Continuous Strategy Markov Equilibrium in a Dynamic Duopoly with Capacity Constraints*

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The paper deals with an infinite horizon dynamic duopoly composed of price setting firms, producing differentiated products, with sales constrained by capacities that are increased or maintained by investments. We analyze continuous strategy Markov perfect equilibria, in which strategies are continuous functions of (only) current capacities. The weakest possible criterion of a renegotiation-proofness, called renegotiation-quasi-proofness, eliminates all equilibria in which some continuation equilibrium path in the capacity space does not converge to a Pareto efficient capacity vector giving both firms no lower single period net profit than some (but the same for both firms) capacity unconstrained Bertrand equilibrium.

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1. INTRODUCTION

Many infinite horizon, discrete time, deterministic oligopoly models involve physical links between periods, i.e., they are oligopolistic difference games. These links can stem, for example, from investment or advertising. In difference games, a current state, which is payoff relevant, should be taken into account by rational players when deciding on a current period action. If strategies depend only on a current payoff relevant state and not on which history led to it, they are called Markov. Application of the requirement of subgame perfection to Markov strategies forming a Nash equilibrium leads to the solution concept of a Markov perfect equilibrium. In many oligopolistic infinite horizon difference games (the unique) Markov perfect equilibrium is non-collusive. (See [6] for a typical example. Maskin and Tirole’s model of a dynamic Bertrand duopoly [7] is a notable exception.)

In the present paper we analyze Markov perfect equilibria in an infinite horizon, discrete time, dynamic duopoly, composed of price setting firms, discounting future profits, producing differentiated products, with sales constrained by capacities. In order to maintain next period capacities on their current level or to increase them, current period investments are needed. Thus, current capacities are payoff relevant state variables. We impose an additional requirement on firms’ strategies: they should be continuous in their arguments. In a Markov setting it means that each firm’s current period investment into capacity and price charged is a continuous function of a current period capacity vector. Nevertheless, an equilibrium strategy profile has to be immune against every possible unilateral deviation to any closed loop strategy.

The requirement of continuity of strategies was introduced into the analysis of infinite horizon games with discounting of payoffs by J. W. Friedman and L. Samuelson ([2], [3]). It is based on the view that punishments should "fit the crime." Two main arguments in favour of continuous strategies are [3]: they are more appealing to real human players than a Draconian punishment after a very small deviation; following a deviation, the convergence of continuous strategies to the original action profile (or, in a Markov setting, to the limit of the original sequence of action profiles) reflects an intuitively appealing rebuilding of trust. We think that in the case of oligopolistic infinite horizon games there is an additional, policy related, argument: since collusive behaviour can be viewed as a violation of antitrust laws, continuous strategies reduce the risk of attracting attention of an agency responsible for protection of competition.

Continuous Markov strategies have the plausible property that large changes in payoff relevant variables have large effects on current actions, minor changes in payoff relevant variables have minor effects on current actions, and changes in variables that are not payoff relevant have no effect on current actions. This is an improvement in comparison with Markov strategies without the requirement of continuity for which there is only the distinction between effect and no effect according to whether a variable that has changed is payoff relevant or not, so that minor changes in payoff relevant variables can have large effects on current actions. (See the discussion of minor causes and minor effects in [8]). It is an improvement also in comparison with continuous strategies without the Markov property, which allow changes in variables that are not payoff relevant to effect current actions.

The requirement that strategies be Markov is an application of Harsanyi’s and Selten’s [4] principle of invariance of (selected) equilibrium strategies with respect to isomorphism of games. (This fact is pointed out by Maskin and Tirole [8].) The latter principle requires that strategically equivalent games have identical solution (i.e., selected equilibrium or subset of equilibria). Applying it to a subgame perfect equilibrium of the analyzed difference game, it implies that any element of selected subset of equilibrium strategy profiles should prescribe the same play in all subgames that are strategically equivalent, i.e., in all subgames with the same initial state. The requirement of continuity of strategies is a strengthening of this principle. When two subgames are, from the strategic point of view, "close", i.e., an initial state of one of them is in a neighbourhood of an initial state of the other, a sequence of vectors of actions prescribed for the former should be in a neighbourhood (on the element-wise basis) of a sequence of vectors of actions prescribed for the latter. (More precisely: Consider two subgames with initial states, i.e. capacity vectors, $y \in Y$ and $y' \in Y$. Let the sequences of actions profiles that an equilibrium strategy profile prescribes in them if neither firm deviates be $\{(\rho_1^k, \eta_1^k, \rho_2^k, \eta_2^k)\}_{k=1}^{\infty}$ and $\{(\rho_1'^k, \eta_1'^k, \rho_2'^k, \eta_2'^k)\}_{k=1}^{\infty}$, where, for each positive integer k, ρ_1^k, ρ_2^k, $\rho_1'^k$, $\rho_2'^k$ are prices and η_1^k, η_2^k, $\eta_1'^k$, $\eta_2'^k$ are investment expenditures. Then, for each $k \in \{1, 2, ...\}$, $(\rho_1^k, \eta_1^k, \rho_2^k, \eta_2^k)$ converges to $(\rho_1'^k, \eta_1'^k, \rho_2'^k, \eta_2'^k)$ as y converges to y'.)

Restriction of attention to Markov perfect equilibria imposes three limitations on equilibrium strategies. First, counting of repetitions of a certain action vector is impossible. Therefore, after a profitable deviation (i.e., a deviation that, if the other firm ignored it, would increase a deviator's continuation average discounted net profit) the play has to pass (in general) through different action vectors lying on a punishment path before reaching a collusive action profile. Second, an action vector prescribed (by an equilibrium strategy profile) for the first period must be the same as an action vector prescribed when the initial state reappears after a deviation. Thus, even at the beginning of the game a collusive action vector can be only gradually approached (unless the initial state cannot result from a profitable unilateral deviation from any continuation equilibrium and does not lie on a punishment path). Third, the punishment path must be the same for both firms. Otherwise, if there were two different punishment paths, after some (unilateral) deviations from them it would be impossible to determine, only on the basis of a current capacity vector, from which of them a deviation took place. Adding the requirement of continuity of strategies leads to the fourth limitation. After a deviation as well as at the beginning of the game, the play can (in general) only converge to a collusive action vector, without reaching it in any finite time.

In equilibrium firms charge prices given by a capacity constrained Bertrand equilibrium. The reason for this is that a deviation in a price (when a capacity vector is consistent with the play prescribed by an equilibrium strategy profile) cannot be punished, because strategies depend only on capacities. Therefore, what matters most in any subgame, is an initial capacity vector and a sequence of investments by both firms, determining a sequence of capacity vectors, i.e., a "reduced Cournot form" of a subgame. This is a typical feature of models of oligopoly with capacity constrained price setters, first pointed out (in the framework of two-stage homogeneous good duopoly model) by Kreps and Scheinkman [5].

For discount factors close to one, the analyzed dynamic duopoly game has a multiplicity (indeed, a continuum) of continuous strategy Markov perfect equilibria. The reason for this is that, unlike in Maskin and Tirole’s [6] model of a dynamic Cournot duopoly, in every period there is a (nontrivial) payoff relevant state variable for each firm. This makes detection of a unilateral deviation possible (although identification of a deviator and an action vector from which a deviation took place is not possible).

