# SELECTION OF MARKOV EQUILIBRIUM IN A DYNAMIC OLIGOPOLY WITH PRODUCTION TO ORDER

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We use the requirement of continuity of strategies and the weakest possible criterion of renegotiation-proofness, called renegotiation-quasiproofness, to select a (limit of continuation equilibrium paths of a) Markov perfect equilibrium in an infinite horizon dynamic oligopoly with costly changes of output between the periods, producing to order a homogeneous good. In each renegotiation-quasiproof continuous strategy Markov perfect equilibrium every continuation equilibrium path converges in the price space to the symmetric price vector with each component equal to the monopoly price of the firm(s) with the lowest marginal costs of production.

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## 1. INTRODUCTION

Game theoretic results dealing with folk theorems were applied to the study of infinite horizon, discrete time, deterministic oligopoly models since publication of the paper by James Friedman (1971). For the models without physical links between periods, i.e., for oligopolistic supergames, in the case of discounting of future profits, the folk theorem of Fudenberg and Maskin (1986) implies that any feasible strictly individually rational payoff vector of the stage game is sustainable as the average discounted payoff vector in a subgame perfect equilibrium. (The full dimensionality condition is satisfied in oligopolistic supergames, provided that monopoly profit of each firm is positive.) Folk theorems for supergames rely on dependence of equilibrium strategies on whole histories of previous actions, or on actions in a finite number of preceding periods. This feature can be justified on the grounds that it leads to equilibrium outcomes Pareto dominating those that can be obtained when past actions are ignored in making current choices. On the other hand, it contradicts the fact that the past is sunk (i.e., it is not payoff relevant).

Many infinite horizon, discrete time oligopoly models involve physical links between periods, i.e., they are oligopolistic difference games. These links can stem, for example, from investment, advertising, or costs of changing outputs or prices.[2] In difference games, a current state, which is payoff relevant, should be taken into account by rational players when deciding on a current period action. If strategies depend only on a current state (which is payoff relevant), they are called Markov. Application of the requirement of subgame perfection to Markov strategies forming a Nash equilibrium leads to the solution concept

[2] These costs can be used to justify sequential moves. See Maskin and Tirole (1987, 1988) for examples.

of a Markov perfect equilibrium. In many oligopolistic infinite horizon difference games Markov perfect equilibrium is non-collusive.[3] Maskin and Tirole's (1988) model of a dynamic Bertrand duopoly is a notable exception. In the latter, the existence of payoff relevant states stems from sequential naming of a price by duopolists, one of them naming a price in odd periods, the other in even periods. Price set in a period t remains in effect also in a period t + 1. Unfortunately, it is hard to imagine how to generalize that model to more than two firms.

In the present paper we deal with the selection from Markov perfect equilibria in an infinite horizon dynamic oligopoly with costly changes of output between periods and discounting of profits, composed of firms with linear cost functions, producing to order a single homogeneous non-durable good. Costs of changes in output can stem, for example, from additional charges when ordered quantities of inputs are changed shortly before delivery or, in the case of a reduction of an output, from legal restrictions (mainly with respect to labour) on immediate reduction of used amount of an input. Thus, these costs are, by their economic substance, different from investments into enlarging or maintaining a capacity.

In a period t firms, taking into account their output in a period t - 1, name prices, on the basis of which customers place their orders (with rationing of orders if more than one firm charges the lowest price). Paying attention to costs of output changes, firms decide what portions of orders they confirm and unconfirmed portions of orders are available to other firms. Outputs equal to (thus modified) confirmed orders are produced, with costs of a change of output in comparison with the previous period incurred by each firm, and sold. A state in a period t is identified with a vector of outputs in a period t - 1. (This is payoff relevant, because changes of output are costly.)

[3] See Maskin and Tirole (1987) for a typical example.

