in G^{n+1}(\phi),$$
$$\delta^n\phi = G(\phi).$$

Then the cohomology is defined as the factor space

$$H^n = \mathcal{C}on^n_{cl;} / G^{n-1},$$

of closed multi-point connections with respect to the space of connection forms.

6.2. **Double complex spaces for ordinary Čech-de Rham cohomology.** Let us recall the setup for ordinary Čech-de Rham cohomology [3,18]. Consider a smooth manifold $\mathcal{M}$. Consider the double complex

$$C^{k,l} = \prod_{U_1 \overset{h_1}{\hookrightarrow} \cdots \overset{h_k}{\hookrightarrow} U_{k-1}} \Omega^l(U_1), \tag{6.7}$$

where $\Omega^l(U_1)$ is the space of differential l-forms on U_1, and the product ranges over all k-tuples of holonomy embeddings between open domains U_i, $i = 1, \ldots, k$. Component of $\varpi \in C^{k,l}$ are denoted by $\varpi(h_1, \ldots, h_l) \in \Omega^l(U_1)$. The vertical differential is defined as

$$(-1)^k d : C^{k,l} \to C^{k,l+1},$$

where d is the usual de Rham differential. The horizontal differential

$$\delta : C^{k,l} \to C^{k+1,l},$$

is given by

$$\delta = \sum_{i=1}^{k} (-1)^i \delta_i,$$

$$\delta_i \varpi(h_1, \ldots, h_{k+1}) = G(h_1, \ldots, h_{k+1}), \tag{6.8}$$

where $G(h_1, \ldots, h_{k+1})$ is the multi-point connection of the form (6.1), i.e.,

$$\delta_i \varpi(h_1, \ldots, h_{p+1}) = \begin{cases} h_1^* \varpi(h_2, \ldots, h_{p+1}), & \text{if } i = 0, \\ \varpi(h_1, \ldots, h_{i+1} h_i, \ldots, h_{p+1}), & \text{if } 0 < i < p+1, \\ \varpi(h_1, \ldots, h_p), & \text{if } i = p+1. \end{cases} \tag{6.9}$$

This double complex is actually a bigraded differential algebra, with the usual product

$$(\varpi \cdot \eta)(h_1, \ldots, h_{k+k'}) = (-1)^{kk'} \varpi(h_1, \ldots, h_k)\ h_1^* \ldots h_k^*\ .\eta(h_{k+1}, \ldots h_{k+k'}), \tag{6.10}$$

for $\varpi \in C^{k,l}$ and $\eta \in C^{k',l'}$, thus, $(\varpi \cdot \eta)(h_1, \ldots, h_{k+k'}) \in C^{k+k', l+l'}$. Let us denote by $\mathbf{U}$ a collection of open domains on $\mathcal{M}$. The cohomology $\check{H}^*_{\mathbf{U}}(M)$ of this complex is called the Čech-de Rham cohomology of $\mathcal{M}$ with respect to $\mathbf{U}$. It is defined by

$$\check{H}^*_{\mathbf{U}}(M) = \mathcal{C}on^{k+1}_{cl}(h_1, \ldots, h_{k+1})|_{\mathbf{U}} / G^k(h_1, \ldots, h_k)|_{\mathbf{U}},$$

where $\mathcal{C}on^{k+1}_{cl}(h_1, \ldots, h_{k+1})$ is the space of closed multi-point connections, and G^k $(h_1, \ldots, h_k)$ is the space of k-point connection forms on $\mathbf{U}$.

7. Appendix: Proofs of Lemmas 1, 2, and Proposition 3

In this Appendix we provide proofs of Lemmas 1, 2, and Proposition 3. We start with the proof of Lemma 1.

Proof. From the construction of spaces for double complex for a grading-restricted vertex algebra cohomology, it is clear that the spaces $C^n(V, \mathcal{W}, \mathcal{M})(U_j)$, $1 \le s \le m$ in Definition 8 are non-empty. On each domain U_s, $1 \le s \le m$, $\Phi(v_1, c_j(p_1); \ldots; v_n, c_j(p_n))$ belongs to the space $\mathcal{W}_{c_j(p_1), \ldots, c_j(p_n)}$, and satisfy the $L(-1)$-derivative (2.5) and $L(0)$-conjugation (2.10) properties. A map $\Phi(v_1, c_j(p_1); \ldots; v_n, c_j(p_n))$ is composable with m vertex operators with formal parameters identified with local coordinates $c_j(p'_j)$, on each domain U_j. Note that on each domain U, n and m the spaces (3.6) remain the same. The only difference may be constituted by the composibility conditions (2.14) and (2.16) for Φ.