Multiplicity of equilibria raises the issue of reduction of the set of them. We approach the latter on the basis of the weakest possible concept of a renegotiation-proofness, which we call renegotiation-quasi-proofness. A continuous strategy Markov perfect equilibrium s^0 is renegotiation-quasi-proof if there does not exist another continuous strategy Markov perfect equilibrium s such that renegotiation from s^0 to s makes both firms better off (i.e., increases their continuation average discounted net profit) in every subgame. (Since this is really a weak concept, we do not find it appropriate to call it simply "renegotiation-proofness." On the other hand, we cannot call it "weak renegotiation-proofness" because the latter name was used by Farrell and Maskin [1] for a different concept.) For discount factors close to one this concept is similar to the definition of a renegotiation-proofness used in Maskin's and Tirole's [7] paper on dynamic Bertrand duopoly. In their model a renegotiation is based on a change of the price vector that is, after a finite number of periods, infinitely repeated in every subgame. (A renegotiation away from an Edgeworth cycle is a switching to a collusive strategy profile prescribing, after a finite number of periods, an infinite repetition of a certain price vector.) In our case a renegotiation is based on a change of the capacity and the price vector to which the play in every subgame converges.

The set of strategy profiles to which firms are allowed to renegotiate is restricted here. We confine attention to renegotiation to other continuous strategy Markov perfect equilibria. This is natural. Since we assume (for reasons explained above) that firms will coordinate on an equilibrium of this type, there is no reason to assume that they will renegotiate to another equilibrium that is not of this type. Application of the criterion of renegotiation-quasi-proofness eliminates all (continuous strategy Markov perfect) equilibria in which some continuation equilibrium path (in the capacity space) does not converge to a Pareto efficient capacity vector giving both firms no lower single period net profit than they earn in some (but the same for both firms) capacity unconstrained Bertrand equilibrium. (Pareto efficiency of a capacity vector is defined here with respect to a profit vector resulting from selling outputs equal to capacities at prices determined by an inverse demand function and incurring costs of maintaining current capacities. See the following section for technical details.) Then we make an additional assumption that, for each capacity unconstrained Bertrand equilibrium, there is some neighbourhood of the vector of demands at it (in the capacity space) in which a capacity constrained Bertrand equilibrium is unique. Under this additional assumption, we show that for each Pareto efficient capacity vector y^*, giving both firms single period gross profit that is no lower than their single period gross profit in the capacity unconstrained Bertrand equilibrium (which is unique under the above additional assumption), there exists a renegotiation-quasi-proof continuous strategy Markov perfect equilibrium in which all continuation equilibrium paths (in the capacity space) converge to y^*. Thus, we obtain a characterization of the set of equilibria.

The paper is organized as follows. In the next section we formally describe the analyzed dynamic duopoly. In Section 3 we prove necessary conditions of the existence of a renegotiation-quasi-proof continuous strategy Markov perfect equilibrium of the analyzed duopolistic game. In Section 4 we prove sufficient conditions of its existence. Section 5 concludes.

2. THE DYNAMIC DUOPOLY

In this section we specify assumptions on the analyzed duopolistic industry and give definitions of game theoretic solution concepts that we use.

Assumption 1. The analyzed dynamic duopoly is composed of two firms, indexed by subscripts 1 and 2, each of them producing in every period $t \in \{1, 2, ...\}$ a single non-durable product differentiated from the product of the competitor. The two products are gross substitutes. Buyers will not become biased towards one of the firms.

Assumption 2. Output of each firm in every period is constrained by its capacity in that period. The set of all feasible capacities of firm $i \in \{1, 2\}$ is $Y_i = [0, y_i^{max}]$, $y_i^{max} > 0$.

We set $y^{max} = (y_1^{max}, y_2^{max})$. $Y = Y_1 \times Y_2$ is the capacity space, identified with the state space, and $y(t) = (y_1(t), y_2(t)) \in Y$ is a capacity (state) vector in period t. The initial capacity vector $y(1) \in Y$ is given. Capacity is increased or (fully or partially) maintained by capacity creating investments. We denote firm i's capacity creating investment expenditures in period t by $k_i(t)$. The set of all feasible investment expenditures of firm i is $K_i = [0, k_i^{max}]$, where k_i^{max} is the investment expenditure needed to increase capacity from 0 to y_i^{max}. (Such an increase in one period need not be economically sound. If it is not economically sound, this only confirms that firm i will not incur investment expenditures exceeding k_i^{max}.) The (minimum) investment expenditures needed to achieve capacity $y_i(t+1)$ when the current capacity is $y_i(t)$ are $\alpha_i(y_i(t), y_i(t+1))$, where α_i:$\{(y_i(t), y_i(t+1)) \in Y_i^2 | \; y_i(t+1) \geq \gamma_i y_i(t)\} \rightarrow [0, k_i^{max}]$ is firm i's investment function (γ_i is defined below).

Assumption 3. For each $i \in \{1, 2\}$, α_i is strictly increasing in its second argument, strictly decreasing in its first argument, and continuously differentiable in each argument (with left or right hand side derivatives at boundary of its domain) on its domain.

Assumption 4. For every $i \in \{1, 2\}$, each $y_i(t) \in Y_i$, and every $k_i(t) \in K_i$ the function $\alpha_i(y_i(t), y_i(t+1))$ is continuously invertible with respect to $y_i(t+1)$ and the inverse $\alpha_i^{-1}(y_i(t),$

$k_i(t))$ reflects the upper bound on firm i's capacity, i.e., $\alpha_i^{-1}(y_i(t), k_i(t)) \leq y_i^{max}$.

We set $\alpha^{-1}(y(t), (k_1(t), k_2(t))) = (\alpha_1^{-1}(y_1(t), k_1(t)), \alpha_2^{-1}(y_2(t), k_2(t)))$.

Assumption 5. For each $i \in \{1, 2\}$, there is $\delta_i \in (0, 1)$ such that

$$\alpha_i(y_i, y_i) = \delta_i y_i, \ \forall \ y_i \in Y_i, \ i \in \{1, 2\}, \tag{1}$$

i.e., δ_i is a depreciation rate.

Assumption 6. For each $i \in \{1, 2\}$, there is $\gamma_i \in (0, 1)$ such that $\alpha_i(y_i(t), \gamma_i y_i(t)) = 0$ for all $y_i(t) \in Y_i$.

Assumptions 3-6 are satisfied for investment functions usually employed in the literature, like linear investment function, or (if $y_1(1) > 0$ and $y_2(1) > 0$) Prescott's [9] investment function.

Let Q_i be the set of all feasible outputs of firm i and set $Q = Q_1 \times Q_2$. Since outputs are constrained by capacities, we have $Q_i = [0, y_i^{max}]$, $i \in \{1, 2\}$.

Assumption 7. For $i \in \{1, 2\}$, firm i's costs of producing output $q_i \in Q_i$ are $c_i q_i$, where $c_i > 0$.

In each period $t \in \{1, 2, ...\}$, firm $i \in \{1, 2\}$ charges price $p_i(t) \in P_i = [c_i, p_i^{max}]$ and incurs investment costs $k_i(t)$. (The prices p_i^{max} are defined below.) Thus, firm i's action vector in period t is $(p_i(t), k_i(t))$ and its action space is $P_i \times K_i$. We set $P = P_1 \times P_2$ and $p(t) = (p_1(t), p_2(t))$. Firm i discounts future revenues and costs by discount factor $\beta_i \in (0, 1)$ and we let $\beta = (\beta_1, \beta_2)$.

For $i \in \{1, 2\}$, $Z_i: P \times Y \rightarrow Q_i$ is firm i's capacity constrained demand function. We view the vector function $Z(p, y) = (Z_1(p, y), Z_2(p, y))$ as a primitive of the model.

Assumption 8. For each $i \in \{1, 2\}$, $Z_i(p, y) \in [0, y_i]$ and it is continuous in (p, y) on its domain. For every (p, y) from the closure of $P^+ \times Y^+$, the subset of $P \times Y$ on which $0 < Z_k(p, y) < y_k$ for both $k \in \{1, 2\}$, $Z_i(p, y)$ is strictly decreasing and concave in p_i, strictly increasing and concave in p_j, $j \neq i$, twice continuously differentiable with respect to p_1 and p_2 (with right or left hand side derivatives at boundaries of the closure of $P^+ \times Y^+$), mixed second partial derivative is non-positive, and $|\partial Z_i(p, y)/\partial p_i| > \partial Z_i(p, y)/\partial p_j$, $j \neq i$.

