From Generic Differentiability to Galvin's Conjecture

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Outline

1. Generic differentiability
2. The Radon-Nikodym property
3. Fragmentability and σ-fragmentability
4. Generic continuity
5. Universal meagerness
6. Mazur's ordinal
7. Sierpinski's equivalence
8. Galvin's conjecture
9. Ideas from the proof
10. Things left to do

Generic differentiability

Let U be a nonempty open convex subset of a Banach space X. A convex continuous function $\varphi : U \to \mathbb{R}$ is *Fréchet differentiable* at $x \in U$ if there is $f \in X^*$ such that

$$\lim_{t \to 0} \frac{\varphi(x + th) - \varphi(x)}{t} = f(h)$$

uniformly on $\|h\| = 1$. We write $f = \varphi'(x)$.

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Theorem (Asplund 1968, Lindenstrauss 1962)

If a Banach space X has separable dual then every continuous convex function $\varphi : X \to \mathbb{R}$ is Fréchet differentiable at every point of a dense G_δ-subset of X.

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When $\varphi : X \to \mathbb{R}$ satisfies the conclusion of this theorem, we say φ is *generically Fréchet differentiable*.

Radon-Nikodym property

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We say that a Banach space X has the *Radon-Nikodym property* if for every finite measure space (S, Σ, μ) and every vector measure $\gamma : (S, \Sigma) \to X$ of bounded variation with $\gamma \ll \mu$ there is a Bochner integrable function $g : S \to X$ such that

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Theorem (Namioka-Phelps 1975, Stegall 1975)

The following are equivalent for a Banach space X:

1. *Every continuous convex real function on X is generically Fréchet differentiable.*
2. *The dual space X^* has the Radon-Nikodym property.*
3. *The weak* topology of every bounded subset of X^* is fragmented by the norm.*

Fragmentability

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A topological space (X, τ) is *fragmented by* a pseudo-metric ρ on X, if for every nonempty subset A of X and $\epsilon > 0$ there is $U \in \tau$ such that $U \cap A \neq \emptyset$ and ρ-diam$(U \cap A) < \epsilon$.

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If for every $\epsilon > 0$ we can find a countable partition $X = \bigcup_{n=0}^{\infty} X_n^\epsilon$ with the property the for every n and $A \subseteq X_n^\epsilon$ there is $U \in \tau$ such that $U \cap A \neq \emptyset$ and ρ-diam$(U \cap A) < \epsilon$ then X is said to be σ*-fragmented* by ρ.

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Remark

The weak topology w of the space c_0 is σ-fragmented by the norm but c_0 cannot be covered by countably many subsets Y with $w \restriction Y$ fragmented by the norm.

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BQ3. Which compact spaces K have the topology that the topology τ_p on $C(K)$ of pointwise convergence on K is σ-fragmented by the norm.

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Theorem (Namioka 1974)

Let K be a compact space and X either a complete metric space of a compact space. Then for every separately continuous function $f : X \times K \to \mathbb{R}$ there is a dense G_δ-subset G of X such that f is jointly continuous at every point of $G \times K$.

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The right assumption on X in this theorem is that it is *Čech complete*, i.e., that it is a G_δ-subset of (one, equivalently, all) of its compactification.

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Theorem (Jayne, Namioka, Rogers 1993)

Let (X, τ) be a topological space which is σ-fragmented by a lower-semicontinuous metric ρ. Then for every continuous map $f : Y \to (X, \tau)$ where Y is a Baire space, the corresponding map $f : Y \to (X, \rho)$ is continuous at each point of a dense G_δ-subset of Y.

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Definition

We say that the triple (X, τ, ρ) satisfies Namioka's generic continuity principle if the conclusion of the previous theorem holds.

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Conjecture (Namioka 1990-1992)

For every compact space K the generic continuity requirement on $(C(K), \tau_p, \|\cdot\|_\infty)$ implies that τ_p is σ-fragmented by the norm.

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Remark

If the function space $(C(K), \| \cdot \|_\infty$ has the generic Fréchet differentiability property then the compactum K must be scattered. So it is natural to first examine this case.

Universal meagerness

Theorem (Haydon 1989)

The topology τ_p *of pointwise convergence of* $C(K)$ *for* K *a scattered compactum is* σ*-fragmented by the norm if and only if the restriction of* τ_p *on the function subspace* $C(K,2)$ *of* $\{0,1\}$*-valued continuous maps on* K *is* σ*-scattered.*

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Theorem (Haydon 1989, Namioka-Pol 1992)

The function space $C(K)$ over a scattered compactum K satisfies the Namioka generic continuity requirement if and only if for every Baire space B every continuous map $f : B \to (C(K, 2), \tau_p)$ must be constant on a nonempty open subset of B.

