On the measurement of polarization for ordinal data

Martyna Kobus*†

Abstract

Atkinson's Theorem (Atkinson, 1970) is a classic result in inequality measurement. It establishes Lorenz dominance as a useful criterion for comparative judgements of inequality between distributions. If a Lorenz distribution A dominates distribution B, then *all* indices in a broad class of measures must confirm A as less unequal than B. Recent research, however, shows that standard inequality theory cannot be applied to ordinal data (Zheng, 2008), such as self-reported health status or educational attainment. A new theory in development (Apouey, 2007; Abul Naga and Yalcin, 2008) measures disparity of ordinal data as polarization. Typically a criterion used to compare distributions is the polarization relation as proposed by Allison and Foster (AF) (2004). We characterize classes of polarization measures equivalent to the AF relation analogously to Atkinson's original approach.

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1 Introduction

For several decades now, studies of well-being have sought to come to grips with measuring non-income dimensions. Increasingly, this stance has also pervaded policy making, as exemplified by the announcement of a happiness index by the British Prime Minister in November 2011, and by the launching of the OECD Better Life index in May 2012. A great many non-income data, such as the widely used self-reported health status data (Apouey, 2007; Zheng, 2011) and the happiness data (Di Tella and McCulloch, 2006; Diener and Lucas, 1999; Frey and Stutzer, 2002; Kahneman and Krueger, 2006; Layard, 2005; Oswald, 1997), are ordinal. To be precise, these are data that are ordinal and discrete. By ordinal we mean invariant with respect to monotone transformations (as opposed to cardinal, when particular numbers are meaningful). Discrete means that values of variables are concentrated on a fixed number of points, as opposed to continuous variables which

*Corresponding author; Institute of Economics, Polish Academy of Sciences, Palac Staszica, Nowy Swiat 72, 00-330 Warsaw, Poland; email: mkobus@wne.uw.edu.pl; Phone + 48226572707; Fax + 48228267254

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accord a particular value with probability zero.[1] In health surveys, for example, individuals are asked to choose one of five categories to describe their health, namely, very bad, bad, fine, good, very good. The standard procedure for constructing measures of concentration from ordinal variables, such as health status, is to assign numerical values to categories in a manner consistent with the ordering of the preferences; this procedure is referred to as scaling. Clearly, any increasing transformation of a scale reflects the same ordering of categories. The numerical data lend themselves to polarization indices in order to measure polarization, but such a tempting procedure is flawed. Allison and Foster (2004) and Kobus and Miłoś (2012) provided examples of inequality measures that change the ranking of distributions depending on the scale of an ordinal indicator. Standard polarization measures (Wolfson 1994, 1997; Wang and Tsui 2000) suffer from similar problems as the following example illustrates. As Apouey (2007) notes, such measures are also only applicable to cardinal data.

Let the distributions of self-reported health status among men, π^M, and among women, π^W, be $\pi^M = (0.01, 0.48, 0.02, 0.48, 0.01)$ and $\pi^W = (0.31, 0.08, 0.22, 0.08, 0.31)$; that is, forty eight percent of men are in the second health category and thirty one percent of women in the first category and so on. By construction, a higher category number indicates better health status. Category $m = 3$ is the common median. Consider Wang and Tsui index (Wang and Tsui 2000) $P^{WT} = \theta \sum_{i=1}^{n} |c_i - c_m|^r p_i$, where $c = (c_1, \ldots, c_i, \ldots, c_n)$ denotes a scale i.e., a sequence of numbers assigned to an ordinal variable, $r \in (0, 1)$, and θ is a constant which we fix to one. We put $r = 0.5$. We now show that there exist two different scales, c and $\tilde{c}$, such that under scale c health inequality is higher among women than men ($P^{WT}(\pi^W) > P^{WT}(\pi^M)$) and under scale $\tilde{c}$ health inequality is higher among men than women ($P^{WT}(\pi^W) < P^{WT}(\pi^M)$). We have $P^{WT}(\pi^W) = 0.31\sqrt{|c_1 - c_3|} + 0.08\sqrt{|c_2 - c_3|} + 0.08\sqrt{|c_4 - c_3|} + 0.31\sqrt{|c_5 - c_3|} > P^{WT}(\pi^M) = 0.01\sqrt{|c_1 - c_3|} + 0.48\sqrt{|c_2 - c_3|} + 0.48\sqrt{|c_4 - c_3|} + 0.01\sqrt{|c_5 - c_3|}$, that is, $3/4\left(\sqrt{|c_1 - c_3|} + \sqrt{|c_5 - c_3|}\right) > \sqrt{|c_2 - c_3|} + \sqrt{|c_4 - c_3|}$. This holds for e.g., $c = (1, 5, 9, 13, 17)$, because $3\sqrt{2} > 4$. On the other hand, for $P^{WT}(\pi^W) < P^{WT}(\pi^M)$ to hold we need $\sqrt{|c_2 - c_3|} + \sqrt{|c_4 - c_3|} > 3/4\left(\sqrt{|c_1 - c_3|} + \sqrt{|c_5 - c_3|}\right)$. Since $|c_1 - c_3| > |c_2 - c_3|$ and $|c_5 - c_3| > |c_4 - c_3|$, we also have $\sqrt{|c_1 - c_3|} + \sqrt{|c_5 - c_3|} > \sqrt{|c_2 - c_3|} + \sqrt{|c_4 - c_3|}$. Both inequalities are fulfilled for e.g., $\tilde{c} = (1, 2, 6, 10, 11)$, since $2\sqrt{5} > 2 + 2 > 3/2(\sqrt{5})$.

Recently, researchers have acknowledged this problem, and a new theory for measuring dispersion of ordinal data has evolved, notably Blair and Lacy (2000), Allison and Foster (2004), Apouey (2007), Abul Naga and Yalcin (2008), Zheng (2010, 2008), and Kobus and Miłoś (2012). This body of work recognizes that, in the case of ordinal data, one should work with probability distributions rather than values assigned to categories. Then, however, a problem arises with the distinction between inequality and polarization. Polarization refers to the phenomenon of the "disappearing

[1] Other types of data are e.g., ordinal and continuous (such as the Body Mass Index (BMI) which, as the ratio of two continuous variables, is continuous but the differences between two BMI's are meaningful only in an ordinal sense); cardinal and discrete data (the distribution of the number of cars in households, there is a fixed number of values, and particular values are meaningful).

middle class" (Wolfson, 1994) and the emergence of a divided population. There is already broad literature on polarization measurement, with particular emphasis on how it differs from inequality measurement (Esteban and Ray, 2012). For cardinal variables such as income this difference is clear; inequality decreases following the Pigou-Dalton transfer, whereas polarization may increase i.e. if the transfer happens within groups, then it increases group homogeneity. The Pigou-Dalton transfer is a rank-preserving transfer from a "rich" individual to a "poor" individual. In an ordinal framework, it is difficult to imagine a meaningful version of a Pigou-Dalton transfer e.g., what would transferring health from a healthy to a less healthy individual mean? Even when one thinks about an underlying good that is transferable, the impact of the Pigou-Dalton transfer may remain indeterminate, depending on whether the transfer is sufficient to move individuals to different categories. As a consequence, the distinction between inequality and polarization for cardinal variables may not be easily translated into an ordinal setting.

In the standard inequality measurement literature, inequality is measured as deviation from the perfectly equal distribution, namely, a distribution in which every individual exhibits the same value for a given cardinal attribute, which then by definition is the mean value of the attribute. A natural candidate for a perfectly equal (and the non polarized distribution) in an ordinal framework is a distribution in which every individual is in the same category e.g., every individual enjoys the same health status. Yet, while in a standard framework a perfectly equal distribution is unique, with ordinal data there are as many perfectly equal distributions as there are categories. The choice of perfectly equal distribution is arbitrary, but has an influence on results. Researchers avoid the problem using deviation from a perfectly unequal distribution. A typical approach in the literature (Leik, 1966; Berry and Mielke, 1992; Blair and Lacy, 2000; Allison and Foster, 2004) is to treat the distribution in which half the probability mass is concentrated in the lowest category and half in the highest category as the most unequal distribution. Yet while such a distribution is uniquely defined, it reflects polarization rather than inequality because it measures concentration around the tails. Clearly, more research is needed to determine any meaningful difference between inequality and polarization in an ordinal framework. So far "a noteworthy approach is to measure the disparity of ordinal data as polarization" (Zheng, 2008). We follow this approach here.

A criterion, typically used to compare distributions of ordinal variables in terms of dispersion, is the partial ordering proposed by Allison and Foster (2004). A natural question arises about robustness of comparisons based on their criterion. Formally, robustness here means that the ranking of distributions induced by the AF condition is consistent with a class of polarization indices. In a standard framework, Atkinson's Theorem (Atkinson, 1970) provides an answer to this question. The Theorem puts Lorenz dominance in the centre of inequality measurement theory making it a robust criterion for evaluating income distributions. Formally, let $x \in \mathbb{R}^n$ denote a vector of incomes; the Lorenz curve displays the percentage of income accruing to the $100l$ percent poorest individuals in x for all $l \in (0, 1)$. We say that distribution x Lorenz dominates distribution y if the Lorenz curve for distribution x is the same as or lies above the Lorenz curve for distribution

y for every $l \in (0, 1)$ and strictly lies above distribution y for some l. Atkinson (1970) shows that Lorenz dominance is the largest relation[2] such that *all* symmetric inequality indices fulfilling the Pigou-Dalton Transfer Principle do not decrease. Symmetry means that an index is invariant to permutations of individuals, namely, individual labels do not matter for inequality measurement. As a consequence, the measurement of inequality becomes largely independent of the arbitrariness involved in using specific inequality indices. Instead of calculating inequalities in two distributions using different inequality indices, one can compare two distributions on the basis of the Lorenz criterion and the obtained result is compatible with many indices of inequality.

In this paper we prove a similar result in an ordinal setting. In the Atkinsonian spirit we relate the Allison-Foster relation (which the authors view as analogous to Lorenz dominance in a standard framework) to polarization measures. We start by generalizing the AF relation. In their paper Allison and Foster (2004) assumed that two distributions had a unique and common median. This assumption is not needed and can be generalized to include distributions with multiple medians that share at least one median. Then, as main results we characterize two classes of polarization measures which are equivalent to such a generalized AF relation. The AF is the largest relation with which all polarization indices fulfilling "a median-preserving spread" principle and all T-convex polarization indices agree.