Assumption 9. $Z_i(p, y^{max}) < y_i^{max}$ for both $i \in \{1, 2\}$ and every $p \in P$.

Let $D_i:P \to Q_i$ be the (capacity unconstrained) demand function of firm $i \in \{1, 2\}$, i.e., $D(p) = (D_1(p), D_2(p)) = Z(p, y^{max})$.

Assumption 10. For each $i \in \{1, 2\}$, every $p \in P$, and each $y \in Y$ for which $D_i(p) > Z_i(p, y)$, $\partial Z_i(p, y)/\partial p_i = 0$. (An infinitesimal change of a price changes the capacity unconstrained demand only infinitesimally, so the capacity constraint remains binding).

The prices p_1^{max} and p_2^{max} are the lowest prices satisfying $D_1(p_1^{max}, p_2^{max}) = D_2(p_1^{max}, p_2^{max}) = 0$.

Since we restrict attention to Markov strategies, in equilibrium, for any vector of capacities, prices charged by the firms are equal to capacity constrained Bertrand equilibrium prices, at which capacity constrained demands equal capacity unconstrained ones (see the proof of Lemma 1 in Section 3 for details).

Assumption 8 implies that the inverse vector demand function $D^{-1}:Q \to [0, p_1^{max}] \times$

$[0, p_2^{max}]$, continuous in q, exists for each strictly positive $q \in Q$. For any other $q \in Q$, we set $D^{-1}(q)$ equal to the limit of the sequence $\{D^{-1}(q^n)\}_{n=1}^{\infty}$ for the sequence $\{q^n\}_{n=1}^{\infty}$ of strictly positive elements of Q converging to q. Taking into account Assumption 8, for each $i \in \{1, 2\}$, $D_i^{-1}(q)$ is twice continuously differentiable with respect to q_1 and q_2, strictly decreasing and concave in both q_1 and q_2, and mixed second partial derivative is non-positive at each $q \in Q$ with $D_i^{-1}(q) > 0$ (with left or right hand side derivatives at boundaries of Q).

Assumption 11. For $i \in \{1, 2\}$, $D_i(c_i, c_j) > 0$,

$$\text{if } D_2^{-1}(0, y_2) > c_2, \text{ then } D_1^{-1}(0, y_2) - c_1 - \delta_1 > 0, \tag{2a}$$

and

$$\text{if } D_1^{-1}(y_1, 0) > c_1, \text{ then } D_2^{-1}(y_1, 0) - c_2 - \delta_2 > 0. \tag{2b}$$

Assumption 12. Binding agreements between the firms are not possible.

The conditions (2a)-(2b) ensure that no firm can eliminate the other firm from the market. In other words, each firm, if it was originally a monopolist, would have to accommodate an entry of the other. Therefore, no firm can avoid an infinite horizon strategic interaction with the other.

Firm i's gross (of investment expenditures) profit in period t is

$$\pi_i[p(t), y(t)] = (p_i(t) - c_i)Z_i[p(t), y(t)], \qquad i \in \{1, 2\}, \tag{3}$$

and its net profit in period t is $\pi_i[p(t), y(t)] - k_i(t)$. We set $\pi[p(t), y(t)] = (\pi_1[p(t), y(t)], \pi_2[p(t), y(t)])$.

In the following text we refer to this dynamic duopoly as "G_{DD}" or "the game G_{DD}."

Assumption 13. For each firm, average discounted net profit is its payoff function in

G_{DD}.

We say that a capacity vector $y^+ \in Y$ is Pareto efficient if the net profit vector $\pi[D^{-1}(y^+), y^+] - (\delta_1 y_1^+, \delta_2 y_2^+)$ is Pareto efficient with respect to the set $\{\pi[D^{-1}(y), y] - (\delta_1 y_1, \delta_2 y_2) | y \in Y\}$. In other words, a capacity vector y^+ is Pareto efficient if there is no other feasible capacity vector y such that an infinite repetition of y and a price vector $D^{-1}(y)$ increases, in comparison with the infinite repetition of y^+ and the price vector $D^{-1}(y^+)$, average discounted net profit of both firms. (The last sentence expresses the criterion of weak Pareto efficiency, but in this case weak and strict Pareto efficiency coincide.) This interpretation of Pareto efficiency of a capacity vector plays important role in the arguments in the following sections.

A Markov strategy of firm i, s_i, is a function that assigns to each feasible state a price charged and an investment expenditures incurred by firm i, i.e., $s_i: Y \rightarrow P_i \times K_i$. Thus, a Markov strategy is a special case of a closed loop strategy. (We restrict here attention to pure strategies. We think that allowing for randomization between investment expenditures would not improve realism of the model. Since a pure strategy capacity constrained Bertrand equilibrium exists for all feasible capacity vectors, randomization between prices is not needed.) For each $i \in \{1, 2\}$, the function $\rho_i: Y \rightarrow P_i$ defines the first component and the function $\eta_i: Y \rightarrow K_i$ the second component of the vector function s_i. That is, $s_i = (\rho_i, \eta_i)$. We let $\rho = (\rho_1, \rho_2)$ and $\eta = (\eta_1, \eta_2)$. The set of all Markov strategies of firm i is denoted by S_i. We set $S = S_1 \times S_2$ and $s = (s_1, s_2) \in S$.

Average discounted net profit of firm i in a subgame with an initial state y, when firms follow a Markov strategy profile s, is

∞

$$\Pi_i(s, y) = (1 - \beta_i) \sum_{t=1} \beta_i^{t-1}\{\pi_i[\rho(y(t)), y(t)]$$

$$- \eta_i[y(t)]\}, \; i \in \{1, 2\}, \tag{4}$$

where $y(1) = y$ and $y_i(t+1) = \alpha_i^{-1}[y_i(t), \eta_i(y(t))]$ for each $i \in \{1, 2\}$ and every positive integer t. (Without loss of generality, we can number the first period of a subgame by one.) We set $\Pi(s, y) = (\Pi_1(s, y), \Pi_2(s, y))$.

A Markov perfect equilibrium of G_{DD} is a profile of Markov strategies that yields a Nash equilibrium in every subgame of G_{DD}. The following definition expresses this formally.

DEFINITION 1. A profile of Markov strategies $s \in S$ is a Markov perfect equilibrium of G_{DD} if, for every state $y \in Y$, for each firm $i \in \{1, 2\}$, for $j \in \{1, 2\}\backslash\{i\}$, and for every strategy $s_i' \in S_i$, $\Pi_i(s, y) \geq \Pi_i((s_i', s_j), y)$.

If one of the firms uses Markov strategy then there is a best response of the other firm against it (chosen from the whole set of its closed loop strategies) that is also a Markov strategy. Therefore, a Markov perfect equilibrium is still a subgame perfect equilibrium when the Markov restriction is not imposed.

Firm i's Markov strategy s_i is continuous if it is a continuous function from Y to P_i x K_i. The set of all continuous Markov strategies of firm i is denoted by S_i^* and we let $S^* = S_1^*$ x S_2^*.

A continuous strategy Markov perfect equilibrium of G_{DD} is a profile of continuous Markov strategies that yields a Nash equilibrium in every subgame of G_{DD}. For the sake of completeness we state the following formal definition.

DEFINITION 2. A profile of continuous Markov strategies $s \in S^*$ is a continuous strategy Markov perfect equilibrium of G_{DD} if, for every state $y \in Y$, for each firm $i \in \{1, 2\}$, for $j \in \{1, 2\} \backslash \{i\}$, and for every strategy $s_i' \in S_i$, $\Pi_i(s, y) \geq \Pi_i((s_i', s_j), y)$.

Note that here we (have to) require explicitly that a continuous strategy Markov perfect equilibrium is immune to all unilateral deviations to a Markov strategy that is not continuous.

