Sunspot Fluctuations: A Way out of a Development Trap?

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First version: 06/20/1999 This version: 03/30/2001

Abstract

This paper contains a study of stochastic stability of the development trap in a model of economic growth when the production function is subject to externalities and, as a result, the development trap steady state is indeterminate. In the presence of indeterminacy, sunspot equilibria can exist. I study the stability of the trap, subject to continuous-time sunspot shocks, modeled as a Wiener process. Global dynamics of the deterministic and stochastic versions of the model are completely characterized. Numerical simulations of the process of escape from the poverty trap caused by the presence of sunspot fluctuations are conducted. Escape time and probabilities are estimated analytically and numerically as a function of initial conditions and the model's parameters.

Abstrakt

Článek studuje stochastickou stabilitu růstové pasti v modelu ekonomického růstu, ve kterém výrobní funkce závisí na externatitách. V důsledku externalit je stacionární stav odpovídající růstové pasti nedeterministický, což vede k existenci stacionárních stavù podmíněných "skvrnami na Slunci" (tzv. "sunspot equilibria"). Článek se zabývá analýzou stability růstové pasti vzhledem k časově spojitým "skvrnám na Slunci" (náhodným šokům) modelovaným jako Wienerův proces. Poté co kompletně charakterizujeme globální dynamiku deterministické i stochastické verze modelu, simulujeme numericky únik z růstové pasti zapříčiněný náhodnými šoky. Střední doba úniku a pravděpodobnost úniku jsou vyčísleny jak analyticky, tak numericky v závislosti na počátečních podmínkách a parametrech modelu.

JEL Classification: E32, O41

Keywords: development trap, indeterminacy, stochastic stability, sunspots

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1 Introduction

1.1 Development Traps and Indeterminacy

There are several types of models that produce "development traps" or "poverty traps". One group, best represented by Azariadis and Drazen (1990), relies on "thresholds" to generate poverty traps. In this model, investing a non-zero amount of effort into accumulating human capital can lead to a balanced growth path with unlimited growth of all per capita quantities. Due to the presence of externalities, however, it is not optimal to invest in human capital accumulation until the average stock of it in the economy reaches some threshold value. Any economy that starts below threshold remains there forever. If, due to errors, some human capital is accumulated, it does not depreciate. The time of crossing the threshold is, therefore, a function of the magnitude of errors, but the crossing is inevitable if the magnitude is bounded above zero[1].

Other papers with similar dynamics include Lee (1996), where financial intermediaries accumulate information about investment opportunities by making loans. In a low information equilibrium, nobody lends. The paper proposes credit subsidies or inflow of relatively cheap foreign capital to overcome the trap. In Ciccone and Matsuyama (1996), an insufficient number of intermediate inputs hinders adoption of modern technologies. High start-up costs required to establish the production of necessary inputs mean that reallocating scarce resources from traditional production is inefficient, locking the economy in the poverty trap. It is sometimes possible for a large number of entrepreneurs expecting future growth to enter the specialized inputs markets, escaping from the poverty trap due to self-fulfilling prophecy, but for other parameter values the trap is inescapable. In another application of the same idea, Burguet and Fernandez-Ruiz (1998) construct a development trap in an economy with publicly provided goods and public capital; a sufficiently low world interest rate might be needed for escape.

[1] Arifovic, Bullard, and Duffy (1997) use a revised version of the model. Instead of errors, it is random mutations forming part of the genetic algorithm learning mechanism that lead to the accumulation of human capital. Eventually, the threshold is passed.

A general characteristics of the papers cited above is the existence of a certain threshold that separates poverty-trap-locked economies from developing ones. For an economy in the trap, there is no way out other than some change in parameters: consistent non-optimal accumulation of human capital, credit subsidy, or supply of external funds at low world interest rates.

The other strand of models with poverty traps has some kind of dynamic coordination failure or pessimistic expectations built in. Examples of such models include Matsuyama (1991), Gali and Zilibotti (1995), Gans (1998), Baland and Francois (1996), and Skiba (1978). In this type of models, non-convexity in production function due to increasing returns, externalities, and/or market power leads to a possibility of multiple steady states. In these models, indeterminacy exists — for given values of stock variables like capital there are different choices of control variables like consumption, work effort, etc., such that a perfect foresight equilibrium trajectory converges to a steady state. Different choices of control variables might imply convergence to different steady states, and initial conditions do not necessarily determine to which steady state the economy converges. One can say that in the above models the economy might be consigned to a poverty trap by the failure of economic agents to agree on the control variable value leading to the best equilibrium. The distinction between the two groups is not strict, though, as a majority of models in the second group allow parameter values leading to a threshold-type poverty trap.

The major goal of the current paper is to discuss an additional mechanism for overcoming coordination failures or pessimistic expectations in the models of the second type. As noted above, these models exhibit indeterminacy. There are two types of indeterminacy. One situation is when there are two (or more) saddle path stable steady states, and there are corresponding unique trajectories converging to them. This case is sometimes referred to as global indeterminacy. In this case, pessimistic or optimistic expectations simply select one trajectory out of two or other small number. This happens for some parameter values in Gali and Zilibotti (1995), for example. On the other hand, it may happen that for one or more steady states the linearization of the law of motion has fewer unstable roots than “free” or control variables. In this case, the stable manifold of the steady state has fewer dimensions than the number of control variables, and there exists a continuum of values of control variables that put the system onto the stable manifold. Therefore, there exists a continuum of perfect foresight trajectories satisfying all the conditions for being an equilibrium trajectory, including the transversality condition. This case is referred to as local indeterminacy, and it is the subject of the current paper.

What happens if the system exhibits local indeterminacy? Suppose that I have a decentralized economy. Agents are free to choose initial values of the control variable(s) from some large set. Once the initial conditions are agreed upon and the dynamics of the system unfolds, none of the agents has an incentive to deviate from the optimal trajectory, which depends on the initial conditions[2]. However, the trajectory chosen can be a very bad one: it could include a very low level of, say, work effort, and a low growth rate as a result. Choosing a different initial condition with higher level of work effort could increase the growth rate and provide higher utility to every agent and thus be Pareto improving[3]. A different starting point can even imply convergence to a much better steady state with unbounded growth of all per capita variables, as in endogenous growth models. A classic case of coordination failure can exist in situations with local indeterminacy of the steady state.

Imagine the situation where a low growth state is locally indeterminate. The decentralized economy develops along one of the trajectories leading to the low

[2]If agents are small compared to the size of the economy, their deviation will not significantly change variables that are arguments of their decision rules - interest rate and wage rate, for example. Thus, individual deviation from the optimal trajectory will reduce an agent’s payoff.

[3]Note that in the presence of increasing returns and/or externalities, the initial trajectory not necessary was Pareto optimal. In the process of solving such models, one usually assumes that every agent takes the current level of externality as exogenously given; every agent then faces a concave production function, and this decision problem is easily solved. Alternatively, increasing returns to scale can be supported by monopolistic competition. In any case, every agent makes a decision under incomplete information and/or some market failures. Therefore, the solution is not required to be Pareto optimal to begin with.

growth state, that is, the economy is in the development trap. Assume that there exists a high growth steady state which can also be locally indeterminate or saddle path stable. In any case, agents need some device to help them coordinate on a trajectory converging to the high growth steady state.

1.2 Sunspots as a Coordinating Mechanism

"Sunspot equilibria" are "rational expectations equilibria in which purely extrinsic uncertainty affects equilibrium prices and allocations", Woodford (1990). "Purely extrinsic uncertainty" denotes some random variable which has no effect on preferences, endowments, or production possibilities. If this random variable and the resulting allocations and prices are stationary, one speaks about stationary sunspot equilibria, or SSE. In discrete time, one of the ways in which SSE are constructed is the randomization between different non-sunspot equilibria; alternatively, SSE can be a randomization over different trajectories converging to a non-sunspot steady state. This procedure can be performed when a non-sunspot steady state is indeterminate. Indeed, in a simple OLG economy with a constant supply of money, as in Azariadis (1981), a necessary condition for the existence of a particular kind of SSE is exactly the condition for the indeterminacy of the non-sunspot rational expectations equilibrium; see Woodford (1990). This connection between indeterminacy of a rational expectations equilibrium and the existence of some SSE (known as "Woodford's Conjecture") was established for a broad class of discrete time models, for example in Woodford (1986), Grandmont (1986), and Spear, Srivastava, and Woodford (1990).

Existence of sunspots is by no means limited to OLG or OLG-like discrete time models. Spear (1991) showed the existence of sunspot equilibria in a pure capital accumulation model where production is subject to externality. Switching to continuous time models allows complete understanding of the model's global dynamics, especially when the model reduces to a two-dimensional system of differential equations. In Drugeon and Wigniolle (1996) a continuous-time endogenous growth model was studied. It was shown that when a balanced growth path is locally stable (indeterminate), a sunspot equilibrium with a Poisson process as a sunspot variable exists. Finally, Shigoka (1994) constructed a continuous time SSE in a variety of growth models (including the one used here), where a sunspot variable is a continuous-time Markov process with finitely many states. Woodford's Conjecture holds in all three cases.

Stability under the equilibrium learning dynamics was proposed in Lucas (1986) as a criterion in deciding which of the many equilibria in the OLG should be considered as more likely to occur. Lucas's conjecture was that only a limited number of equilibria, and in particular locally determinate steady states, will survive such a test. If this conjecture were always true, sunspot equilibria could be considered esoteric theoretical constructs having no practical importance. Using a simple adaptive learning rule, Duffy (1994) has shown that an indeterminate monetary steady state can be selected over a determinate one in an OLG economy with fiat money, thus rejecting Lucas's conjecture. Furthermore, as was shown in Woodford (1990), Evans and Honkapohja (1999), and Evans and Honkapohja (2001), adaptive learning can converge to determinate, indeterminate, and SSE equilbria. Indeterminate and SSE equilibria are more than a theoretical curiosity; one can observe them[4].