In particular, for $l_1, \ldots, l_n \in \mathbb{Z}_+$ such that $l_1 + \cdots + l_n = n + m$, $v_1, \ldots, v_{m+n} \in V$ and $w' \in W'$, recall (2.11) that

$$\Psi_i \quad = \quad \omega_V(v_{k_1}, c_{k_1}(p_{k_1}) - \zeta_i) \ldots \omega_V(v_{k_i}, c_{k_i}(p_{k_i}) - \zeta_i)\ \mathbf{1}_V, \tag{7.1}$$

where k_i is defined in (2.12), for $i = 1, \ldots, n$, depend on coordinates of points on domains. At the same time, in the first composibility condition (2.14) depends on projections $P_r(\Psi_i)$, $r \in \mathbb{C}$, of $\mathcal{W}_{c(p_1), \ldots, c(p_n)}$ to W, and on arbitrary variables ζ_i, $1 \le i \le m$. On each transversal connection U_s, $1 \le s \le m$, the absolute convergence is assumed for the series (2.14) (cf. Appendix 2.2). Positive integers $N^n_m(v_i, v_j)$, (depending only on v_i and v_j) as well as ζ_i, for i, $j = 1, \ldots, k$, $i \ne j$, may vary for domains U. Nevertheless, the domains of convergence determined by the conditions (2.15) which have the form

$$|c_{m_i}(p_{m_i}) - \zeta_i| + |c_{n_i}(p_{n_i}) - \zeta_i| < |\zeta_i - \zeta_j|, \tag{7.2}$$

for $m_i = l_1 + \cdots + l_{i-1} + p$, $n = l_1 + \cdots + l_{j-1} + q$, i, $j = 1, \ldots, k$, $i \neq j$ and for $p = 1, \ldots, l_i$ and $q = 1, \ldots, l_j$, are limited by $|\zeta_i - \zeta_j|$ in (7.2) from above. Thus, for the intersection variation of sets of homology embeddings in (3.6), the absolute convergence condition for (2.14) is still fulfilled. Under intersection in (3.6) by choosing appropriate $N^n_m(v_i, v_j)$, one can analytically extend (2.14) to a rational function in $(c_1(p_1), \ldots, c_{n+m}(p_{n+m}))$, independent of $(\zeta_1, \ldots, \zeta_n)$, with the only possible poles at $c_i(p_i) = c_j(p_j)$, of order less than or equal to $N^n_m(v_i, v_j)$, for $i, j = 1, \ldots, k$, $i \neq j$.

As for the second condition in Definition of composibility, we note that, on each domain of $\mathbf{U}$, the domains of absolute convergense $c_i(p_i) \neq c_j(p_j)$, $i \neq j$ $|c_i(p_i)| > |c_k(p_j)| > 0$, for $i = 1, \ldots, m$, and $k = 1 + m, \ldots, n + m$, for

$$\mathcal{J}^n_m(\Phi) \quad = \quad \sum_{q \in \mathbb{C}} R_{\omega_W \ldots \omega_W \; P_q(\Phi)(v_1, \ldots, v_m)} \left(p_1, \ldots, p_m, p_{1+m}, \ldots, p_{n+m}\right), \tag{7.3}$$

are limited from below by the same set ot absolute values of local coordinates on a domain of $\mathbf{U}$. Thus, under intersection in (3.6) this condition is preserved, and the sum (2.16) can be analytically extended to a rational function in $(c_1(p_1), \ldots, c_{m+n}(p_{m+n}))$ with the only possible poles at $c_i(p_i) = c_j(p_j)$, of orders less than or equal to $N^n_m(v_i, v_j)$, for $i, j = 1, \ldots, k$, $i \neq j$. Thus, we proved the lemma. □

Next, we prove Proposition 3.