A median-preserving spread is a (single) transfer of probability mass away from the median such that the median remains unchanged.[3] As already mentioned the disappearing middle class phenomenon lies at the heart of polarization literature (Levy and Murnane, 1992; Wolfson, 1994). Most studies define the middle by the median income and high concentration around the median corresponds to low polarization (Blackburn and Bloom 1985; Foster and Wolfson 1992, 2009; Wolfson 1994; Wang and Tsui 2000).[4] Therefore it is natural to assume that transferring probability away from the median cannot decrease polarization. For instance, the Wang and Tsui (2000) index is essentially a measure of the distance from the distribution for which all mass is concentrated in the median category, which is assumed to represent the case of minimum polarization. Theorem 1 states that the unanimous ranking, rendered by the class of indices that do not decrease following a median-preserving spread, is the AF relation. That the AF relation is sufficient to guarantee that an index fulfills median-preserving spread principle is hardly surprising; however, it is not obvious that the AF gives us the widest class of distributions[5] that can be ranked without further restriction on the indices.

A T-convex polarization index does not decrease after multiplication via a T-convex matrix. A T-convex matrix is a column stochastic matrix which is formed from an upper triangular matrix

[2] Note that relations can be identified with sets; largest here is used as largest in the sense of inclusion.

[3] With two and more medians, such an assertion is not necessary, that is, transferring mass away from medians does not remove medians (cf. Remark 6).

[4] Another strand of literature treats polarization as a clustering over arbitrary number of groups. Bimodal distribution, however, emerges as the most polarized distribution (Esteban and Ray 1994).

[5] To be precise, as a two-argument relation the AF gives us the widest (in the sense of inclusion) subset of the set of pairs of distributions.

for columns below the median and a lower triangular matrix for columns above the median. A multiplication via such a matrix increases the concentration of probability around the tails. If the median is unique, then it needs to be fixed; otherwise multiplication via a T-concave matrix may change the median. In the case of two and more medians, multiplication by a T-concave matrix does not remove medians. Analogously, although in a reverse direction, in a standard framework, multiplication by a bistochastic matrix is conceived as a process that concentrates a distribution around the mean (e.g. Tsui (1999)). A function is S-convex (S-concave) if it does not increase (decreases) after its arguments are multiplied by a bistochastic matrix. S-convex (S-concave) functions accord a lower (higher) score to distributions that are spread more equally. Given two distributions, there are many T-concave matrices that transform from one distribution to the other, but each such matrix can be effectively constructed using the sequence of median-preserving spreads. There is thus a close link between the two concepts. T-convex matrices are related to generalized majorization i.e. majorization on sequences proposed by Parker and Ram (1997). The set of $n \times n$ matrices with parameters $m_1, \ldots, m_l$ is denoted $T_{n,m_1,\ldots,m_l}$. Parameters $m_1, \ldots, m_l$ denote columns for which there are no restrictions on the distribution of values other than they total to 1 i.e t is a column stochastic matrix. For distributions with medians $m_1 < \ldots < m_l$ that are multiplied by a T-convex matrix, parameters $m_1, \ldots, m_l$ correspond to the medians. The set $T_{n,m_1,\ldots,m_l}$ forms a semigroup i.e. the set of matrices closed under multiplication that includes the identity. Utilizinng this fact the preorder T-majorization can be defined. This preorder when restricted to the domain of distributions with medians $m_1 < \ldots < m_l$ is the AF relation, but in general it is a richer than the AF in the sense that it compares more distributions than the AF does.

Apart from showing the usefulness of Lorenz dominance in inequality measurement, Atkinson's Theorem also links inequality and welfare. On the basis of Atkinson's Theorem, inequality measurement is not merely a statistical exercise, but it also conveys a normative meaning. To be precise, when the mean is fixed, inequality and welfare are the same (up to a sign). On the other hand, when the mean rises, ceteris paribus, social welfare rises too. The mean income and the distribution of income (relative to the mean) can be thus distinguished in the welfare analysis. In an ordinal framework with dispersion measured using the AF relation, such a distinction does not seem to be equally meaningful. If there is unique median, then it needs to be fixed in order for the AF criterion to apply. Social welfare can then be represented as a function of the median and a polarization measure. The median is, however, a poor indicator of potential efficiency gains. For instance, if the median is in the third category, we may transfer probability mass from the fourth category to the fifth category without changing the median. While this is a clear Pareto improvement (i.e., ceteris paribus, people with good health become healthier), the social welfare function does not indicate any change. More generally, as already noted by Allison and Foster (2004), the AF ranking is incompatible with the ranking of distributions induced by first order stochastic dominance (other than in special cases). This suggests problems with inclusion of the AF criterion in welfare considerations with ordinal data.

The paper is organized as follows. In Section 2 we provide definitions and notation. Since in a discrete framework a median need not be unique, we characterize discrete distributions with several medians. In Section 3 we link the AF relation to a class of measures fulfilling a median-preserving spread principle. Section 4 constitutes the central part of the paper and is divided into three subsections. In Section 4.1 we define T-convex matrices and study some of their properties. In Section 4.2 we state the main result of the paper, namely, we link the AF relation to a class of T-convex polarization measures. In Section 4.3 the theory of T-convex matrices is developed further, i.e., decomposition into elementary matrices and the relation between T-convexity and majorization theory. In Section 5 we apply our methodology to data on educational attainment for men and women taken from the US General Social Survey. Using three different inequality indices we show that, when there is dominance in the sense of the AF of the women's over the men's distribution, all indices assign a higher polarization score to the men's distribution. Otherwise, when there is no dominance, the three measures generate inconsistent rankings, rendering robust conclusions about polarization impossible. When the AF dominance holds, we show a sequence of median-preserving spreads and a T-convex matrix that links the two distributions.

2 Basic definitions and notation

Let $c = (c_1, \ldots, c_n)$ be a scale whenever $c_1 < \ldots < c_n$; let C denote the set of all such ordered scales. Since it makes no sense to work with single-category scales, we assume that $n \geq 2$. In what follows c, n are fixed. If for instance, we have ordered responses concerning health status, such that $c = (1, 2, 3, 4, 5)$, this means that the first health category is assigned number 1, the second is assigned 2 and so on. Let p_i denote the share of individuals in category c_i; obviously, we require $p_i \in [0, 1]$ and $\sum_{i=1}^{n} p_i = 1$. A frequency distribution and an associated cumulative distribution function are, $\pi := (p_1, \ldots, p_n)$ and $\Pi = (P_1, \ldots, P_n)$, where $P_i := \sum_{k=1}^{i} p_k$.[6] Furthermore, π is an element of λ and Π is an element of Λ, which denote the sets of all distributions and cumulative distribution functions defined over n discrete states. Let $P : \lambda \times C \mapsto \mathbb{R}$ be a polarization index.[7]

A distribution is degenerate if $p_i = 0$ for some i and non-degenerate otherwise. We let category m be a median of π if for $m > 1$, $P_{m-1} \leq 0.5$ and $P_m \geq 0.5$, and for $m = 1$, $P_1 \geq 0.5$. Thus defined, a median does not have to be unique. For example, in the six-category distribution $(0.25, 0.25, 0, 0, 0.25, 0.25)$ the definition of a median is met by the second, third, fourth and fifth categories. For clarity, sometimes we write $m(\pi)$ to underlie that m is a median of a particular distribution, namely, distribution π.[8] In the case of multiple medians we adhere to the following

[6] A cumulative distribution can also be identified with $\Pi := (P_1(c_1), \ldots, P_n(c_n))$. Slightly abusing the notation we set $P_i(c_i) = P_i$.

[7] We put P to denote a polarization index in order to be consistent with the notation in the received literature. To avoid confusion with the notation of a cumulative distribution function, whenever we consider two different indices we use an upper subscript, here namely, P^1 and P^2. Since the scale is fixed, in fact we work with $P : \lambda \mapsto \mathbb{R}$. The polarization index can also be defined as $P : \Lambda \mapsto \mathbb{R}$.

[8] One can think of $m(\pi)$ as a function which returns precisely median m.

conventions. By l we denote the number of medians and by $m_1 < m_2 < \ldots < m_l$ the set of medians, always in the ordered fashion i.e., m_1 is the first median as it corresponds to the lowest category number among median categories.

We characterize discrete distributions with respect to the number of medians.

Remark 1. *Let $\pi = (p_1, \ldots, p_n)$ be a distribution with $l \geq 1$ medians. Then, the following statements are true.*

(i) If $l > 2$, the distribution is degenerate.

(ii) π is of one of three types.

(a) π has unique median

(b) π has two medians $m_1 < m_2$ if and only if they are not separated by any category (i.e. $m_2 = m_1 + 1$ for clarification)

(c) π has three and more medians $m_1 < \ldots < m_l$ if and only if $p_{m_j} = 0$ for all $m_2 \leq m_j \leq m_{l-1}$.

(iii) If $l \geq 2$, then $P_{m_1} = \ldots = P_{m_{l-1}} = 0.5$.

Proof. From the definition of a median, it follows that $P_{m_1} \geq 0.5$, $P_{m_l - 1} \leq 0.5$. P_i is non-decreasing with i, therefore for $l \geq 2$ we obtain $P_{m_1} = \ldots = P_{m_l - 1} = 0.5$. Note that from this it follows that all categories between m_1 and m_l are medians. This yields, for example, that $m_l - 1$ is the $(l-1)$-th median m_{l-1}. Therefore a distribution has two medians if and only if they are not separated by any category. For $l > 2$ we get $p_{m_2} = \ldots = p_{m_{l-1}} = 0$ i.e. the distribution is necessarily degenerate. □

In general, degeneracy of the distribution does not determine the number of medians. There are both degenerate and non-degenerate distributions with one unique median e.g. $\Pi = (0.2, 0.4, 0.6, 0.6, 0.8, 1)$ and $\tilde{\Pi} = (0.2, 0.4, 0.6, 0.7, 0.8, 1)$. Yet when there are more than two medians, the distribution is necessarily degenerate, so there are no non-degenerate distributions with three or more medians. Non-degenerate distributions can have at most one or two medians. Furthermore, the only distributions with two medians are such that the medians are not separated by any category and the only distributions with three and more medians are such that except for the lowest and the highest median, all other medians are empty categories.

As we mentioned in the Introduction, we draw on the Allison-Foster (AF) (Allison and Foster, 2004) partial ordering for evaluating the degree of polarization of given distributions.