In this paper, I postulate the existence of SSE in a continuous-time model in which the sunspot variable is a sample-path continuous stochastic process[5]. Production technology in the model is subject to externality. It is also postulated that the learning mechanism, like that described in Woodford (1990), has taken place and has converged to a sunspot equilibrium. Agents simply add the sunspot variable to their optimal decision, and this is the SSE[6]. As a result, instead of simply moving along a particular trajectory, and, according

[4]Extracting information on belief shocks from financial markets data, Salyer and Sheffrin (1998) show that a model with self-fulfilling beliefs has incremental predictive power for key US economic time series.

[5]Taking into account Shigoka (1994), this assumption does not seem to be too overstretched.

[6]A more detailed description of the construction of a sunspot equilibrium is given in Section 3.

to the assumptions about the agents being uninformed about the nature of the externality, choosing actions based on incomplete information about the state space outside of that trajectory, agents coordinate on the sunspot and get to explore new regions of the state space.

Suppose the economy starts in the perfect foresight development trap. In the model used here, it means that consumption (and the work effort) are chosen to be too low because of the pessimistic expectations of the future wages and interest rates. It is possible to select a level of initial consumption which will push the system out of the trap and into the region of attraction of the positive steady state. However, no individual agent has an incentive to experiment, and everyone is coordinating on a trajectory leading to the origin. This coordination failure could be fixed if agents could form expectations corresponding to a trajectory converging to the positive steady state. Agents are unaware of the existence of such a trajectory because the externality is unknown and they cannot calculate the whole phase portrait of the system. If a sunspot variable, modeled as a Wiener process, is included in the model, agents could take it into account when making their decisions. Coordinating on a sunspot white noise allows exploring new regions of the state space and can eventually move the trajectory of the system out of the trap. As soon as the economy leaves the trap, agents become aware of the existence of a new non-stochastic steady state. It is assumed that in this case a regime change takes place and the agents stop taking the sunspot variable into account. Therefore the further dynamics of the system reduces to convergence to the positive steady state[7]. This will happen if a zero steady state is stochastically unstable under sunspot fluctuations or an

[7] Why didn't the agents choose this trajectory earlier? They could only have learned to predict the perfect foresight trajectory starting from today's initial conditions, but not from other points, because of the externality. Alternatively, they could have learned only the optimal decision rule (the Euler equation), as in Flam and Mirman (1998), but suppose there is an outside body, a government, capable of calculating perfect foresight trajectories and announcing them. The government's announcement of a perfect foresight path starting from today's initial conditions and leading to the superior steady state generates the regime change. If the government knows the whole phase portrait, its announcements could still be useless while inside the trap, because choosing a "good" path involves a discontinuous jump and is prevented by coordination problems. Only the actual escape makes the announced perfect foresight path feasible.

initial condition lies outside the region of stochastic stability that might not coincide with the development trap of the deterministic system. In the case when the economy eventually leaves the trap, it is possible to calculate expected first exit times from the region of attraction of a zero steady state depending on the initial conditions and magnitude of the sunspot process.

The rest of the paper is organized as follows. In Section 2, a brief summary of the model described in Benhabib and Farmer (1994) is given, construction of the phase portrait of the model is performed and existence of the poverty trap is proven. Properties of the model subject to sunspot fluctuations, in particular stochastic stability of the development trap, are studied in Section 3. Section 4 provides numerical estimates of escape probabilities and times; analytical approximations and comparisons to the numerical results are given in Section 5; and Section 6 concludes.

2 The Model

As a basis for analysis, I use a slightly modified model from Benhabib and Farmer (1994). This deterministic continuous-time model with infinitely lived agents is characterized by increasing social returns to scale due to externality in the production function of which the agents are assumed to be unaware. There are two steady states. One has zero capital and zero consumption (the origin), while the other has positive levels of both capital and consumption. For some parameter values, both steady states are indeterminate, and the whole state space is separated into two regions of attraction of the steady states. The region of attraction of the origin is a development trap[8].

The economy consists of a large number of identical consumers seeking to

[8] For expositional clarity, I have chosen the model reduced to the most simple mathematical form possible. There are other models with indeterminate interior poverty traps, like Gali (1994), Gali (1995), and Perli (1998). Those models provide a more realistic description of economic phenomena leading to the poverty trap. Further work will focus on more realistic models.

In endogenous growth models, a development trap is a balanced growth path with lower growth rate. In a model without exogenous or endogenous technological change, such as the one used in this paper, a steady state with a lower level of contemporaneous utility is studied instead.

maximize

$$\int_0^\infty \left(\frac{C^{1-\sigma}}{1-\sigma} - \frac{N^{1-\chi}}{1-\chi}\right)e^{-\rho t}dt,$$

subject to

$$\dot{K} = (r-\delta)K + wN - C,$$

where C is consumption, K capital, N work effort, r interest rate, and w the wage rate. There are a large number of identical firms with the production function

$$Y = K^a N^b \overline{K}^{\alpha-a}\overline{N}^{\beta-b}, \tag{1}$$

where $a+b=1$, $\alpha > a$, $\beta > b$, and $\overline{K}$ and $\overline{N}$ are economywide averages of K and N per firm, which are taken as given by every individual firm. From the profit maximization, the interest rate and the wage rate are given by

$$wN = bY, \tag{2a}$$

$$rK = aY. \tag{2b}$$

Identical consumers take trajectories of wage and interest rates as given and solve their maximization problem. In a symmetric equilibrium, all firms employ the same amount of labor and capital, and thus $K = \overline{K}$, $N = \overline{N}$. In the perfect foresight equilibrium, consumers know the correct trajectories of r and w. Solving the problem and switching to logs, one gets the following system of equations, almost equivalent to the one derived in the original Benhabib and Farmer (1994) model:

$$\dot{c} = \left[\frac{a}{\sigma}\exp(w - vk + uc) - \frac{\delta+\rho}{\sigma}\right], \tag{3a}$$

$$\dot{k} = [\exp(w - vk + uc) - \exp(c-k) - \delta]. \tag{3b}$$

where w, v, and u are given by

$$\begin{aligned} w &= -\frac{\beta\log(b)}{\beta+\chi-1}, \\ v &= \frac{\beta-(1-\alpha)(1-\chi)}{\beta+\chi-1}, \\ u &= \frac{\sigma\beta}{\beta+\chi-1}. \end{aligned} \tag{4}$$

The system (3) is extremely hard to analyze. It is therefore useful to change the coordinates to

$$x = \exp(w - vk + uc), \tag{5a}$$

$$y = \exp(c - k). \tag{5b}$$

After this change of variables, the system transforms into

$$\dot{x} = x[(\frac{a}{\sigma}u - v)x + vy + v\delta - u\frac{\delta + \rho}{\sigma}], \tag{6a}$$

$$\dot{y} = y[((\frac{a}{\sigma} - 1)x + y + \delta - \frac{\delta + \rho}{\sigma})]. \tag{6b}$$

By construction, x and y are nonnegative; therefore only the first quadrant of the (x, y) space should be considered.

2.1 Steady States and Their Stability

The positive steady state of (6) is **A**$= (x^*, y^*) = (\frac{\delta+\rho}{a}, \frac{\delta+\rho}{a} - \delta)$. Linearization of (6) around this steady state produces

$$\mathbf{J}^* = \begin{bmatrix} x^*(\frac{a}{\sigma}u - v) & x^*v \\ y^*(\frac{a}{\sigma} - 1) & y^* \end{bmatrix},$$

and

$$Det\ (\mathbf{J}^*) = \frac{a}{\sigma}x^*y^*(u - v), \qquad trace(\mathbf{J}^*) = x^*(\frac{a}{\sigma}u - v) + y^*.$$

To get indeterminacy, one needs 2 stable roots, which means $Det\ (\mathbf{J}^*) > 0$, $trace(\mathbf{J}^*) < 0$. Recalling definitions of u and v and simplifying, one gets:

$$u - v = \frac{(\sigma - 1)\beta + (1 - \alpha)(1 - \chi)}{\beta + \chi - 1}. \tag{7}$$

Following the original paper, where $\alpha < 1$, $\chi < 0$, and assuming σ is not too far away from 1 ($\sigma = 1$ means utility logarithmic in consumption), the necessary condition for indeterminacy is still $\beta + \chi - 1 > 0$ as in Benhabib and Farmer (1994), because the numerator is positive. The trace is given by

$$\frac{\delta + \rho}{a}(\frac{a}{\sigma}u - v + 1) - \delta = \frac{\delta + \rho}{a}\frac{a\beta - \alpha(1 - \chi)}{\beta + \chi - 1} - \delta = \tag{8}$$
$$\frac{\delta + \rho}{a}\frac{a(\beta + \chi - 1) - (\alpha - a)(1 - \chi)}{\beta + \chi - 1} - \delta = \rho - \frac{\delta + \rho}{a}\frac{(\alpha - a)(1 - \chi)}{\beta + \chi - 1}.$$

If there is no capital externality ($\alpha = a$), trace equals ρ and is positive. The lowest α that makes trace negative is given by $\alpha = a(1+\frac{\rho}{\delta+\rho}\frac{\beta+\chi-1}{1-\chi})$. Combining all the conditions together, I see that if

$$\begin{aligned} \beta+\chi-1 &> 0, \\ a(1+\frac{\rho}{\delta+\rho}\frac{\beta+\chi-1}{1-\chi}) &< \alpha < 1, \\ (\sigma-1)\beta+(1-\alpha)(1-\chi) &> 0, \end{aligned} \tag{9}$$

then the positive steady state **A** is indeterminate. From now on, only parameter values satisfying conditions (9) will be considered.