Proof. Here we prove that for generic elements of a quasi-conformal grading-restricted vertex algebra Φ and $\omega_W \in \mathcal{W}_{z_1, \ldots, z_n}$ and are canonical, i.e., independent on changes

$$z_i \mapsto w_i = \rho_i(z_i), \quad 1 \leq i \leq n, \tag{7.4}$$

of local coordinates of $c_i(p_i)$ at points p_i $1 \leq i \leq n$. Thus, the construction of the double complex spaces (3.6) is proved to be canonical too. Let us denote by

$$\xi_i = \left(\left(\beta_0^{(i)}\right)^{-1} \; dw_i \right)^{\mathrm{wt}\ (v_i)}.$$

Recall the linear operator (3.4) (cf. Appendix 5). Introduce the action of the transformations (7.4) as

$$\begin{aligned} &\Phi \left(dw_1^{\mathrm{wt}\ (v_1)} \otimes v_1, w_1; \ldots; dw_n^{\mathrm{wt}\ (v_n)} \otimes v_n, w_n \right) \\ &\quad = \left(\frac{d\widetilde{\rho}_\zeta(\zeta)}{d\zeta} \right)^{-L_W(0)} \; P\left(\widetilde{\rho}_\zeta(\zeta)\right) \; \Phi\left(\xi_1 \otimes v_1, z_1; \ldots; \xi_n \otimes v_n, z_n\right). \end{aligned} \tag{7.5}$$

We then obtain

Lemma 11. *An element* (2.3)

$$R_{\Phi\left(dz_1^{\mathrm{wt}(v_1)} \otimes v_1, \, \ldots, \, dz_n^{\mathrm{wt}(v_n)} \otimes v_n\right)}(z_1, \ldots, z_n),$$

of $\mathcal{W}_{z_1,\ldots,z_n}$ *is canonical, i.e., it is invariant under elements* (7.4) *of* $\left(\mathrm{Aut}_{p_1}\,\mathcal{O}^{(1)}\times \ldots\times \mathrm{Aut}_{p_n}\,\mathcal{O}^{(1)}\right)$.

Proof. Consider (7.5). First, note that

$$\widetilde{\rho}_i'(\zeta)=\frac{d\widetilde{\rho}_i(\zeta)}{d\zeta}=\sum_{m\geq 0}(m+1)\;\beta_m^{(i)}\;\zeta^m.$$

By using the identification (5.12) and and the $L_W(-1)$-properties (2.5) and (2.10) we obtain

$$R_{\Phi\left(dw_1^{\mathrm{wt}(v_1)}\otimes v_1\;\ldots\;dw_n^{\mathrm{wt}(v_n)}\otimes v_n\right)}(w_1,\ldots,w_n)$$

$$=R_{\widetilde{\rho}_\zeta'(\zeta)^{-L_W(0)}\;P(\widetilde{\rho}_\zeta(\zeta))\;\Phi(\xi_1\,\otimes v_1,\ldots;\;\xi_n\otimes v_n)}(z_1,\ldots,z_n)$$

$$=R_{\left(\frac{d\widetilde{\rho}_\zeta(\zeta)}{d\zeta}\right)^{-L_W(0)}\;\Phi\left(dw_1^{\mathrm{wt}(v_1)}\otimes v_1,\ldots,dw_n^{\mathrm{wt}(v_n)}\otimes v_n,\right)}\left(\sum_{m\geq 0}(m+1)\;\beta_m^{(1)}z_1^{m+1},\ldots,\sum_{m\geq 0}(m+1)\;\beta_m^{(n)}z_n^{m+1}\right)$$

$$=R_{\left(\frac{d\widetilde{\rho}_\zeta(\zeta)}{d\zeta}\right)^{-L_W(0)}\;\Phi\left(dw_1^{\mathrm{wt}\;(v_1)}\otimes v_1,\ldots,dw_n^{\mathrm{wt}(v_n)}\otimes v_n\right)}\left(\left(\frac{d\widetilde{\rho}_1(z_1)}{dz_1}\right)\;z_1,\ldots,\left(\frac{d\widetilde{\rho}_n(z_n)}{dz_n}\right)\;z_n\right)$$

