Axiomatization of a Preference for Most Probably Winner

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Abstract:

In binary choice between discrete outcome lotteries, an individual may prefer lottery L_1 to lottery L_2 when the probability that L_1 delivers a better outcome than L_2 is higher than the probability that L_2 delivers a better outcome than L_1. Such a preference can be rationalized by three standard axioms (completeness, continuity and first order stochastic dominance) and two less standard ones (weak independence and a fanning-in). A preference for the most probable winner can be represented by a skew-symmetric bilinear utility function. Such a utility function has the structure of a regret theory when lottery outcomes are perceived as ordinal and the assumption of regret aversion is replaced with a preference for a win. The empirical evidence supporting the proposed system of axioms is discussed.

Abstrakt:

V případě binární volby mezi loteriemi s diskrétním výsledkem, jednotlivec může preferovat loterii L_1 před loterií L_2 když pravděpodobnost že L_1 nastane s lepším výsledkem než L_2 je větší než pravděpodobnost že L_2 nastane s lepším výsledkem než L_1. Takové preference je možné racionalizovat podle třech standardních axiomú (spojitost, první stochastická dominance a kompletnost preferencií) a dvou méně standardních axiomú (slabá nezávislost a sevření). Preference najpravděpodobnejšího vítěze je možné reprezentovat funkcií užitečnosti, která je lineární v obou její argumentech a je symetricky sešikmená. Taková funkce užitečnosti má strukturu podle teorie lítosti když výsledky loterie jsou vnímány jako ordinální a předpoklad averze k lítosti je nahrazen preferencí pro vítězství. Empirické evidence, které podporují tento navrhovaný axiomatický systém jsou diskutovány.

JEL Classification codes: C91, D81

Key words: EUT, axiomatization, stochastic dominance, betweenness, weak independence, fanning-in, regret theory

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1. Introduction

An individual has menu-dependent preferences when his preference between two choice options depends on the availability of additional options (content of the choice set). The literature often describes such preferences as "context-dependent" (*e.g.* Stewart *et al,* 2003, Tversky and Simonson, 1993, Simonson and Tversky, 1992). The context of a choice situation is a very general concept, however, that can be also used to describe aspects other than the content of a choice set. A more suitable term to describe a very specific phenomenon—the dependence of individual preferences on the menu of a choice set—is menu-dependence.

In choice under risk (Knight, 1921), a special type of menu-dependent preference is a preference for a lottery that is most probable to outperform all other feasible lotteries. The literature refers to such a preference as "a preference for probabilistically prevailing lottery" (*e.g.* Bar-Hillel and Margalit, 1988) or "the criterion of the maximum likelihood to be the greatest" (*e.g.* Blyth, 1972). Recent experimental evidence suggests that a preference for most probable winner prevails in binary choice between lottery frequencies of equal expected value (Blavatskyy, 2003) and in small feedback-based problems (*e.g.* Barron and Erev, 2003, Blavatskyy, 2003a). In this paper, I build a system of axioms rationalizing a preference for most probable winner in binary choice.

Given the probability distributions of any two independent lotteries it is always possible to calculate directly the (relative) probability of each lottery to outperform the other. Such calculation requires little cognitive effort when the state space (a joint distribution of lotteries) is available. Blavatskyy (2003) provides experimental evidence that a preference for most probable winner emerges when an individual follows a simple majority rule—to pick up a lottery that gives a better outcome in the majority of (equally probable) states of the world. In such a cognitively undemanding environment it is plausible to assume that an individual follows a simple behavioral rule—he calculates the relative probabilities of each lottery to win over the other lotteries and then maximizes among those probabilities. This behavioral rule (the heuristic of relative probability comparisons) resembles one-reason fast and frugal decision making (*e.g.* Gigerenzer and Goldstein, 1996). Like all heuristics, it ignores some of the available information by treating lottery outcomes as ordinal. Additionally, like all heuristics, this behavioral rule applies only to a bounded subset of decision problems, *e.g.* when lotteries have equal or similar expected values.

In cognitively demanding environments, a straightforward calculation of relative probabilities of a lottery to win over others, however, demands more cognitive effort. Examples include situations when probability information is presented visually (*e.g.* Tversky, 1969) or not presented at all (*e.g.* Barron and Erev, 2003), when lotteries have many outcomes or an individual faces a choice among many lotteries. Nevertheless, assuming an individual preference for most probable winner it is possible to explain observed decision making in such environments, as demonstrated in Blavatskyy (2003a) in his alternative explanation of the data in Barron and Erev (2003).

Since individuals are likely to use only simple rules of thumb (*e.g.* Gigerenzer *et al*., 1999), a descriptive fit of a preference for most probable winner in cognitively demanding decision environments can be explained only through a general theory of preference. Unlike a heuristic approach that describes a plausible psychological process underlying observed decision making (*e.g.* Newell and Shanks, 2003), a theory of preference states that an individual has an underlying preference for most probable winner. The purpose of this paper is to explore the theoretical properties of an individual's preference for most probable winner, how it is related to various non-expected utility theories (Starmer, 2000), what normative axioms are necessary and sufficient for rationalizing such preference, and how descriptive those axioms are.

The proposed axiomatization provides theoretical insights into an individual's preference for most probable winner and highlights some surprising connections to other decision theories. It also provides "thought experiment" evidence for a descriptive validity of the theory (*e.g.* Friedman and Savage, 1952, Quiggin, 1982, Machina, 1982, Wakker, 2003). However, "thought experiment" evidence can be drastically different from actual decision making (*e.g.* Tversky, 1969). Therefore, the paper also focuses on the experimental evidence supposedly documenting the systematic violation of the proposed axioms.

The remainder of this paper is structured as follows. Section 2 introduces the system of axioms with an emphasis on their normative appeal. The proposed axioms are used in section 3 to derive a utility function representation and family of indifference curves. The descriptive validity of the proposed axioms is discussed in section 4. Section 5 concludes.

2. The system of axioms

2.1. Basic definitions

An option *A* is strictly preferred to option *B*, or $A \succ B$, if an individual chooses *A* and is not willing to choose *B* from the choice set $\{A, B\}$. An individual is indifferent between choice options *A* and *B*, or $A \sim B$, if the choice of *A* and the choice of *B* are equally possible from the choice set $\{A, B\}$.

This paper deals with individuals' binary choices between discrete lotteries. The set of lottery outcomes $\{x_1,...x_n\}$ is finite and ordered in such a way that $x_1 \prec ... \prec x_n$. Note that outcomes are not necessarily monetary (measured in reals). We only require the outcomes to be strictly ordered in terms of individual preference. A lottery $L(p_1,..., p_{n-1})$ is defined as a mapping $L:\{x_1,..., x_{n-1}\} \mapsto [0,1]^{n-1}$, where $p_i \in [0,1]$ is the probability of occurrence of outcome x_i, $i \in [1, n-1]$. The best outcome x_n is mapped to the residual probability $1 - p_1 - ... - p_{n-1} \geq 0$.

In a joint independent distribution of any two lotteries L_1, L_2 only three events are possible: L_1 delivers a better outcome than L_2 (state s_1), L_2 delivers a better outcome than L_1 (state s_2) and lotteries L_1, L_2 deliver the same outcome (state s_3). An individual has a preference for most probable winner when $L_1 \succ L_2 \Leftrightarrow prob(s_1) > prob(s_2)$. This decision rule is rationalized below.

2.2. Standard axioms

This section introduces a set of standard axioms that are routinely employed in one form or another in nearly all axiomatizations of decision theories (*e.g.* Abdellaoui, 2002, Segal, 1990). Notably, this set of standard axioms does not contain the transitivity axiom. Indeed, a preference for most probable winner can be intransitive (*e.g.* Blyth, 1972).

Axiom 1 (Completeness) For any two lotteries L_1, L_2 either $L_1 \succ L_2$ or $L_1 \prec L_2$ or $L_1 \sim L_2$.

Axiom 2 (Continuity) For any three lotteries L_1, L_2, L_3 such that $L_1 \succ L_2$ and $L_2 \succ L_3$ $\exists \alpha \in (0,1) : \alpha L_1 + (1-\alpha) L_3 \sim L_2$

Axiom 3 (First order stochastic dominance) For any two lotteries $L_1(p_1,..., p_{n-1})$, $L_2(q_1,..., q_{n-1})$ such that $\sum_{j=1}^{i} p_j \le \sum_{j=1}^{i} q_j, \forall i \in [1, n-1]$ and $\exists i \in [1, n-1] : \sum_{j=1}^{i} p_j < \sum_{j=1}^{i} q_j \Rightarrow L_1 \succ L_2$

2.3. Weak independence axiom

The cornerstone of EUT—the independence axiom—states that, if an individual has a preference between two lotteries L_1 and L_2, then he has the same preference between probability mixtures of L_1 and L_2 with an arbitrary third lottery L_3 (von Neumann Morgenstern, 1944). However, unlike the standard axioms (notably completeness and continuity), the independence axiom has a context-dependent normative appeal. Similar to heuristics, the independence axiom is intuitively appealing in the environments where its application requires little cognitive effort (probability mixtures are presented in a split format with transparent common consequences whose cancellation appears obvious) (*e.g.* Kahneman and Tversky, 1979, and Conlisk, 1989). In contrast, in cognitively demanding environments (probability mixtures are presented in a coalesced format), the independence axiom needs a reference to normatively appealing principles (reduction axiom, Segal 1990). It also looses an immediate intuitive appeal, for example, to considerations of similarity (*e.g.* Kahneman and

Tversky, 1979). Fortunately, a weak version of independence axiom is sufficient to characterize an individual's preference for most probable winner.

Definition 1 (Difference in probability distributions) For any two lotteries $L_1(p_1,\ldots,p_{n-1})$ and $L_2(q_1,\ldots,q_{n-1})$ a difference in probability distributions $\overrightarrow{L_1L_2}$ is defined as $\overrightarrow{L_1L_2}=(q_1-p_1,\ldots,q_{n-1}-p_{n-1})$.

A difference in probability distributions $\overrightarrow{L_1L_2}$ effectively measures the degree of similarity between lotteries L_1 and L_2. It shows what probability changes are required for a lottery L_1 to become a lottery L_2.

Axiom 4 (weak independence) For any four lotteries $L_1 \sim L_2$ and $L_1' \sim L_2'$ such that

$$\exists \alpha \in \text{ú}: \overrightarrow{L_1L_1'}=\alpha\, \overrightarrow{L_2L_2'} \;\Rightarrow \forall \beta \in (0,1): \beta L_1+(1-\beta)L_1' \sim \beta L_2+(1-\beta)L_2'$$

Axiom 4 states that an individual's preference within two lottery pairs extends to the cross-pair probability mixtures if the cross-pair differences in probability distributions are *proportionate*. An analogue of the independence axiom would be more restrictive—an individual's preference within lottery pairs is preserved for the cross-pair probability mixtures if the cross-pair differences in probability distributions are *the same* ($\alpha=1$). For lotteries defined on a common three-outcome structure axiom 4 effectively guarantees that an individual's indifference curves are either parallel lines (when $\alpha=1$) or they have a unique point of intersection (when $\alpha \neq 1$)[1]. Chew (1983) and Fishburn (1983) offer an alternative but functionally equivalent formulation of the weak independence axiom. However, Starmer (2000) argues that their formulation lacks an intuitive appeal.