There are other steady states of (6), given by

$$\mathbf{B} = (0,0),\ \mathbf{C} = (0, \frac{\delta+\rho}{\sigma} - \delta),\ \text{and}\ \mathbf{D} = (\frac{u\frac{\delta+\rho}{\sigma} - v\delta}{\frac{a}{\sigma}u - v}, 0).$$

For σ not too large, $\frac{\delta+\rho}{\sigma} - \delta$ is positive. In the expression for abscissa of **D**, the denominator is given by

$$\frac{a}{\sigma}u - v = \frac{a\beta - \beta + (1-\alpha)(1-\chi)}{\beta+\chi-1} = \frac{a\beta - \alpha(1-\chi)}{\beta+\chi-1} - 1 = \alpha - 1 - \frac{\alpha - a}{\beta+\chi-1}, \tag{10}$$

which is always negative if conditions (9) are true. For the numerator, one gets

$$u\frac{\delta+\rho}{\sigma} - v\delta = \frac{(\delta+\rho)\beta - \delta(\beta-(1-\alpha)(1-\chi))}{\beta+\chi-1} = \frac{\rho\beta + \delta(1-\alpha)(1-\chi)}{\beta+\chi-1}, \tag{11}$$

which is always positive given (9). Therefore, the third equilibrium lies in the second quadrant and does not interest me[9].

Linearizing (6) around the origin, one gets the following Jacobian:

$$\mathbf{J} = \begin{bmatrix} v\delta - u\frac{\delta+\rho}{\sigma} & 0 \\ 0 & \delta - \frac{\delta+\rho}{\sigma} \end{bmatrix}. \tag{12}$$

The first non-zero element was estimated in (11) and is always negative, while the second is negative for σ not too large (and negative for $\sigma = 1$). Therefore, the origin is also stable in (6). Finally, for the steady state **C**, one gets

$$\mathbf{J} = \begin{bmatrix} \frac{\delta+\rho}{\sigma}(v-u) & 0 \\ (\frac{a}{\sigma} - 1)(\frac{\delta+\rho}{\sigma} - \delta) & \frac{\delta+\rho}{\sigma} - \delta \end{bmatrix}.$$

[9] Steady states **B** and **C** both represent trajectories diverging to ($-\infty, -\infty$) in the (c,k) space with different asymptotic behavior. The change of variables collapses infinity points from the lower half of the (c,k) space onto the vertical half-axis in the (x,y) space. Trajectories with different asymptotic behavior at minus infinity are mapped onto different points on the axis.

Here, the (2,2) element of $\mathbf{J}$ is positive, and taking into account (7) I conclude that the (1,1) element of $\mathbf{J}$ is negative. Therefore, $\mathbf{C}$ is a saddle.

2.2 Dulac Criterion and Limit Cycles

To characterize the global dynamics of the system it is necessary to know whether limit cycles exist. The Dulac criterion states that if for the analytical two-dimensional system

$$\begin{aligned} \dot{x} &= P(x,y), \\ \dot{y} &= Q(x,y), \end{aligned}$$

in a simply connected region G there exists a continuously differentiable function $B(x,y)$, such that $\frac{\partial(PB)}{\partial x} + \frac{\partial(QB)}{\partial y}$ does not change sign in G, then there are no simple closed curves in G which are unions of paths of the system[10]. In particular, there are no limit cycles, see Andronov, Leontovich, Gordon, and Maier (1973). For a system of the type (6),

$$\dot{x} = x(a_1x + b_1y + c_1), \tag{13a}$$

$$\dot{y} = y(a_2x + b_2y + c_2), \tag{13b}$$

the Dulac function is $B(x,y) = x^{k-1}y^{h-1}$, where $k = \frac{b_2(a_2-a_1)}{\Delta}$, $h = \frac{a_1(b_1-b_2)}{\Delta}$, $\Delta = a_1b_2 - a_2b_1 \neq 0$. Then

$$\frac{\partial(PB)}{\partial x} + \frac{\partial(QB)}{\partial y} = (\frac{a_1c_2(b_1-b_2)}{\Delta} + \frac{b_2c_1(a_2-a_1)}{\Delta})x^{k-1}y^{h-1}. \tag{14}$$

When $\xi = a_1c_2(b_1-b_2) + b_2c_1(a_2-a_1) \neq 0$, $\frac{\partial(PB)}{\partial x} + \frac{\partial(QB)}{\partial y}$ vanishes only along the integral curves $x = 0$ and $y = 0$. It does not change sign in the interior of any of the four quadrants. Also, it can be shown that there can be no closed contours which are unions of paths in this case. After some algebraic transformations, it can be shown that the condition on ξ amounts to $a_1x^* + b_2y^* = trace(\mathbf{J}^*) = 0$, where (x^*,y^*) denotes the non-trivial steady state. When $trace(\mathbf{J}^*) = 0$, all trajectories of the system are closed orbits.

[10] Note that Bendixson's criterion is a special case of Dulac's with $B(x,y) = 1$.

It is possible to have $Det\ (\mathbf{J}^*) > 0$ and $trace(\mathbf{J}^*) = 0$ with two complex conjugate eigenvalues having zero real part. However, the system does not undergo Hopf bifurcation because there are no limit cycles when $trace(\mathbf{J}^*) \neq 0$.

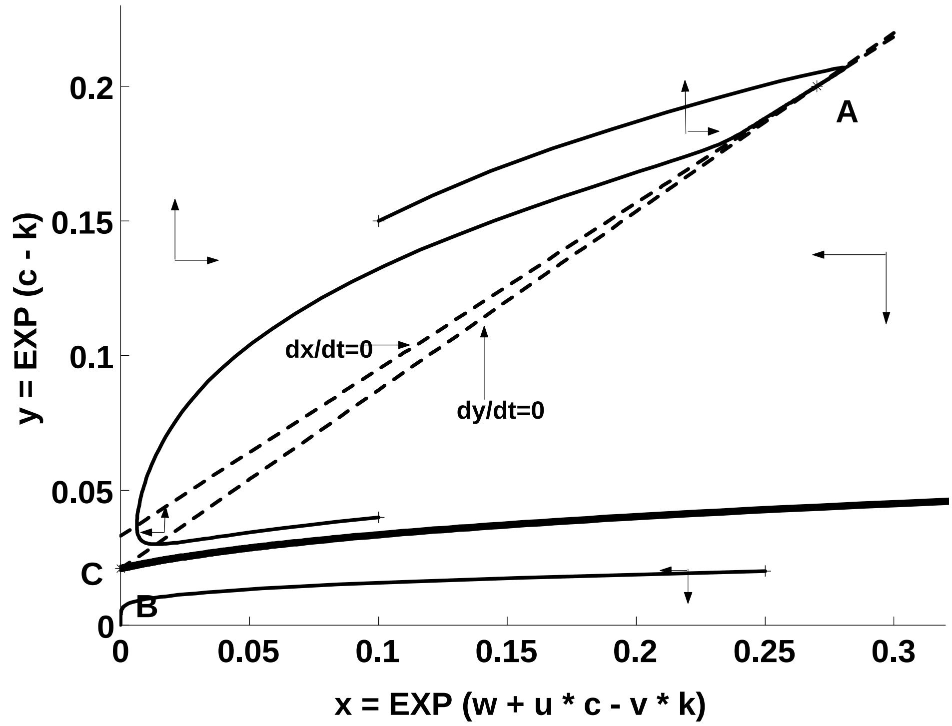
2.3 Global Behavior

The phase portrait of (6) is presented in Figure 1. The whole first quadrant is divided into 2 regions of attraction[11]. The only trajectories that diverge to infinity are those that start on the vertical axis above **C**. The stable manifold of **C** serves as a separatrix between the regions of attraction. In logged consumption and capital, the phase portrait is given by Figure 2. All trajectories that start above the transformed stable manifold of **C** converge to the positive steady state corresponding to **A**. Trajectories with the initial conditions below it diverge to minus infinity. In the original (C, K) variables (Figure 3), the phase portrait looks very similar to that of (6), the only difference being that now the separatrix of the two regions of attraction starts at the origin rather than on the vertical axis. The stable manifold approaches the origin as a ray of constant positive tangent. Any other trajectory of the system which approaches the origin behaves asymptotically as $C \sim K \exp(-\rho t)$. The distance between the stable manifold and any such trajectory expressed as a percentage of actual consumption level grows exponentially with time.

To obtain a point on the vertical axis $\{(x, y) : x = 0,\ y > 0\}$ of Figure 1, the following should be true: $uc - vk = u(c-k) + (u-v)k \to -\infty,\ c-k = const$. This means that $k \to -\infty, c \to -\infty$, but $c - k$ is finite. This corresponds to going to the origin in the non-logged (C, K) space along a ray with finite tangent. In the (c, k) space any trajectory asymptotically linearly diverging to minus infinity satisfies the condition. A point on the horizontal axis $\{(x, y) : x > 0,\ y = 0\}$ is obtained when $uc - vk = u(c - k) + (u - v)k = const,\ c - k \to -\infty$. This is possible only when $k \to \infty$, and c is arbitrary, but c goes to infinity slower than

[11] There were previous attempts to obtain the region of stability of the positive steady state in this model; see, for example, Russell and Zecevic (1998) for the Lyapunov function approach. The approach used here is much broader. I am able to study the global dynamics of the model instead of the compact neighborhood of the steady state, as in the reference.

Figure 1: Phase portrait of the transformed system in (x,y) variables



k or converges to a nonzero constant.