Definition 1. ***AF partial ordering***
Let π, ω be two probability distributions and m_k be a median of π. Let $\Pi, \Omega := (Q_1, \ldots, Q_n)$ be their cumulative distribution functions. We write $\pi \precsim_{AF} \omega$ if and only if the following three conditions are met:

(AF1) *m_k is a median of ω;*

(AF2) $P_i \leq Q_i$ *for any* $i < m_k$*;*

(AF3) $P_i \geq Q_i$ *for any* $i \geq m_k$*.*

The interpretation of $\pi \precsim_{AF} \omega$ ordering is intuitive. In particular, we have that $\pi \precsim_{AF} \omega$ when π is better concentrated (i.e., when there is more probability mass) around the median than ω. For example, the cumulative distribution functions corresponding to distributions $\pi = (0.2, 0.2, 0.2, 0.2, 0.2)$ and $\omega = (0.3, 0.15, 0.15, 0.1, 0.3)$ are, respectively, $\Pi = (0.2, 0.4, 0.6, 0.8, 1)$ and $\Omega = (0.3, 0.45, 0.6, 0.7, 1)$. The common, and in this case unique, median of Π and Ω is the third category. Because $0.2 < 0.3; 0.4 < 0.45$ and $0.8 > 0.7$, by Definition 1, $\pi \precsim_{AF} \omega$. The AF partial ordering is similar to the single - crossing criterion of Hemming and Keen (1983). Originally, in the definition of the AF relation Allison and Foster (2004) assume that π, ω have a unique and common median. This assumption can be substantially relaxed to cover distributions with several medians and at least one common median and transitivity is still preserved.

Remark 2. *Relation defined in Definition 1 is transitive.*

Proof. Let $\pi \precsim_{AF} \omega$ and $\omega \precsim_{AF} \delta$, where $\Delta := (S_1, \ldots, S_n)$ is the cumulative distribution function associated with δ. We now show that $\pi \precsim_{AF} \delta$ i.e., (AF1)-(AF3) are met with respect to π, δ.

We start with (AF1). According to Definition 1 $m_k(\pi)$ is a median of ω and $m_h(\omega)$ is a median of δ. If $m_k(\pi) = m_h(\omega)$, then $m_k(\pi)$ is a median of δ and (AF1) is fulfilled. If $m_h(\omega) < m_k(\pi)$, then we have the following set of observations.

(a) Given Remark 1 from $\pi \precsim_{AF} \omega$ (AF1) we have $Q_{m_h(\omega)} = \ldots = Q_{m_k(\pi)-1} = 0.5$.[9]

(b) From $\omega \precsim_{AF} \delta$ (AF3) we have that for $i \geq m_h(\omega)$, $S_i \leq Q_i$.

(c) From $\omega \precsim_{AF} \delta$ (AF1) we have that $S_{m_h(\omega)} \geq 0.5$.

(d) S_i is non-decreasing with i.

Altogether these observations imply that $S_{m_h(\omega)} = \ldots = S_{m_k(\pi)-1} = 0.5$ i.e., $m_k(\pi)$ is a median of δ. We treat the case of $m_h(\omega) > m_k(\pi)$ similarly. We note the following.

(a) Given Remark 1 from $\pi \precsim_{AF} \omega$ (AF1) we have $Q_{m_k(\pi)} = \ldots = Q_{m_h(\omega)-1} = 0.5$.

(b) From $\omega \precsim_{AF} \delta$ (AF2) we have that for $i < m_h(\omega)$, $S_i \geq Q_i$.

(c) From $\omega \precsim_{AF} \delta$ (AF1) we have that $S_{m_h(\omega)-1} \geq 0.5$.

(d) S_i is non-decreasing with i.

[9] Recall that $m_k(\pi)$ is a category number (which is also the k-th median of π). Therefore, $Q_{m_k(\pi)}$ denotes a cumulative mass concentrated in the category $m_k(\pi)$ of a distribution ω, which a priori is not necessarily a median of ω.

Again $S_{m_k(\pi)} = \ldots = S_{m_h(\omega)-1} = 0.5$ and $m_k(\pi)$ is a median of δ.

We now focus on (AF2). We have $P_i \leq Q_i$ for $i < m_k(\pi)$ and $Q_i \leq S_i$ for $i < m_h(\omega)$. If $m_k(\pi) \leq m_h(\omega)$, then $P_i \leq S_i$ for $i < m_k(\pi)$ and (AF2) holds. If $m_h(\omega) < m_k(\pi)$, then $P_i \leq S_i$ for $i < m_h(\omega)$. We recall $S_{m_h(\omega)} = \ldots = S_{m_k(\pi)-1} = 0.5$ and that for $m_h(\omega) \leq i < m_k(\pi)$ we have $P_i \leq 0.5$ since $m_k(\pi)$ is a median of π. Therefore, $P_i \leq S_i$ as required.

Finally we deal with (AF3). We have $P_i \geq Q_i$ for $i \geq m_k(\pi)$ and $Q_i \geq S_i$ for $i \geq m_h(\omega)$. If $m_h(\omega) \leq m_k(\pi)$, then $P_i \geq S_i$ for $i \geq m_k(\pi)$ and (AF3) is fulfilled. If $m_h(\omega) > m_k(\pi)$, we have $P_i \geq S_i$ for $i \geq m_h(\omega)$. We recall $S_{m_h(\omega)} = \ldots = S_{m_k(\pi)-1} = 0.5$ and that for $i \geq m_k(\pi)$ we have $P_i \geq 0.5$ since $m_k(\pi)$ is a median of π. Therefore, $P_i \leq S_i$ as required.

We conclude that $\pi \precsim_{AF} \delta$. □

Although in the condition (AF1) we require that at least one median is common, further conditions ((AF2) and (AF3)) imply that all medians of π are also medians of ω. Formally, the set of medians of ω is a superset of the set of medians of π. Through the AF relation it is not possible to remove existing, but only to add new medians. Also the choice of m_k does not affect the ranking of distributions.

Remark 3. *Let π, ω be two distributions such that $\pi \precsim_{AF} \omega$, and let M_π, M_ω denote their sets of medians. Then, $M_\pi \subseteq M_\omega$.*

Proof. Let ω be a distribution with s medians. We first show that $m_1(\omega) \leq m_1(\pi)$ by contradiction. If $m_1(\pi) < m_1(\omega) \leq m_k(\pi)$, then using the notation as in Definition 1 we have the following.

(a) From the definition of a median $m_1(\omega)$, $Q_i \leq 0.5$ for $i \leq m_1(\omega) - 1$.

(b) From (AF2) $P_i \leq Q_i$ for $i < m_k(\pi)$.

(c) From Remark 1 $P_i = 0.5$ for $m_1(\pi) \leq i \leq m_k(\pi) - 1$.

Therefore, $Q_i = 0.5$ for $m_1(\pi) \leq i \leq m_1(\omega) - 1$ i.e., categories $m_1(\pi) \leq i \leq m_1(\omega) - 1$ are medians of ω, which contradicts the assumption $m_1(\omega) > m_1(\pi)$. The cases of $m_1(\pi) < m_k(\pi) < m_1(\omega)$ and $m_s(\omega) \geq m_l(\pi)$ can be treated similarly.

Finally, all categories between $m_1(\omega)$ and $m_s(\omega)$ are medians of ω (Remark 1), in particular, $m_1(\pi) < \ldots < m_l(\pi)$ are such in-between categories. □

Remark 4. *The choice of m_k in the Definition 1 is arbitrary.*

Proof. Our goal is to show that if we put $m_d \neq m_k$ in the Definition 1 we still have $\pi \precsim_{AF} \omega$. From Remark 3 m_d is a median of ω. Then, if $m_d(\pi) < m_k(\pi)$ we have $P_{m_d(\pi)} = \ldots = P_{m_k(\pi)-1} = 0.5$ and also $Q_{m_d(\pi)} = \ldots = Q_{m_k(\pi)-1} = 0.5$, and if $m_d(\pi) > m_k(\pi)$ we have $P_{m_k(\pi)} = \ldots = P_{m_d(\pi-1)} = 0.5$ and also $Q_{m_k(\pi)} = \ldots = Q_{m_d(\pi)-1} = 0.5$. Thus, $P_i \leq Q_i$ for $i < m_d$ and $P_i \geq Q_i$ for $i \geq m_d$. □

3 Polarization and median-preserving spreads

In this section we link the AF partial ordering to a class of polarization measures that do not decrease following a spread away from the median. We show that when all polarization measures from this class increase, it is equivalent to an increase in terms of the AF relation. That is to say, this is the class of functions which are order-preserving for the AF relation.

Definition 2. ***Median-preserving spread***
Let $\eta > 0$ and π, ω be two distributions and let m_1, m_l denote, respectively, the first and the last median of π. We say that ω is obtained from π via a median-preserving spread if and only if

- *For some i, j such that $i < j \leq m_1$ or $m_l \leq j < i$ there is $q_j = p_j - \eta$, $q_i = p_i + \eta$ and $q_k = p_k$ otherwise.*
- *If π has unique median m, then m is also median of ω.*

It is straightforward to see how a median-preserving spread changes a cumulative distribution function.

Remark 5. *Let ω be obtained from π via a median-preserving spread. Then we have the following cumulative distribution function $\Omega := (Q_1, \ldots, Q_n)$.*

- *If a transfer is for i, j such that $i < j \leq m_1$ then, $Q_r = P_r$ for $r < i$, $r \geq j$ and $r > m_1$, and $Q_r = P_r + \eta$ for $i \leq r < j$*
- *If a transfer is for i, j such that $m_l \leq j < i$ then, $Q_r = P_r$ for $r < j$, $r < m_l$ and $r \geq i$, and $Q_r = P_r - \eta$ for $j \leq r < i$.*

Proof. Let us check the first case i.e., $r < i$. Then, $P_r = p_1 + \ldots + p_r = q_1 + \ldots + q_r = Q_r$. Now let $r \geq j$ and $r > m_1$. Then, $P_r = p_1 + \ldots + p_i + \ldots + p_j + \ldots + p_r = p_1 + \ldots + (p_i + \eta) + \ldots + (p_j - \eta) + \ldots + p_r = q_1 + \ldots + q_i + \ldots + q_j + \ldots + q_r = Q_r$. Further, let $i \leq r < j$. Then, $P_r = p_1 + \ldots + p_i + \ldots + p_r$ and $Q_r = q_1 + \ldots + (p_i + \eta) + \ldots + q_r = P_r + \eta$. The case of $m_l \leq j < i$ can be treated similarly. □

In the case of more than one median, there is no need to fix any median, because the described transfers do not change medians, although new medians may be created. Indeed, for a median-preserving spread we can prove an analogue to Remark 3.

Remark 6. *Let π, ω be two distributions such that ω is obtained from π via a median preserving spread and let M_π, M_ω denote their sets of medians. Then, $M_\pi \subseteq M_\omega$.*

Proof. For $l = 1$ this is asserted in the Definition 2. Let $l > 1$. First we show that transfers for categories $i < j \leq m_1(\pi)$ do not remove medians. Given Remark 5, $P_{m_1(\pi)} = Q_{m_1(\pi)} = 0.5$, and $P_i = Q_i$ for $i \geq m_1(\pi)$, so $m_1(\pi) < \ldots < m_l(\pi)$ are medians of ω. Transfers for categories

$m_l(\pi) \leq j < i$ also do not remove any of the medians since $P_i = Q_i$ for $i \leq m_l(\pi) - 1$, which is enough to assert that $m_1(\pi) < \ldots < m_l(\pi)$ are medians of ω.