$$=R_{\Phi\left(\left(\frac{d\widetilde{\rho}_1(z_1)}{dz_1}\;dw_1\right)^{-\mathrm{wt}\;(v_1)}\otimes v_1,\;\ldots,\;\left(\frac{d\widetilde{\rho}_n(z_n)}{dz_n}\;dw_n\right)^{-\mathrm{wt}\;(v_n)}\otimes v_n\right)}(z_1,\ldots,z_n)$$

$$=R_{\Phi\left(dz_1^{\mathrm{wt}(v_1)}\otimes v_1,\;\ldots,\;dz_n^{\mathrm{wt}(v_n)}\otimes v_n\right)}(z_1,\ldots,z_n).$$

Thus, we proved the Lemma. □

The elements $\Phi(v_1,z_1;\ldots;v_n,z_n)$ of $C_m^n(\mathbf{U})$ belong to the space $\mathcal{W}_{z_1,\ldots,z_n}$ and assumed to be composable with a set of vertex operators $\omega_W(v_j,c_j(p_j))$, $1\leq j\leq m$. Vertex operators $\omega_W(dc_j(p_j)^{\mathrm{wt}\;(v_j)}\otimes v_j,c_j(p_j))$ constitute particular examples of mapping of $C_\infty^1(\mathbf{U})$ and, therefore, are invariant with respect to (7.4). Thus, the construction of spaces (3.6) is invariant under the action of the group □

Finally, we give a proof of Lemma 2.

Proof. Since n is the same for both spaces in (3.8), it only remains to check that the conditions for (2.14) and (2.16) for $\Phi(v_1,c_j(p_1);\;\ldots;\;v_n,c_j(p_n))$ of composibility Definition 2.2 with vertex operators are stronger for $C_m^n(\mathbf{U})$ then for $C_{m-1}^n(\mathbf{U})$. In particular, in the first condition for (2.14) in definition of composability 5 the difference between the spaces in (3.8) is in indeces. Consider (7.1). For $C_{m-1}^n(\mathbf{U})$, the summations in idexes $k_1=l_1+\cdots+l_{i-1}+1$, ..., $k_i=l_1+\cdots+l_{i-1}+l_i$, for the coordinates $c_j(p_1)$, ..., $c_j(p_n)$ with $l_1,\ldots,l_n\in\mathbb{Z}_+$, such that $l_1+\cdots+l_n=n+(m-1)$, and vertex algebra elements $v_1,\ldots,v_{n+(m-1)}$ are included in summation for indexes for $C^n_m(\mathbf{U})$. The conditions for the domains of absolute convergence for $\mathcal{M}$, i.e., $|c_{l_1+\cdots+l_{i-1}+p} - \zeta_i| + |c_{l_1+\cdots+l_{j-1}+q} - \zeta_i| < |\zeta_i - \zeta_j|$, for $i,\ j = 1, \ldots, k,\ i \neq j$, and for $p = 1, \ldots, l_i$ and $q = 1, \ldots, l_j$, for the series (2.14) are more restrictive then for $m-1$. The conditions for $\mathcal{I}^n_{m-1}(\Phi)$ to be extended analytically to a rational function in $(c_1(p_1),\ \ldots,\ c_{n+(m-1)}(p_{n+(m-1)}))$, with positive integers $N^n_{m-1}(v_i, v_j)$, depending only on v_i and v_j for $i, j = 1, \ldots, k,\ i \neq j$, are included in the conditions for $\mathcal{I}^n_m(\Phi)$.

Similarly, the second condition for (2.16), of is absolute convergence and analytical extension to a rational function in $(c_1(p_1), \ldots, c_{m+n}(p_{m+n}))$, with the only possible poles at $c_i(p_i) = c_j(p_j)$, of orders less than or equal to $N^n_m(v_i, v_j)$, for $i, j = 1, \ldots, k$, $i \neq j$, for (2.16) when $c_i(p_i) \neq c_j(p_j),\ i \neq j\ |c_i(p_i)| > |c_k(p_k)| > 0$ for $i = 1, \ldots, m$, and $k = m+1, \ldots, m+n$ includes the same condition for $\mathcal{J}^n_{m-1}(\Phi)$. Thus, we obtain the conclusion of Lemma. $\square$

8. Appendix: Cohomological classes and characterization of complex curves

In this appendix we describe certain classes associated to the first and the second vertex algebra cohomologies of complex curves. It is a separate geometrical problem to introduce a product defined among elements of spaces $C^n_m(\mathbf{U})$ of (3.6). Nevertheless, even with such a product yet missing, it is possible to introduce the lower-level cohomological classes of the form $[\delta\eta]$. Let us give some further definitions.