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A topological space X is universally meager if every continuous map $f : B \to X$ from a Baire space B must be somewhere constant (constant on a nonempty open subset of B).

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Remark

Haydon [1989] writes:
"...it seems overly optimistic to think that the answer might be affirmative...".

Mazur's game and Baire spaces

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For a topological space X the *Mazur game* on X, $MG(X)$, is an infinite game with perfect information

E	U_0		U_2		$\cdots$
N		U_1		U_3	$\cdots$

where $U_0 \supseteq U_1 \supseteq U_2 \supseteq U_3 \supseteq \cdots$ are nonempty open subset of X, and where E wins if $\bigcap_{n=0}^{\infty} U_n = \emptyset$; otherwise N wins the infinite play.

Theorem (Banach-Mazur 1935, Oxtoby 1957)

A topological space X is a Baire space if and only if the empty player E does not have a winning strategy in the Mazur game played on X.

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$$x \cap \gamma = y \cap \gamma \ \text{ implies } \ x \in S \text{ iff } y \in S.$$

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S in $\Sigma(\delta)$ is *positive* if for every $f : \gamma^{<\omega} \to \gamma$ with $\gamma < \delta$ there is $x \in S$ such that $f[(x \cap \gamma)^{<\omega}] \subseteq x$.

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Let $MG(\delta)$ denote the game

I	S_0		S_2		$\cdots$
II		S_1		S_3	$\cdots$

where $S_0 \supseteq S_1 \supseteq S_2 \supseteq S_3 \supseteq \cdots$ forms an infinite decreasing sequence of positive elements of $\Sigma(\delta)$ and where we proclaim player I a winner if $\bigcap_{i=0}^{\infty} S_i = \emptyset$.

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An ordinal δ is a Mazur ordinal if it is regular and uncountable and if I does not have a winning strategy in $MG(\delta)$.

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Remark

Deep work of Foreman, Magidor, Shelah and Woodin can be used to show that many of the standard large cardinals are Mazur. For example, the large cardinals used in the 1980's by Martin and Steel to prove the Projective Determinacy are also Mazur.

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Theorem (T., 2007)

If a universally meager space X with point-countable base is dominated by some Mazur ordinal then it can be well-odered $(X, <_{wo})$ in such a way that all initial segments $\{x \in X : x <_{wo} y\}$ $(y \in X)$ are closed in X

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if there is a Mazur ordinal then every universally meager separable metric space must be countable.

A variety of Baire spaces

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Fix a space X with point-countable base. We may assume $X = (\gamma, \tau)$ for some $\gamma < \delta$ where δ is a Mazur ordinal. Consider the following element of the field $\Sigma(\delta)$

$$T = \{x \in [\delta]^\omega : \overline{x \cap \gamma} \neq x \cap \gamma\},$$

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Lemma (T., 2007)

If T is positive then there is a Baire space B and a nowhere constant continuous map $f : B \to X$.

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If T is positive then there is a Baire space B and a nowhere constant continuous map $f : B \to X$.

Lemma (Fleissner 1986)

If T is not positive there is a well-ordering of X with all initial segments closed.

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Theorem (T., 2007)

If a function space $\mathcal{C}(K)$ is smaller that some Mazur ordinal and if it satisfies the Namioka generic continuity requirement then the norm-density of every subset X of $\mathcal{C}(K)$ is equal to the supremum of lengths of well-ordered chains of pointwise-open subsets of X.

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Corollary (T., 2007)

If a function space $\mathcal{C}(K)$ is bounded by some Mazur ordinal and if it satisfies the Namioka generic continuity requirement then every pointwise hereditarily Lindelöf subspace of $\mathcal{C}(K)$ is norm-separable.