New medians can be created e.g., $j = m_1(\pi)$ and $p_{m_1(\pi)} = \eta$ after a transfer of η we have $q_j = 0$, $Q_{m_1(\pi)-1} = 0.5$ and $Q_{m_1(\pi)-2} = P_{m_1(\pi)-2} < 0.5$, therefore $m_1(\pi) - 1$ is the first median of ω. $\square$

We now establish the equivalence between the AF relation and a finite sequence of median-preserving spreads, which we further use in proving Theorems 1 and 2.

Lemma 1. *$\pi \precsim_{AF} \omega$ if and only if ω can be obtained from π via a finite sequence of median-preserving spreads.*

Proof. We prove first the "if" part of Lemma 1. We first focus on $i < m_k(\pi)$ (AF2). If $m_k(\pi) = 1$ then this case is redundant. From the definition of the AF ordering (Definition 1), we have that $P_i \leq Q_i$. Let i_0 be the smallest category for which $P_{i_0} < Q_{i_0}$ (if such i_0 does not exist then the following transfers are not necessary and two distributions agree up to $m_k(\pi)$). We make a median-preserving spread from i_0+1 to i_0 obtaining a new distribution $\tilde{p}$ such that $\tilde{P}_{i_0} > P_{i_0}$ and $\tilde{P}_i = P_i$ for $i \neq i_0$. Two cases are possible. The mass p_{i_0+1} suffices to make $\tilde{P}_{i_0} = Q_{i_0}$. If the mass transferred is insufficient, we make a median-preserving spread from $i_0 + 2$ to i_0, obtaining a distribution $\tilde{\tilde{\pi}}$ such that (cf. Remark 5) $\tilde{\tilde{P}}_{i_0} > \tilde{P}_{i_0}$, $\tilde{\tilde{P}}_{i_0+1} > \tilde{P}_{i_0+1}$ and $\tilde{\tilde{P}}_i = \tilde{P}_i$ otherwise. If $\tilde{\tilde{P}}_{i_0} < Q_{i_0}$, then we make a median-preserving spread from $i_0 + 3$ and so on, possibly up to $m_k(\pi)$. This procedure ends before we reach $m_k(\pi)$ as $P_{m_k(\pi)} = Q_{m_k(\pi)} \geq 0.5$.[10] Continuing the procedure, we can make all states $i < m_k(\pi)$ equal.

Let us now focus on categories $i \geq m_k(\pi)$. Let i_0 denote the smallest category number such that $P_i > Q_i$. If $l > 1$ and if $m_l(\pi) > m_k(\pi)$, then $P_i = Q_i$ for $m_k(\pi) \leq i \leq m_l(\pi) - 1$, therefore the smallest possible i_0 is $m_l(\pi)$, which in case of $l = 1$ is simply m i.e., the unique median. Via median-preserving spreads from i_0 to $i_0 + 1$ similar to the case of $i < m_k(\pi)$, we can equalize all categories.

To prove the converse implication ("only if") part of Lemma 1 we first notice that relation $\precsim_{AF}$ is transitive (Remark 2). We show that each median preserving spread implies an increase in terms of $\precsim_{AF}$. Let ω be obtained from π via a median-preserving spread. Given Remark 6 $m_k(\pi)$ is a median of ω, hence (AF1) holds. Given Remark 5, for $i < m_k(\pi)$ we have $P_i \leq Q_i$ i.e., (AF2) holds, and for $i \geq m_k(\pi)$ we have $P_i \geq Q_i$ i.e., (AF3) holds. This together with transitivity establishes the claim.

$\square$

A direct consequence of Lemma 1 is the following.

Remark 7. *Formally, the AF relation is a transitive closure of the relation induced by the median-preserving spread.*

[10] In fact, for $l > 1$, this procedure ends before we reach $m_1(\pi)$ because $P_{m_1(\pi)} = Q_{m_1(\pi)} = 0.5$. There can be no transfers from medians $m_2 < \ldots < m_{l-1}$, because $p_{m_1(\pi)} = \ldots = p_{m_{l-1}} = 0$ (cf. Remark 1).

The transitive closure in question is the intersect of all transitive relations that contain a relation induced by the median-preserving spread.

The following axiom defines the class of measures consistent with the AF partial ordering.

Definition 3. ***Median-preserving spread principle***
A polarization index $P : \lambda \mapsto \mathbb{R}$ satisfies a median-preserving spread principle if for any ω, π such that ω is obtained from π via a median-preserving spread we have $P(\pi) \leq P(\omega)$.

Essentially, Definition 3 requires that the polarization index does not decrease consequent to the spread of probability away from the median. The intuition behind Definition 3 is the following. It is typically assumed that the least polarized distribution is obtained when all individuals are in the same category, that is, "by default," all are in the median category. At the other extreme, the most polarized distribution is equally concentrated around the tails. Therefore, shifting probability mass away from the median to the tails without removing the median cannot plausibly reduce polarization.

The main result of this section is that the AF partial ordering is the largest (in the sense of inclusion) relation compatible with each polarization index fulfilling the "median-preserving spread" principle. The key word in Theorem 1 is "all". While it is quite straightforward that the AF condition implies that indices fulfilling median preserving spread show less polarization in π than ω[11], it is not at all obvious that when we take *all* indices fulfilling median-preserving spread, they jointly imply the AF relation.

Theorem 1.

$$\pi \precsim_{AF} \omega \quad \textit{iff} \quad P(\pi) \leq P(\omega) \ \textit{ for all } P \textit{ satisfying Definition 3 .}$$

Proof. Assume that $\pi \precsim_{AF} \omega$ and that P satisfies Definition 3. Taking into account Lemma 1, ω is obtained from π via a finite sequence of median-preserving spreads. We note that after each median-preserving spread, P does not decrease (Definition 3), hence $P(\omega) \leq P(\pi)$.

The next part of the proof follows by contradiction. Assume that Definition 1 is not true. Then, either (AF1) or (AF2) or (AF3) fails. We start with (AF1); $m_k(\pi)$ is a median of π but not a median of ω. Let $m_h(\omega)$ be a median of ω. We have $m_h(\omega) \neq m_k(\pi)$. Let P^1 be the following polarization index. We have $P^1(\delta) = 1$ if $m_k(\pi)$ is a median of δ and $P^1(\delta) = 0$ otherwise.[12] P^1 fulfils Definition 3 i.e., it is constant if $m_k(\pi)$ is a median of ψ and it increases when $m_k(\pi)$ becomes a median of δ via a median-preserving spread. We have $P^1(\pi) > P^1(\omega)$, which contradicts the assumption that $P(\pi) \leq P(\omega)$ for all P satisfying Definition 3.

If $m_k(\pi) = 1$, then (AF2) is always fulfilled. We now assume that (AF2) fails, thus it makes sense to consider $m_k(\pi) > 1$. There exists $i < m_k(\pi)$ such that $P_i > Q_i$. Let δ be a distribution with

[11]Indeed, Allison and Foster (2004) even write explicitly that their AF relation might be termed a "median-preserving spread."

[12]Given Remark 4 one may ask what happens to P^1 if we put a different median in the Definition 1. P^1 changes then from 1 to 0. Indeed, if π has l medians, we have a family of indices $P^1_{m_j(\pi)}$, where $1 \leq j \leq l$, such that $P^1_{m_j(\pi)}(\delta) = 1$ if $m_j(\pi)$ is a median of δ and 0 otherwise.

cumulative distribution function $\Delta := (S_1, \ldots, S_n)$ and let $P^2(\delta) := S_i$ if $i < m_k(\pi)$ and 0 otherwise. We show that P^2 fulfils Definition 3. From Remark 5 transfers within $k < j \leq m_1$ either increase or leave S_i unchanged. For transfers $m_l \leq j < k$, P^2 is constant. Then, $P^2(\pi) > P^2(\omega)$, which contradicts the assumption that $P(\pi) \leq P(\omega)$ for all P satisfying Definition 3. The case of negation of (AF3) can be treated similarly by taking $P^3(\delta) := 1 - S_i$ for $i \geq m_k(\pi)$ and 0 otherwise. □

Theorem 1 points to a potential trade-off. One could further restrict the class of indices/welfare functions to then rank more pairs of distributions than the AF partial ordering achieves. Yet the conclusions from this ranking are less robust, applying only to the restricted class of measures. For a class of measures that fulfill only median-preserving spread principle, nothing beyond the AF relation can be ordered.

4 Polarization and T-convex matrices

We now impose a property called T-convexity on polarization measures to ensure compatibility with Definition 1. The theory and results are presented in three sections. In Section 4.1 we define and study some properties of T-convex matrices. In Section 4.2 we state and prove the main result of this paper, namely, the equivalence between the AF and the set of all T-convex polarization measures. In Section 4.3 we show that every T-convex matrix can be decomposed into elementary T-convex matrices. We show that the set of T-convex matrices forms a semigroup. We use this observation to define a majorization preorder based on T-convex matrices.

4.1 T-convex matrices

Definition 4. ***T-convex matrices***
Given parameters $m_1 < \ldots < m_l \leq n$ we define a set of $n \times n$ matrices t with the following properties (i denote rows, j denote columns).

1. *For any $i, j \in \{1, 2, \ldots, n\}$ we have $t_{ij} \geq 0$.*
2. *For any $j \in \{1, 2, \ldots, n\}$ we have $\sum_{i=1}^{n} t_{ij} = 1$.*
3. *If $l = 1$, then for any $j < m$ we have that for any $i > j$ there is $t_{ij} = 0$ and for any $j > m$ we have that for any $i < j$ there is $t_{ij} = 0$.*
4. *If $l > 1$, then for any $j \leq m_1$ we have that for any $i > j$ there is $t_{ij} = 0$ and for any $j \geq m_l$ we have that for any $i < j$ there is $t_{ij} = 0$.*

A set of all such matrices is denoted by $T_{n,m_1,\ldots,m_l}$. An element $t \in T_{n,m_1,\ldots,m_l}$ is called a T-convex matrix.

Matrices in the set $T_{n,m_1,\ldots,m_l}$ are column stochastic matrices; that is, elements in the columns sum up to one. Elements $i < m_1, j \leq m_1$ form an upper triangular matrix and elements $i >$

$m_l, j \geq m_l$ form a lower triangular matrix. Parameters $m_1 < \ldots < m_l \leq n$ can be thought of as medians when t is applied to π. If there is unique median, then for a column which corresponds to a median, the elements can be distributed in any way, as long as they sum up to unity. If there are more medians, the elements can be distributed in any way only for columns which correspond to in-between medians i.e., $m_2, \ldots, m_{l-1}$.