We conclude this section by the definition of the concept of a renegotiation-quasi-proof continuous strategy Markov perfect equilibrium.

DEFINITION 3. A continuous strategy Markov perfect equilibrium $s^0 \in S^*$ of G_{DD} is renegotiation-quasi-proof if there does not exist a continuous strategy Markov perfect equilibrium $s \in S^*$ of G_{DD} such that $\Pi_i(s, y) > \Pi_i(s^0, y)$ for both $i \in \{1, 2\}$ and for every $y \in Y$.

That is, $s^0 \in S^*$ is renegotiation-quasi-proof if there is no continuous strategy Markov perfect equilibrium $s \in S^*$ such that, by jointly switching from s^0 to s, both firms increase their average discounted net profit in every subgame of G_{DD}. As it is usual in the literature on renegotiation-proofness (e.g. [1], [7]), the set of strategy profiles to which firms are allowed to renegotiate is restricted here. Since we assume (for reasons explained in the Introduction) that firms will coordinate on a continuous strategy Markov perfect equilibrium, there is no reason to assume that they will renegotiate to an equilibrium that is not of this type. The use of weak rather than strict Pareto efficiency in Definition 3 is also in line with the bulk of the literature on renegotiation-proofness. (Nevertheless, our results would still hold if we used strict Pareto efficiency.)

The concept of a renegotiation-quasi-proofness used here is the weakest possible notion of a renegotiation-proofness. A renegotiation increasing average discounted net profit of both firms in every subgame is the most tempting of all imaginable renegotiation opportunities. If it is possible then we can hardly expect that firms will stick to the original equilibrium. Nevertheless, this weak concept is strong enough to ensure that all continuation equilibrium paths (in the capacity space) converge to a Pareto efficient capacity vector. Moreover, the use of a stronger concept of renegotiation-proofness would not strengthen our results.

Since the concept of a renegotiation-proofness that we use is so weak we do not find it appropriate to call it simply "renegotiation-proofness." Nor we can use the term "weak renegotiation-proofness" because the latter was already coined by Farrell and Maskin [1] for a different concept. Therefore we use (somewhat unwieldy) term "renegotiation-quasi-proofness."

For discount factors close to one the concept of a renegotiation-quasi-proofness is similar to the concept of renegotiation-proofness in Maskin’s and Tirole’s [7] paper on dynamic Bertrand duopoly. In their model a renegotiation is based on a change of the price vector that is, after a finite number of periods, infinitely repeated in every subgame. (A renegotiation away from an Edgeworth cycle is a switching to a collusive strategy profile prescribing, after a finite number of periods, an infinite repetition of a certain price vector.) In our case a renegotiation is based on a change of the capacity and price vectors to which the play in every subgame converges.

3. NECESSARY CONDITIONS

In this section we give necessary conditions (in Proposition 1) of the existence of a renegotiation-quasi-proof continuous strategy Markov perfect equilibrium of G_{DD}. We show that, in each renegotiation-quasi-proof continuous strategy Markov perfect equilibrium, all continuation equilibrium paths (in the capacity space) converge to a Pareto efficient capacity vector. We specify also other requirements that a limit of all continuation equilibrium paths (in the capacity space) of a renegotiation-quasi-proof continuous strategy Markov perfect equilibrium must satisfy.

As already noted in the Introduction, at each state prices charged form a capacity constrained Bertrand equilibrium. Therefore we first define the latter concept and give (in Lemma 1) some of its properties.

For each $i \in \{1, 2\}$ the function $\zeta_i{:}Q_j \to Q_i$, $j \neq i$, is firm i's Cournot reaction function, i.e., $\zeta_i(q_j) = \text{argmax}\{(D_i^{-1}(q_i, q_j) - c_i)q_i \,|\, q_i \in Q_i\}$. Assumptions 7, 8, 9, and 11 imply that, for both $i \in \{1, 2\}$, a function ζ_i is well defined, strictly decreasing, and $\zeta_i(q_j) \in (0, y_i^{max})$ for each $q_j \in Q_j$ such that $D_j^{-1}(q_j, 0) > c_j$.

For any $y \in Y$, a capacity constrained Bertrand equilibrium $p^B(y)$ is a price vector that is a pure strategy Nash equilibrium of the one shot duopoly game $G_D'(y)$ with the space of feasible strategy profiles equal to P and profits $(p_i - c_i)Z_i(p, y)$, $i \in \{1, 2\}$, as payoff functions. We use the term "unconstrained Bertrand equilibrium" to refer to a standard concept of Bertrand equilibrium, i.e., to a pure strategy Nash equilibrium of the game $G_D'(y^{max})$. (Assumption 9 implies that, for both $i \in \{1, 2\}$, firm i's output in any capacity unconstrained Bertrand equilibrium is lower than y_i^{max}. Also, for each $p^B(y^{max})$, $D_i[p^B(y^{max})] < y_i^{max}$ for both $i \in \{1, 2\}$, so every $p^B(y^{max})$ is a capacity unconstrained Bertrand equilibrium.) The symbol P^B denotes the correspondence P^B:Y $\rightarrow$ P, which assigns to each capacity vector y $\in$ Y the set of capacity constrained Bertrand equilibria (i.e., the set of Nash equilibria of $G'_D(y)$ in pure strategies). The properties of a capacity constrained Bertrand equilibrium needed for our arguments are given by the following lemma.

LEMMA 1. *(a) $P^B(y)$ is nonempty for each $y \in Y$.*

(b) The correspondence P is upper hemicontinuous at each $y \in Y$.

(c) $D_i[p^B(y)] \in [\min\{y_i, \zeta_i(D_j(p^B(y)))\}, y_i]$ for every $y \in Y$, every $p^B(y) \in P^B(y)$, each $i \in \{1, 2\}$, and $j \in \{1, 2\}\backslash\{i\}$.

(d) If, at some $y \in Y$, $y_i \leq \zeta_i(y_j)$, $j \neq i$, for both $i \in \{1, 2\}$, then $P^B(y)$ is a singleton and its only element is $p^B(y) = D^{-1}(y)$.

(e) For every $y \in Y$, for which $p^B(y) \in P^B(y)$ satisfying $D_i[p^B(y)] = y_i$ for both $i \in \{1, 2\}$ exists, $P^B(y)$ is a singleton.

(f) If, for some $y \in Y$ and $p^B(y) \in P^B(y)$,

$$D_i[p^B(y)] + [p_i^B(y) - c_i]\frac{\partial D_i[p^B(y)]}{\partial p_i} = 0, \qquad i = 1, 2, \qquad (5)$$

then $p^B(y) \in P^B(y^{max})$.

(g) If $y' \in Y$, $y'' \in Y$, $D^{-1}(y') \in P^B(y')$, $D^{-1}(y'') \in P^B(y'')$, and $y'_i \leq y''_i$ for both $i \in \{1, 2\}$ with at least one of these inequalities strict, then, for any $y \in [y',y'']$, $P^B(y) = \{D^{-1}(y)\}$. Moreover, for each $y \in [y', y'']\backslash\{y''\}$, $D^{-1}(y)$ satisfies

Proof. (a) The existence follows from the fact that strategy spaces are non-empty,

$$y_i + [D_i^{-1}(y) - c_i]\frac{\partial D_i[D^{-1}(y)]}{\partial p_i} < 0, \quad i = 1, 2. \quad (6)$$

compact, and convex intervals and payoff functions are continuous in strategy profiles and quasiconcave in firms' own strategies on P. (Each firm's payoff function is strictly concave in its own price at every $p \in P$ such that $(p, y) \in P^+ \times Y^+$. This implies quasiconcavity in firm's own price on whole P.)