[1] The point of intersection may be either a well-defined lottery (figure 9 in Machina, 1987), figure 3.2 in Fishburn (1988) p. 71) or a triple of real numbers summing up to one but not restricted to belong to interval [0,1] individually. In the latter case the point of intersection is outside the lottery space and the transitivity of preferences is preserved (*e.g.* Starmer (2000), p.343). Khrennikov (1999) builds a non-Kolmogorov probability theory restricting probabilities to sum up to one but not necessarily to belong to interval [0,1] individually.

By setting $L_1' = L_2$ and $L_2' = L_1$ axiom 4 implies a strong mixture symmetry ($\forall L_1 \sim L_2, \forall \beta \in (0,1)$: $\beta L_1 + (1-\beta)L_2 \sim (1-\beta)L_1 + \beta L_2$), which is the only possible form of mixture symmetry consistent with axioms 2 and 3 (*e.g.* Chew *et al.*, 1991). If we allow $\alpha = 0$ then by setting $L_1' = L_2' = L_1$ axiom 4 implies a betweenness axiom 4a.

Axiom 4a (Betweenness) For any lotteries $L_1 \sim L_2$ and $\alpha \in (0,1) \Rightarrow \alpha L_1 + (1-\alpha)L_2 \sim L_1$

Proposition 1 If axioms 1-3 and 4a hold then for any lottery $L_0(q_1,...,q_{n-1})$ it is possible to find a vector of outcome-associated "utilities" $\{a_1,...,a_{n-1}\} \in ú^{n-1}$ such that $\forall L(p_1,..., p_{n-1})$: $L \sim L_0 \Leftrightarrow a_1 p_1 + ... + a_{n-1} p_{n-1} = 1$. ■ Proof is presented in appendix I ■

Proposition 1 defines the set of all lotteries such that an individual is indifferent between them and a given lottery L_0. It is not a conventional indifference set because without transitivity an indifference relation is not an equivalence relation ($L_1 \sim L_0, L_2 \sim L_0 \not\Rightarrow L_1 \sim L_2$). Proposition 1 defines this set as a $(n-2)$-dimensional hyperplane (written in canonical form) exactly as an implicit weighted utility (Chew, 1989) or implicit expected utility (Dekel, 1986) without the assumption of transitivity. EUT satisfies proposition 1 as a special case when $a_i = \frac{u_n - u_i}{u_n - u}, i \in [1, n-1]$, where $u = \sum_{j=1}^{n-1} q_j u_j + (1 - q_1 - ... - q_{n-1})u_n$ is an individual's utility of lottery $L_0(q_1,...,q_{n-1})$ and $u_i, i \in [1, n]$ is an individual's utility of an outcome x_i. Cumulative prospect theory or CPT (Tversky and Kahneman, 1992) violates the betweenness axiom and consequently its utility function does not belong to the class defined by proposition 1 (apart from a trivial case when CPT simplifies into EUT). Interestingly, a preference for most probable winner also belongs to the betweenness class defined by proposition 1 although it can lead to intransitive preference ordering. Thus, an individual's preference for most probable winner is a special case of implicit weighted (expected) utility theory without the transitivity axiom.

2.4. A fanning-in axiom

The last axiom, and arguably the least intuitively appealing, assumes that individuals have a particular type of diminishing sensitivity to probability. Specifically, when probability mass is largely shifted to the best or the worst outcome, tiny probabilities attached to the intermediate outcomes become progressively unimportant to individuals. Unlike the previous four axioms, this axiom has not been used in other decision theory axiomatizations in the literature.

Axiom 5 (A fanning-in) If $L_1(0,...,0,p_i,0,...,0) \sim L_2(0,...0,p_j,0,...,0)$, $j > i \Rightarrow \lim_{p_i \to 0} \frac{p_j - p_i}{p_j} = 0$,

if $L_3(1-p_k,...,0,p_k,0,...,0) \sim L_4(1-p_l,...0,p_l,0,...,0)$, $l > k \Rightarrow \lim_{p_l \to 0} \frac{p_k - p_l}{p_l} = 0$

To understand the logic behind axiom 5 let us consider first the situation when $p_i \to 0$. First of all, notice that $p_j > p_i$ because $x_i \prec x_j \prec x_n$. If $p_i \to 0$ then lottery L_1 approaches to the lottery $\overline{L}(0,...,0)$, which gives the best possible outcome x_n for sure. Since there could be no other lottery $\widetilde{L}_2$ such that $\widetilde{L}_2 \sim \overline{L}(0,...,0)$ it must be the case that $\lim_{p_i \to 0} p_j = 0$. When two lotteries L_1, L_2 approach to the lottery $\overline{L}(0,...,0)$ the absolute differences in tiny probabilities attached to the not-best outcome disappear $\lim_{p_i \to 0}(p_j - p_i) = 0$. Axiom 5 additionally requires that the relative differences in probabilities attached to the not-best outcome also disappear as L_1 and L_2 become increasingly similar to $\overline{L}(0,...,0)$. In other words $\lim_{p_i \to 0} \frac{p_j - p_i}{p_j} = 0$.

EUT violates axiom 5 since it implies $\lim_{p_i \to 0} \frac{p_j - p_i}{p_j} = const$. CPT satisfies axiom 5 only in a special case when $\lim_{p \to 1} \frac{\partial \pi^{-1}}{\partial p} \frac{\partial \pi}{\partial p} = \frac{u_n - u_i}{u_n - u_j}$, where $\pi(p)$ is the probability weighting function (*e.g.* Tversky and Kahneman, 1992, Abdellaoui, 2000). When $\lim_{p_i \to 0} \frac{p_j - p_i}{p_j} = 0$ then an individual's indifference curves plotted in the probability triangle[2] are not parallel but fanning-in, which explains the name of the axiom. The assumption $\lim_{p_i \to 0} \frac{p_j - p_i}{p_j} = 0$ also implies that an individual becomes infinitely risk seeking when probability mass is largely shifted to the best outcome.

The second part of axiom 5 assumes that the above logical argument applies as well to the situation when lotteries $L_3(1 - p_k, 0, \ldots, 0, p_k, 0, \ldots, 0)$ and $L_4(1 - p_l, 0, \ldots, 0, p_l, 0, \ldots, 0)$, $L_3 \sim L_4$, approach to the lottery $\underline{L}(1, 0, \ldots, 0)$, which gives the worst possible outcome x_1 for sure. The only difference is that an individual becomes infinitely risk averse when probability mass is largely shifted to the worst outcome. The implication of axiom 5 that an individual becomes risk seeking (averse) when probability mass is largely shifted to the best (worst) outcome is the counterpart of Machina's intuition for universal fanning out (Machina, 1987 pp. 129-130)

[2] Note that lotteries L_1, L_2, although defined as probability distributions over *n* outcomes, have non-zero probabilities attached only to three outcomes x_i, x_j, x_n. Therefore, lotteries L_1, L_2 can be plotted in the probability triangle based only on outcomes x_i, x_j, x_n.

3. Utility representation and indifference curves

3.1. Lotteries defined on a common three-outcome structure

This section derives a utility function representation and the family of indifference curves representing an individual's preferences satisfying the system of axioms from section 2. First, let us consider only lotteries defined on a common three-outcome structure. This case allows for a convenient representation of indifference curves in two-dimensional simplex (probability triangle). Additionally, $n = 3$ is the only case when transitivity holds and a conventional family of indifference curves (neo-classical utility function) can be constructed.

Proposition 2 (Transitivity) If $n = 3$ and axioms 1-3 and 4a hold then for any three lotteries L_1, L_2, L_3 such that $L_1 \sim L_2$ and $L_2 \sim L_3 \Rightarrow L_3 \sim L_1$.

■ Proof of proposition 2, and all subsequent propositions, is presented in appendix II ■

Proposition 3 (Indifference curve) If $n = 3$ and axioms 1-5 hold then for any arbitrary lottery $L_0(s_1, s_2)$ the set $\{L : L \sim L_0\}$ is $\left\{L(r_1, r_2) : \frac{1+s_2}{s_1+s_2} r_1 + \frac{1-s_1}{s_1+s_2} r_2 = 1\right\}$ and we call this set the indifference curve of lottery $L_0(s_1, s_2)$.

Proposition 3 demonstrates that for lotteries defined on a common three-outcome structure an individual's preference for most probable winner can be captured by an utility function that has the structure of the weighted utility theory (*e.g.* Chew and MacCrimmon, 1979, Chew, 1983). However, this result is not generic. Proposition 2 shows that an individual's preference for most probable winner is transitive when $n = 3$. Fishburn (1982, 1983, 1984) proved that the weighted utility theory coincides with the skew-symmetric bilinear utility theory when preferences are transitive. In general an individual's preference for most probable winner may be intransitive (*e.g.* Blyth, 1972). Thus, although the weighted utility theory captures such preference when $n = 3$, this result may not hold in general. In fact, in subsection 3.2, it is demonstrated that when $n \geq 4$ a binary individual's preference for most probable winner is still captured by the skew-symmetric bilinear utility but not by the weighted utility theory.

Figure 1 plots the family of indifference curves from proposition 3 inside the probability triangle (Machina, 1982). The same family of indifference curves is obtainable from the weighted utility theory when $w(x_M) > 1$ (*e.g.* Chew, 1983, Chew and Waller, 1986). Notice that this indifference map is independent of individual-specific parameters (functions) and cardinal measures of lottery outcomes *i.e.* the map is invariant for all triples of lottery outcomes such that $x_1 \prec x_2 \prec x_3$. The family of indifference curves implied by axioms 1-5 consists of straight lines with different slopes reflecting a changing individual attitude towards risk (Humphrey and Verschoor, 2002). Specifically, a universal fanning-in, as in figure 1, shows that an individual becomes more risk seeking (averse) when probability mass is shifted to the best (worst) outcome, which Chew and Waller (1986) call "the heavy hypothesis". Proposition 3 also implies that an individual is risk neutral along the 45° line on figure 1 *i.e.* he is exactly indifferent between a medium outcome for sure and a 50%-50% chance of the best and the worst outcome. More generally axioms 1-5 imply a symmetry axiom (*e.g.* Fishburn (1982, 1983, 1984).