Remark 8. *$T_{n,m_1,\ldots,m_l}$ is a convex set.*

Proof. Let $\alpha \in [0,1]$ and $t, \tilde{t} \in T_{n,m_1,\ldots,m_l}$ be two matrices. We need to show that $\alpha t + (1-\alpha)\tilde{t} \in T_{n,m_1,\ldots,m_l}$. Zero elements do not change, because $\alpha 0 = 0$. Further, $\sum_{i=1}^{n} \alpha t_{ij} + \sum_{i=1}^{n}(1-\alpha)\tilde{t}_{ij} = \alpha \sum_{i=1}^{n} t_{ij} + (1-\alpha)\sum_{i=1}^{n} \tilde{t}_{ij} = \alpha + (1-\alpha) = 1$. □

Example 1

Let $n = 5$ and $m = 3$ be unique median. An example of a matrix from the set $T_{5,3}$ is

$$\tilde{t} = \begin{pmatrix} 1 & 0.8 & 0 & 0 & 0 \\ 0 & 0.2 & 0.3 & 0 & 0 \\ 0 & 0 & 0.3 & 0 & 0 \\ 0 & 0 & 0.4 & 0.9 & 0 \\ 0 & 0 & 0 & 0.1 & 1 \end{pmatrix}.$$

Multiplying vector $\pi = (0.2, 0.2, 0.2, 0.2, 0.2)$ by $\tilde{t}$ yields $\omega = (0.36, 0.1, 0.06, 0.26, 0.22)$; that is, by Definition 1 ω is more spread than π. The probability mass concentrated in the median category is now lower and correspondingly the mass concentrated in the tails of the distribution is higher. The best way to understand how matrix $\tilde{t}$ "operates" is by looking at its columns. For example, column 2 reveals how the probability mass from category 2 is distributed after multiplication. Eighty percent of this mass is shifted to category 1, and twenty percent of the initial mass is retained by category 2, hence spread is increased. Matrix $\tilde{t}$ is not unique. For example, for the following matrix we also have $\pi = \bar{t}\omega$.

$$\bar{t} = \begin{pmatrix} 1 & 0.5 & 0.3 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0.3 & 0 & 0 \\ 0 & 0 & 0.3 & 1 & 0 \\ 0 & 0 & 0.1 & 0 & 1 \end{pmatrix}$$

Comparing to matrix $\tilde{t}$ multiplication via matrix $\bar{t}$ leads to more median-preserving spreads from the median to other categories and fewer spreads between non-median categories. If the median is unique then multiplication via a T-convex matrix from the set $T_{n,m}$ can change the median as the following example shows.

Example 2

Let $n = 5$ and $m = 3$ be unique median. An example of a matrix from the set $T_{5,3}$ is

$$\bar{t} = \begin{pmatrix} 1 & 0.8 & 0 & 0 & 0 \\ 0 & 0.2 & 0.6 & 0 & 0 \\ 0 & 0 & 0.3 & 0 & 0 \\ 0 & 0 & 0.1 & 0.9 & 0 \\ 0 & 0 & 0 & 0.1 & 1 \end{pmatrix}.$$

Multiplying vector $\pi = (0.2, 0.2, 0.2, 0.2, 0.2)$ by $\tilde{t}$ yields $\omega = (0.36, 0.16, 0.06, 0.20, 0.22)$. Cumulative distribution functions change accordingly, namely, from $\Pi = (0.2, 0.4, 0.6, 0.8, 1)$ to $\Omega = (0.36, 0.52, 0.58, 0.78, 1)$. The median is changed from $m = 3$ to $m = 2$. On the other hand, for a given distribution π with $l > 1$ medians, matrices that preserve medians constitute the whole set $T_{n,m_1,\ldots,m_l}$.

Example 3

Let $n = 6$ and $m_1 = 2, m_2 = 3, m_3 = 4$ be three medians. An example of a matrix from the set $T_{6,2,3,4}$ is

$$\tilde{t} = \begin{pmatrix} 1 & 0.8 & 0.2 & 0 & 0 & 0 \\ 0 & 0.2 & 0.2 & 0 & 0 & 0 \\ 0 & 0 & 0.2 & 0 & 0 & 0 \\ 0 & 0 & 0.2 & 0.3 & 0 & 0 \\ 0 & 0 & 0.2 & 0.4 & 0.9 & 0 \\ 0 & 0 & 0 & 0.3 & 0.1 & 1 \end{pmatrix}.$$

Multiplying vector $\pi = (0.2, 0.3, 0, 0.1, 0.2, 0.2)$ via $\tilde{t}$ results in $\omega = (0.44, 0.06, 0, 0.03, 0.22, 0.25)$. Medians remain unchanged, because no mass concentrated in m_1 is distributed to the higher categories and no mass concentrated in m_l is distributed to the lower categories, analogously to spreads that preserve medians. The elements can be distributed in any way in columns that correspond to median m_2 (and more generally, to all medians between m_1 and m_l), because probability is zero for those categories and therefore multiplication cannot change anything. We can show an analogue to Remark 3.

Remark 9. *Let $l > 1$ and let π, ω be two distributions such that $\omega = t\pi$. Further, let M_π, M_ω denote their sets of medians. Then, $M_\pi \subseteq M_\omega$.*

Proof. We start by showing that $m_1(\pi)$ is a median of ω. For $i \leq m_1$ we have

$$q_1 = 1 \times p_1 + t_{12}p_2 + t_{13}p_3 + \ldots + t_{1m_1}p_{m_1} + \ldots + t_{1m_{l-1}}p_{m_{l-1}} + 0 \times p_{m_{l-1}} + \ldots + 0 \times p_n$$

$$q_2 = (1 - t_{12})p_2 + t_{23}p_3 + t_{24}p_4 + \ldots + t_{2m_1}p_{m_1} + \ldots + t_{2m_{l-1}}p_{m_{l-1}}$$

$$q_3 = (1 - t_{13} - t_{23})p_3 + t_{34}p_4 + \ldots + t_{3m_1}p_{m_1} + \ldots + t_{3m_{l-1}}p_{m_{l-1}}$$

$$\ldots\ldots\ldots$$

$$q_i = \left(1 - \sum_{j=1}^{i-1} t_{ji}\right) p_i + t_{i(i+1)p_{(i+1)}} + \ldots + t_{im_1}p_{m_1} + \ldots + t_{im_{l-1}}p_{m_{l-1}}.$$

Given that $p_{m_2} = \ldots = p_{m_{l-1}} = 0$, we have $Q_i = \sum_{j=1}^{i} p_j + \sum_{k=i+1}^{m_1} p_k \left(\sum_{j=1}^{i} t_{jk}\right)$, where $\sum_{j=1}^{i} t_{jk} \leq 1$, because conditions (1)-(4) in Definition 4 jointly imply $\sum_{j=1}^{k} t_{jk} = 1$ and $\sum_{j=1}^{i} t_{jk}$ is obviously non-decreasing with respect to j. Thus, $P_i \leq Q_i = P_i + \sum_{k=i+1}^{m_1} \sum_{j=1}^{i} t_{jk} \leq P_{m_1} = 0.5$ for all $i \leq m_1$. For $i = m_1$ we have $Q_{m_1} = P_{m_1-1} + p_{m_1}\left(\sum_{j=1}^{m_1} t_{jm_1}\right) = P_{m_1} = 0.5$. Therefore, $m_1(\pi)$ is a median of ω.

For $m_1 < i \leq m_{l-1}$ we have $q_i = \sum_{k=i}^{m_{l-1}} t_{ik}p_k = 0$, because $p_{m_2} = \ldots = p_{m_{l-1}} = 0$. Thus $Q_{m_2} = \ldots = Q_{m_{l-1}} = 0.5$ are medians of ω, which is enough to assert that m_l is also a median of ω.

New medians can be created though. Let $t_{1m_1} = 1$. Then $Q_{m_1-1} = P_{m_1-1} + p_{m_1}\left(\sum_{j=1}^{m_1} t_{jm_1}\right)$ and $\left(\sum_{j=1}^{m_1} t_{jm_1}\right) = 1$, because $t_{1m_1} = 1$, that is, all mass p_{m_1} is transferred to the first category.

□

4.2 T-convex polarization indices

We now define a class of T-convex polarization measures in the following way.

Definition 5.

A polarization index $P : \lambda \mapsto \mathbb{R}$ is T-convex if and only if the following conditions hold.

- *(i) Let $l = 1$ i.e., π is a distribution with unique median m. Then, for $\pi, \omega, t \in T_{n,m}$ such that m is a median of ω and $w = t\pi$ we have that $P(\pi) \leq P(\omega)$.*
- *(ii) Let $l > 1$ i.e., π is a distribution with medians $m_1 < \ldots < m_l$. Then, for $\pi, \omega, t \in T_{n,m_1,\ldots,m_l}$ such that $w = t\pi$ we have that $P(\pi) \leq P(\omega)$.*

T-convex functions are functions that do not decrease after their arguments are multiplied by a t matrix. Because a t matrix increases the spread of a distribution, for T-convex polarization indices, a higher spread infers greater polarization.

We now state and prove the main result of this section.

Theorem 2.

$$\pi \precsim_{AF} \omega \quad \textit{iff} \quad P(\pi) \leq P(\omega) \ \textit{for all } P \textit{ that are } T\textit{-convex.}$$

Proof. Assume that $\pi \precsim_{AF} \omega$ and that P satisfies Definition 5. We first prove that $P(\pi) \leq P(\omega)$. By Lemma 1 there exists a sequence $\psi_1, \ldots, \psi_n$ such that $\pi = \psi_1$, $\omega = \psi_n$ and for each j we have that ψ_{j+1} is obtained from ψ_j by means of a median preserving spread. To proceed, it suffices to show that for any j we have $P(\psi_j) \leq P(\psi_{j+1})$, namely, a T-convex index does not decrease following a median-preserving spread. If ψ_{j+1} is obtained from ψ_j by means of a median-preserving spread, then the two distributions differ by two categories.[13] Without loss of generality, we assume that they differ for the first and the second category, that is, $\psi_j = (a_1, a_2, a_3, \ldots, a_n)$, $\psi_{j+1} = (b_1, b_2, a_3, \ldots, a_n)$ and $b_1 > a_1$ since the spread increases the mass concentrated in the tails (recall Definition 2). Obviously $a_1 + a_2 = b_1 + b_2$. We put

$$\hat{t} = \begin{bmatrix} 1 & \frac{b_1 - a_1}{a_2} \\ 0 & 1 - \frac{b_1 - a_1}{a_2} \end{bmatrix}.$$

We obtain $(b_1, b_2) = \hat{t}(a_1, a_2)$, and $\hat{t}$ that can be easily extended to t such that $\psi_{j+1} = t\psi_j$ via the identity matrix. For example, for $n = 7$, it is

$$t = \begin{pmatrix} 1 & \frac{b_1-a_1}{a_2} & 0 & \ldots & \ldots & \ldots & 0 \\ 0 & 1 - \frac{b_1-a_1}{a_2} & 0 & \ldots & \ldots & \ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots & \ldots & 0 \\ 0 & 0 & 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 0 & 1 & \ldots & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Moreover, matrix t is just as described in Definition 5, and hence function P does not decrease.