(b) For every $y \in Y$, each $p^B(y) \in P^B(y)$ satisfies, for both $i \in \{1, 2\}$, $p_i^B(y) > c_i$, $D_i[p^B(y)] > 0$ (because $D_i(c_i, c_j) > 0$), $D_i[p^B(y)] \leq y_i$ (because otherwise firm i could increase its price without decreasing a capacity constrained demand for its product), $p_i^B(y) < p_i^{max}$ (because $D_i(p_1^{max}, p_2^{max}) = 0$), and first order conditions (which are both necessary and sufficient for $p^B(y)$ being a capacity constrained Bertrand equilibrium)

$$D_i[p^B(y)] + [p_i^B - c_i]\frac{\partial D_i[p^B(y)]}{\partial p_i} \leq 0, \qquad (7a)$$

and

$$[D_i[p^B(y)] + [p_i^B(y) - c_i]\frac{\partial D_i[p^B(y)]}{\partial p_i}][D_i[p^B(y)] - y_i] = 0. \quad (7b)$$

Consider a sequence $\{y^n\}_{n=1}^{\infty} \in Y^{\infty}$ converging to $y' \in Y$ and a sequence $\{p^{Bn}\}_{n=1}^{\infty} \in P^{\infty}$ converging to $p^{B'} \in P$, with $p^{Bn} \in P^B(y^n)$ for each finite positive integer n. Each p^{Bn} must satisfy conditions (7a)-(7b). Since left hand sides of both of them are continuous in (y^n, p^{Bn}), (7a) is in the form of weak inequality and (7b) in the form of equality, they must be satisfied also at $(p^{B'}, y')$.

(c) If, for some $i \in \{1, 2\}$ and $j \in \{1, 2\}\backslash\{i\}$, we had $D_i[p^B(y)] < \min\{y_i, \zeta_i[D_j(p^B(y))]\}$ then, for arbitrarily small $\varepsilon > 0$, firm i's upward deviation in output to $D_i[p^B(y)] + \varepsilon$ in a Cournot setting would increase its profit. Firm i's downward deviation in price in a Bertrand setting to p_i' satisfying $D_i(p_i', p_j^B(y)) = D_i[p^B(y)] + \varepsilon$ would give it higher profit than the Cournot deviation to $D_i[p^B(y)] + \varepsilon$, because $D_j(p_i', p_j^B(y)) < D_j[p^B(y)]$, which implies that $p_i' > D_i^{-1}[D_i(p^B(y)) + \varepsilon, D_j(p^B(y))]$. Therefore, we must have $D_i[p^B(y)] \geq \min\{y_i, \zeta_i[D_j(p^B(y))]\}$. The upper bound on $D_i[p^B(y)]$ is obvious.

(d) The claim follows straightforwardly from part (c) of this lemma and the fact that, for both $i \in \{1, 2\}$, ζ_i is strictly decreasing in y_j, $j \neq i$, at each $y_j \in Y_j$ with $D_j^{-1}(y_j, 0) > c_j$.

(e) Suppose that $P^B(y)$ has at least two elements and let $p^{B}{}'(y) \in P^B(y)\backslash\{p^B(y)\}$. Then there is $i \in \{1, 2\}$ with $D_i[p^{B}{}'(y)] < y_i$. This implies that $p_j^{B}{}'(y) > p_j^B(y) = D_j^{-1}(y)$ for both $j \in \{1, 2\}$. Nevertheless, then (with respect to Assumption 8 and its implications for derivatives of capacity unconstrained demand functions) for firm i the left hand side of (7a) is strictly negative and the left hand side of (7b) is strictly positive, which is a violation of first order conditions for capacity constrained Bertrand equilibrium. Thus, $P^B(y) = \{D^{-1}(y)\}$.

(f) The claim follows from the fact that the conditions (5) are sufficient for $p^B(y) \in P^B(y^{max})$. (g) Both $D^{-1}(y')$ and $D^{-1}(y'')$ must satisfy (7a). Moving along the line segment $[y', y'']$, both coordinates of $D^{-1}(y)$ are strictly decreasing, which implies that the right hand side of (7a) is strictly increasing for both firms. This and the fact that (7a) holds for both firms at $D^{-1}(y'')$ imply that (7a) must hold for both firms at $D^{-1}(y)$ at any y from the interior of $[y', y'']$, as well as the last claim of this part of the lemma. For every $y \in [y', y'']$, $D^{-1}(y)$ satisfies (7b). Therefore, $D^{-1}(y) \in P^B(y)$ for each $y \in [y, y'']$. Uniqueness follows from part (e) of this lemma. ■

The following Proposition establishes, for the case of both discount factors close to one, several properties of a renegotiation-quasi-proof continuous strategy Markov perfect equilibrium. It shows that all continuation equilibrium paths in the capacity space converge to the same (but, in general, different for different renegotiation-quasi-proof continuous strategy Markov perfect equilibria) capacity vector. It gives lower bounds on equilibrium average discounted net profits. Finally, it states restrictions on the shape of continuation equilibrium paths.

PROPOSITION 1. *There exists a vector of discount factors* $\beta^0 \in (0, 1)^2$ *such that for all* $\beta \in (\beta_1^0, 1) \times (\beta_2^0, 1)$ *the following property holds: Let* s^* *be a renegotiation-quasi-proof continuous strategy Markov perfect equilibrium of* G_{DD}. *Then:*

(a) All its continuation equilibrium paths in the capacity space converge to a capacity vector y^* *that is Pareto efficient.*

(b) There exists $p^B(y^{max}) \in P^B(y^{max})$ *such that*

$$\pi_i[D^{-1}(y^*), y^*] - \delta_i y_i^* \geq \pi_i[p^B(y^{max}), D(p^B(y^{max}))]$$
$$- \delta_i D_i[p^B(y^{max})], \qquad i \in \{1, 2\}, \tag{8}$$

and

$$\Pi_i(s^*, y(1)) \geq \pi_i[p^B(y^{max}), D(p^B(y^{max}))]$$
$$- \delta_i D_i[p^B(y^{max})], \; i \in \{1, 2\}. \tag{9}$$

(c) In each continuation equilibrium, the relation between the two firm's capacities in some small right hand side neighbourhood $O(y_1^*)$ *of* y_1^* *can be described by a strictly increasing continuous function* $f{:}O(y_1^*) \rightarrow Y_2$. *The functions* $[D_1^{-1}(y_1, f(y_1)) - c_1 - \delta_1]y_1$ *and* $[D_2^{-1}(y_1, f(y_1)) - c_2 - \delta_2]f(y_1)$ *(whose domain is* $O(y_1^*)$*) are strictly decreasing at* y_1^*.

Proof. (a) Let Θ be the union of all continuation equilibrium paths in the capacity space (i.e., $\Theta \subset Y$) with the following property: After a unilateral deviation from any $y \in \Theta$, s^* prescribes a movement back along a continuation equilibrium path from which a deviation took place. (If a unilateral deviation from $y \in Y$ cannot be punished without an increase in capacity of the firm that did not deviate, then $y \in \Theta$. This implies that $\Theta \neq \emptyset$.) Denote by cl(Θ) the closure of Θ. The set cl(Θ) must be a single connected compact curve in Y, not containing loops. If it was a union of two or more disjoint connected curves in Y (or if it contained a loop), it would not be possible to determine, only on the basis of a current state, from which of them (or from which part of the loop) a unilateral deviation took place. (Thus, an action profile prescribed by an equilibrium strategy profile would not depend on the state from which a unilateral deviation took place.) Therefore, for at least one of the connected subsets of cl(Θ) (or for at least one part of the loop), the set of states from which the play switches to it would have to be open, which would contradict continuity of strategies. The claim that cl(Θ) must be connected follows from continuity of strategies. The fact that cl(Θ) is a single connected compact curve in Y implies that all continuation equilibrium paths in Y must converge to the same capacity vector. Denote it by y^*.