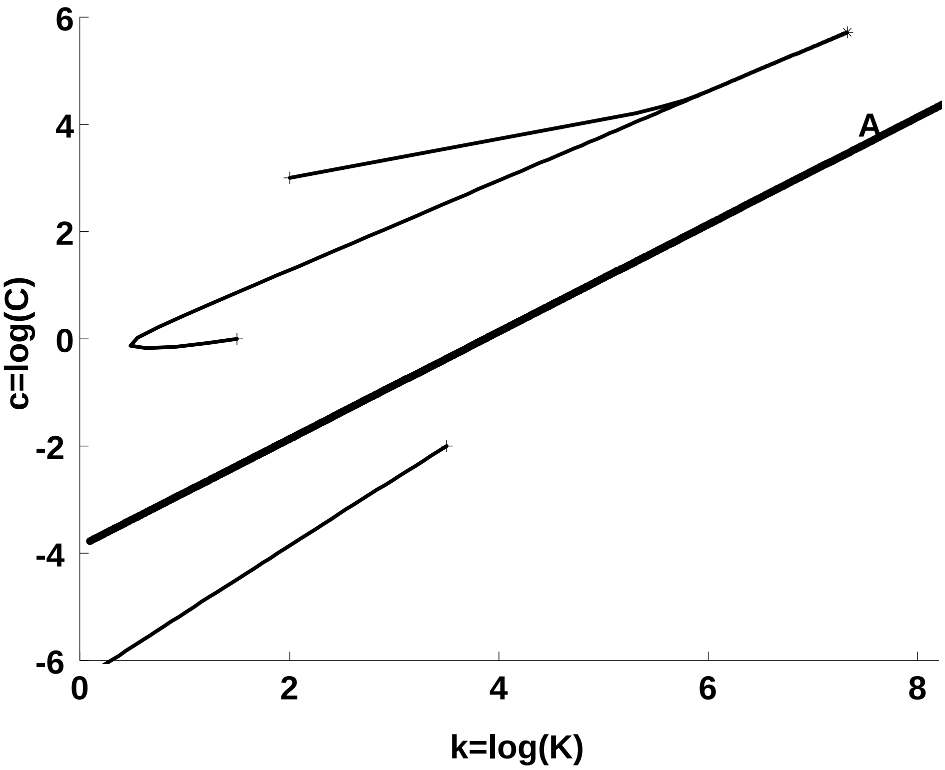
What does the origin in the (x, y) space correspond to? Writing the change of coordinates (5) as $x = \left(\frac{C}{K}\right)^v C^{u-v}$, $y = \frac{C}{K}$, it is easy to see that the origin corresponds to $C < \infty$, $\frac{C}{K} = 0$. Any trajectory in the (C, K) space such that $C = o(K)$, $C \to 0$ corresponds to a trajectory converging to the origin in the (x, y) space. The trivial solution of (6) corresponds to a poverty trap, or imploding economy.

3 Stochastic Dynamics

3.1 Constructing SDE

From the previous section, I know that the system (6) has two stable steady states, there are no limit cycles, and no trajectory starting in the interior of the first quadrant escapes to infinity in the (x, y) space. A trajectory of (6) that

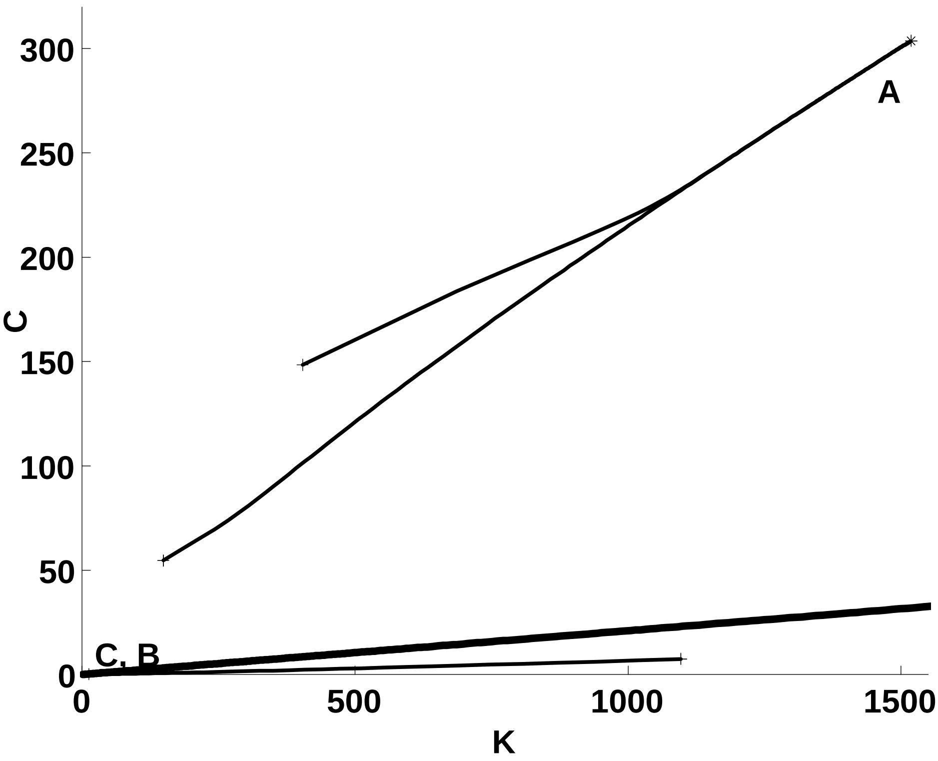
Figure 2: Phase portrait in log capital and log consumption



starts on the vertical axis above **C** escapes to infinity; however, in the (C,K) space this corresponds to a trajectory going to the origin with ever increasing slope. Now I introduce a stochastic process into the system - the sunspot process. A key behavioral assumption is that agents observe a sunspot variable, a Wiener process. They simply add a "derivative" of the process to their decision rule. To justify such an approach, one has to remember that an Itô stochastic differential equation can be obtained as a limit in probability of difference equations if the driving noise is a Markov process with independent increments[12]. Existence of

[12] Construction of the SDE is very similar to that reported in Shigoka (1994). Introducing a sunspot disturbance in this way has a simple justification. Adding $\widetilde{\sigma} dW_t$ to the equation for $log(C)$ is approximately equivalent to adding $C\widetilde{\sigma} dW_t$ to the equation for C. C is the share of the net present wealth (future wages and interest income) agents choose to consume at time t. If agents consider increments of the sunspot variable as fluctuations in their present discounted wealth, $C\widetilde{\sigma} dW_t$ is simply an adjustment of this share due to the fact that the perceived wealth has changed.

Figure 3: Dynamics of the system in original, nonlogged variables



the SSE of this form was shown in the current model by Shigoka (1994) for a continuous time Markov process with finitely many states. A Wiener process is a continuous time Markov process with infinitely many states. For a formal and rigorous introduction to the concept of stochastic differential equation (SDE), see, for example, Karatzas and Shreve (1991). Description of the limiting argument that allows going from the difference equation with Markov variable to SDE can be found in Khasminskii (1980).

I start with a deterministic differential equation (3) and formally add a "differential" of the Wiener process to the RHS of the equation for consumption. The result is

$$dc = [\frac{a}{\sigma}\exp(w - vk + uc) - \frac{\delta + \rho}{\sigma}]dt + \widetilde{\sigma} dW_t, \quad (15a)$$

$$dk = [\exp(w - vk + uc) - \exp(c - k) - \delta]dt. \quad (15b)$$

Doing the same change of variables as in the previous section and applying the Itô theorem as described in Appendix A, one arrives at the following system of SDEs:

$$dx = [x((\frac{a}{\sigma}u - v)x + vy + v\delta - u\frac{\delta+\rho}{\sigma} + \frac{1}{2}\widetilde{\sigma}^2 u^2)]dt + ux\widetilde{\sigma}dW_t, \quad (16a)$$

$$dy = [y((\frac{a}{\sigma} - 1)x + y + \delta - \frac{\delta+\rho}{\sigma} + \frac{1}{2}\widetilde{\sigma}^2)]dt + y\widetilde{\sigma}dW_t. \quad (16b)$$

3.2 Global Stochastic Dynamics

It is necessary to specify the behavior of the stochastic process defined by (16). As was mentioned above, in the present paper I assume that the economy is evolving according to (16) for as long as it is located in the deterministic development trap, the area below the stable manifold in Figure 1. If the trajectory hits the trap boundary, it is assumed that the sunspot process dies out. Economic justification of this assumption is rather simple. Agents in an economy that has spent all its history inside the development trap were unable to see a perfect foresight deterministic trajectory converging to anything but the origin, point **B**. After hitting the boundary, however, they immediately observe another steady state **A** with much better welfare properties. A reasonable assumption would be that a regime change occurs in this situation. I model this change of regime by assuming that the agents stop following the sunspot process and coordinate on some trajectory converging to the positive steady state **A**[13]. It is possible to drop the assumption of the regime change and study the invariant measure, dependent on the initial condition, that arises in this case. However, this extension is beyond the scope of the present paper.

The process that solves (16) and starts in the development trap either stays in the trap forever or exits through the upper trap boundary, crossing the deterministic stable manifold of **C**. In the former case it exists from $t = t_0$ to infinity;

[13] Strictly speaking, if the sunspot process stops immediately at the boundary, then the economy converges to the steady state **C** in Figure 1 which has, again, zero consumption and capital. To avoid this, I have to assume that the sunspot stops at the distance of ϵ *above* the boundary. Then, after the sunspot dies out, I have the deterministic dynamics, and there is only one possibility left: convergence to the steady state **A** in Figure 1.

in the latter it becomes deterministic at some finite time and then converges to the steady state **A**, thus also being defined up to $t = \infty$. If the process stays forever in the trap then it converges to the origin with probability one. Neither of the axes can be attained by the process in a finite time. Therefore, only two possibilities remain in the limit $t \to \infty$: convergence to the origin or to the positive steady state **A**. A sketch of the argument can be found in Appendix B.

The introduction of self-fulfilling fluctuations does not change the long term properties of the model. The same two outcomes observed in the deterministic case (without sunspots) are achieved. My interest, however, lies in studying the probability with which an unfavorable outcome can be transformed into a better one.

3.3 Stochastic Stability of the Origin

A very special structure of the transformed system (16) allows me to derive some analytical results on the stability of the origin under self-fulfilling beliefs driven fluctuations. To prove the asymptotic stability of the origin I will use the stability in first approximation.