Restriction of attention to Markov perfect equilibria imposes three limitations on equilibrium strategies. First, counting of repetitions of a certain price vector is impossible. Therefore, after a profitable unilateral deviation (i.e., a unilateral deviation that would increase a deviator’s continuation average discounted net profit if the other firms ignored it) the play has to pass (in general) through several different price vectors lying on a punishment path. Second, a price vector prescribed (by an equilibrium strategy profile) for the first period must be the same as a price vector prescribed when the initial state reappears after a deviation. Thus, at the beginning of the game a collusive price vector can be only gradually approached (unless the initial state cannot result from a profitable unilateral deviation from any continuation equilibrium and does not lie on a punishment path). Third, the punishment path (containing, for each unilateral deviation, a continuation equilibrium path triggered by it) must be the same for all firms. Otherwise, after some unilateral deviations it would not be possible to determine, only on the basis of a previous period output vector, from which of the punishment paths a deviation took place.

The analyzed dynamic oligopoly has many (a continuum of) Markov perfect equilibria. We approach the problem of equilibrium selection on the basis of two additional restrictions on Markov strategies. First, we require them to be continuous (functions of a current state). Second, we impose the weakest possible requirement of renegotiation-proofness, which we call renegotiation-quasiproofness.

The requirement of continuity of strategies was introduced into the analysis of infinite horizon games with discounting of payoffs by J. W. Friedman and L. Samuelson (1994a, 1994b). It is based on the view that punishments should "fit the crime." Two main arguments in favour of continuous strategies are (Friedman and Samuelson 1994b): they are more appealing to real human players than a Draconian punishment after a very small deviation;

following a deviation, the convergence of continuous strategies to the original action profile (or, in a Markov setting, to the limit of the original sequence of action profiles) reflects an intuitively appealing rebuilding of trust.

Continuous Markov strategies have the plausible property that large changes in payoff relevant variables have large effects on current actions, minor changes in payoff relevant variables have minor effects on current actions, and changes in variables that are not payoff relevant have no effects on current actions. This is an improvement in comparison with Markov strategies without the requirement of continuity for which there is only the distinction between effect and no effect according to whether a variable that has changed is payoff relevant or not, so that minor changes in payoff relevant variables can have large effects on current actions. (See the discussion of minor causes and minor effects in Maskin and Tirole 1994.) It is an improvement also in comparison with continuous strategies without the Markov property, which allow changes in variables that are not payoff relevant to effect current actions.

The requirement that strategies be Markov is an application of Harsanyi's and Selten's (1992) principle of invariance of (selected) equilibrium strategies with respect to isomorphism of games. (This fact is pointed out by Maskin and Tirole 1994.) The latter principle requires that strategically equivalent games have identical solution (i.e., selected equilibrium or subset of equilibria). Applying it to a subgame perfect equilibrium of the analyzed difference game, it implies that selected equilibrium strategy profile(s) should prescribe the same play in all subgames that are strategically equivalent, i.e., in all subgames with the same initial state. The requirement of continuity of strategies is a strengthening of this principle. When two subgames are, from the strategic point of view, "close", i.e., an initial state of one of them is in a neighbourhood of an initial state of the other, a sequence of vectors of actions prescribed for the former should be in a neighbourhood (on the element-wise basis) of a sequence of vectors of actions prescribed for the latter.[4]

The requirement of continuity of Markov strategies imposes further two limitations on an equilibrium strategy profile. First, the play cannot approach (or even reach) a cycle between different price vectors. The play would have to approach them by switching between several paths (in the price and output spaces), one for each element of the cycle. After some profitable unilateral deviations from these paths it would not be possible to determine, just on the basis of an output vector, from which of them a deviation took place. For a similar reason an equilibrium strategy profile cannot prescribe a non-convergent sequence of price vectors. Thus, each continuation equilibrium path must converge and (since the punishment path is the same for all firms) all continuation equilibrium paths must converge to the same limit. Second, the convergence of the play to this limit can only have the form of approaching it, without reaching it in any finite time.