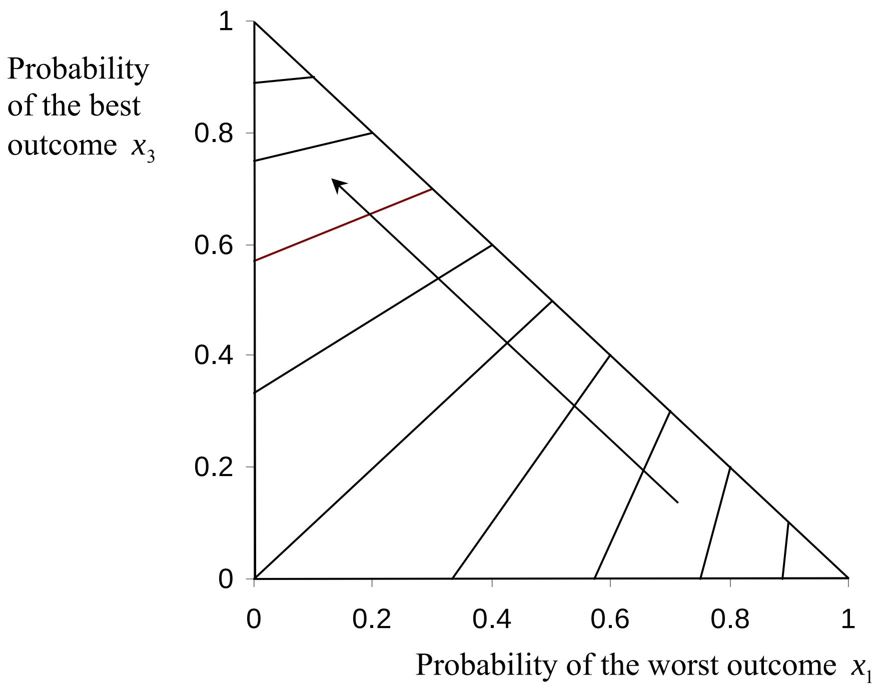

Figure 1 Family of indifference curves inside the probability triangle

The family of indifference curves from figure 1 represents the following preference:

$$L(r_1,r_2)\overset{\succ}{\underset{\prec}{\sim}}L_0(s_1,s_2)\Leftrightarrow\frac{1+s_2}{s_1+s_2}r_1+\frac{1-s_1}{s_1+s_2}r_2\overset{<}{\underset{>}{=}}1\Leftrightarrow r_2s_1+(1-r_1-r_2)(s_1+s_2)\overset{>}{\underset{<}{=}}r_1s_2+(1-s_1-s_2)(r_1+r_2)$$

The last equation states that an individual prefers lottery L over lottery L_0 if and only if the probability of lottery L to outperform lottery L_0 is greater than the probability of lottery L_0 to outperform lottery L. This is exactly the definition of a preference for most probable winner. Thus, an individual preferring most probable winner respects axioms 1-5 when lotteries are defined on a common three-outcome structure.

Since each indifference curve is uniquely mapped to some lottery $\widetilde{L}_0(p_0,0)$ we can think of probability $1-p_0$ as a utility measure (normalized between zero and unity). Then for any three-outcome lottery $L_0(s_1,s_2)$: $U(L_0)=(1-s_1)/(1+s_2)$, where $U:L\to\mathbb{R}$ is the conventional utility function. Of course, any monotone transformation of $U:L\to\mathbb{R}$ will also be a utility measure. Interestingly, $U:L\to\mathbb{R}$ is a cardinal utility measure, which allows us to make interpersonal comparisons of lottery utilities.

3.2. Lotteries with four and more outcomes

In case of discrete lotteries with $n\geq 4$ outcomes axioms 1-5 contradict the transitivity of individual preferences over lotteries. For an arbitrary lottery L_0 the set $\{L:L\sim L_0\}$ can contain lotteries L_1,L_2 such that $L_1\succ L_2$. Thus, we cannot regard this set as the neoclassical indifference curve. If we attempt to plot the family of such curves they will intersect reflecting intransitive preferences (*e.g.* Starmer, 2000, p.343). However, we can still represent such preferences by a *menu-dependent* (or *relative*) *utility function*. A menu-dependent (or relative) utility function evaluates a lottery not on the basis of its absolute characteristics alone but also takes into account the characteristics of other available lotteries, which can be chosen instead (*e.g.* Chechile and Cooke, 1997, pp. 90-91). Such a utility function is the natural extension of the neoclassical single-argument utility to represent intransitive individual preferences (*e.g.* Fishburn, 1988, p. 67).

Proposition 4 If axioms 1-5 hold then for any lottery $L_0(q_1, q_2, ..., q_{n-1})$ the set $\{L : L \sim L_0\}$ is

$$\left\{L(p_1, p_2, ..., p_{n-1}) : \sum_{i=1}^{n-1} \frac{1 - q_1 - ... - q_{i-1} + q_{i+1} + ... + q_{n-1}}{q_1 + q_2 + ... + q_{n-1}} p_i = 1\right\}.$$

Proposition 4 states that an individual respecting axioms 1-5 has preferences $L(p_1, p_2, ..., p_{n-1}) \underset{\prec}{\overset{\succ}{\sim}} L_0(q_1, q_2, ..., q_{n-1}) \Leftrightarrow \sum_{i=1}^{n-1} \frac{1 - q_1 - ... - q_{i-1} + q_{i+1} + ... + q_{n-1}}{q_1 + q_2 + ... + q_{n-1}} p_i \underset{>}{\overset{<}{=}} 1$. Let us denote by $prob(L_0 > L) = \sum_{i=1}^{n-1} (1 - q_1 - ... - q_i) p_i$ the probability of lottery L_0 to deliver a higher outcome than lottery L. Consequently, $prob(L > L_0) = \sum_{i=1}^{n-1} (1 - p_1 - ... - p_i) q_i$ is the probability of lottery L to deliver a higher outcome than lottery L_0. Then, simple algebra implies that $\sum_{i=1}^{n-1} \frac{1 - q_1 - ... - q_{i-1} + q_{i+1} + ... + q_{n-1}}{q_1 + q_2 + ... + q_{n-1}} p_i \underset{>}{\overset{<}{=}} 1 \Leftrightarrow prob(L > L_0) \underset{>}{\overset{<}{=}} prob(L_0 > L)$. Note that $prob(L > L_0)$ satisfies the requirement of a menu-dependent (or relative) utility function since $\forall L, L_0 : L \underset{\prec}{\overset{\succ}{\sim}} L_0 \Leftrightarrow prob(L > L_0) \underset{>}{\overset{<}{=}} prob(L_0 > L)$. This last equation is also nothing but the definition of a preference for most probable winner. Proposition 4 indicates that axioms 1-5 are sufficient to characterize an individual's preference for most probable winner. It is relatively straightforward to demonstrate that an individual's preference for most probable winner also satisfies axioms 1-5. Thus, axioms 1-5 are both necessary and sufficient.

In binary choice a menu-dependent (or relative) utility function $U(L_1, L_2) = prob(L_1 > L_2)$ is skew-symmetric in the sense that $U(L_1, L_2) = 1 - U(L_2, L_1)$ and it is also bilinear, which follows from a betweenness axiom 4a. Thus, in binary choice an individual's preference for most probable winner is captured by the skew-symmetric bilinear utility theory (*e.g.* Fishburn, 1982, 1983, 1984). Fishburn (1988) (Chapter 4) derives the skew-symmetric bilinear utility functional mainly from the symmetry axiom. The symmetry axiom imposes the same restrictions on a transitive individual preference relation over lotteries as the weak independence axiom does in this paper. To represent an individual's preference for most probable winner by a skew-symmetric bilinear utility, Fishburn's theory has to be further restricted by axiom 5.

In fact, an individual's preference for most probable winner may be alternatively rationalized by axioms 1-3, 5 and an analogue of Fishburn (1982) symmetry axiom instead of the weak independence axiom 4. Then, axioms 1-3 and the symmetry axiom enable us to represent an individual's preference over lotteries by a skew-symmetric bilinear utility function (theorem 1 in Fishburn (1982): $L_1(p_1,..,p_n)\underset{\prec}{\overset{\succ}{\sim}}L_2(q_1,...,q_n)\Leftrightarrow\sum_{i=1}^{n}\sum_{j=1}^{n}p_iq_j\psi(x_i,x_j)\underset{<}{\overset{>}{=}}0$. Additionally, the first part of axiom 5 implies that $\psi(x_i,x_n)=\psi(x_j,x_n),\forall i\neq j\in[1,n-1]$ and the second part of axiom 5 implies that $\psi(x_1,x_i)=\psi(x_1,x_j),\forall i\neq j\in[2,n]$. Intuitively, an addition of axiom 5 imposes ordinality on Fishburn's function $\psi:\{x_1,...x_n\}\times\{x_1,...x_n\}\to$ ú. Since function $\psi(.,.)$ is also skew-symmetric, we can write $\psi(x_1,x_n)=-\psi(x_n,x_1)$. Thus, when an individual's preference over lotteries satisfies axioms 1-3, 5 and the symmetry axiom, it can be represented as $L_1(p_1,..,p_n)\underset{\prec}{\overset{\succ}{\sim}}L_2(q_1,...,q_n)\Leftrightarrow\sum_{i=1}^{n}\sum_{j=1}^{n}p_iq_j\psi_{ij}\underset{<}{\overset{>}{=}}0$, where $\psi_{ij}=\begin{cases}a & i>j\\ 0 & i=j\\ -a & i<j\end{cases}$, $a\neq 0, a=const$.

Then again simple algebra implies $L_1\underset{\prec}{\overset{\succ}{\sim}}L_2\Leftrightarrow prob(L_1>L_2)\underset{<}{\overset{>}{=}}prob(L_2>L_1)$.

Alternatively, an individual's preference for most probable winner may be captured within the structure of the utility function implied by regret theory, when lottery outcomes are perceived as ordinal. Regret theory proposed by Loomes and Sugden (1982) represents a binary individual's preference over lotteries through the following functional (in our notation): $L_1(p_1,\ldots,p_{n-1}) \underset{\prec}{\overset{\succ}{\sim}} L_2(q_1,\ldots,q_{n-1}) \Leftrightarrow \sum_{i=1}^{n}\sum_{j=1}^{n} p_i q_j \psi(x_i,x_j) \underset{<}{\overset{>}{=}} 0$, where $\psi(x_i,x_j)$ is an anticipated net advantage of having chosen L_1 rather than L_2 if L_1 yields an outcome x_i and L_2 yields an outcome x_j (Loomes and Sugden, 1987).