If y^* was not Pareto efficient then, for discount factors less than but close enough to one, average discounted net profit of both firms in every subgame could be increased by replacing the limit capacity vector by a capacity vector in its neighbourhood strictly Pareto dominating it. (Pareto efficiency of the limit capacity vector implies that at any capacity vector y' from a small neighbourhood of the latter the condition of part (d) of Lemma 1 is satisfied, so at y' the price vector prescribed by an equilibrium strategy profile, which must be a capacity constrained Bertrand equilibrium, is $D^{-1}(y')$.)

(b) Denote the other (than y^*) end point of the curve cl(Θ) by y^0. When β_i

approaches one, firm i's continuation equilibrium average discounted net profit in any subgame approaches the left hand side of (8). (Pareto efficiency of y^* and part (d) of Lemma 1 imply that $P^B(y^*) = \{D^{-1}(y^*)\}$.) A unilateral deviation by firm i at state y^0 must be punished without an increase in capacity of firm $j \neq i$. (It follows from the definition of Θ.) For the sake of an argument leading to a proof of this part of Proposition 1, we can assume that there is a firm $k \in \{1, 2\}$ for which at $\rho(y^0)$ (i.e., at a capacity constrained Bertrand equilibrium prescribed by s^* at y^0) (7a) holds as equality. (If it is not the case we can prolong cl(Θ) by adding a line segment with a strictly positive slope to it. An argument similar to the one in the proof of part (g) of Lemma 1 shows that for some length of an added line segment we reach a new y^0 at which, for some $k \in \{1, 2\}$, $\rho(y^0)$ satisfies (7a) as equality.) For firm $i \neq k$, when β_i approaches one, its average discounted net profit from a permanent deviation at the state y^0 (starting at the state y^0) would approach a number no lower than

$$\min\{\pi_i[p^B(y_i^{max}, y_j^0), D(p^B(y_i^{max}, y_j^0))] - \delta_i D_i[p^B(y_i^{max}, y_j^0)] \mid p^B(y_i^{max}, y_j^0) \in P^B(y_i^{max}, y_j^0), j \neq i\}. \quad (10)$$

There is $p^{B'} \in P^B(y^{max})$ with $D_i(p^{B'}) \geq y_i^0$ such that the expression (10) is no lower than $\pi_i[p^{B'}, D(p^{B'})] - \delta_i D_i(p^{B'})$. If (8) did not hold for this $p^{B'}$ and firm i, for β_i close enough to one firm i could increase its continuation average discounted net profit (in comparison with its continuation equilibrium average discounted net profit) in a subgame of G_{DD} with the initial state y^0 by a permanent unilateral deviation. If (9) did not hold for that $p^{B'}$ and firm i, firm i could increase, for β_i close enough to one, its average discounted net profit in the whole game above $\Pi_i(s^*, y(1))$.

For firm k, when β_k approaches one, its average discounted net profit from a permanent (upward or downward) deviation in capacity at the state y^0 (starting at the state y^0) could approach a number no lower than $\max\{\pi_k(\rho(y^0), y^0) - \delta_k y_k^0, \pi_k[p^{B'}, D(p^{B'})] - \delta_k D_k(p^{B'})\}$.

Thus, (8) and (9) follow.

(c) Small unilateral upward deviations in capacity at the state y^* must be punished by switching to a capacity vector with both coordinates strictly exceeding those of y^*. Thus, with respect to continuity of strategies, in some small right hand side neighbourhood of y_1^*, $O(y_1^*)$, $cl(\Theta)$ must lie above the Pareto efficient frontier in the capacity space and it must have strictly positive slope. Therefore, in $O(y_1^*)$ $cl(\Theta)$ can be described by a strictly increasing function f. Since $cl(\Theta)$ is a single connected compact curve, $O(y_1^*)$ can contain neither points at which a limit of f does not exist nor points at which a limit of f differs from a value of f. Also, f is defined at each point of $O(y_1^*)$. Thus, f is a continuous function. The last claim of this part of the proposition is obvious. Otherwise, small unilateral upward deviations in capacity at the state y^* could not be punished. ■

4. SUFFICIENT CONDITIONS

In this section we give sufficient conditions of the existence of a renegotiation-quasi-proof continuous strategy Markov perfect equilibrium of G_{DD}. We characterize the set of limits of all continuation equilibrium paths of all such equilibria. We show that, under one additional assumption, Pareto efficiency and conditions analogous to (8), but expressed in terms of gross rather than net profits, are sufficient for a capacity vector to be a limit of all continuation equilibrium paths in the capacity space of some renegotiation-quasi-proof continuous strategy Markov perfect equilibrium of G_{DD}.

PROPOSITION 2. *Assume that* $y^* \in Y$ *is a Pareto efficient capacity vector and there* *exists* $p^B(y^{max}) \in P^B(y^{max})$ *such that*

$$\pi_i[D^{-1}(y^*), y^*] \geq \pi_i[p^B(y^{max}), y^{max}], \qquad i \in \{1, 2\}. \tag{11}$$

Also assume that, for each $p^B(y^{max}) \in P^B(y^{max})$*, there is a neighbourhood* $O[D(p^B(y^{max}))]$ *of* $D[p^B(y^{max})]$ *such that, for every* $y \in O[D(p^B(y^{max}))]$*,* $P^B(y)$ *is a singleton. Then there exists a vector of discount factors* $\beta^0 \in (0, 1)^2$ *such that, for all vectors of discount factors* $\beta \in [\beta^0_1, 1) \times [\beta^0_2, 1)$*, there exists a renegotiation-quasi-proof continuous strategy Markov perfect equilibrium of* G_{DD}*, whose all continuation equilibrium paths in the capacity space converge to* y^*.

Proof. Preliminaries. Let $y^B = D[p^B(y^{max})]$, where $p^B(y^{max})$ satisfies (11). Note that y^B must lie above the Pareto efficient frontier. It cannot lie on the Pareto efficient frontier. (If y is Pareto efficient, it lies below both firms' Cournot reaction curves, so (7a) is satisfied as a strict inequality at $D^{-1}(y)$ for both firms. Assumption 9 rules out the case $y^B = y^{max}$.) Suppose that it lies below the Pareto efficient frontier. Then at $p^B(y^{max})$ (7a) must be satisfied as equality for both firms. Take a Pareto efficient capacity vector y' with $y'_j > y^B_j$ for both $j \in \{1, 2\}$. We have $P^B(y') = \{D^{-1}(y')\}$. (Pareto efficiency of y' implies that the conditions of part (d) of Lemma 1 are satisfied.) This contradicts the last claim of part (g) of Lemma 1.

Let $f: Y_1 \to Y_2$, be a linear function with $f(y^*_1) = y^*_2$ whose first derivative equals

$$-\frac{D_1^{-1}(y^*) - c_1 - \delta_1 + y_1^* \dfrac{\partial D_1^{-1}(y^*)}{\partial y_1}}{y_1^* \dfrac{\partial D_1^{-1}(y^*)}{\partial y_2}} =$$

$$- \frac{y_2^* \dfrac{\partial D_2^{-1}(y^*)}{\partial y_1}}{D_2^{-1}(y^*) - c_2 - \delta_2 + y_2^* \dfrac{\partial D_2^{-1}(y^*)}{y_2}}. \qquad (12)$$

The equality of the two expressions follows from Pareto efficiency of y^*. Pareto efficiency of y^* also implies that f is strictly increasing. The fact that the derivative of f equals the expressions given in (12) ensures that the functions

$$[D_1^{-1}(y_1, f(y_1)) - c_1 - \delta_1]y_1 \qquad (13a)$$

and

$$[D_2^{-1}(y_1, f(y_1)) - c_2 - \delta_2]f(y_1), \qquad (13b)$$

whose domain is $[y_1^*, y_1^{max}]$, are strictly decreasing at y_1^*. Moreover, they are strictly decreasing on their whole domain. This follows from properties of partial derivatives of inverse demand functions implied by Assumption 8.