Definition 1 *The solution $x(t) \equiv 0$ is said to be stable in probability if, for every $\varepsilon > 0$ and every $t > t_0$,*
$\lim_{x_0 \to 0} \mathbf{P}\{\sup_{t>t_0} |x(t, \omega, t_0, x_0)| > \varepsilon\} = 0$. *It is said to be asymptotically stable in probability if it is stable in probability and moreover* $\lim_{x_0 \to 0} \mathbf{P}\{\lim_{t \to \infty} x(t, \omega, t_0, x_0) = 0\} = 1$.

In plain English, according to the definition, the origin is asymptotically stochastically stable if one can choose the δ-neighborhood of the origin such that all trajectories starting in it will remain inside a given ϵ-neighborhood of the origin with probability going to 1 as δ goes to 0. This definition is analogous to the definition of stability in the deterministic case. Moreover, one wants all such trajectories to converge to the origin as δ goes to 0, which has a close counterpart in the asymptotic stability in the deterministic case.

Consider two systems of SDEs, one linear, another nonlinear:

$$dX_t = BXdt + \sigma X dW_t, \tag{17a}$$

$$dX_t = b(t, X)dt + \sigma(t, X)dW_t. \tag{17b}$$

Suppose that coefficients $b(t, x)$ and $\sigma(t, x)$ are "close" to B and σ. Can one deduce the stability or instability of the origin for the nonlinear system from the stability of the origin for (17a)? It turns out that if the system (17a) is obtained from (17b) by linearization around the origin, and the origin is asymptotically stochastically stable in (17a), then it is asymptotically stable in (17b) as well. This result is known as stability in the first (linear) approximation.

The trivial solution of the system (16) is asymptotically stable in probability in a sufficiently small neighborhood of the origin. Appendix C contains relevant theorems and calculations.

Note the difference between the last result and stability in the large discussed in Appendix B. Stability in the large was restricted to the set of events such that sample paths never left the trap. The last result does not use this restriction and is in this sense a broader one.

Such a result means that for the economy that started very close to the origin, probability of escape from the trap is low and goes to zero as the initial point approaches the origin. There is no way out if expectations are very pessimistic. The sunspot variable cannot fix expectations if they are too low to begin with. The result should not come as a surprise considering the specification of the process that governs expectations. An addition to the derivative of consumption due to the sunspot variable is proportional to the current level of consumption itself. In the model, low expectations mean low consumption. Therefore, in a pessimistic state the sunspot variable exercises very little influence in absolute terms. As stated previously, for the economy converging to the origin the distance to the boundary of the poverty trap becomes very large as a percentage of the current level of consumption. The influence of the sunspot variable gets smaller as the level of consumption gets small. The only realistic chance of escape comes when the distance to the boundary is not exponentially large and the sunspot influence is not negligible. Both requirements are satisfied when the consumption level is not too low, which means expectations are not too pessimistic.

Now I have to make a distinction between the stability of the origin in the deterministic system (6) and the stochastic system (16). The basin of attraction of the origin in the former system is a set in (x, y) space that for some values of y is unbounded in x. The solution of (16) is guaranteed to converge to the origin only as the initial condition converges to zero. For any non-zero initial condition, there is a positive probability that the trajectory will not converge to the origin. A solution of (16) that started outside the "sufficiently small neighborhood" of the origin is not guaranteed to converge to it or to remain near it at all. Therefore, following a sunspot variable leaves the possibility that the economy will escape the poverty trap.

4 How Good is the Chance?

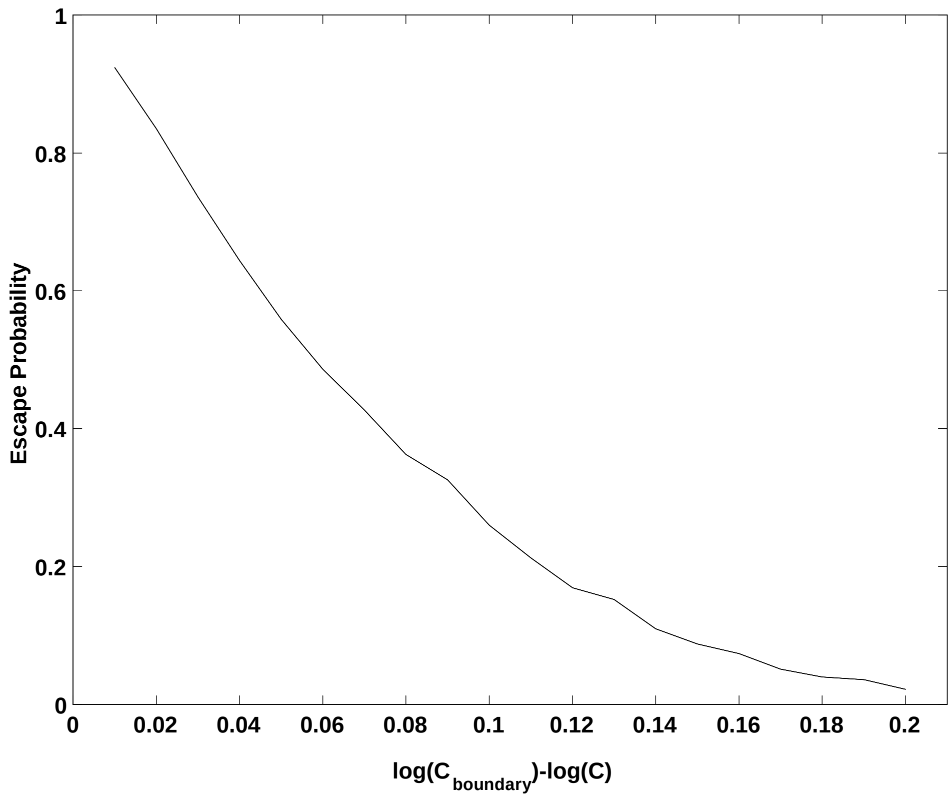
To understand how important sunspot-driven fluctuations could be for the economy's escape from the poverty trap, some numerical simulations of stochastic differential equation (15) were performed. First, to obtain a "realistic" noise magnitude, batches of 100 trajectories each with different noise intensities starting from the positive steady state **A** of the *deterministic* system were run for 300 time units (years). A noise intensity that resulted in approximately 14% standard deviation of the log consumption was chosen. This number is close to the average reported for several developing countries by Mendoza (1995). The second step was to calculate the separatrix of the two regions of attraction. This separatrix is the stable manifold of the steady state **C** of the transformed system (6). A standard procedure was employed: calculate the eigenvector corresponding to the stable eigenvalue at **C** and run the system of differential equations (6) backwards in time from a point close to **C** in the direction of the eigenvector. Matlab5 procedure `ode45` was used to calculate the trajectory. Using the transformation inverse to (5), this trajectory was transformed into (c, k) space in which further simulations were made. The separatrix is the thick solid line in Figures 1-3.

Numerical simulations of SDE are based on a stochastic Taylor expansion. For a brief exposition of the numerical solution of SDEs based on Kloeden, Platen, and Schurz (1994) the reader is referred to Appendix D. I run batches of 100 trajectories with initial points inside the deterministic poverty trap. The share of trajectories crossing the trap boundary is interpreted as a probability that sunspot-driven fluctuations of a given magnitude will lead to the escape from the trap. For the purposes of the simulations, the time interval from 0 to 300 was chosen. All trajectories either crossed the boundary or moved very close to the origin in the (C, K) space during this time interval[14].

The basic result of this section can be stated as follows: for the chosen level of noise intensity, the probability of escaping the trap is not negligible only when the initial condition is very close to the trap boundary. The initial level of consumption, C, should not be less than 80% of the boundary level in order to see at least a couple of escapes in a batch of 100 trajectories. The probability is not very sensitive to the initial level of capital. Figure 4 plots the probability of escape averaged over initial capital level versus the difference between the initial and borderline levels of consumption. As expected, it increases as expectations become more optimistic (the difference becomes smaller). Figure 5 presents similarly averaged mean and median escape times for trajectories that eventually leave the trap. For very optimistic expectations (initial consumption very close to the boundary) an absolute majority of escapes happen within the first year. For the few trajectories that escape from pessimistic initial conditions (consumption far from the boundary) the time is much longer, 30 years or more.

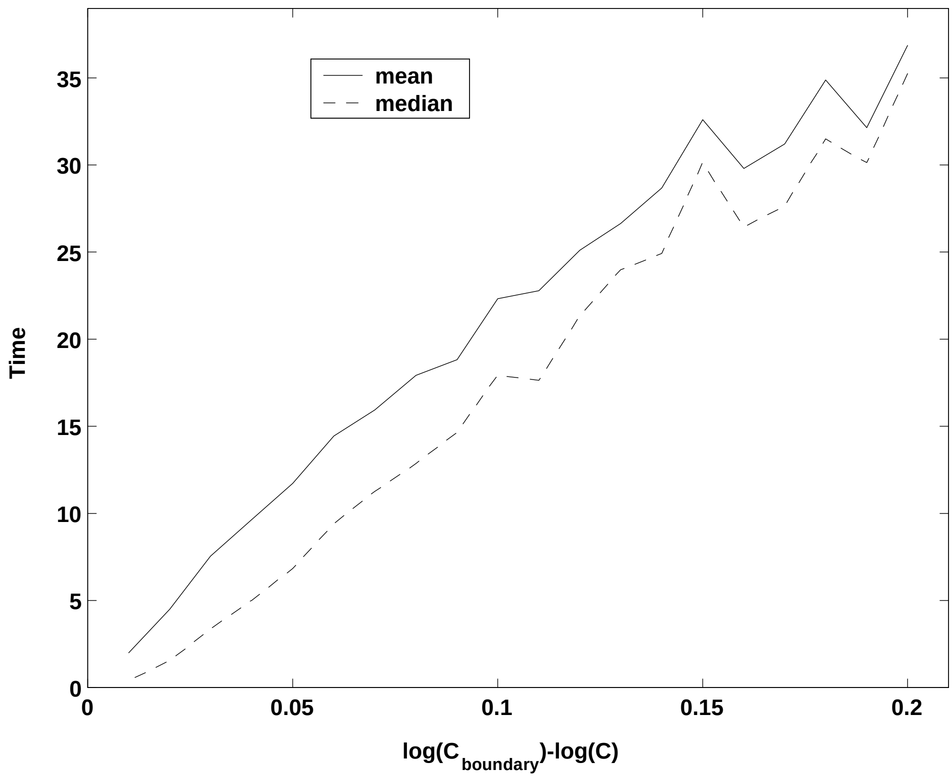
[14] Conditional on not crossing the boundary, the origin is stochastically stable in the large. A trajectory was considered as having converged to the origin in the (x,y) space and correspondingly in the (C, K) space if $\log(K)$ fell below 0. For the positive steady state **A**, $\log(K)$=7.32. Initial points for simulation purposes varied from $\log(K)$=4 to $\log(K)$=8. In practice, after approximately 100 years the non-escaped solutions would become numerically indistinguishable from the origin in the (C,K) space.