Clearly, when the function $\psi(x_i,x_j)$ is ordinal in outcomes $\psi(x_i,x_j)=\begin{cases} 1, & x_i > x_j \\ 0, & x_i = x_j \\ -1, & x_i < x_j \end{cases}$, the decision rule of regret theory reduces to a preference for most probable winner. However, such "ordinal" function $\psi(x_i,x_j)$ always violates a key assumption of regret theory, regret aversion, which requires $\forall x > y > z \Rightarrow \psi(x,z) > \psi(x,y) + \psi(y,z)$ (*e.g.* Loomes *et al*., 1992). In terms of regret theory the "ordinal" function $\psi(x_i,x_j)$ of a preference for most probable winner always reflects regret seeking: $\forall x > y > z \Rightarrow \psi(x,z) < \psi(x,y) + \psi(y,z)$.

4. Descriptive validity of proposed axioms

This section critically reviews the existing experimental evidence on the alleged violations of the five proposed axioms. To the best of my knowledge, there are no studies documenting any systematic violations of axioms 1 and 2 (completeness and continuity).

4.1. Violations of first-order stochastic dominance

A small number of experimental studies documents systematic violations of axiom 3 (first-order stochastic dominance). Tversky and Kahneman (1986), Loomes *et al.* (1992), Birnbaum (1997), Birnbaum and Navarrete (1998) and Birnbaum *et al.* (1999) constitute perhaps a complete list. Apparently, when outcomes of two lotteries are tied together in a joint distribution over the states of the world, the concept of a first-order stochastic dominance loses its immediate intuitive appeal to the concept of a state-wise dominance (as defined in Loomes et al. (1992) p. 18). An individual often chooses a stochastically dominated lottery when it appears as a seemingly state-wise dominant.

Tversky and Kahneman (1986) and Loomes *et al.* (1992) introduce a new state dimension of the decision problem (color of a drawn marble and a lottery ticket number, respectively) that ousts the probability dimension of the decision problem. An individual's seeking for a dominance relation in a state-wise manner but not probabilistically is moreover surprising in a study of Loomes *et al.* (1992) that employs a visual presentation of the probability information under which the more probable states of the world should have received more attention than the less probable states. Violations of a first-order stochastic dominance found in Tversky and Kahneman (1986) and Loomes *et al.* (1992) are typically attributed to framing and event-splitting effects.

Birnbaum (1997), Birnbaum and Navarrete (1998) and Birnbaum et al (1999) demonstrate that an individual's preference for a seemingly state-wise dominant lottery prevails over an individual's preference for a stochastically dominant lottery even when the states of the world are endogenous (not exogenously framed): “worst outcome”, “medium outcome” and “best outcome”. In sum, when it is possible to establish a common state dimension between two lotteries, an individual may substitute a first-order stochastic dominance for a seemingly state-wise dominance. However, when first order stochastic dominance is transparent, it is an extremely robust descriptive axiom.

4.2. Violations of betweenness

Axiom 4 (weak independence) implies a betweenness axiom 4a. A casual survey of the experimental literature testing betweenness suggests that it is not a descriptive axiom. However, when this empirical evidence is thoroughly examined, a more favorable picture emerges. The literature on stochastic utility (Loomes and Sugden 1998) reached a generic conclusion that some behavioral patterns, which appear as a systematic violation of a certain principle when taken at a face value, may actually support the principle once a stochastic specification is allowed. This generic conclusion applies to the case of betweenness. In section 4.2.1. I review the alleged systematic violations of betweenness documented in the experimental literature. In section 4.2.2. this evidence is reconciled with a stochastic version of betweenness *i.e.* when an individual obeys betweenness with an occasional random error.

4.2.1. Experimental evidence on betweenness

A relatively large number of experimental studies in the late 1980’s and early 1990’s documents an alleged systematic violation of betweenness. The main findings from that literature can be summarized as follows. Coombs and Huang (1976) (experiment 1), Chew and Waller (1986), Camerer (1989), Battalio *et al.* (1990), Gigliotti and Sopher (1993) (experiments 1 and 3) and Camerer and Ho (1994) all find that approximately 68% of subjects respect betweenness. The remaining subjects are split between quasi-convex (*i.e.* they dislike randomization) and quasi-concave (*i.e.* they like randomization) preferences approximately in a (non-corresponding) proportion of 24% to 8%. This alleged systematic violation of betweenness emerges when some lotteries used in the experiment are located on the edges of the probability triangle.

Coombs and Huang (1976) (experiment 2), Camerer (1992), Starmer (1992) and Gigliotti and Sopher (1993) (experiment 2) find that approximately 76% of subjects respect betweenness and a split between quasi-convex and quasi-concave preferences is non-systematic (approximately in a non-corresponding proportion of 14% to 10%) when all of the lotteries used in the experiment are located inside the probability triangle. Additionally, Camerer and Ho (1994) find that the asymmetry of alleged betweenness violations disappears and Bernasconi (1994) finds that the number of betweenness violations decreases (though not their asymmetry) when a probability mixture of two lotteries is presented in a compound and not a reduced form.

Finally, Prelec (1990) finds that 76% of subjects reveal quasi-concave preferences and only 24% of subjects respect betweenness when probability mass of the hypothetical lotteries is largely shifted to the worst outcome. Camerer and Ho (1994) find the same result for one lottery triple "TUV" with real payoffs. A similar strong asymmetric violation of betweenness when betweenness is not a modal choice pattern is found in two lottery pairs (1 and 3) in Bernasconi (1994).

To sum up, an asymmetric split between quasi-convex and quasi-concave preferences is frequently found in experimental studies but only in very few lottery pairs is betweenness not a modal choice. Since the concept of stochastic utility was formalized only in the mid 1990's, this empirical evidence initially had been accepted as strong support for frequent violations of betweenness (*e.g.* Camerer (1992). Section 4.2.2 shows that this experimental evidence is actually consistent with stochastic betweenness theories.

4.2.2. A reexamination of experimental methodology

The studies reviewed in the above section 4.2.1 employ an experimental methodology that is questionable if an individual is assumed to respect betweenness in a noisy manner. All experimental studies mentioned above use the same method to test for betweenness violations. An experimenter determines two lotteries L_1 and L_2 and asks the subjects to choose one lottery from $\{L_1, L_2\}$, $\{L_1, \alpha L_1 + (1-\alpha)L_2\}$ and $\{L_2, \alpha L_1 + (1-\alpha)L_2\}$, $\alpha \in (0,1)$. Earlier studies typically consider only the first and the second pair-wise choices. A particular fallacy of such a truncated experimental procedure is discussed below. If a probability mixture $\alpha L_1 + (1-\alpha)L_2$ is frequently (almost never) chosen in the second and third pair-wise choices it has been interpreted as an evidence of quasi-concave (quasi-convex) preferences.

Harless and Camerer (1994), Hey and Orme (1994) and Loomes and Sugden (1995) propose different ways of incorporating a stochastic element into deterministic decision theories. The following argument is built upon a more general stochastic specification that is consistent with the stochastic specifications of Hey and Orme (1994) and Loomes and Sugden (1995) but not with Harless and Camerer (1994). Harless and Camerer (1994) p. 1261 propose a constant choice-independent error rate, which apparently has a less intuitive appeal apart from its simplicity.

Suppose that the true preference of an individual is governed by the betweenness axiom. Hence, a probability mixture $\alpha L_1 + (1-\alpha)L_2$ is located between lotteries L_1 and L_2 in terms of utility. An individual's actual (observed) choice is obscured by a random error. Lotteries L_1 and L_2 are more distinct in terms of utility than lotteries $\alpha L_1 + (1-\alpha)L_2$ and L_1 (L_2). Therefore, the impact of a random error is more significant in the second and the third pair-wise choices than in the first pair-wise choice. In terms of the approach of Loomes and Sugden (1995), the strength of an individual's preference relation is greater in the first pair-wise choice than in the second and the third pair-wise choices.

A stochastic betweenness axiom implies either $\text{Prob}(L_1 \succ L_2) > \text{Prob}(\alpha L_1 + (1-\alpha)L_2 \succ L_2), \text{Prob}(L_1 \succ \alpha L_1 + (1-\alpha)L_2) > 0.5$ or $\text{Prob}(L_1 \succ L_2) < \text{Prob}(\alpha L_1 + (1-\alpha)L_2 \succ L_2), \text{Prob}(L_1 \succ \alpha L_1 + (1-\alpha)L_2) < 0.5$. Therefore, if an individual respects stochastic betweenness, a choice pattern when a probability mixture $\alpha L_1 + (1-\alpha)L_2$ is chosen only once in the second and third pair-wise choices should be a modal (most frequent) choice pattern. Additionally, when $\text{Prob}(\alpha L_1 + (1-\alpha)L_2 \succ L_2) > \text{Prob}(L_1 \succ \alpha L_1 + (1-\alpha)L_2)$, a more frequent choice of a probability mixture $\alpha L_1 + (1-\alpha)L_2$ is observed in the second and third pair-wise choices, which appears as evidence of quasi-concave preferences. When $\text{Prob}(\alpha L_1 + (1-\alpha)L_2 \succ L_2) < \text{Prob}(L_1 \succ \alpha L_1 + (1-\alpha)L_2)$, a rare choice of a probability mixture $\alpha L_1 + (1-\alpha)L_2$ is observed in the second and third pair-wise choices, which appears as evidence of quasi-convex preferences. However, both of these choice pattern are consistent with stochastic betweenness that does not impose any restrictions on the relationship between $\text{Prob}(\alpha L_1 + (1-\alpha)L_2 \succ L_2)$ and $\text{Prob}(L_1 \succ \alpha L_1 + (1-\alpha)L_2)$.

For example, let us suppose that lotteries L_1 and L_2 are sufficiently distinct in terms of an individual utility in the sense that a random error reverses very rarely a true individual's preference $L_1 \succ L_2$ so that $\text{Prob}(L_1 \succ L_2) = 0.99$. However, a probability mixture $\alpha L_1 + (1-\alpha)L_2$ is not that distinct from either lottery L_1 or L_2 in terms of individual utility. An occasional random error can reverse a true underlying individual preference $L_1 \succ \alpha L_1 + (1-\alpha)L_2$ and $\alpha L_1 + (1-\alpha)L_2 \succ L_2$. Furthermore, let us assume a mixture $\alpha L_1 + (1-\alpha)L_2$ is closer to lottery L_2 than to lottery L_1 in terms of an individual utility. Thus, a random error obscures an individual's preference $\alpha L_1 + (1-\alpha)L_2 \succ L_2$ more often so that $\text{Prob}(\alpha L_1 + (1-\alpha)L_2 \succ L_2) = 0.73$ is closer to a chance performance (50%) than

$\text{Prob}(L_1 \succ \alpha L_1 + (1-\alpha)L_2) = 0.89$[3]. In terms of the approach of Loomes and Sugden (1995), an individual's preference $L_1 \succ L_2$ is stronger than the preference $L_1 \succ \alpha L_1 + (1-\alpha)L_2$, which in turn is stronger than the preference $\alpha L_1 + (1-\alpha)L_2 \succ L_2$. Then, the above experimental procedure reveals that an individual apparently respects betweenness with probability 65%, apparently has quasi-convex preferences with probability 24% and apparently has concave preferences with probability 8%.