The second part of the proof goes along similar lines as the second part of the proof of Lemma 1, namely, by contradiction. (AF1) fails i.e., let $m_h(\omega)$ be a median of ω and $m_h(\omega) \neq m_k(\pi)$. Let P^1 be the following polarization index. We have $P^1(\delta) = 1$ if $m_k(\pi)$ is a median of δ and 0 otherwise. Multiplication via a T-convex matrix does not remove medians (cf. Remark 9), therefore P^1 is T-convex. Yet we have $P^1(\pi) > P^1(\omega)$, which contradicts the assumption $P(\pi) \leq P(\omega)$ for all T-convex functions P.

If $m_k(\pi) = 1$, then (AF2) is always satisfied, therefore it makes sense to consider $m_k(\pi) > 1$. (AF2) fails. There exists $i < m_k(\pi)$ such that $P_i > Q_i$. We define $P^2(\delta) := S_i$ for $i < m_k(\pi)$ and 0 otherwise. Obviously, $P^2(\pi) > P^2(\omega)$, which contradicts the statement $P(\pi) \leq P(\omega)$ for all T-convex functions P. We need to show, however, that P^2 is indeed T-convex. We define

$$v = (\underbrace{1, \ldots, 1}_{i}, \underbrace{0, \ldots, 0}_{n-i}).$$

Obviously we have $S_i = v.\delta$, where a dot denotes the standard dot product. Using this operator, we can write the cumulative distribution function for $\omega = t\delta$ as

$$Q_i = v.(t\delta) = (t^{Tr}v).\delta,$$

where Tr denotes the matrix transposition. Let $\tilde{v} := t^{Tr}v$ and $\tilde{v} = (\tilde{v}_h)_{h=1}^n$. Our goal now is to characterize vector $\tilde{v}$. T-convex matrix transposition fulfills the following conditions (cf. Definition 4).

[13] In Definition 2 ω is obtained from π via a median-preserving spread and q_j, p_j and q_i, p_i differ by the amount of spread η.

(i) $t_{ij}^{Tr} \geq 0$.

(ii) $\sum_{j=1}^{n} t_{ij}^{Tr} = 1$.

(iii) If $l = 1$, then for $i < m$ we have that for $j > i$ there is $t_{ij}^{Tr} = 0$ and for any $i > m$ we have that for $j < i$, $t_{ij}^{Tr} = 0$.

(iv) If $l > 1$, then for $i \leq m_1$ we have that for $j > i$ there is $t_{ij}^{Tr} = 0$ and for any $i \geq m_l$ we have that for any $j < i$ there is $t_{ij}^{Tr} = 0$.

One can show that for $h \leq \min(m_1, i)$ (and in the case of unique median for $h \leq \min(m-1, i)$) we have $\tilde{v}_h = 1$. In general, $\tilde{v}_h = \sum_{j=1}^{n} t_{hj}^{Tr} v_j$. For $j > h$ there is $t_{ij}^{Tr} = 0$. Further, $\sum_{j=1}^{h} t_{ij}^{Tr} = 1$ (implied by conditions $(i) - (iv)$ above) and $v_h = 1$ for $h \leq i$ (from the definition of vector v). On the other hand, for $h \geq \max(m_l, i)$ (and in the case of unique median for $h \geq \max(m+1, i)$) we obtain $\tilde{v}_h = 0$, because $\tilde{v}_h = \sum_{j=h}^{n} t_{hj}^{Tr} v_j$ and for $j > i$ there is $v_j = 0$. Finally, for $\min(m_1, i) < h < \max(m_l, i)$ (and in the case of unique median for $h = m$) we have $\tilde{v}_h \leq 1$. Indeed, $\tilde{v}_h = \sum_{j=1}^{n} t_{hj}^{Tr} v_j$, but $v_h = 0$ for $h > i$, hence $\tilde{v}_h = \sum_{j=1}^{i} t_{hj}^{Tr} v_j \leq \sum_{j=1}^{n} t_{hj}^{Tr} = 1$. Further, $\tilde{v}_h = 1$ if and only if $\sum_{j=1}^{i} t_{hj}^{Tr} = 1$.

Returning to P^2, we have for $h \leq \min(m_1, i)$, $\tilde{v}_h = 1$ and for $m_2 \leq i < m_k$, $S_i = 0$, thus $S_i \leq Q_i$ and $P^2(\delta) \leq P^2(\omega)$ consistent with Definition 5. The case of $i \geq m_k(\delta)$ can be treated similarly by putting $P^3 := 1 - S_i$.

□

The proof of Theorem 2 shows how to effectively construct a T-convex matrix given two distributions (i.e. matrix $\hat{t}$). Let η_{ij} denote a median-preserving spread from category j to category i. If ω is obtained from π via a sequence of median-preserving spreads, then a T-convex matrix t such that $\omega = t\pi$ can be constructed by putting $t_{ij} = \frac{\eta_{ij}}{p_j}$ or $t_{ij} = 0$ if there is no transfer between categories.

4.3 Simple T-convex matrices and $\mathcal{T}$-majorization

Every T-convex matrix can be decomposed into elementary matrices. We call such elementary matrices *simple* T-convex matrices.

Definition 6. *A matrix s is called a simple T-convex matrix if it is T-convex and is diagonal, except for (possibly) one column. Formally, given a T-convex matrix $t \in T_{n,m_1,\ldots,m_l}$, where $t = (t_{ij})_{i,j\in\{1,2,\ldots,n\}}$, we define a sequence of simple T-convex matrices $t^k = (t_{ij}^k)_{i,j\in\{1,2,\ldots,n\}}$, $k \in \{1, 2, \ldots, n\}$ by*

$$t_{ij}^k = \begin{cases} 1 & \text{if } i = j, j \neq k, \\ 0 & \text{if } i \neq j, j \neq k, \\ t_{ik} & \text{if } j = k. \end{cases}$$

Intuitively, we modify a diagonal matrix by replacing its k-th column with the k-th column of the matrix t. Applying such a matrix to a given distribution, we change the probability mass in a single category only. An example will clarify.

Example 4

Given matrix t from *Example 1* we have the following sequence of simple matrices: $t^1 = id, t^5 = id$,

$$t^2 = \begin{pmatrix} 1 & 0.8 & 0 & 0 & 0 \\ 0 & 0.2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad t^3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0.3 & 0 & 0 \\ 0 & 0 & 0.3 & 0 & 0 \\ 0 & 0 & 0.4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad t^4 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0.9 & 0 \\ 0 & 0 & 0 & 0.1 & 1 \end{pmatrix}$$

Multiplying t^2 by $\pi = (0.2, 0.2, 0.2, 0.2, 0.2)$ yields $t^2\pi = (0.36, 0.04, 0.2, 0.2, 0.2)$. In other words, we redistributed probability mass concentrated in the second category , transferring 80 percent of this mass to the first category, leaving only 20 percent in the second category.

As Lemma 2 shows, simple matrices are building blocks of any T-convex matrix.

Lemma 2. *Let $t \in T_{n,m_1,\ldots,m_l}$ and t^k be defined as above. Then*

$$t = t^{m_{l-1}} \ldots t^{m_1} t^{m_1 - 1} \ldots t^1 t^{m_l} t^{m_l+1} \ldots t^n.$$

In the case of unique median m we have

$$t = t^m t^{m-1} \ldots t^1 t^{m+1} t^{m+2} \ldots t^n.$$

Proof. We prove the result inductively. Let t be a T-convex matrix and t^k be a sequence of simple T-convex matrices as in Definition 6. As a preliminary, we define, for any set of indices $A \subset \{1, 2, \ldots, n\}$, a matrix t^A by $t^A = (t^A_{ij})_{i,j \in \{1,2,\ldots,n\}}$, where

$$t^A_{ij} = \begin{cases} 1 & \text{if } i = j, j \notin A, \\ 0 & \text{if } i \neq j, j \notin A, \\ t_{ik} & \text{if } j \in A. \end{cases}$$

Obviously, when A is a singleton, we obtain simple matrices introduced in Definition 6. We claim that the following equalities hold.

$$t^{\{m_l+k, m_l+k+1, \ldots, n\}} = t^{m_l+k} t^{m_l+k+1} \ldots t^n, \tag{1}$$

$$t^{\{k, \ldots, 1, m_l+1, m_l+2, \ldots, n\}} = t^k \ldots t^1 t^{m+1} t^{m+2} \ldots t^n. \tag{2}$$

We note that (2) suffices to conclude the proof of the theorem, as we have $t = t^{\{m_{l-1}, \ldots, 1, m+1, m+2, \ldots, n\}}$.

We now resort to the induction. We start with (1). The base step $t^{\{n\}} = t^n$ is obvious. We next assume (1) for some $k \geq 1$, and we consider

$$h := t^{m_l+k-1} t^{\{m_l+k, m+k+1, \ldots, n\}}.$$

We refer to the j-th column of h. We have the following cases:

- $j \leq m_l + k - 1$. Such a column is constructed by taking the j-th column of $t^{\{m_l+k, m+k+1, \ldots, n\}}$, which is e_j (the unit vector with 1 on the j-th coordinate) and scalar product it with the consecutive rows of t^{m_l+k-1}. The result is the j-th column of t^{m_l+k-1}.

- $j > m_l + k - 1$. The j-th column of $t^{\{m_l+k, m+k+1, \ldots, n\}}$, denoted by w, is the same as the j-th column of t. Hence, by the fact that t is T-convex, we have $w_i = 0$ for $i \leq m_l + k - 1$. We note that the lower $(n - (m_l + k - 1), n - (m_l + k - 1))$ sub-matrix of t^{m_l+k-1} is an identity matrix. The last two observations amount to stating that the result of the multiplication of w by the consecutive rows of t^{m_l+k-1} is still w.