Definition 33. We call a map

$$\Phi \in C^n_k(\mathbf{U}),$$

closed if it is a closed connection:

$$\delta^n_k \Phi = G(\Phi) = 0.$$

For $k \geq 1$, we call it exact if there exists $\Psi \in C^{n+1}_{k-1}(\mathbf{U})$ such that $\Psi = \delta^n_k \Phi$, i.e., Ψ is a form of connection.

For $\Phi \in C^n_k(V, W, \mathcal{M})$ we call the cohomology class of mappings $[\Phi]$ the set of all closed forms that differ from Φ by an exact mapping, i.e., for $\chi \in C^{n-1}_{k+1}(V, W, \mathcal{M})$,

$$[\Phi] = \Phi + \delta^{n-1}_{k+1}\chi.$$

As we will see in this section, there are cohomological classes, (i.e., $[\Phi]$, $\Phi \in C^1_m(\mathbf{U})$, $m \geq 0$), associated with two-point connections and the first cohomology $H^1_m(\mathbf{U})$, and classes (i.e., $[\Phi]$, $\Phi \in C^2_{ex}(\mathbf{U})$), associated with transversal connections and the second cohomology $H^2_{ex}(\mathbf{U})$, of $\mathcal{M}$.

Remark 4. That means that the actual functional form of $\Phi(v, z)$ (and therefore $\langle w', \Phi \rangle$, for $w' \in W'$) varies with various choices of $v \in V$.

In this section we consider a general formulation of characterizasion of a complex curve $\mathcal{M}$ by means of rational functions of invariants. Let us introduce further notations, for $n \geq 0$,

$$\mathbf{x} = (x_1, \ldots, x_n),$$

for n vertex algebra element, formal parameters, points, etc. Introduce

Definition 34. For an element $\Phi(\mathbf{v}, \mathbf{c}(\mathbf{p})) \in \mathcal{W}_{c_1(p_1),\dots,c_n(p_n)}$ let us call n-variable rational function valued form

$$\mathcal{R}(\mathbf{z}) = R_{\Phi(\mathbf{v})}(\mathbf{c}(\mathbf{p})), \tag{8.1}$$

the characteristic form.

We have used this form for the construction of chain complexes in Section 3.2. In certain cases, depending on properties of $\Phi(\mathbf{v}, \mathbf{z})$, one is able to compute this matrix element explicitely.

By varying vertex algebra elements v_i, one can vary the the form of dependence of $\Phi(\mathbf{v}, \mathbf{c}(\mathbf{p}))$ on $\mathbf{v}$, and, therefore, obtain various functions of $R(\mathbf{z})$. By using the freedom of choice of $v \in V$, we could try to find a suitable pattern for of $\Phi(\mathbf{v}, \mathbf{c}(\mathbf{p}))$ (as a functional of v), in such a way (8.1) would result to a specific differential form. Since $\Phi(\mathbf{v}, \mathbf{c}(\mathbf{p}))$ belongs to $C^n_m(\mathbf{U})$ for some n, m, it is important to mention that, due to our formulation in terms of matrix elements, (8.1), associated to cohomological invariants are supposed to be absolutely convergent in suitable domains of $\mathcal{M}$. One can also integrate (8.1) along (closed) paths either on $\mathcal{M}$. For that purpose we introduce

Definition 35. We call a multiple integral

$$F(\mathbf{z}') = \int_{(p_1)}^{(p_2)} \mathcal{R}\left(\mathbf{c}(\mathbf{p})\right), \tag{8.2}$$

the characteristic function for $\mathcal{M}$, where (p_i), $i = 1, 2$ denote limiting points of integration.