Sierpinski's expansion

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Definition

The Sierpinski expansion of $(\mathbb{R}, <)$ *is a structure of the form* $(\mathbb{R}, <, <_{wo})$ *where* $<_{wo}$ *is a well-ordering of* $\mathbb{R}$*. For a positive integer* k *the Sierpinski equivalence* $\mathcal{SE}_k$ *is an equivalence relation on* $[\mathbb{R}]^k$ *defined by letting two* k*-elements subsets* a *and* b *of* $\mathbb{R}$ *equivalent if they generate isomorphic substructures of the Sierpinski structure* $(\mathbb{R}, <, <_{wo})$*.*

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Theorem (Sierpinski 1933)

For every positive integer k *every one of the* $k!(k-1)!$ *equivalence class of* $\mathcal{SE}_k$ *on* $[\mathbb{R}]^k$ *is realized on every topological copy of* $\mathbb{Q}$ *in* $\mathbb{R}$*.*

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Question (Galvin 1970, 1986)

Does the Sierpinski expansion $(\mathbb{R}, <, <_{wo})$ solve the expansion problem for $(\mathbb{R}, <)$ relative to the topological copies of $\mathbb{Q}$ in $\mathbb{R}$?

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Problem (Galvin 1970, 1986)

Is it true that for every positive integer k and every finite colouring of $[\mathbb{R}]^k$ there is a subset X homeomorphic to $\mathcal{Q}$ such that $[X]^k$ has at most $k!(k-1)!$ colours

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Problem (Galvin 1970, 1986)

Is it true that for every finite colouring of $[\mathbb{R}]^2$ there is a subset X homeomorphic to $\mathcal{Q}$ such that $[X]^2$ has at most 2 colours

A general colouring problem

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Theorem (Baumgartner 1986, T.-Weiss 1995)

For every σ-discrete metric space X there is $c : [X]^2 \to \mathbb{N}$ such that $c[Y]^2 = \mathbb{N}$ for every $Y \subseteq X$ homeomorphic to $\mathbb{Q}$.

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Suposse X is a non σ-discrete metric space. Is it true that for every finite colouring of $[X]^2$ there is a topological copy $Y \subseteq X$ of $\mathbb{Q}$ such that $[Y]^2$ has at most 2 colours?

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Is this true when X is an uncountable separable metric space?

Arrow notation

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Let X and Y be topological spaces. For natural numbers $k, l, t \geq 1$, we write

$$X \to (Y)^k_{l,t}$$

to mean that for every set L of cardinality l and every coloring $c : [X]^k \to L$, there exist a subspace $Y' \subseteq X$ homeomorphic to Y and a subset $T \subseteq L$ of cardinality at most t such that $c[Y']^k \subseteq T$. If $t = 1$, then it is not recorded in this notation, i.e., we write $X \to (Y)^k_l$ instead of $X \to (Y)^k_{l,1}$.

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1. *There is no well-ordering $<_{wo}$ of X with all initial segments $\{x \in X : x <_{wo} y\}$ $(y \in X)$ closed.*
2. $X \to (\omega + 1)^2_2$.
3. $X \to (\omega + 1)^k_l$ *for all natural numbers $k, l \geq 1$.*

Problem

Suppose X is a space with point-countable bases with no well-ordering with all initial segments closed. Is it true that $X \to (\mathbb{Q})^2_{l,2}$ for every positive integer l?

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Corollary (Raghavan-T. 2018)

If there is a Mazur ordinal then for every uncountable separable metric space X, we have that $X \to (\mathbb{Q})^2_{l,2}$ for every integer $l \geq 1$.

Ideas from the proof

We start with $c : [X]^2 \to \{0, 1, ..., l\}$ and assume $X = (\gamma, \tau)$ for some $\gamma < \delta$ with δ a Mazur ordinal.

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For $i \leq l$, set

$$K_i = \{\{a, b\} \in [T]^2 : \gamma_a \neq \gamma_b \text{ and } c(\{\gamma_a, \gamma_b\}) = i\}.$$

From now on we let the variables a, b, c,... range over elements of T and variables A, B, C,... range over positive elements of the field $\Sigma(\delta)$ that are included in T.

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Definition

We say that (A, B) is (i, j)-saturated if for all $A' \subseteq A$ and $B' \subseteq B$, the sets

$$\{a \in A' : a \text{ is } i\text{-large on } B'\} \text{ and } \{b \in B' : b \text{ is } j\text{-large on } A'\}$$

are both positive.

Lemma

There exist (i, j) *and* C *such that for every* $D \subseteq C$ *there exist* $A, B \subseteq D$ *such that the pair* (A, B) *is* (i, j)*-saturated.*

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Definition

We say that $a \in A$ is a (i,j)-winner in A if there is an infinite sequence (A_n, B_n) of subsets of A such that for all n :

1. $A_{n+1}, B_{n+1} \subseteq A_n$,
2. $A_n, B_n \subseteq U_n^T(a)$,
3. (A_n, B_n) *is* (i,j)*-saturated,*
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For all positive A, the set

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Lemma

If a point $a \in A$ is an (i,j)-winner in A, then there is an infinite sequence B_n of subsets of A such that for all n:

1. $B_n \subseteq U_n^T(a)$,
2. (B_m, B_n) *is (i,j)-saturated for all* $m < n$,
3. $B_n \subseteq K_i(a)$.