We can conclude that $h = t^{\{m_l+k-1, m_l+k, \ldots, n\}}$. This completes the proof of (1). (2)is proved similarly. □

T-convex matrices are related to generalized majorization formulated by Parker and Ram (1997). For $x, y \in \mathbb{R}^n_+$, define the majorization ordering $x \precsim y$ by

$$\sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i, \quad k = 1, \ldots, n-1 \quad \text{and} \quad \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i.$$

For instance, $(2,4) \precsim (3,3) \precsim (4,2)$. If x, y are descendingly-sorted vectors, then generalized majorization becomes classical majorization. Let π be a distribution with l medians and let π_L, π_H denote, respectively, $\pi_L = (p_1, \ldots, p_{m_1})$, $\pi_H = (p_{m_l}, \ldots, p_n)$ and in case of $l = 1$, $\pi_L = (p_1, \ldots, p_{m-1})$, $\pi_H = (p_{m+1}, \ldots, p_n)$. Note that the AF relation is such that for π_L generalized majorizes ω_L and ω_H generalized majorizes π_H. A necessary and sufficient condition for $x \precsim y$ is that there exists a lower triangular matrix L such that $x = Ly$, or equivalently, an upper triangular matrix U such that $Ux = y$. Coming back to the AF, we have $U\pi_L = \omega_L$ and $L\pi_H = \omega_H$. Thus the definition of a T-convex matrix becomes clear.

We have the following result.

Lemma 3. *The set $T_{n,m_1,\ldots,m_l}$ is a semigroup i.e. a set of matrices closed under multiplication that includes the identity.*

Proof. The identity matrix is a T-convex matrix, so it belongs to $T_{n,m_1,\ldots,m_l}$. Let $t, \tilde{t} \in T_{n,m_1,\ldots,m_l}$ and $\bar{t} = t\tilde{t}$, where $\bar{t}_{ij} = \sum_{k=1}^n t_{ik}\tilde{t}_{kj}$. Obviously, $\bar{t}_{ij} \geq 0$. Further,for any j we have $\sum_{i=1}^n \bar{t}_i j = \sum_{i=1}^n \sum_{k=1}^n t_{ik}\tilde{t}_{kj} = \sum_{i=1}^n \tilde{t}_{ij} \sum_{k=1}^i t_{ki} = 1$, because $t, \tilde{t}$ fulfill (2) in the Definition 4. Let $j \leq m_1$. For any $i > j$ we have $\bar{t}_{ij} = 0$, because $\tilde{t}_{kj} = 0$ for $k > j$ and similarly for $j \geq m_l$. □

Following Parker and Ram (1997) we define the preorder $T_{n,m_1,\ldots,m_l}$-majorization.

Definition 7. *The preorder $T_{n,m_1,\ldots,m_l}$-majorization $\omega \precsim_T \pi$ holds whenever there exists a matrix $t \in T_{n,m_1,\ldots,m_l}$ such that $\omega = t\pi$.*

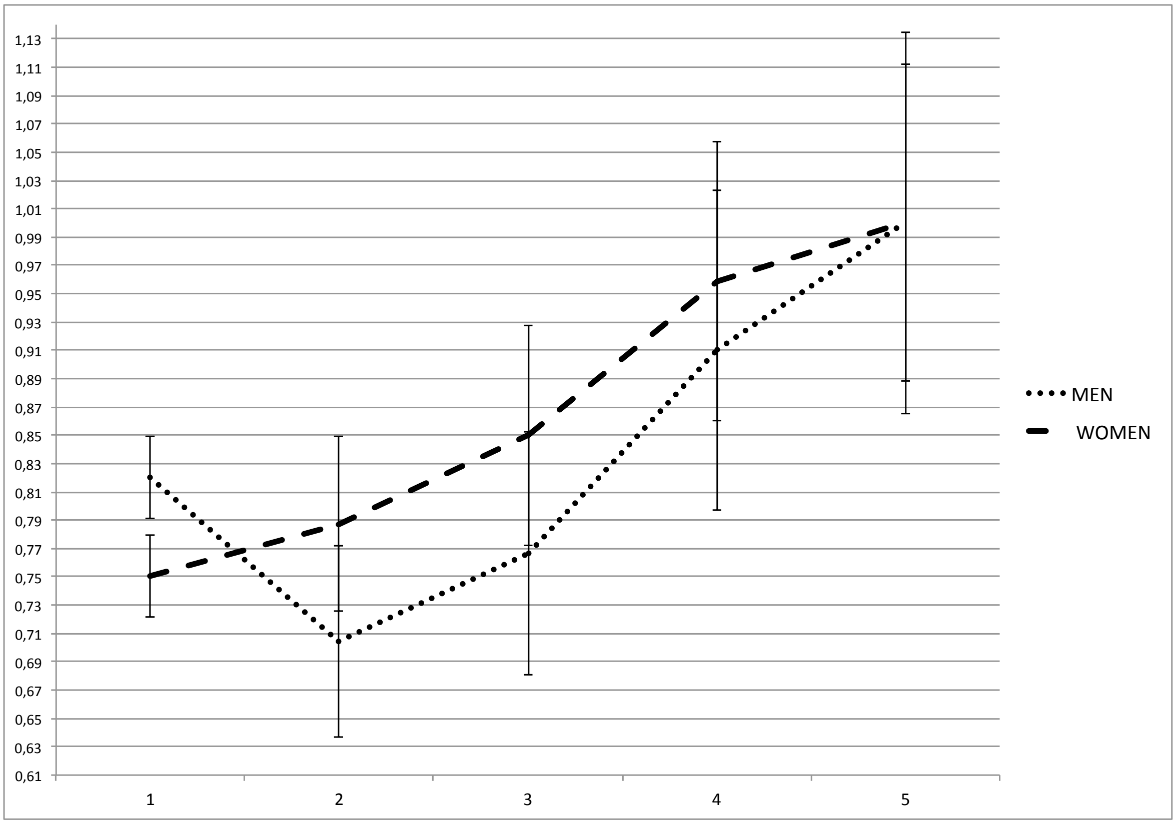
This is a preorder because $T_{n,m_1,\ldots,m_l}$ is a semigroup i.e. the identity matrix ensures reflexivity and closure under multiplication ensures transitivity. Note that in the Definition 7 it is not required that π have medians in categories corresponding to parameters $m_1, \ldots, m_l$. For instance, t is a matrix from the set $T_{4,2}$, whereas π is such that $\Pi = (0.1, 0.4, 0.6, 1)$, so $m = 3$. On the other hand, this preorder restricted to the set of distributions with medians $m_1, \ldots, m_l$ is the AF partial ordering.

5 Empirical application: educational polarization among men and women

In previous sections we showed that the AF relation is the partial ordering with which a broad class of polarization measures is in agreement. Thus, the AF relation can be used to rank distributions in a robust way, namely, to a great extent the comparison of polarization between two distributions does not depend on the choice of polarization measure because all polarization measures belonging to particular classes rank distributions in the same way. To illustrate the usefulness of this property, we compare the educational polarization among men with the educational polarization among women in years 1989 and 2004 in the United States, using the General Social Survey data that contain information on a sample of adults aged 18 and over. Note that these two particular years are only chosen as a clear example. In particular, we show that in cases when AF dominance does not hold, the comparison of health polarization among men and women is inconclusive and depends on the choice of polarization measure. We calculate polarization using three indices and obtain inconsistent results in the case of no dominance. On the other hand, these measures fulfill the median-preserving spread principle and are T-convex, therefore according to Theorems 1 and 2, if only the AF dominance holds, they all should rank the distributions in a consistent manner, which is indeed the case in our dataset. In cases when the AF dominance holds, we show the sequence of median-preserving spreads and T-convex matrices that increase the spread of the less polarized distribution.

There are five educational levels: adult high school, high school, junior college, bachelor, and graduate. The median category for both groups in both years is high school. An appealing way of presenting the AF ordering is through the use of dominance curves. Let $U : \Lambda \mapsto \mathbb{R}$ be defined as follows: $U(P_i) = 1 - P_i$ for $i < m$ (or $i < m_1$ for multiple medians) and $U(P_i) = P_i$ for $i \geq m$ (or $i \geq m_l$). Then, $\pi \precsim_{AF} \omega$ if and only if $U(P_i) \geq U(Q_i)$ for all i, that is, we have a monotone representation of the AF relation. In particular, when π AF dominates ω, then the U curve for π is equal to or above the U curve for ω. We utilize U curves in what follows. As shown in Table 1 and as depicted in Figure 1, in year 1989 there is no clear dominance of the women's over the men's distribution in terms of educational polarization; the two U curves cross each other. Furthermore, the two curves are significantly different from each other at the lower end of the distribution.

Figure 1: Educational polarization in the US in 1989 for men and women: U curves



Source: Own calculations based on the US General Social Survey, 1989.

As seen in Figure 2, in year 2004 the U curve for women lies above the U curve for men, hence there is less educational polarization among women than men, and this conclusion is robust in the sense that all polarization indices that fulfill the median-preserving spread principle and at the same time are T-convex, rank the two distributions in the same way. The difference between the two U curves based on bootstrap confidence intervals is, however, not statistically significant.