Let y_1^0 be the lowest y_1 satisfying for some $k \in \{1, 2\}$

$$D_k(p) + (p_k - c_k)\frac{\partial D_k(p)}{\partial p_k} = 0, \qquad (14)$$

where $p = D^{-1}[y_1, f(y_1)]$. Set $y^0 = (y_1^0, f(y_1^0))$. The last claim of part (g) of Lemma 1 implies that $y_i^0 > y_i^B$ and $y_j^0 < y_j^B$, $i \neq j$, or $y^0 = y^B$.

For each $j \in \{1, 2\}$ and $i \neq j$, there is a left hand side neighbourhood $\Omega_L(y_j^B)$ of y_j^B, a right hand side neighbourhood $\Omega_R(y_i^B)$ of y_i^B, and a continuous strictly decreasing function $\omega_j:\Omega_L(y_j^B) \to \Omega_R(y_i^B)$ such that, for each $y_j \in \Omega_L(y_j^B)$, $P^B((y_j, \omega_j(y_j))) = \{D^{-1}((y_j, \omega_j(y_j)))\}$, and at $D^{-1}((y_j, \omega(y_j)))$ (7a) is satisfied as equality for firm i. This can be seen as follows. Assumption 9 implies that, for each $y_j \in \Omega_L(y_j^B)\backslash\{y_j^B\}$, there is $y_i \in (y_i^B, y_i^{max})$ such that at

$D^{-1}((y_j, y_i))$ (7a) is satisfied as equality for one of the firms. It cannot be satisfied as equality for firm j, because then we would have $D^{-1}((y_j, y_i)) \in P^B((y_j^B, y_i))$, contradicting the assumption of the uniqueness of a capacity constrained Bertrand equilibrium in some small neighbourhood of y^B. Thus, at $D^{-1}((y_j, y_i))$ (7a) must be satisfied as equality for firm i. We set $\omega_j(y_j) = y_i$. Assumption 8 implies that when y_j is decreasing, $\omega_j(y_j)$ must be increasing. Uniqueness of a capacity constrained Bertrand equilibrium at $(y_j, \omega_j(y_j))$ follows from part (e) of Lemma 1. To see continuity of ω_j, consider a sequence $\{y_j^n\}_{n=1}^{\infty}$ of numbers from the domain of ω_j converging to y_j from its domain. A corresponding sequence $\{\omega_j(y_j^n)\}_{n=1}^{\infty}$ has a converging subsequence (because it is bounded). Denote its limit by y_i'. Parts (b) and (e) of Lemma 1 imply that $\{D^{-1}((y_j, y_i'))\} = P^B((y_j, y_i'))$. The last claim in part (g) of Lemma 1 implies that we must have $y_i' = \omega_j(y_j)$.

Obviously, the domain of ω_j can be extended leftwards (with all its properties established above holding) until we reach a point y_j such that either $D^{-1}((y_j, \omega_j(y_j))) \in P^B(y^{max})$ or $y_j = 0$.

The assumption of the uniqueness of a capacity constrained Bertrand equilibrium in some small neighbourhood of $D(p^{B\prime})$ for each $p^{B\prime} \in P^B(y^{max})$ implies that $P^B(y^{max})$ is a singleton. If $P^B(y^{max})$ had at least two different elements, p^B and $p^{B\prime}$, then (with respect to part (g) of Lemma 1) the graphs of functions ω_j and ω_i, $i \neq j$, joining $D(p^B)$ and $D(p^{B\prime})$ would have to coincide. Thus, each point of this graph would be a demand vector at some element of $P^B(y^{max})$, contradicting the above assumption of local uniqueness of a capacity constrained Bertrand equilibrium.

The above considerations further imply that $P^B(y)$ is a singleton for each $y \in Y$. For capacity vectors lying below and to the left from curves ω_j and ω_i or on these curves the latter claim follows from part (e) of Lemma 1. For other capacity vectors it follows from part (g) of Lemma 1. Thus, taking into account part (b) of Lemma 1, P^B is a continuous function.

If $y^0 = D[p^B(y^{max})]$, the arguments in the following parts of the proof are significantly simplified (in an obvious way). Therefore, in the remainder of this proof we assume that $y_k^0 > y_k^B$ and $y_j^0 < y_j^B$, $j \neq k$, where (14) holds for firm k. Construct function ω_j, $j \neq k$, for $p^B(y^{max})$. Its properties imply that $y_k^0 = \omega_j(y_j^0)$.

Description of equilibrium strategy profile s^*. The line segment $[y^*, y^0]$ corresponds to cl(Θ) from the proof of Proposition 1. We define function $L:[y^*, y^0] \rightarrow [0, 1]$ by $L(y) = (y_j - y_j^*)(y_j^0 - y_j^*)^{-1}$. Obviously, L is continuously invertible. We use the symbol $L_i^{-1}(x)$ to denote i-th coordinate of the capacity vector $y \in [y^*, y^0]$ satisfying $L(y) = x$.

We further define two parameters, used in the definition of equilibrium strategies: $\psi > 0$ and $r \in (0, 1)$. We assume that ψ is close to zero and r is close to one. For $y \in Y$ we set $\lambda(y) = \psi|f(y_1) - y_2|$. The variable ξ(y), used in the definition of equilibrium strategies, is defined as follows. If $y_2 \leq f(y_1)$, then

$$\xi(y) = rL[(\max\{y_1^*, \min\{(1 + \lambda(y))y_1, y_1^0\}\}, f(\max\{y_1^*, \min\{(1 + \lambda(y))y_1, y_1^0\}\}))]. \tag{15a}$$

If $y_2 \geq f(y_1)$, then

$$\xi(y) = rL[f^{-1}(\max\{y_2^*, \min\{(1 + \lambda(y))y_2, y_2^0\}\}), \max\{y_2^*, \min\{(1 + \lambda(y))y_2, y_2^0\}\})]. \tag{15b}$$

The investment expenditures part of s^* is given by

$$\eta_i^*(y) = \alpha_i[y_i, \max\{L_i^{-1}[\xi(y)], \alpha_i^{-1}(y_i, 0)\}], \quad i \in \{1, 2\}. \tag{16}$$

We set the parameter r close enough to one to ensure that

$\alpha_i^{-1}(y_i, 0) \leq L_i^{-1}[\xi(y)]$ for each $y \in [y^*, y^0]$ and for both $i \in \{1, 2\}$. (We identify other factors restricting the lower bound on r below.)

Let k be the index of the firm for which (14) holds and let $j \neq k$. The price part of

s^* is given by $\rho^*(y) = D^{-1}(y)$ for all $y \in Y$ lying below and to the left from the graphs of functions ω_j and ω_k or on them (i.e., for all $y \in Y$ with $y_j \le y_j^B$ and $y_k \le \omega_j(y_j)$, and for all $y \in Y$ with $y_k \le y_k^B$ and $y_j \le \omega_k(y_k)$), by $\rho^*(y) = D^{-1}((y_j, \omega_j(y_j)))$ for all $y \in Y$ with $y_j \le y_j^B$ and $y_k > \omega_j(y_j)$ by $\rho^*(y) = D^{-1}((y_k, \omega_k(y_k)))$ for all $y \in Y$ with $y_k \le y_k^B$ and $y_j > \omega_k(y_k)$, and by $\rho^*(y) = p^B(y^{max}) = D^{-1}(y^B)$ for all $y \in Y$ with $y_i \ge y_i^B$ for both $i \in \{1, 2\}$.

Of course, $s_i^* = (\rho_i^*, \eta_i^*)$ for both $i \in \{1, 2\}$ and $s^* = (s_1^*, s_2^*)$.