Figure 4: Probability of escape as a function of the distance to the poverty trap boundary



The answer to the question posted at the beginning of the section then is — "Not very good". It is possible to miss the target level of consumption (and work effort) and still avoid falling into the poverty trap, but the gap should not be large. Expected escape time and escape probability are inversely related, and the prognosis for chronically trapped economies is not good. Highly probable escape happens very fast; if there is no escape, the imploding economies will probably continue the downward spiral.

Figure 5: Mean and median escape time as a function of the distance to the poverty trap boundary



5 Analytical Estimates of the Escape Probability

In this section I will exploit the specific geometric structure of the model (15) to derive some very approximate estimates of the escape probabilities. For values of the initial capital that are not larger than in the positive steady state, consumption values that put the system into the development trap generate a very low value of x, usually less than 0.0005, which is significantly lower than the value of y, which is approximately 0.025. For a trajectory in the development trap, x can only decrease.

Consider again the system of SDE (15). Formally subtract the second equation from the first to obtain the following relationship:

$$d(c-k) = [(\frac{a}{\sigma} - 1)\exp(w - vk + uc) + \exp(c-k) + \delta - \frac{\delta+\rho}{\sigma}]dt + \widetilde{\sigma} dW_t. \quad (18)$$

Notice that the same result could be derived by changing to the variables $(c - k, k)$ and keeping only the first equation. Using the fact that $x = \exp(w - vk + uc)$ is much less than $min(y = \exp(c-k)\ , \delta - \frac{\delta+\rho}{\sigma})$, I can derive the following approximate SDE:

$$dZ_t \approx [\exp(Z_t) + \delta - \frac{\delta+\rho}{\sigma}]dt + \widetilde{\sigma} dW_t. \quad (19)$$

This is a one-dimensional equation in $Z = c - k$. Initial condition Z_0 is given by $(c-k)_g - d$, where d denotes initial distance to the boundary. The boundary is almost horizontal for small $x = \exp(w - vk + uc)$ and is given by $(c-k)_g = \log(\frac{\delta+\rho}{\sigma} - \delta)$. Another approximation would be to substitute $\exp(Z_0)$ for $\exp(Z_t)$ in the right-hand side of (19). After the substitution I arrive at the following approximate SDE:

$$d\frac{Z_t}{\widetilde{\sigma}} \approx \frac{[1-e^{-d}](\delta - \frac{\delta+\rho}{\sigma})}{\widetilde{\sigma}} dt + dW_t. \quad (20)$$

Consider a typical trajectory that starts in the development trap in Figure 1. Initial x is very small, and y is almost unchanged on the stable manifold for $x \in [0, x_0]$, $y \approx y^* = \frac{\delta+\rho}{\sigma} - \delta$. A trajectory escapes the trap if its y component increases by the amount given by the difference between y^* and y_0. Also note that $Z = c - k$ is simply a monotonic transformation of y, $Z = \ln(y)$. Therefore, the trajectory leaves the trap if Z_t achieves some fixed value. The process Z_t is a well-known Brownian motion with drift $b = \frac{[1-e^{-d}](\delta - \frac{\delta+\rho}{\sigma})}{\widetilde{\sigma}}$. I want to know the probability with which $\frac{Z_t}{\widetilde{\sigma}}$ increases by $\frac{d}{\widetilde{\sigma}}$ over its initial value, just hitting the boundary of the trap. Denoting the first passage time to $\frac{d}{\widetilde{\sigma}}$ by $T_d = \inf\{t \geq 0 : Z_t - Z_0 = \frac{d}{\widetilde{\sigma}}\}$, one can write the probability of hitting $\frac{d}{\widetilde{\sigma}}$ in a finite time, $P[T_d < \infty] = \exp(\frac{bd}{\widetilde{\sigma}^2} - \left|\frac{bd}{\widetilde{\sigma}^2}\right|)$; see Karatzas and Shreve (1991, p.197). Finally, substituting the value of b, I derive the probability of escape as

$$P[T_d < \infty] = \exp\left[-2(\frac{\delta+\rho}{\sigma} - \delta)\frac{d(1-e^{-d})}{\widetilde{\sigma}^2}\right]. \quad (21)$$

The expression just derived demonstrates that the escape probability should be decreasing in the distance to the boundary and increasing in the magnitude of the sunspot noise, as expected. Quadratic dependence on the noise guarantees that even the most pessimistic initial conditions do not preclude an escape given high enough noise magnitude. In short, it pays to behave non-optimally.

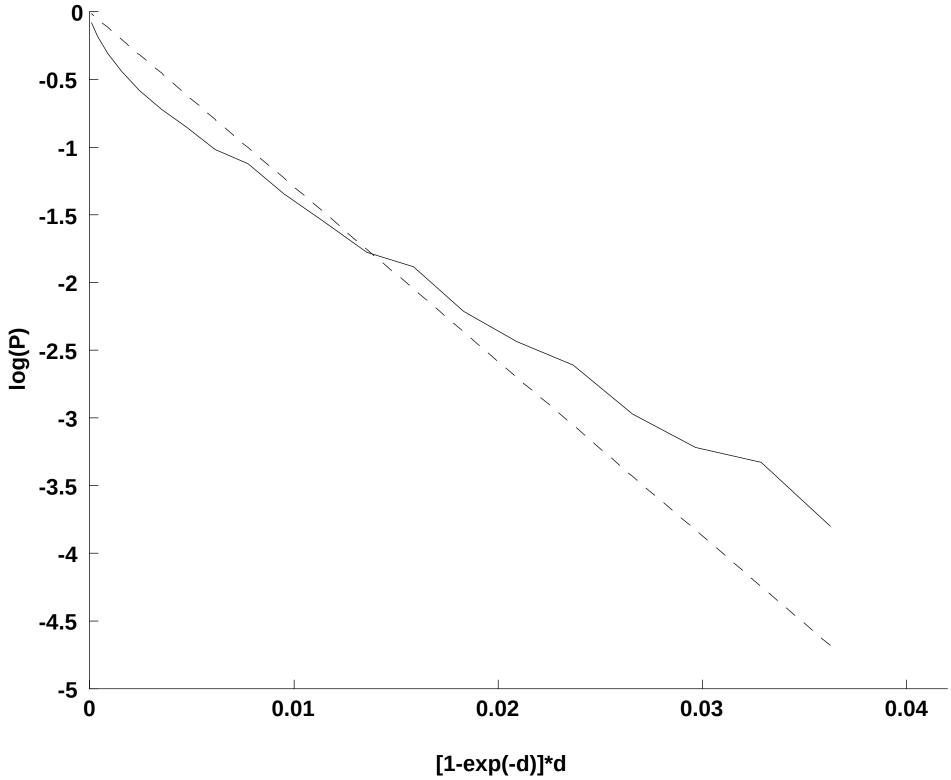
It is very likely that (21) is not a very accurate description of the escape probability. One source of the discrepancy is that for large d, when the sample path starts climbing towards the boundary, it enters the region where drift b is less in absolute value than at the initial point. True probability of escape will be higher than given by the formula. For small d, on the other hand, it is possible to have a sample path initially move away from the boundary and then come back. Such a path spends some time in the region with drift b higher than the one given by initial conditions, and the true escape probability will probably be smaller than predicted by (21).

Figure 6 plots $log(P)$ vs $d(1 - e^{-d})$. The agreement is surprisingly good given the number of assumptions I had to make to arrive at the expression for the escape probability. The difference between numerical results and (21) has expected sign — positive for large d and negative for small ones.

6 Conclusion

Poverty traps and indeterminacy in macroeconomic models may be caused by the same set of reasons, like externalities or increasing returns to scale. Wooford's conjecture, proven to hold in a broad set of discrete-time and continuous-time models, allows one to expect the presence of sunspot fluctuations whenever indeterminacy of the steady state is present. However, the traditional approach to sunspot fluctuations is strictly local: the sunspot variable is assumed to behave in such a way that the economy subject to self-fulfilling beliefs shocks does not leave the region of the state space where the dynamics without sunspots takes place. This is usually achieved by choosing a random variable with bounded support as the sunspot variable. Considering a two-dimensional

Figure 6: Probability of escape as a function of the distance to the trap boundary



continuous-time model allows one to describe fully both deterministic and stochastic dynamics of the system. Ability to discuss global properties of the stochastic process allowed me to raise a new question, that of the connection between the sunspot driven fluctuations and escape from the poverty trap.