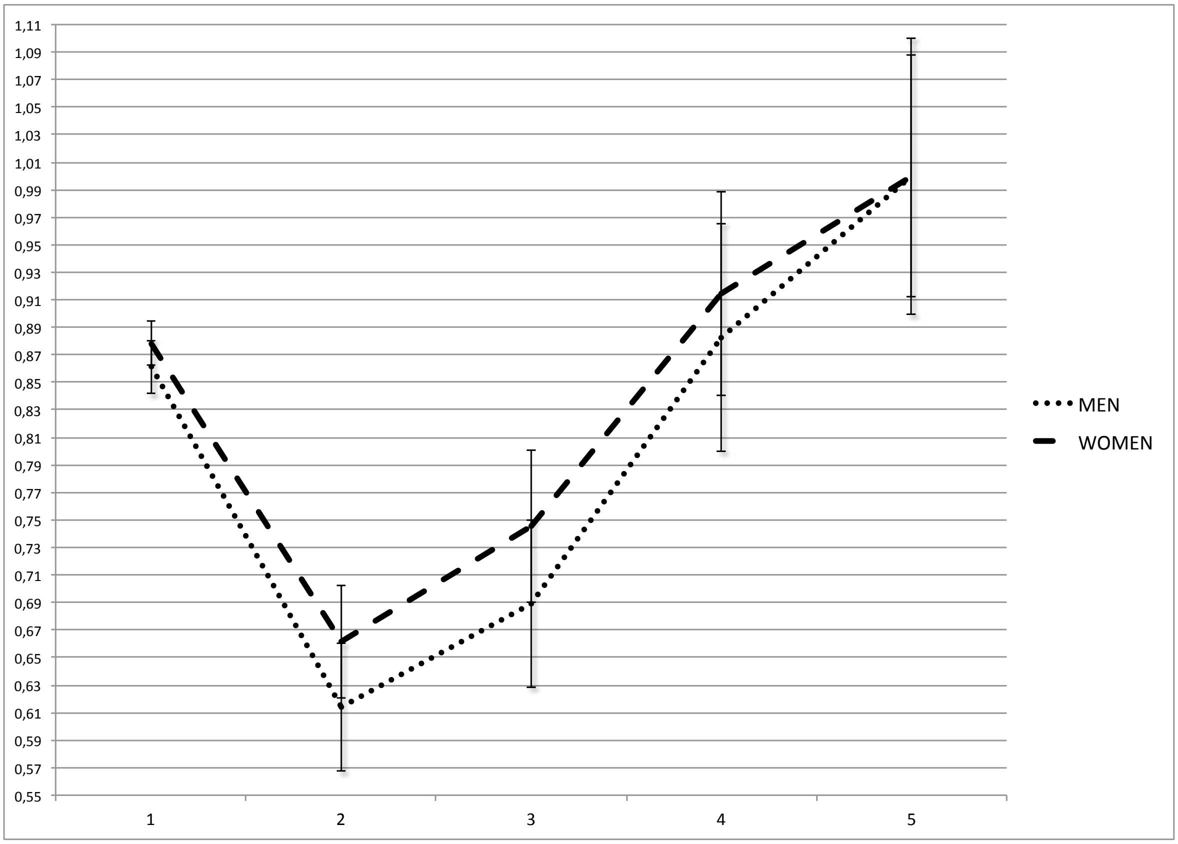
We consider the following family of indices:

$$P_{a,b} = \frac{a \sum_{i<m} P_i - b \sum_{i \geq m} P_i + b(n+1-m)}{\frac{a(m-1)+b(n-m)}{2}}. \tag{3}$$

To recall, n is the number of categories, m is the median, and a, and b are parameters.[14] When $a > b$ the index $P_{a,b}$ is more sensitive to dispersion below the median, whereas the opposite holds when $a < b$, as more weight is attached to dispersion above the median. When $a = b = 1$, dispersion below and above the median are weighted equally. In this case, we get an absolute value index as

[14] This can be easily generalized to accommodate multiple medians noting that $P_i = 0.5$ for $m_1 \leq i \leq m_{l-1}$.

Figure 2: Educational polarization in the US in 2004 for men and women: U curves



Source: Own calculations based on the US General Social Survey, 2004.

proposed by Abul Naga and Yalcin (2008) and Apouey (2007). Clearly, $P_{a,b}$ fulfills Definitions 2 and 5. When the probability mass is transferred away from the median or when the distribution is multiplied by a T-convex matrix, the mass changes for either categories below or above the median. If the transfer takes places below the median, then (cf. Remark 5) for $i < m$ P_i cannot decrease, therefore $\sum_{i<m} P_i$ does not decrease and given $a > 0$ the index does not decrease either. If the transfer takes place above the median, then $i \geq m$ P_i cannot increase, therefore $\sum_{i \geq m} P_i$ does not increase and given $b > 0$ the index does not decrease.

We now compare the polarization rankings in years 1989 and 2004 according to three indices: $P_{1,10}, P_{1,1}, P_{10,1}$, where the choice of parameters is for demonstration purposes only. In particular, we would like to work with indices for which it is evident that dispersions below and above the median are treated differently (hence, the spread of weights, 1 and 10), and an absolute value index. Before we present the results, we note key observations that can be inferred from looking at the raw data. First, the most notable change in completed educational level between 1989 and 2004 occurred for women at the lower end of the distribution. The percentage of women obtaining the lowest educational level (adult high school) reduced by 13 percentage points: from 24 in 1989 to 10.8 in 2004. For men this also dropped, but by 5 points (from 18.6 in 1991 to 13.8 in 2004). Thus, while

Table 1: A comparison of educational polarization among men and among women in the US in 1989 and in 2004

Year	Index	Men vs. Women	Verbal description
1989	$P_{1,10}$	$0.41 > 0.27$	Less educational polarization among women than among men
	$P_{1,1}$	$0.39 > 0.32$	Less educational polarization among women than among men
	$P_{10,1}$	$\mathbf{0.37 < 0.44}$	**More educational polarization among women than among men**
2004	$P_{1,10}$	$0.53 > 0.45$	Less educational polarization among women than among men
	$P_{1,1}$	$0.47 > 0.40$	Less educational polarization among women than among men
	$P_{10,1}$	$0.34 > 0.29$	Less educational polarization among women than among men

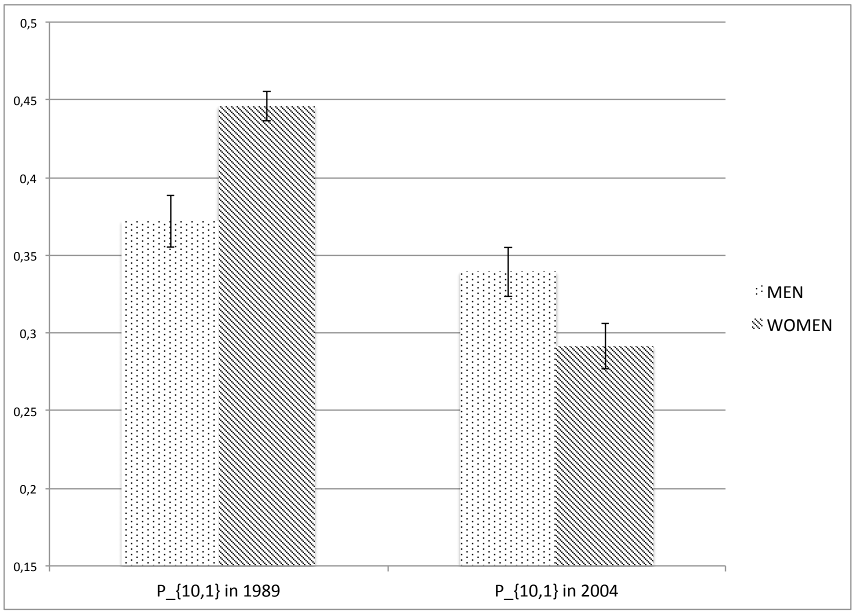
Source: Own calculations based on the US General Social Survey, 1989 and 2004.

in 1989 the percentage of women at the lowest educational level was higher than the comparable percentage of men (24 vs. 18.6), in 2004 this trend reversed (10.8 vs. 13.8). This largely accounts for the fact that the distribution is more compressed for women, which we will also observe using polarization indices. Second, for high school graduates the decrease is more pronounced for men (6 percentage points) than for women (almost no change). Third, the percentage of women holding junior college degrees increased by a little more than the comparable percentage for men (2 vs. 1.3). Fourth, bachelor level trends are similar for both groups, namely, we observe a 6 percentage point rise between 1989 and 2004. Finally, for graduate level the increase is more visible among women than men (5 vs 3.3).

The results are in Table 1. In year 1989 there is no consistent dominance (the two U curves cross); the $P_{10,1}$ index delivers a different verdict than the other two indices. An explanation for this "dissonance" is that it preferentially weights the percentage of individuals below the median, which in 1989 is higher for women (24) than for men (18.6). In year 1989 polarization as measured by the $P_{10,1}$ is significantly greater for women than for men whereas in year 2004 the opposite holds (Figure 3).

Let π^{2004} and ω^{2004} denote distributions of educational attainment in year 2004 for women and men, respectively. Since we find that in this year $\pi^{2004} \precsim_{AF} \omega^{2004}$, there is a sequence of transfers (cf. Lemma 1) that allows transformation of π^{2004} into ω^{2004}. In particular, $\pi^{2004} = (0.108, 0.546, 0.082, 0.178, 0.086)$ and $\omega^{2004} = (0.138, 0.476, 0.075, 0.197, 0.114)$. Using the algorithm

Figure 3: Educational polarization in the US in 1989 and in 2004 for men and women: $P_{10,1}$ and its confidence intervals



Source: Own calculations based on the US General Social Survey, 1989 and 2004.

in the proof of Lemma 1 we start by transferring $\eta_1 = 0.03$ from the second (unique median) to the first category, which results in $\psi_1 = (0.138, 0.516, 0.082, 0.178, 0.086)$. Then, transferring $\eta_2 = 0.028$ from the fourth to the fifth category we obtain $\psi_2 = (0.138, 0.516, 0.082, 0.150, 0.114)$. Next, shifting $\eta_3 = 0.047$ from the third to the fourth category results in $\psi_3 = (0.138, 0.516, 0.035, 0.197, 0.114)$. Finally, a transfer of $\eta_4 = 0.04$ from the median to the third category gives us $\psi_4 = \omega^{2004} = (0.138, 0.476, 0.075, 0.197, 0.114)$. Having the sequence of median-preserving spreads it is easy to construct simple T-convex matrices. For example, the overall mass transferred from the median category is $\eta_1 + \eta_4 = 0.07$, which contributes $\frac{0.07}{0.546} = 0.128$ to the overall mass concentrated in the median of π^{2004}. From this, $\frac{0.03}{0.546} = 0.055$ goes to the first category, $\frac{0.04}{0.546} = 0.073$ goes to the third category, and 0.872 is left in the median category. Therefore, matrix t^2 is the following.

$$t^2 = \begin{pmatrix} 1 & 0.055 & 0 & 0 & 0 \\ 0 & 0.872 & 0 & 0 & 0 \\ 0 & 0.073 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Multiplying all simple matrices in a manner shown in Lemma 2, we obtain the following T-convex matrix.

$$t = \begin{pmatrix} 1 & 0.055 & 0 & 0 & 0 \\ 0 & 0.872 & 0 & 0 & 0 \\ 0 & 0.073 & 0.426 & 0 & 0 \\ 0 & 0 & 0.573 & 0.842 & 0 \\ 0 & 0 & 0 & 0.158 & 1 \end{pmatrix}$$

and $\pi^{2004} = t\omega^{2004}$. The largest relative transfer between the women's and men's distributions is a transfer of almost 60 percent of mass from the third to the fourth category, with a similar mass in the median and below median categories representing relatively more men with the bachelor degrees, increasing the spread.

To sum up, the AF relation is a fairly robust criterion to evaluate dispersion in ordinal data. It is consistent with measures that do not decrease following a median-preserving spread and multiplication by a T-concave matrix. On the other hand, the rankings induced by both concepts are richer than the AF relation, because they allow comparison of distributions with different medians. Naturally future research should address comparisons between distributions which do not share a common median. Some effort in this vein have been made by Abul Naga and Yalcin (2010); more is called for.

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