Strategy profile s^* forms a Markov perfect equilibrium. Clearly, strategies depend only on a current capacity vector, so they are Markov. To show that they form a subgame perfect equilibrium, we first consider a unilateral single period deviation in investment expenditures by firm $i \in \{1, 2\}$ at state $y \in Y$ with $\alpha^{-1}[y, \eta^*(y)] \in [y^*, L^{-1}(r - d)]$, where d is a very small positive real number. It causes an upward shift in (or start of) movement along the line segment $[y^*, y^0]$ in the capacity space. (A continuation equilibrium path triggered by a deviation can start outside this line segment. The important point is that it reaches $[y^*, L^{-1}(r)]$ in a finite number of periods bounded from above independently of the value of the parameter r.) In the limit case $r = \beta_i = 1$ (fixing r to one since the period when a deviation takes place) this reduces firm i's continuation average discounted net profit in comparison with the situation without a deviation. Due to the continuity of firm i's continuation average discounted net profit in r and β_i, the same holds also for $r < 1$ close enough to one and $\beta_i < 1$ close enough to one.

Now consider a unilateral single period deviation in investment expenditures by firm k at $y \in Y$ with $\alpha^{-1}[y, \eta^*(y)] \notin [y^*, L^{-1}(r - d)]$. In a continuation equilibrium triggered by a deviation the play in one period reaches a subgame with firm k's continuation equilibrium average discounted net profit no higher than its continuation equilibrium average discounted net profit in a subgame in which a deviation took place. With respect to definition of y^0 and

ρ^*, for $\beta_k \in (0, 1)$ close enough to one, for $r \in (0, 1)$ close enough to one, and for $d > 0$ sufficiently small, if firm k incurred investment costs $\delta_k L_k^{-1}(r - d)$ or higher in each of the two periods (including a period in which a deviation took place) before the play reaches that subgame, a deviation would decrease its continuation average discounted net profit. In reality of a deviation firm k incurs in a period when a deviation takes place investment costs larger than (in the case of an upward deviation in capacity) or in the following period investment costs no smaller than (in the case of a downward deviation in capacity) $\delta_k L_k^{-1}(r - d)$. Thus, for $\beta_k < 1$ close enough to one (so that $(1 - \beta_k)$ is close to $(1 - \beta_k^2)$) firm k cannot increase its continuation average discounted net profit by a deviation of the type analyzed here.

Finally, consider a single period unilateral deviation in investment expenditures by firm j at $y \in Y$ with $\alpha^{-1}[y, \eta^*(y)] \notin [y^*, L^{-1}(r - d)]$. In a continuation equilibrium triggered by a deviation the play in a finite number of periods, bounded from above independently from r and β_j, reaches a subgame with firm j's continuation equilibrium average discounted net profit no higher than its continuation equilibrium average discounted net profit in a subgame in which a deviation took place. For $\beta_j \in (0, 1)$ close enough to one, for $r \in (0, 1)$ close enough to one, for $d > 0$ sufficiently small, and with respect to (11), if firm j incurred investment costs $\delta_j L_j^{-1}(r - d)$ or higher in every period before reaching that subgame (including the period in which a deviation took place), a deviation would decrease its continuation average discounted net profit. An argument completely analogous to the one at the end of the immediately preceding paragraph shows that even in the real situation created by a deviation (with investment costs incurred only in the period of a deviation or in the immediately following period), a deviation does not increase firm j's continuation average discounted net profit.

Continuity of s^*. The choice of ρ^* implies that $(\rho_1^*(y), \rho_2^*(y))$ is continuous in y.

Continuity of $(\eta_1^*(y), \eta_2^*(y))$ in y follows from continuity of the functions α_i, α_i^{-1}, ($i \in \{1, 2\}$), L, L^{-1}, f^{-1}, maximum, minimum, $\xi(y)$, and $\lambda(y)$.

Renegotiation-quasi-proofness of s^* follows from the fact that y^* is Pareto efficient and that $s^*(y^*) = y^*$. ■

REMARK 1. Proposition 2 does not directly imply that the set of renegotiation-quasi-proof continuous strategy Markov perfect equilibria of G_{DD} is nonempty. Nevertheless, the method of its proof enables us to show that the latter set is nonempty. Clearly, there is a Pareto efficient capacity vector y^* such that the half-line (in the non-negative orthant of the two-dimensional Euclidean space) with y^* as its origin and slope given by (12) passes through $D[p^B(y^{max})]$. Then the method used in the proof of Proposition 2 to handle deviations by firm k at $y \in Y$ with $\alpha^{-1}[y, \eta^*(y)] \notin [y^*, L^{-1}(r - d)]$ can be used for both firms. The condition (11) is not needed, it is enough that

$$\pi_i[D^{-1}(y^*), y^*] - \delta_i y_i^* > \pi_i[p^B(y^{max}), D(p^B(y^{max}))] - \delta_i D_i[p^B(y^{max})], \quad \forall\, i \in \{1, 2\}, \quad (17)$$

which is guaranteed by (12).

REMARK 2. We have used the condition (11) in the formulation of Proposition 2 in order to minimize number of requirements contained in sufficient conditions of the existence of a renegotiation-quasi-proof continuous strategy Markov perfect equilibrium of G_{DD} and to use only requirements that are easily comprehensible and based on notions clearly related to well-known concepts in economics (like a capacity unconstrained Bertrand equilibrium, coinciding with a standard Bertrand equilibrium). We could replace (11) by a less restrictive but more cumbersome requirement that, for each $y_j \in [y_j^0, y_j^B]$,

$$[D_j^{-1}((y_j, \omega_j(y_j))) - c_j - \delta_j]y_j < \pi_j[D^{-1}(y^*), y^*] - \delta_j y_j^*, \quad (18)$$

where capacity vector y^0, index j, and the function ω_j are defined as in the "Preliminaries" part of the proof of Proposition 2.

5. CONCLUSIONS

We analyzed a dynamic duopoly in which simultaneously moving firms choose in each period investment expenditures, affecting next period capacities, and prices charged. (The results can be applied to other dynamic games in economics or political science where players simultaneously choose actions affecting future state variables and other actions whose consequences are constrained by state variables.) A reduction of the set of equilibria was based on imposing three requirements on subgame perfect equilibrium strategies. First, we required them to be Markov. Second, we required that equilibrium strategies be continuous (functions of their arguments). Third, we required them to be renegotiation-quasi-proof. This still leaves a continuum of equilibria, even a continuum of limits of continuation equilibrium paths. Nevertheless, the set of equilibria (and especially of limits of continuation equilibrium paths) is restricted by the lower bound on continuation equilibrium average discounted net profits derived from single period net profits at the capacity unconstrained static Bertrand equilibrium of the analyzed duopoly.

We have derived (in part (c) of Proposition 1) a necessary condition that a function describing continuation equilibrium paths must satisfy in a neighbourhood of their limit. Thus, although we did not identify the unique limit of all continuation equilibrium paths of all (renegotiation-quasi-proof continuous strategy Markov perfect) equilibria, evaluation of them (and, possibly, choice between them) can be based on judgement on how reasonable continuation equilibrium paths, and equilibrium strategies generating them, are. (See also the discussion of strategies versus payoffs in [10].) For example, if $y_1^* = y_2^* = 5$, then the linear punishment path $y_2 = 5y_1 - 20$ (implying that a deviation by firm one to capacity 6 would increase difference $y_2 - y_1$ to slightly more than 4, whereas a deviation by firm two to capacity 6 would increase it only to slightly more than 0.8) does not seem to be plausible. (Under the assumptions made in this paper, for each Pareto efficient capacity vector satisfying (11) there is exactly one linear punishment path.)

We have made the assumption of constant marginal costs of production only for the sake of simplicity of exposition. All qualitative results continue to hold also for strictly convex production costs functions, but a capacity constrained vector demand function Z(p, y) must be redefined, taking into account that, at a given price, a firm will not produce and sell more than an amount equal to its competitive supply.

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