Taking a simple model that exhibits indeterminacy of both the positive steady state and zero steady state I was able to prove that the development trap is asymptotically stochastically stable under the chosen specification of the sunspot variable, which can be interpreted as a change in perceived present discounted wealth. Therefore the economy that starts with a very low initial capital and very pessimistic expectations of future interest rates and wages gets trapped. However, this analytical result is valid only asymptotically and economies starting with finite levels of capital and consumption have nonzero probability of escape. To estimate numerically this probability as a function of initial conditions, I assumed that the economies of several developing countries operated around the positive steady state with business cycle fluctuations caused by the sunspots described in the model. Allowing the sunspots of similar magnitude to act in the economy with initial conditions in the poverty trap, I was able to map the trap for initial conditions providing non-negligible probability of escape. The set of those initial conditions is not very large and is restricted to an initial level of consumption within 20% of the level necessary to put the system right on the boundary between the poverty trap and the region of attraction of the positive steady state. At every finite level of the capital stock, there exists a level of consumption (and, accordingly, of the work effort) that withdraws the system from the poverty trap. However, for very low levels of capital the change from a "pessimistic" optimal level of consumption to the "optimistic" one may constitute hundreds and thousands of percent of the "pessimistic" level. Analytical approximations showed good agreement with the numerical results.

The pessimistic outcome is brought about by the sunspot with magnitude proportional to the current level of consumption. A different specification of the sunspot variable might lead to more optimistic results in this model. Even a sunspot variable that is proportional to the current consumption can be more effective in models where a poverty trap is not the origin. Then the magnitude of the sunspot does not converge to zero as time goes to infinity, and the escape can be inevitable given enough time.

There are several obvious extensions of the work presented here. Work is in progress on the models with interior poverty traps, like the ones described in Matsuyama (1991), Gali and Zilibotti (1995), Gans (1998), or Baland and Francois (1996). Another interesting question is whether the sunspot equilibrium described here can be learned as a result of some learning process. At present it is unclear how to extend the work done in Evans and Honkapohja (1999) to continuous time. However, after this is done, one could study the learnability of indeterminate Pareto inferior steady states and of sunspot-driven equilibrium dynamics that exists in the poverty trap. It might be possible to start with a discrete time approximation to the model and ask the question on learnability. Going to the limit as time period converges to zero will allow one to demonstrate which continuous-time equilibria can be thought to be stable under the learning and which cannot. Finally, a test of the escape mechanism presented here might be possible if one could demonstrate the existence of, and measure the importance of, sunspots in the developing economies. However, current methods of testing for sunspots require either high frequency economic data, as in Farmer and Guo (1995), or the existence of highly developed financial markets and the corresponding time series, as in Salyer and Sheffrin (1998). Both conditions are unlikely to be met in the economies for which the question of escape from the poverty trap is relevant.

A Derivation of the Stochastic Differential Equation

The Itô theorem, found in any standard textbook on stochastic calculus like Karatzas and Shreve (1991), states the following:

Theorem 1 *Let stochastic process X_t be written as a d-dimensional stochastic differential,*

$$dX_t = b_t dt + F_t dW_t.$$

Let $U : [0,T] \times R^d \to R$ have continuous partial derivatives $\frac{\partial U}{\partial t}, \frac{\partial U}{\partial x_k}, \frac{\partial^2 U}{\partial x_k \partial x_i}$ for $k, i = 1, 2...d$, and define a scalar process $\{Y_t, 0 \leq t \leq T\}$ by

$$Y_t = U(t, X_t)$$

with probability 1. Then the stochastic differential for Y_t is given by

$$\begin{aligned} dY_t &= \left\{ \frac{\partial U}{\partial t} + \sum_{k=1}^{d} b_t^k \frac{\partial U}{\partial x_k} + \frac{1}{2} \sum_{i,k=1}^{d} F_t^i F_t^k \frac{\partial^2 U}{\partial x_k \partial x_i} \right\} dt + \\ & \sum_{i=1}^{d} F_t^i \frac{\partial U}{\partial x_i} dW_t. \end{aligned}$$

As is easy to see, the only difference between a usual chain rule and the Itô formula is the presence of the term $\frac{1}{2} \sum_{i,k=1}^{d} F_t^i F_t^k \frac{\partial^2 U}{\partial x_k \partial x_i}$. Taking $x = \exp(w + uc - vk)$ and $y = \exp(c - k)$ as Y_t in the formula, I obtain the system of equations presented in (16).

B Global Stochastic Dynamics

First, it is necessary to demonstrate that the stochastic process is unique and exists for all $t \geq t_0$. Then I will show that a particular trajectory that started in the development trap and evolved subject to the sunspot fluctuations according to (16) can evolve in one of two possible ways. One is convergence to the positive steady state **A** and another, convergence to the origin **B,** exactly like in the deterministic version of the model. All the theorems and definitions cited here are taken from Khasminskii (1980), where all the proofs can be found, and are presented here only to generate a relatively self-contained account.

Write a general multidimensional SDE in the integral form,

$$X_t = X_{t_0} + \int_{t_0}^{t} b(s, X_s)ds + \int_{t_0}^{t} \sigma(s, X_s)dW_s. \tag{22}$$

Define Lipschitz continuity and linear growth conditions as follows:

$$|b(s,x) - b(s,y)| + |\sigma(s,x) - \sigma(s,y)| \leq B\,|x-y|\,, \tag{23a}$$

$$|b(s,x)| + |\sigma(s,x)| \leq B(1+|x|). \tag{23b}$$

Let C_2 be the class of functions on $I \times R^n \to R$ which are twice continuously differentiable with respect to $x_1 ... x_n$ and continuously differentiable with respect to t. Let $V \in C_2$. For the process given by (22) $LV(s,x)$ is defined as

$$LV(s,x) = \frac{\partial V(s,x)}{\partial s} + \sum_{i=1}^{N} b_i(s,x)\frac{\partial V(s,x)}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{N} \sigma_i(s,x)\sigma_j(s,x)\frac{\partial^2 V(s,x)}{\partial x_i \partial x_j}.$$

Consider the following restrictions on the function V:

$$LV \leq cV, \tag{24a}$$

$$V_R = \inf_{|x|>R} V(t,x) \to \infty \; as \; R \to \infty. \tag{24b}$$

The following theorem considers the existence, uniqueness, and certain properties of the solution to (22).

Theorem 2 *(Theorem 3.4.1) Suppose that conditions (23) are valid in every cylinder $I \times U_R$ and, moreover, that there exists a nonnegative function $V \in C_2$ on the domain R^n such that for some constant $c > 0$ conditions (24) hold.*

Then:

i) For every random variable X_{t_0} independent of the process $W_t - W_{t_0}$ there exists a solution X_t, unique up to equivalence[15], of equation (22) which is an almost surely continuous stochastic process and is unique up to equivalence.

ii) This solution is a Markov process.

[15] Two solutions X_t^1 and X_t^2 are said to be equivalent if $P\{X_t^1 = X_t^2 \; for \; all \; t \in [t_0, T] = 1$.

Conclusions iii), iv), and v) of the Theorem do not concern me here and are omitted.

The process satisfying all the conditions of the Theorem is a regular process[16]. A regular process is almost surely defined for all $t \geq t_0$.

In the following discussion, I will restrict attention to the set of trajectories that start in the development trap and never leave it. More formally, denote $\tau_g(s,x)$ time of the first exit through the upper boundary (stable manifold of **C**) for the trajectory that started at (s,x) in the trap. Consider only the set of events U such that $\tau_g(s,x) = \infty$. A trivial application of all the arguments in Khasminskii (1980) then demonstrates that all the results proved there for R^n hold for U.

For the system of equations (16) a simple energy function will suffice. Set $V(t,x) = x^2 + y^2$. Then

$$\begin{aligned} LV &= 2x^2 \left[(\frac{a}{\sigma}u - v)x + vy + v\delta - u\frac{\delta+\rho}{\sigma} + \widetilde{\sigma}^2 u^2 \right] + \\ &\quad +2y^2 \left[(\frac{a}{\sigma} - 1)x + y + \delta - \frac{\delta+\rho}{\sigma} + \widetilde{\sigma}^2) \right]. \end{aligned}$$

V is obviously nonnegative and LV is negative everywhere but at the origin and in the neighborhood of the steady state **C**. Take $2\widetilde{\sigma}^2$ as the constant c in (24). I have just shown that the process defined by (16) exists for all $t \geq t_0$ and is unique if it does not leave the trap. If it does leave the trap, by assumption the sample path becomes a solution to the deterministic equation (6) and converges to the positive steady state **A**.

Let us now consider the set of events such that the trajectory never leaves the region in the (x,y) space where $LV < 0$[17]. This region is the whole development trap minus a small neighborhood of the steady state **C**. Denote this set U_1.

[16]Informally, a regular process is a process for which the first exit time from a union of bounded domains $\bigcup_{n=1}^{\infty} U_n = \{|x| < n\}$ is infinite with probability one.

[17]In the next Appendix I will show that the origin is asymptotically stochastically stable, which means that trajectories that start close enough converge to the origin. Such trajectories never leave the region where $LV < 0$ and this subset of U is thus not empty.

Definition 2 *The solution* $X_t = 0$ *of equation (22) is said to be (asymptotically) stable in the large if it is stable in probability and also for all* s, x $P[\lim_{t\to\infty} X_t^{s,x} = 0] = 1$.

I want to show that for all events in U_1 the origin is stable in the large. The following theorem is used to prove the result.

Theorem 3 *(Theorem 5.4.5) The following conditions are sufficient for the solution* $X_t = 0$ *of equation (22) to be stable in the large:*

i) the process X_t *is regular;*

ii) there exists a nonnegative function $V_1(t, x) \in C_2^0$ *such that the function* LV_1 *is negative definite;*

iii) there exists a positive definite function $V_2(t, x) \in C_2^0$*, having an infinitesimal upper limit, such that* $LV_2 \le 0$*.*

Class C_2^0 differs from class C_2 by dropping differentiability at $x = 0$. Infinitesimal upper limit means $\lim_{x\to 0} \sup_{t>0} V(t, x) = 0$. If V does not depend on t explicitly, only $\lim_{x\to 0} V(x) = 0$ is needed. Using the same Lyapunov function as above, $V(t, x) = x^2 + y^2$, I see that in U_1 all the conditions of the theorem are satisfied. Therefore, any trajectory that started in the development trap, never left it, and never wandered close to the steady state **C**, will converge to the origin. The other two possibilities for this trajectory are leaving the trap through the upper boundary or moving into the neighborhood of the steady state **C** where x is very small.