However, it is misleading to interpret these results as a systematic violation of betweenness. To illustrate, Camerer and Ho (1994) p.176 found a similar asymmetric split between quasi-convex and quasi-concave preferences in 11 different experiments and concluded that "betweenness is clearly violated". However, when these authors tried to fit different stochastic theories to the same 11 data sets, a disappointment aversion theory (Gul 1991), which is a betweenness theory, fit best in 5 out of 11 studies and on aggregate it accommodated data better than CPT and EUT (Camerer and Ho 1994 p. 189).

An asymmetric pattern of betweenness violations arises because lotteries L_1, L_2 and $\alpha L_1 + (1-\alpha)L_2$ are not equally spaced in terms of individual utility (Hey and Orme, 1994) or, alternatively, the strength of an individual's preference between lotteries L_1, L_2 and $\alpha L_1 + (1-\alpha)L_2$ is not uniform (Loomes and Sugden, 1995). Therefore, random errors obscure an elicited individual's binary preference relation more severely for some lottery pairs than for others. Persuasive evidence of systematic betweenness violations would have been an asymmetric split between quasi-convex and quasi-concave preferences when betweenness is not a modal choice pattern. However, as documented before, such violations are rare.

Notice that if an asymmetric split between quasi-convex and quasi-concave preferences is caused by random errors, the observed violations of betweenness are likely to

[3] For demonstration purposes the numerical values of probabilities were chosen deliberately to match a typically observed proportion between quasi-convex and quasi-concave preferences in the experimental literature.

be more symmetric for mixtures with α close to 0.5 and they are likely to be highly asymmetric with α close to 0 or 1. This prediction is confirmed exactly by experimental evidence. A highly asymmetric split between quasi-convex and quasi-concave preferences reported in Prelec (1990) and Camerer and Ho (1994) (triple TUV) is elicited for $\alpha = 1/17$, in Bernasconi (1994)—for $\alpha = 0.05$ and $\alpha = 0.95$.

Moreover, all empirical studies using $\alpha = 0.5$ except Camerer and Ho (1994) elicit an individual's choice only from $\{L_1, L_2\}$ and $\{L_1, \alpha L_1 + (1-\alpha)L_2\}$. Under such truncated experimental procedure an individual obeying stochastic betweenness exhibits more often a preference $\alpha L_1 + (1-\alpha)L_2 \succ L_1 \succ L_2$, which appears as if a quasi-concave preference, when $\mathrm{Prob}(L_1 \succ L_2) > \mathrm{Prob}(L_1 \succ \alpha L_1 + (1-\alpha)L_2)$. The same individual exhibits more often a preference $L_2 \succ L_1 \succ \alpha L_1 + (1-\alpha)L_2$, which appears as if a quasi-convex preference, when $\mathrm{Prob}(L_1 \succ L_2) < \mathrm{Prob}(L_1 \succ \alpha L_1 + (1-\alpha)L_2)$. Thus, stochastic betweenness implies a more frequent incidence of quasi-concave (quasi-convex) preferences when $\mathrm{Prob}(L_1 \succ L_2) > 0.5$ ($\mathrm{Prob}(L_1 \succ L_2) < 0.5$) but this asymmetry is due to the fact that an individual's preference between lotteries L_1 and L_2 is stronger *i.e.* less vulnerable to random error than an individual's preference between L_1 and $\alpha L_1 + (1-\alpha)L_2$. In the extreme case when $\mathrm{Prob}(L_1 \succ L_2) = 0.99$ there may be still some chance to observe "as if" quasi-concave preferences $\alpha L_1 + (1-\alpha)L_2 \succ L_1 \succ L_2$ but almost no chance at all to observe "as if quasi-convex" preferences $L_2 \succ L_1 \succ \alpha L_1 + (1-\alpha)L_2$.

4.3. Evidence for fanning-in

A survey of experiments testing the shape of individuals' indifference curves suggests that there is non-negligible evidence for fanning-out going back to the Allais paradox (Allais, 1953) and common consequence and common ratio effects (Starmer, 2000). However, it appears that a universal fanning-out hypothesis (Machina, 1982) is rejected and that there is growing evidence that supports a universal fanning-in. That growing evidence suggests that an individual's indifference curves tend to fan in when probability mass is associated with the best and the worst outcome and tend to fan out when probability mass is associated with outcomes in between. In addition, evidence for fanning-in in all regions of the probability triangle has recently emerged.

Conlisk (1989) finds strong experimental support for the type of fanning-in implied by axiom 5 — 53% and 80% of subjects choose a more risky gamble in a common consequence problem when probability mass is largely shifted to the medium and the best outcome, correspondingly. This finding can be interpreted as an individual's indifference curves becoming almost horizontal when probability mass is largely shifted to the best outcome. Analogously, so-called vertical fanning-in is documented in Starmer and Sugden (1989), Camerer (1989) p.92 and Battalio *et al* (1990). Wu and Gonzalez (1998) p.119 report vertical fanning-in when the probability of the best outcome is above 0.33 and vertical fanning-out when it is below 0.33[4].

Prelec (1990) and Kagel *et al.* (1990) find fanning-in when probability mass is largely shifted to the worst outcome. Wu and Gonzalez (1996) report so-called horizontal fanning-in when a probability of the worst outcome is above 0.63 and horizontal fanning-out when it is below 0.63. Camerer (1989) p.92 finds similar evidence for small gains[5].

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[4] Neilson (1992) proposed a mixed-fan hypothesis for fanning-in when the best outcome is highly probable and fanning-out otherwise.

[5] Trying to match this empirical evidence Jia *et al* (2001) developed a generalized disappointment model that can imply fanning-in in the neighborhood of the best and the worst lottery outcome.

Bernasconi (1994) p.63 finds experimental evidence for fanning-in by observing a reverse common ratio effect. Cubitt and Sugden (2001), Bosman and van Winden (2001) and Cubitt *et al* (2002) find indirect evidence for a reverse common ratio effect in dynamic decision making under risk. Barron and Erev (2003) find a reverse common ratio effect in small feedback-based decision making. Battalio *et al* (1990) and Thaler and Johnson (1990) find evidence for fanning-in *i.e.* an increased risk seeking for stochastically dominant lotteries when lotteries involve only guaranteed gains. Finally, Starmer (1992) and Humphrey and Verschoor (2002) find strong evidence consistent with universal fanning-in in all regions of the probability triangle.

The above literature elicits fanning-in/out of an individual's indifference curves from an observed binary choice in a common consequence or common ratio problem involving lotteries typically defined on a common three-outcome structure. The main findings from this literature can be summarized in the probability triangle (figure 2).

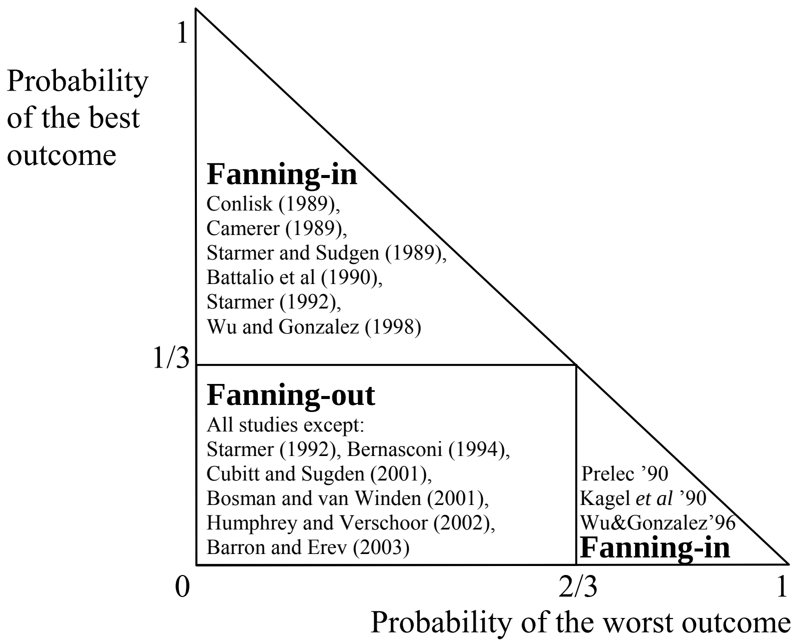

Figure 2 Empirical evidence for fanning-in

5. Conclusions

The proposed axiomatization explores theoretical features of an individual's preference for most probable winner in a binary choice under risk. Although such preference is implied by a simplistic behavioral rule (the heuristic of relative probability comparisons), I find some surprising perhaps even unexpected connections with other decision theories (implicit weighted utility, implicit expected utility, skew-symmetric bilinear utility, weighted utility and regret theory). A preference for most probable winner can be rationalized by a system of five axioms: three standard axioms (completeness, continuity and first order stochastic dominance) and two less standard ones (weak independence and a fanning-in). Notably, transitivity of preferences is not required. The present paper deals with binary choice; a natural extension of this work is to axiomatize a preference for most probable winner in a choice among many lotteries.

A preference for most probable winner falls into the betweenness class of decision theories that assume linearity in probability of sets $\{L : L \sim L_0\}, \forall L_0$. Thus, it is a special case of implicit weighted (expected) utility theory when the assumption of transitivity is relaxed and individual indifference curves exhibit a universal fanning-in. The alleged systematic violations of betweenness found in the experimental literature in the late 1980's and early 1990's can be explained within the concept of a stochastic utility developed in the mid 1990's. If the experimental evidence is reevaluated in the light of notions of stochastic utility, the betweenness axiom turns out to be quite descriptive.

All in all, four of the proposed axioms (completeness, continuity, first order stochastic dominance, and betweenness) appear to be empirically sound. Additionally, experimental evidence is emerging for universal fanning-in of individuals' indifference curves. However, this evidence seems to be stronger for some areas of the probability triangle than for others. The experimental evidence for the system of axioms proposed here to rationalize an individual's preference for most probable winner provides indirect evidence for the domain of applicability of the heuristic of relative probability comparisons.

The five proposed axioms are necessary and sufficient conditions to characterize an individual preference for most probable winner. This preference can be captured by a menu-dependent (relative) utility function $U(L_1, L_2) = prob(L_1 > L_2)$. This utility function is a special case of a skew-symmetric bilinear utility theory that coincides with the weighted utility theory when lotteries are defined on a common three-outcome structure (when an individual's preference for most probable winner is transitive). In general, an individual's preference for most probable winner is intransitive but it is captured within the structure of a regret theory utility function when outcomes are perceived as ordinal and the assumption of regret aversion is replaced with a preference for a win. Thus, an individual's preference for most probable winner is a simplified mirror image of regret theory.