Here, $U_n^T(a) = (U_n(\gamma_a))^T$

Construction scheme

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There is $Y \subseteq X$ *homeomorphic to* $\mathbb{Q}$ *such that* $c[Y]^2 \subseteq \{i, j\}$.

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$\mathbb{N}^{<\mathbb{N}}$ is the tree of finite sequence of integers ordered by endextension and $<_{lex}$ is its lexicographical ordering where $\sigma \supset \tau$ implies $\sigma <_{lex} \tau$. For a subtree P of $\mathbb{N}^{<\mathbb{N}}$ by $L(P)$ we denote its end nodes and

$$N(P) = P \setminus L(P).$$

A *construction scheme* for the required set Y homeomorphic to $\mathbb{Q}$ is an infinite increasing sequence P_m of downwards closed subtrees of $\mathbb{N}^{<\mathbb{N}}$ such that for all m :

(1) $P_0 = \{\emptyset\}$ and $\mathbb{N}^{<\mathbb{N}} = \bigcup_{m=0}^{\infty} P_m$,

(2) P_m is nonempty and has finite height,

(3) there is $\sigma_m \in L(P_m)$ such that

$$P_{m+1} = \{\sigma_m^\frown n : n \in \mathbb{N}\}.$$

We also recursively construct sequences

$$\{a_m : m \in \mathbb{N} \setminus \{0\}\} \subseteq [\delta]^\omega \text{ and } F_m : L(P_m) \to \Sigma(\delta) \ (m \in \mathbb{N})$$

such that for all m :

(4) $F_m(\sigma)$ is a positive subset of T for all $\sigma \in L(P_m,)$

(5) $a_{m+1} \in F_m(\sigma_m)$ is a (i, j)-winner in $F_m(\sigma_m)$,

(6) $m' \leq m$ and $\sigma \in L(P_{m'}) \cap L(P_m)$ imply $F_m(\sigma) \subseteq F_{m'}(\sigma)$.

(7) $\sigma, \tau \in L(P_m)$ and $\sigma <_{lex} \tau$ imply that $(F_m(\sigma), F_m(\tau))$ is (i, j)-saturated.

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Note that by the previous lemmas there is no difficulties constructing these objects. However, we need extra conditions that would guarantee that

$$[\{a_m : m \geq 1\}]^2 \subseteq K_i \cup K_j,$$

that $\gamma_{a_m} \neq \gamma_{a_{m'}}$ for $m \neq m'$, and that

$$Y = \{\gamma_{a_m} : m \geq 1\}$$

is a topolgical copy of $\mathbb{Q}$ in X.

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(8) $F_{m+1}(\sigma_m^\frown n) \subseteq F_m(\sigma_m) \cap U_m^T(a_{m+1})$ for all $n \in \mathbb{N}$,

(9) If $m' < m$, if $\sigma \in L(P_m)$ and if $\sigma <_{lex} \sigma_{m'}$ then

$$F_m(\sigma) \subseteq K_i(a_{m'+1}),$$

(10) if $m' < m$, if $\sigma \in L(P_m)$, and if $\sigma_{m'} <_{lex} \sigma$, then

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Remark

Note that in (9) and (10) we are proving that on our copy of $\mathbb{Q}$ the given colouring c is coarser than the Sierpinski colouring.

An application

Theorem (Raghavan-T., 2018)

Assume that there is a Mazur ordinal and let X be any uncountable set of reals.

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Assume that there is a Mazur ordinal and let X be any uncountable set of reals.Let $<$ be the usual ordering between the reals and let $<_{\texttt{wo}}$ be any well-ordering of X. Then for every binary relation $R \subseteq X^2$, there exists $Y \subseteq X$ homeomorphic to $\mathbb{Q}$ such that $R \cap Y^2$ is equal to one of the following relations restricted to Y :

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Things left to do

Problem

If X is an uncountable separable metric space, does

$$X \to (\mathbb{Q})^{k}_{l,k!(k-1)!}$$

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Prove that every universally meager space is σ-scattered.

Problem

Prove that the generic continuity requirement on a compact space K guarantees that the topology on $C(K)$ of pointwise convergence on K is σ-fragmented by the norm.

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