Arguments presented in Section 5 show that a process near **C** can be approximated by a Brownian motion with drift along the y direction. Therefore, the process can hit the trap boundary with a probability calculated in that Section. Alternatively, it can leave the lower boundary of the neighborhood of **C** and enter the region where U_1 is negative. The probability of doing so in a finite time, conditioned on the fact that the process does not leave through the upper boundary first, is one. The process cannot hit the y axis in a finite time. A theorem from Friedman and Pinsky (1973) states that a process that solves (22) cannot hit a closed domain Ω if normal components of both drift and diffusion vanish on $\partial\Omega$. It is immediately obvious that on the y axis there is no drift or diffusion component in the x direction for (16). Therefore, the y axis cannot be attained by our stochastic process[18].

C Asymptotic Stability in the First Approximation

Theorem 4 *(Khasminskii (1980), Theorem 7.1.1) If the linear system with constant coefficients (17a) is asymptotically stable in probability, and the coefficients of the system (17b) satisfy an inequality*

$$|b(t,x) - Bx| + |\sigma(t,X) - \sigma x| < \gamma|x|, \tag{25}$$

in a sufficiently small neighborhood of the point $x = 0$ and with sufficiently small constant γ, then the solution $X = 0$ of the nonlinear system is asymptotically stable in probability.

Remark 1 *In the proof of Theorem 7.1.1, Khasminskii actually shows that if the origin in (17a) is exponentially p-stable for sufficiently small p and (25) holds, then the Theorem is true. For linear systems with constant coefficients, asymptotic stability in probability implies exponential p-stability for sufficiently small p (Theorem 6.4.1 in Khasminskii 1980).*

Definition 3 *Exponential p-stability, Khasminskii (1980). The solution $X \equiv 0$ of the system (22) is said to be exponentially p-stable for $t \geq 0$, if for some positive constants A and α the following condition is true:*

$$\mathbf{E}|\mathbf{x}(\mathbf{t}, \boldsymbol{\omega}, \mathbf{x}_0, t_0)|^p \leq A|x_0|^p \exp\{-\alpha(t - t_0)\}.$$

Theorem 5 *(Khasminskii (1980), Theorem 6.3.1) The solution $X \equiv 0$ of the linear system with constant coefficients is exponentially p-stable if and only if*

[18] Actually, both x and y axes are unattainable in a finite time.

there exists a function V(t,x), homogeneous of degree p in x, such that for some constants $k_i > 0$

$$\begin{aligned} k_1|x|^p &\leq V(t,x) \leq k_2|x|^p;\ LV(t,x) \leq -k_3|x|^p, \\ |\frac{\partial V}{\partial x_i}| &\leq k_4|x|^{p-1};\ |\frac{\partial^2 V}{\partial x_i \partial x_j}| \leq k_4|x|^{p-2}. \end{aligned} \tag{26}$$

Applying Theorem 4, I can see that the stability of the origin in (16) depends on the stability of the origin in the following linear system:

$$\begin{aligned} dx &= x(v\delta - u\frac{\delta+\rho}{\sigma} + \frac{1}{2}\widetilde{\sigma}^2 u^2)dt + ux\widetilde{\sigma}dW_t \\ dy &= y(\delta - \frac{\delta+\rho}{\sigma} + \frac{1}{2}\widetilde{\sigma}^2)dt + y\widetilde{\sigma}dW_t. \end{aligned} \tag{27}$$

To establish stability of (27), set $V(t,x) = |x|^p + |y|^p$. Then

$$\begin{aligned} LV &= p|x|^p\left[v\delta - u\frac{\delta+\rho}{\sigma} + \frac{1}{2}\widetilde{\sigma}^2 u^2 + \frac{1}{2}\widetilde{\sigma}^2 u^2(p-1)\right] + \\ &\quad + p|y|^p\left[\delta - \frac{\delta+\rho}{\sigma} + \frac{1}{2}\widetilde{\sigma}^2 + \frac{1}{2}\widetilde{\sigma}^2(p-1)\right] \end{aligned}$$

or

$$LV = p|x|^p\left[v\delta - u\frac{\delta+\rho}{\sigma} + \frac{1}{2}\widetilde{\sigma}^2 u^2 p\right] + p|y|^p\left[\delta - \frac{\delta+\rho}{\sigma} + \frac{1}{2}\widetilde{\sigma}^2 p\right]. \tag{28}$$

A quick look at (12) assures one that $LV(t,x) \leq -k_3|x|^p$ for p small enough. Therefore, by Theorem 5 the solution $X \equiv 0$ of the system (27) is exponentially $p - stable$, and by Remark 1, the trivial solution of the system (16) is asymptotically stable in probability in a sufficiently small neighborhood of the origin.

D Numerical Approximation Algorithm

Suppose one is given a one-dimensional SDE (22),

$$X_t = X_{t_0} + \int_{t_0}^{t} b(X_s)ds + \int_{t_0}^{t} \sigma(X_s)dW_s.$$

For any twice continuously differentiable function $f : R \to R$, Itô's formula gives

$$\begin{aligned} f(X_t) &= f(X_{t_0}) + \int_{t_0}^{t} (b(X_s)f'(X_s) + \frac{1}{2}\sigma^2(X_s)f''(X_s))ds + \\ &\quad + \int_{t_0}^{t} \sigma(X_s)f'(X_s)dW_s \qquad (29) \\ &= f(X_{t_0}) + \int_{t_0}^{t} L^0 f(X_s)ds + \int_{t_0}^{t} L^1 f(X_s)dW_s, \end{aligned}$$

where the two operators introduced are

$$\begin{aligned} L^0 f &= bf' + \frac{1}{2}\sigma^2 f'', \\ L^1 f &= \sigma f'. \end{aligned}$$

Now, if one applies Itô's formula to the functions $f = b$ and $f = \sigma$ under integral signs in (29), one gets the following:

$$\begin{aligned} X_t &= X_{t_0} + b(X_{t_0})\int_{t_0}^{t} ds + \sigma(X_{t_0})\int_{t_0}^{t} dW_s + \int_{t_0}^{t}\int_{t_0}^{s} L^0 b(X_z)dzds + \\ &\quad + \int_{t_0}^{t}\int_{t_0}^{s} L^1 b(X_z)dW_z ds + \int_{t_0}^{t}\int_{t_0}^{s} L^0\sigma(X_z)dzdW_s + \int_{t_0}^{t}\int_{t_0}^{s} L^1\sigma(X_z)dW_z dW_s. \end{aligned}$$

The procedure can be repeated, for example by applying Itô's formula to $f = L^1\sigma$ in the above expression, and so on. At every step, the expansion will consist of multiple Itô integrals

$$\int_{t_0}^{t} ds, \quad \int_{t_0}^{t} dW_s, \quad \int_{t_0}^{t}\int_{t_0}^{s} dW_z dW_s,$$

multiplied by some constants, and the remainder term involving higher-order multiple Itô integrals. Multiple integrals can be approximated numerically.

A usual problem in the numerical simulation of SDEs is to generate approximate values of the process X_t at given discretization times inside the interval $[0, T]$. For the uniform discretization $\tau_n = n\Delta$, $n = 1...N$ with the step size $\Delta = \frac{T}{N}$ the simplest approximation will look like

$$Y_{n+1} = Y_n + b(Y_n)\Delta_n + \sigma(Y_n)\Delta W_n,\ Y_0 = X_0. \qquad (30)$$

The random variables ΔW_n are the Wiener process increments; they are independently Gaussian distributed with zero mean and variance Δ.

If a particular approximation satisfies the condition

$$E(\left|X_T - Y_N^{\Delta}\right|) \leq K\Delta^{\gamma},$$

for all sufficiently small time steps Δ and some finite constant K, it is said that the approximation Y^{Δ} converges with strong order γ. For example, the stochastic Euler scheme (30) converges with strong order 0.5, while its deterministic counterpart has the order 1.0.

For purposes of the current paper, an explicit strong order 1.5 scheme was used. For a multi-dimensional process X with only one independent Wiener disturbance[19], the formula becomes

$$\begin{aligned} Y_{n+1}^k &= Y_n^k + b^k\Delta_n + \frac{1}{2}L^0 b^k \Delta_n^2 + \\ &+\sigma^k \Delta W_n + L^0\sigma^k(\Delta W_n \Delta_n - \Delta Z_n) + L^1 b^k \Delta Z_n + \\ &+L^1\sigma^k \frac{1}{2}\left((\Delta W_n)^2 - \Delta_n\right) + L^1 L^1 \sigma^k \frac{1}{2}\left(\frac{1}{3}(\Delta W_n)^2 - \Delta_n\right)\Delta_n. \end{aligned} \tag{31}$$

Here Y^k, $k = 1...K$ is the k^{th} component of the multidimensional vector Y and ΔZ_n is a random variable defined by $\Delta Z_n = \int_{\tau_n}^{\tau_{n+1}} \int_{\tau_n}^{s_2} dW_{s_1} ds_2$. This random variable is normally distributed with mean zero, variance $E\left((\Delta Z_n)^2\right) = \frac{1}{3}\Delta_n^3$, and covariance $E(\Delta Z_n \Delta W_n) = \frac{1}{2}\Delta_n^2$. Two random variables ΔW_n and ΔZ_n can be generated from two independent standard normal variables G_1 and G_2 as $\Delta W_n = G_1\sqrt{\Delta_n}$, $\Delta Z_n = \frac{1}{2}\Delta_n^{3/2}(G_1 + \frac{1}{\sqrt{3}}G_2)$.

[19] That is, for a system like system (15). I have chosen to simulate (15) instead of equivalent system (16) because the former has noise intensity independent of state variables. For the approximation scheme chosen, this represented a major simplification in programming.

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