Finally, as a theoretical byproduct this paper shows that betweenness implies transitivity (assuming completeness, continuity and first-order stochastic dominance) in a choice between lotteries defined on common three-outcome structure. Some studies *e.g.* Camerer and Ho (1994) tested separately the violations of transitivity and betweenness inside a probability triangle. Proposition 2 of this paper shows that such an experiment is trivial. It may also explain why many experiments (employing choices inside the probability triangle) document a very low incidence of intransitive preferences. For example, an individual preferring most probable winner has transitive preferences inside a probability triangle but in general he chooses intransitively. Thus, the results from a binary choice between lotteries defined on a common three-outcome structure may be biased in conclusions on the transitivity of preferences.

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Appendix I

Proof of proposition 1

Lemma 1 If axiom 4a holds then for any lotteries $L_0, L_1, ..., L_n$ such that $L_i \sim L_0, i \in [1, n]$ and

$$\forall \alpha_1, ..., \alpha_n \in (0,1),\ \sum_{i=1}^{n} \alpha_i = 1,\ \Rightarrow \alpha_1 L_1 + ... + \alpha_n L_n \sim L_0$$

■ Proof by mathematical induction. If $n = 2$ lemma 1 immediately follows from axiom 4a: $L_1 \sim L_0, L_2 \sim L_0, \overrightarrow{L_0 L_0} = 0 \cdot \overrightarrow{L_1 L_2} \Rightarrow \forall \alpha \in (0,1): \alpha L_1 + (1-\alpha) L_2 \sim L_0$. Let us assume that lemma holds for $n \le N-1$. If $n = N$ then $\alpha_1 L_1 + ... + \alpha_N L_N = \alpha_N L_N + (1-\alpha_N) L^*$, where $L^* = \frac{\alpha_1}{1-\alpha_N} L_1 + ... + \frac{\alpha_{N-1}}{1-\alpha_N} L_{N-1}$. $L^* \sim L_0$ because we assumed that lemma 1 holds for $n \le N-1$. Then it follows from axiom 4 that $\alpha_N L_N + (1-\alpha_N) L^* \sim L_0$. Thus, lemma holds for $n = N$ and accordingly to the principle of mathematical induction it also holds for any *n*. ■

Proof of proposition 1 is organized in several steps. First we find $n-1$ lotteries such that an individual is indifferent between them and a given lottery $L_0(q_1, ...q_{n-1})$. Then we prove that an individual is also indifferent between L_0 and any lottery which is a compound lottery (probability mixture) over these $n-1$ lotteries. Finally we show that any lottery, such that an individual is indifferent between it and L_0, is the reduced form of some compound lottery over the obtained $n-1$ lotteries.

Let us consider lottery $\underline{L_1}(1,0,...,0)$ yielding the worst possible outcome x_1 for sure and lottery $\overline{L_1}(0,...,0)$ giving the best possible outcome x_n for sure. From axiom 1 it follows that either $\underline{L_1} \succ L_0$ or $\underline{L_1} \prec L_0$ or $\underline{L_1} \sim L_0$. Since x_1 is the worst possible outcome it must be the case that either $\underline{L_1} \prec L_0$ or $\underline{L_1} \sim L_0$. The special case when $\underline{L_1} \sim L_0$ will be considered later. Similarly we prove that $L_0 \prec \overline{L_1}$. From axiom 2 it follows that $\exists \alpha_1 \in (0,1): \alpha_1 \underline{L_1} + (1-\alpha_1)\overline{L_1} = L_1(\alpha_1, 0, ..., 0) \sim L_0$.

Now we will consider lottery $\underline{L_2}(0,1,0,...,0)$ and $\overline{L_2}(0,...,0)$. If $\underline{L_2} \prec L_0 \prec \overline{L_2}$ we define $L_2(0, \alpha_2, 0, ..., 0) \sim L_0$ in a similar manner as L_1 and we continue this process until either lottery L_{n-1} is defined or for some $i \in [2, n-1]$: $\underline{L_i}\left(0,...,0,\underbrace{1}_{i-th},0,...,0\right) \succ L_0$ (special case of indifference is considered below). In such case $\underline{L_i}(1,0,...,0) \prec L_0$ by the definition of the worst possible outcome and $\overline{L_i}\left(0,...,0,\underbrace{1}_{i-th},0,...,0\right) \succ L_0$. From axiom 2 it follows that $\exists \alpha_i \in (0,1): \alpha_i \underline{L_i} + (1-\alpha_i)\overline{L_i} = L_i\left(\alpha_i, 0, ..., 0, \underbrace{1-\alpha_i}_{i-th}, 0, ..., 0\right) \sim L_0$. We continue this process further until lottery L_{n-1} is defined. If for some $i \in [1, n-1]$ $\underline{L_i}(0,...,0,1,0,...,0) \sim L_0$ then $L_i = \underline{L_i}$. Thus, for any lottery $L_0(q_1, q_2, ... q_{n-1})$ we obtain a sequence of $n-1$ lotteries $\{L_i\}_{i=1}^{n-1}: L_i \sim L_0$ except for the case when $L_0 \sim \underline{L_1}(1,0,...,0)$ and $L_0 \sim \overline{L_1}(0,...,0)$. If an individual is indifferent between lottery L_0 and a lottery yielding the worst outcome for sure, then L_0 is $L_0(1,0,...,0)$. Since there could not be any lottery such that an individual is indifferent between it and a lottery yielding the worst outcome for sure, proposition 1 is not applicable to the case when L_0 is $L_0(1,0,...,0)$. Similarly the degenerate case $L_0 \sim \overline{L_1}(0,...,0)$ is disregarded.

From lemma 1 it follows that $\forall \beta_1, ..., \beta_{n-1} \in (0,1), \sum_{i=1}^{n-1} \beta_i = 1$: $\beta_1 L_1 + ... + \beta_{n-1} L_{n-1} \sim L_0$. Let us denote the reduced form of lottery $\beta_1 L_1 + ... + \beta_{n-1} L_{n-1}$ as $L(p_1, p_2, ..., p_{n-1})$. Obviously, since there are only $n-2$ free parameters $\beta_1, ..., \beta_{n-2}$ (β_{n-1} is automatically determined from condition $\sum_{i=1}^{n-1}\beta_i = 1$) there is a restriction on $n-1$ reduced form probabilities $p_1, p_2, ..., p_{n-1}$. To find out this restriction we eliminate $\beta_1, ..., \beta_{n-2}$ from the following system of equations:

$$\begin{cases} \alpha_1\beta_1 + \alpha_i\beta_i + \alpha_{i+1}\beta_{i+1} + ... + \alpha_{n-2}\beta_{n-2} + +\alpha_{n-1}(1-\beta_1-\beta_2-...-\beta_{n-2}) = p_1 \\ \alpha_2\beta_2 = p_2 \\ ... \\ \alpha_{i-1}\beta_{i-1} = p_{i-1} \\ (1-\alpha_i)\beta_i = p_i \\ ... \\ (1-\alpha_{n-2})\beta_{n-2} = p_{n-2} \\ (1-\alpha_{n-1})(1-\beta_1-\beta_2-...-\beta_{n-2}) = p_{n-1} \end{cases}$$

The result is: $\frac{1}{\alpha_1}p_1 + ... + \frac{1}{\alpha_{i-1}}p_{i-1} + \frac{1-\alpha_i/\alpha_1}{1-\alpha_i}p_i + ... + \frac{1-\alpha_{n-1}/\alpha_1}{1-\alpha_{n-1}}p_{n-1} = 1$

Or $a_1p_1 + ... + a_{n-1}p_{n-1} = 1$ where $a_j = \frac{1}{\alpha_j}, \forall j \in [1, i-1]$ and $a_j = \frac{1-\alpha_j/\alpha_1}{1-\alpha_j}, \forall j \in [i, n-1]$.

Thus we have found $n-1$ real parameters $a_1, ..., a_{n-1}$ such that for any lottery $L(p_1, ..., p_{n-1})$ satisfying the restriction $a_1p_1 + ... + a_{n-1}p_{n-1} = 1$ it must be the case that $L \sim L_0$. Now we will prove that $\forall \widetilde{L}(\widetilde{p}_1, ..., \widetilde{p}_{n-1}): \widetilde{L} \sim L_0 \Rightarrow a_1\widetilde{p}_1 + ... + a_{n-1}\widetilde{p}_{n-1} = 1$.

First we will prove that the above statement holds for $\widetilde{L} = L_0$ i.e. $a_1q_1 + ... + a_{n-1}q_{n-1} = 1$. Suppose $a_1q_1 + ... + a_{n-1}q_{n-1} \neq 1$. Then it is possible to construct a lottery $L \sim L_0$ such that its reduced form is $L(p_1, q_2, ..., q_{n-1})$ by setting

$$\beta_1 = 1 - \frac{q_2}{\alpha_2} - ... - \frac{q_{i-1}}{\alpha_{i-1}} - \frac{q_i}{1-\alpha_i} - ... - \frac{q_{n-1}}{1-\alpha_{n-1}}$$
$$\beta_2 = \frac{q_2}{\alpha_2}$$
$$...$$
$$\beta_{i-1} = \frac{q_{i-1}}{\alpha_{i-1}}$$
$$\beta_i = \frac{q_i}{1-\alpha_i}$$
$$...$$
$$\beta_{n-2} = \frac{q_{n-1}}{1-\alpha_{n-1}}$$

Then $p_1 = \alpha_1 \left(1 - \frac{q_2}{\alpha_2} - ... - \frac{q_{i-1}}{\alpha_{i-1}} - \frac{q_i}{1-\alpha_i} - ... - \frac{q_{n-1}}{1-\alpha_{n-1}} \right) + \frac{\alpha_i q_i}{1-\alpha_i} + ... + \frac{\alpha_{n-1} q_{n-1}}{1-\alpha_{n-1}} \neq q_1$ If $p_1 > q_1$ (or $p_1 < q_1$) then accordingly to axiom 3 $L \prec L_0$ (or $L \succ L_0$), which contradicts to $L \sim L_0$. Thus, it must be the case that $p_1 = q_1 \Rightarrow a_1 q_1 + ... + a_{n-1} q_{n-1} = 1$.