In Proposition 3 we proved, in particular, that elements of spaces $C^n_m(\mathbf{U}) \in \mathcal{W}_{z_1,\dots,z_n}$ are invariant with respect to changes of formal parameters $(z_1, \dots, z_n)$. In Definition 34 of a characteristic form we use such elements, and, therefore, (8.1), containing wt (v_i), $1 \le i \le n$, of corresponding differentials, is also invariant with respect to action of $\left(\mathrm{Aut}_{p_1}\, \mathcal{O}^{(1)} \times \dots \times \mathrm{Aut}_{p_n}\, \mathcal{O}^{(1)}\right)$.

8.1. **Characterization of curves by composibilty conditions.** Let us start with forms associated to the composibility conditions. For $l_1, \dots, l_n \in \mathbb{Z}_+$ such that $l_1 + \dots + l_n = n + m$, define $k_1 = l_1 + \dots + l_{i-1} + 1$, ..., $k_i = l_1 + \dots + l_{i-1} + l_i$. Consider a set of $p_{k_1}, \dots, p_{k_n}$ with local coordinates $c_{k_1}(p_{k_1}), \dots, c_{k_n}(p_{k_n})$, on $\mathcal{M}$ for points on $\mathcal{M}$. Then, for $v_1, \dots, v_{n+m} \in V$ and $w' \in W'$, one defines (2.11) and there exist positive integers $N^n_m(v_i, v_j)$ depending only on v_i and v_j for $i, j = 1, \dots, k$, $i \ne j$ such that the series (2.14) is absolutely convergent when for $l_p = l_1 + \dots + l_{i-1} + p$, $l_q = l_1 + \dots + l_{j-1} + q$,

$$|c_{l_p}(p_{l_p}) - \zeta_i| + |c_{l_q}(p_{l_q}) - \zeta_i| < |\zeta_i - \zeta_j|, \tag{8.3}$$

for $i, j = 1, \dots, k$, $i \ne j$ and for $p = 1, \dots, l_i$ and $q = 1, \dots, l_j$. Note that in (2.14) the original variables z_i are present in combinations (2.11) only, and the conditions on domains of convergence are express through such combinations $c_{l_p}(p_{l_p})$ and $c_{l_q}(p_{l_q})$, and some ζ_i which could be identified with other local coordinates on $\mathcal{M}$. Thus, we obtain an external (with respect to original coordinates) condition on $\mathcal{I}_m^n(\Phi)$. Geometrically this means that the sum of shifts in domains of convergence with respect to $c_{l_p}(p_{l_p})$ and $c_{l_q}(p_{l_q})$ are smaller than difference for other two points with local coordinates ζ_i and ζ_j. It is also assumed that the sum must be analytically extended to a rational function in $(c_1(p_1), \ldots, c_{m+n}(p_{m+n}))$, independent of $(\zeta_1, \ldots, \zeta_n)$, with the only possible poles at $c_i(p_i) = c_j(p_j)$, of order less than or equal to $N_m^n(v_i, v_j)$, for $i, j = 1, \ldots, k$, $i \neq j$.

Consider the second condition in Definition 5. For $v_1, \ldots, v_{m+n} \in V$, there exist positive integers $N_m^n(v_i, v_j)$, depending only on v_i and v_j, for $i, j = 1, \ldots, k$, $i \neq j$, such that for $w' \in W'$, such that (2.16) is absolutely convergent when $z_i \neq z_j$, $i \neq j$

$$|c_i(p_i)| > |c_k(p_k)| > 0, \tag{8.4}$$

for $i = 1, \ldots, m$, and $k = m+1, \ldots, m+n$, and the sum can be analytically extended to a rational function in $(z_1, \ldots, z_{m+n})$ with the only possible poles at $z_i = z_j$, of orders less than or equal to $N_m^n(v_i, v_j)$, for $i, j = 1, \ldots, k$, $i \neq j$,. Elements Φ of spaces $C_m^n(\mathbf{U})$ (3.6) are composable with m vertex operators, and, therefore possess properies described above. Due to absulute convergence in the regions (8.3) and (8.4) on $\mathcal{M}$, forms $I_m^n(\Phi)$ and $J_m^n(\Phi)$ locally characterize $\mathcal{M}$.

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