Now suppose there exists a lottery $\widetilde{L}(\widetilde{p}_1,...,\widetilde{p}_{n-1})$ such that $\widetilde{L} \sim L_0$ but $a_1 \widetilde{p}_1 + ... + a_{n-1} \widetilde{p}_{n-1} \neq 1$. Since $\widetilde{L} \sim L_0$ and $L_1 \sim L_0$ then accordingly to lemma 1 $\forall \gamma \in (0,1)$: $\gamma \widetilde{L} + (1-\gamma) L_1 \sim L_0$. The reduced form of lottery $\gamma \widetilde{L} + (1-\gamma) L_1$ is $\widetilde{L}_1(\gamma \widetilde{p}_1 + (1-\gamma)\alpha_1, \gamma \widetilde{p}_2,...,\gamma \widetilde{p}_{n-1})$. Similarly we can construct lottery $\widetilde{L}_2(\gamma \widetilde{p}_1, \gamma \widetilde{p}_2 + (1-\gamma)\alpha_2, \gamma \widetilde{p}_3,...,\gamma \widetilde{p}_{n-1}) \sim L_0$ and so forth until lottery $\widetilde{L}_{n-1}$ is constructed.

Lemma 1 guarantees that $\forall \beta_1,...\beta_{n-1} \in (0,1)$, $\sum_{i=1}^{n-1} \beta_i = 1, \Rightarrow \beta_1 \widetilde{L}_1 + ... + \beta_{n-1} \widetilde{L}_{n-1} \sim L_0$. It is possible to construct lottery $\beta_1 \widetilde{L}_1 + ... + \beta_{n-1} \widetilde{L}_{n-1}$ in such a way that its reduced form is $\overline{L}(p_1^*, q_2, q_3,...,q_{n-1})$ by setting

$$\beta_1 = 1 - \frac{q_2}{\alpha_2} - ... - \frac{q_{i-1}}{\alpha_{i-1}} - \frac{q_i}{1-\alpha_i} - \frac{q_{i+1}}{1-\alpha_{i+1}} - ... - \frac{q_{n-1} - \gamma \widetilde{p}_{n-1}}{(1-\gamma)(1-\alpha_{n-1})}$$
$$\beta_2 = \frac{q_2 - \gamma \widetilde{p}_2}{(1-\gamma)\alpha_2}$$
$$...$$
$$\beta_{i-1} = \frac{q_{i-1} - \gamma \widetilde{p}_{i-1}}{(1-\gamma)\alpha_{i-1}}$$
$$\beta_i = \frac{q_i - \gamma \widetilde{p}_i}{(1-\gamma)(1-\alpha_i)}$$
$$...$$
$$\beta_{n-2} = \frac{q_{n-2} - \gamma \widetilde{p}_{n-2}}{(1-\gamma)(1-\alpha_{n-2})}$$

Lottery $\overline{L}$ exists because it is always possible to choose a sufficiently small $\gamma \neq 0$ so that the reduced form probabilities of lottery $\overline{L}$ are all positive. Then $p_1^* = \gamma \widetilde{p}_1 + (1-\gamma) q_1$ and if $\widetilde{p}_1 \neq q_1$, accordingly to axiom 3, either $\overline{L} \succ L_0$ (when $\widetilde{p}_1 < q_1$) or $\overline{L} \prec L_0$ (when $\widetilde{p}_1 > q_1$), which contradicts to $\widetilde{L} \sim L_0$. Thus, it must be the case that $\widetilde{p}_1 = q_1$. In a similar manner we can prove that $\widetilde{p}_j = q_j, \forall j \in [2, n-1]$. Hence, the only lottery $\widetilde{L}(\widetilde{p}_1, \widetilde{p}_2, ..., \widetilde{p}_{n-1})$ that can violate restriction $a_1 \widetilde{p}_1 + a_2 \widetilde{p}_2 + ... + a_{n-1} \widetilde{p}_{n-1} = 1$ is $\widetilde{L} = L_0$. But we already proved that $a_1 q_1 + a_2 q_2 + ... + a_{n-1} q_{n-1} = 1$. Thus, $\forall L(p_1, p_2, ..., p_{n-1}) \sim L_0 \Rightarrow a_1 p_1 + a_2 p_2 + ... + a_{n-1} p_{n-1} = 1$. This completes the proof.

Appendix II

Proof of proposition 2

Without the loss of generality let us assume that lotteries $L_1(q_1, q_2), L_2(r_1, r_2), L_3(s_1, s_2)$ are labeled in such a way that $q_1 \geq s_1 \geq r_1$. Accordingly to proposition 1 the set of lotteries $\{L : L \sim L_2\}$ is $\{L(p_1, p_2) : a_1 p_1 + a_2 p_2 = 1, \quad a_1, a_2 = const\}$. All three lotteries L_1, L_2 and L_3 belong to this set. Therefore, the probability distributions of these lotteries satisfy the restrictions $a_1 q_1 + a_2 q_2 = 1, a_1 r_1 + a_2 r_2 = 1, a_1 s_1 + a_2 s_2 = 1$. Using these restrictions it is possible to show that the reduced form of a compound lottery $\alpha L_1 + (1-\alpha) L_2$ is $L_3(s_1, s_2)$ when $\alpha = \dfrac{s_1 - r_1}{q_1 - r_1}$. Accordingly to axiom 4a, since $L_1 \sim L_2$ and $\alpha \in (0,1)$ then it must be the case that $\alpha L_1 + (1-\alpha) L_2 = L_3 \sim L_1$

Proof of proposition 3

Proof of proposition 3 is organized in several steps. First it is demonstrated that when axioms 1-4 hold then for any distinct lotteries L_1, L_2 the sets $\{L : L \sim L_1\}$ and $\{L : L \sim L_2\}$ have a unique crossover point (x_1, x_2) though this point does not necessary belong to lottery space (see footnote 1 on page 8). Then it is demonstrated that if, additionally, axiom 5 holds this point is $(1, -1)$. This fact allows us to derive the exact functional form of indifference curve as presented in proposition 3.

Let us consider four arbitrary distinct lotteries $L_1(p_1,p_2), L_2(q_1,q_2), L_1'\left(p_1',p_2'\right)$ and $L_2'\left(q_1',q_2'\right)$ such that $L_1 \sim L_2$, $L_1' \sim L_2'$ and $\overrightarrow{L_1L_1'} = \alpha\,\overrightarrow{L_2L_2'}$ for some $\alpha \in ú$. Accordingly to proposition 1 the set of lotteries $\{L : L \sim L_1\}$ is $\{(r_1,r_2) : a_1r_1 + a_2r_2 = 1\}$. Since both lotteries $L_1(p_1,p_2)$ and $L_2(q_1,q_2)$ belong to this set it must be the case that $a_1p_1 + a_2p_2 = 1$ and $a_1q_1 + a_2q_2 = 1$. Solving for parameters a_1, a_2 we can define the set $\{L : L \sim L_1\}$ as $\left\{(r_1,r_2) : \frac{q_2 - p_2}{q_2p_1 - q_1p_2}r_1 + \frac{p_1 - q_1}{q_2p_1 - q_1p_2}r_2 = 1\right\}$. Similarly, the set $\left\{L : L \sim L_1'\right\}$ can be defined as $\left\{(r_1,r_2) : \frac{q_2' - p_2'}{q_2'p_1' - q_1'p_2'}r_1 + \frac{p_1' - q_1'}{q_2'p_1' - q_1'p_2'}r_2 = 1\right\}$.

The sets $\{L : L \sim L_1\}$ and $\left\{L : L \sim L_1'\right\}$ are defined on lottery space (r_1,r_2). If we relax the restriction $r_1, r_2 \in [0,1]$ allowing $r_1, r_2 \in ú$ plus actual infinities then the sets $\{L : L \sim L_1\}$ and $\left\{L : L \sim L_1'\right\}$ necessarily have a common crossover point (x_1,x_2):

$$x_1 = \frac{\left(p_1' - q_1'\right)(q_2p_1 - q_1p_2) - (p_1 - q_1)\left(q_2'p_1' - q_1'p_2'\right)}{\left(p_1' - q_1'\right)(q_2 - p_2) - (p_1 - q_1)\left(q_2' - p_2'\right)}$$

$$x_2 = \frac{(q_2 - p_2)\left(q_2'p_1' - q_1'p_2'\right) - \left(q_2' - p_2'\right)(q_2p_1 - q_1p_2)}{\left(p_1' - q_1'\right)(q_2 - p_2) - (p_1 - q_1)\left(q_2' - p_2'\right)}$$

Since lotteries L_1, L_2, L_1' and L_2' satisfy the requirement of axiom 4 it follows that $\forall \beta \in (0,1) : \beta L_1 + (1-\beta)L_1' \sim \beta L_2 + (1-\beta)L_2'$. Calculating the reduced form of probability mixtures $\beta L_1 + (1-\beta)L_1'$ and $\beta L_2 + (1-\beta)L_2'$ we can demonstrate the set $\left\{L : L \sim \beta L_1 + (1-\beta)L_1'\right\}$ is $\left\{(r_1,r_2) : a_1^\beta r_1 + a_2^\beta r_2 = 1\right\}$ where:

$$a_1^{\beta} = \frac{q_2' - p_2' - \beta\left(q_2' - q_2\right) + \beta\left(p_2' - p_2\right)}{\left(q_2' + \beta\left(q_2 - q_2'\right)\right)\left(p_1' + \beta\left(p_1 - p_1'\right)\right) - \left(q_1' + \beta\left(q_1 - q_1'\right)\right)\left(p_2' + \beta\left(p_2 - p_2'\right)\right)}$$

$$a_2^{\beta} = \frac{p_1' - q_1' + \beta\left(q_1' - q_1\right) - \beta\left(p_1' - p_1\right)}{\left(q_2' + \beta\left(q_2 - q_2'\right)\right)\left(p_1' + \beta\left(p_1 - p_1'\right)\right) - \left(q_1' + \beta\left(q_1 - q_1'\right)\right)\left(p_2' + \beta\left(p_2 - p_2'\right)\right)}$$

The set $\left\{L : L \sim \beta L_1 + (1-\beta)L_1'\right\}$ is defined on lottery space. If we define it on pairs (r_1, r_2) of real numbers plus actual infinities then for $\forall \beta \in (0,1)$ the crossover point (x_1, x_2) as defined above also belongs to the set $\left\{(r_1, r_2) : a_1^{\beta} r_1 + a_2^{\beta} r_2 = 1\right\}$ as long as

$$\frac{p_1' - p_1}{p_2' - p_2} = \frac{q_1' - q_1}{q_2' - q_2}$$

The last equality always holds because $\exists \alpha \in \mathbb{R} : \overrightarrow{L_1 L_1'} = \alpha \overrightarrow{L_2 L_2'}$ *i.e.* the difference between probability distributions of lotteries L_1 and L_1' is "similar" to the difference between probability distributions of lotteries L_2 and L_2'. Thus, for four arbitrary distinct lotteries $L_1(p_1, p_2), L_2(q_1, q_2), L_1'\left(p_1', p_2'\right)$ and $L_2'\left(q_1', q_2'\right)$ such that $L_1 \sim L_2$, $L_1' \sim L_2'$ and $\overrightarrow{L_1 L_1'} = \alpha \overrightarrow{L_2 L_2'}$ for some $\alpha \in \mathbb{R}$ we found a unique crossover point (x_1, x_2) such that for any probability mixture $\beta L_1 + (1-\beta)L_1'$ the set $\left\{L : L \sim \beta L_1 + (1-\beta)L_1'\right\}$ contains the point (x_1, x_2).

Let us pick up lottery $L_1(p_1, 0)$ so that it almost gives the best possible outcome for sure (p_1 approaches zero). By choosing p_1 to be very small we obtain $L_m(0,1) \prec L_1 \prec \overline{L}(0,0)$. From axiom 2 it follows that $\exists L_2(0, q_2) : L_2 \sim L_1$. Note that $\lim_{p_1 \to 0} q_2 = 0$. Consider lottery $L_2'(1 - q_2, q_2)$. Since $\underline{L}(1,0) \prec L_2' \prec \overline{L}(0,0)$ it follows from axiom 2 that

$\exists L_1^{\prime}\left(p_1^{\prime},0\right): L_1^{\prime} \sim L_2^{\prime}$. Since $\lim_{p_1 \to 0} q_2 = 0$ it must also be the case that $\lim_{p_1 \to 0} p_1^{\prime} = 1$. $\overrightarrow{L_1 L_1'} = \left(p_1^{\prime} - p_1, 0\right)$ and $\overrightarrow{L_2 L_2'} = (1 - q_2, 0)$. Obviously, $\exists \alpha = \left(p_1^{\prime} - p_1\right) \big/ (1 - q_2) \in$ ú such that $\overrightarrow{L_1 L_1'} = \alpha \overrightarrow{L_2 L_2'}$. Therefore, there is a unique crossover point (x_1, x_2) such that for any probability mixture $\beta L_1 + (1-\beta) L_1^{\prime} = L_\beta(p_\beta, 0)$ the set $\{L : L \sim L_\beta\}$ contains the point (x_1, x_2).

For any arbitrary lottery L_0: $\underline{L}(1,0) \prec L_0 \prec \overline{L}(0,0)$ and therefore accordingly to axiom 2 there exist a lottery $\widetilde{L}_0(p_0, 0) \sim L_0$. Since $\lim_{p_1 \to 0} p_1^{\prime} = 1$ it is possible to pick up a sufficiently small p_1 such that $L_1^{\prime} \prec \widetilde{L}_0(p_0, 0) \prec L_1$ due to axiom 3. Thus, there exist $\beta \in (0,1)$ such that $\beta L_1 + (1-\beta) L_1^{\prime} = L_\beta(p_\beta, 0) = \widetilde{L}_0(p_0, 0)$. Consequently, lottery L_0 belongs to the set $\{L : L \sim L_\beta\}$ as well as the point (x_1, x_2). Accordingly to proposition 2 transitivity holds and lottery L_β as well as the point (x_1, x_2) also belong to the set $\{L : L \sim L_0\}$. Therefore, if axioms 1-4 hold then it is possible to find a unique point (x_1, x_2) such that for any arbitrary lottery L_0: $(x_1, x_2) \in \{L : L \sim L_0\}$. Effectively this means that there is a unique crossover point at which all indifference sets $\{L : L \sim L_0\}$ intersect.

Specifically, for lotteries $L_1(p_1, 0), L_2(0, q_2), L_2^{\prime}(1 - q_2, q_2)$ and $L_1^{\prime}\left(p_1^{\prime}, 0\right)$ this crossover point (x_1, x_2) is: $x_1 = \dfrac{p_1(q_2 - 1)}{q_2 - 1 + p_1^{\prime} - p_1}, x_2 = \dfrac{q_2\left(p_1^{\prime} - p_1\right)}{q_2 - 1 + p_1^{\prime} - p_1}$. Accordingly to axiom 5 $\lim_{p_1 \to 0} \dfrac{p_1 - q_2}{p_1} = 0, \Rightarrow \lim_{p_1 \to 0} \dfrac{p_1}{q_2} = 1$ and $\lim_{1 - p_1^{\prime} \to 0} \dfrac{q_2 - 1 + p_1^{\prime}}{q_2} = 0, \Rightarrow \lim_{p_1^{\prime} \to 1} \dfrac{q_2 - 1 + p_1^{\prime}}{q_2} = 0$.

Let us explore crossover point (x_1, x_2) when $p_1 \to 0$ (and consequently $p_1^{\prime} \to 1$).

$$\lim_{p_1\to 0} x_1 = x_1 = \lim_{p_1\to 0}\frac{p_1(q_2-1)}{q_2-1+p_1'-p_1} = \frac{\lim\limits_{p_1\to 0}\frac{p_1}{q_2}\cdot\left(\lim\limits_{p_1\to 0} q_2-1\right)}{\lim\limits_{p_1'\to 1}\frac{q_2-1+p_1'}{q_2}-\lim\limits_{p_1\to 0}\frac{p_1}{q_2}} = \frac{1\cdot(0-1)}{0-1} = 1,$$

$$\lim_{p_1\to 0} x_2 = x_2 = \lim_{p_1\to 0}\frac{p_1'-p_1}{\frac{q_2-1+p_1'}{q_2}-\frac{p_1}{q_2}} = \frac{\lim\limits_{p_1\to 0}\left(p_1'-p_1\right)}{\lim\limits_{p_1'\to 1}\frac{q_2-1+p_1'}{q_2}-\lim\limits_{p_1\to 0}\frac{p_1}{q_2}} = \frac{1-0}{0-1} = -1.$$

Therefore, if axioms 1-5 hold then for any arbitrary lottery $L_0(s_1,s_2)$: $(1,-1)\in\{L : L\sim L_0\}$. Accordingly to proposition 1 the set $\{L : L\sim L_0\}$ is linear in probabilities $\{L : L\sim L_0\}=\{L(r_1,r_2) : \hat{a}_1 r_1+\hat{a}_2 r_2=1\}$. Since $L_0(s_1,s_2)\in\{L(r_1,r_2) : \hat{a}_1 r_1+\hat{a}_2 r_2=1\}$ and $(1,-1)\in\{L(r_1,r_2) : \hat{a}_1 r_1+\hat{a}_2 r_2=1\}$ it follows that $\hat{a}_1 s_1+\hat{a}_2 s_2=1$ and $\hat{a}_1-\hat{a}_2=1$. Solving for parameters $\hat{a}_1$ and $\hat{a}_2$ we obtain that the set $\{L : L\sim L_0\}$ is $\left\{L(r_1,r_2) : \frac{1+s_2}{s_1+s_2}r_1+\frac{1-s_1}{s_1+s_2}r_2=1\right\}$.

Then for any arbitrary three-outcome lottery $L_0(s_1,s_2)$ the set $\{L : L\sim L_0\}$, defined on the same three-outcome structure, is $\left\{L(r_1,r_2) : \frac{1+s_2}{s_1+s_2}r_1+\frac{1-s_1}{s_1+s_2}r_2=1\right\}$.

Proof of proposition 4

Proof by mathematical induction. In proposition 3 we demonstrated that proposition 4 holds for $n=3$. Let us assume that the proposition holds for $n\le N-1$. Now we will prove that it also holds for $n=N$. Consider the compound lottery $L_0'(q_1+q_2,q_3,\ldots,q_{N-1})$ defined over the set of outcomes $\{L_{1+2},x_3,\ldots,x_N\}$. L_{1+2} is a lottery yielding outcome x_1 with probability $\frac{q_1}{q_1+q_2}$ and outcome x_2 with probability $\frac{q_2}{q_1+q_2}$. Obviously $L_0'=L_0$. Since $L_1\prec x_3\prec\ldots\prec x_N$ we can treat lottery L_0' as $N-1$ outcome lottery satisfying the prerequisites of axioms 1-5. We assumed that proposition 3 holds for $n \le N-1$. Thus, the set $\{L : L \sim L_0'\}$ is $\left\{L(p_1,...,p_{N-2}) : \frac{1+q_3+...+q_{N-1}}{q_1+...+q_{N-1}} p_1 + \sum_{i=2}^{N-2} \frac{1-q_1-...-q_i+q_{i+2}+...+q_{N-1}}{q_1+...+q_{N-1}} p_i = 1\right\}$. However, the compound lottery $L(p_1,...,p_{N-2})$ is defined over the set of lottery outcomes $\{L_{1+2}, x_3,...,x_N\}$. We will rewrite lottery L in a reduced form as a probability distribution over the prime lottery outcomes $\{x_1,...,x_N\}$—$L\left(\frac{q_1}{q_1+q_2} p_1, \frac{q_2}{q_1+q_2} p_1, p_2,...,p_{N-2}\right)$.

Accordingly to proposition 1 $\exists a_1,...a_{N-1} \in R$ such that $\forall L(r_1,...,r_{N-1}) : L \sim L_0$ $\Leftrightarrow a_1 r_1 + ... + a_{N-1} r_{N-1} = 1$. Since $L \sim L_0' = L_0$ the reduced form probabilities of lottery L should also satisfy the restriction $a_1 r_1 + ... + a_{N-1} r_{N-1} = 1$. Algebraically this corresponds to $\frac{a_1 q_1 + a_2 q_2}{q_1+q_2} p_1 + a_3 p_2 + ... + a_{N-1} p_{N-2} = 1$. But we know already that $\frac{1+q_3+...+q_{N-1}}{q_1+...+q_{N-1}} p_1 + \sum_{i=2}^{N-2} \frac{1-q_1-...-q_i+q_{i+2}+...+q_{N-1}}{q_1+...+q_{N-1}} p_i = 1$. The only possibility when these two equations are mutually compatible for any lottery $L(p_1,...,p_{N-2})$ is when $a_i = \frac{1-q_1-...-q_{i-1}+q_{i+1}+...+q_{N-1}}{q_1+...+q_{N-1}}, i \in [3, N-1]$ and $\frac{a_1 q_1 + a_2 q_2}{q_1+q_2} = \frac{1+q_3+...+q_{N-1}}{q_1+...+q_{N-1}}$. Using the compound lottery $L_0''(q_1, q_2, q_3+q_4, q_5,...,q_{N-1})$ defined over the set of outcomes $\{x_1, x_2, L_{3+4}, x_5,...,x_N\}$[6] we can also demonstrate in the same manner as above that $a_i = \frac{1-q_1-...-q_{i-1}+q_{i+1}+...+q_{N-1}}{q_1+...+q_{N-1}}, i \in [1,2] \cup [5, N-1]$. Thus, we have proven that proposition 4 holds for $n = N$. Then accordingly to the principle of mathematical induction proposition 4 holds for all n outcome lotteries.

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[6] L_{3+4} is a lottery yielding outcome x_3 with probability $\frac{q_3}{q_3+q_4}$ and outcome x_4 with probability $\frac{q_4}{q_3+`q_4}$.

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