The latter limitation of equilibrium strategies implies that the play cannot converge to a symmetric price vector with each component exceeding the monopoly price (maximizing the difference between revenues and production costs of a monopolist, without taking into account costs of changing an output) of the firm(s) with the lowest marginal costs of production. Nor it can converge to an asymmetric price vector. (Proposition 2 in Section 3 gives the argument.) Therefore, it must converge to a symmetric price vector with each component equal to or below the monopoly price of the firm(s) with the lowest marginal costs of production. For all discount factors close to one, each equilibrium based on convergence

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[4] Let q and q’ be initial states in two different subgames and let $\{p(t)\}_{t=1}^{\infty}$ and $\{p'(t)\}_{t=1}^{\infty}$ be sequences of price vectors prescribed in them by a subgame perfect equilibrium profile of continuous Markov strategies. Then, when q converges to q’, p(t) converges to p’(t) for each positive integer t.

to the price vector with each component equal to the monopoly price of the firm(s) with the lowest marginal costs of production is strictly Pareto superior to all continuous strategy Markov perfect equilibria based on convergence to some other symmetric price vector. Thus, intuitively, the former equilibria should be selected. This intuition is supported by imposing the weakest possible requirement of renegotiation-proofness, which we call renegotiation-quasiproofness. A continuous strategy Markov perfect equilibrium is renegotiation-quasiproof if there does not exist another continuous strategy Markov perfect equilibrium such that a switching from the former to the latter increases continuation average discounted net profit of each firm in every subgame. The equilibria based on convergence to the price vector with each component equal to the monopoly price of the firm(s) with the lowest marginal costs of production are the only continuous strategy Markov perfect equilibria that are renegotiation-quasiproof for discount factors arbitrarily close to one.

The paper is organized as follows. In the next Section we describe the analyzed dynamic oligopolistic game. Section 3 contains the statement and proof of a sufficient as well as of a necessary condition of an existence of a continuous strategy Markov perfect equilibrium. Section 4 is devoted to the issue of equilibrium selection. Section 5 concludes.

## 2. THE ANALYZED DYNAMIC OLIGOPOLY

The analyzed industry consists of a finite number of firms, indexed by a subscript $i \in I = \{1, 2, ..., n\}$, $n \geq 2$, producing to order a single homogeneous non-durable good. We analyze this industry in an infinite horizon model with discrete time.

Demand function $D:[0, \infty) \rightarrow [0, D(0)]$, with $D(0)$ finite, is continuous on its domain.

There is $\rho > 0$ such that $D(\lambda) > 0$ for all $\lambda \in [0, \rho)$ and $D(\lambda) = 0$ for all $\lambda \geq \rho$. The function D is strictly decreasing, concave and twice differentiable on the interval $[0, \rho]$ (with right-hand side derivative at 0 and left-hand side derivative at $\rho$).

For each firm $i \in I$, costs of producing output $q_i$ are $c_iq_i$, where $c_i \in (0, \rho)$. We assume, without loss of generality, that firms are indexed in such a way that $c_1 \leq c_2 \leq ... \leq c_n$. This implies $p_1^{mon} \leq p_2^{mon} \leq ... \leq p_n^{mon}$, where $p_i^{mon}$ is firm i's static monopoly price, maximizing difference between revenues and production costs of firm i as a monopolist (without taking into account costs of changing an output).

Besides production costs, there are costs of changing an output in comparison with the previous period, which are a function of the absolute value of a difference between the current and the previous period output. For each firm $i \in I$, these costs are expressed by a function $\alpha_i:[0, D(0)]^2 \rightarrow [0, \infty)$. We have $\alpha_i[q_i(t - 1), q_i(t)] = \gamma_i[|q_i(t - 1) - q_i(t)|]$, where the function $\gamma_i:[0, D(0)] \rightarrow [0, \infty)$ is continuous, twice differentiable, strictly increasing (with possible exception at 0, where it can have derivative equal to zero), and strictly convex[5] on its domain, with $\gamma_i(0) = 0$. Costs of changes in an output can stem, for example, from additional charges when ordered quantities of inputs are changed shortly before delivery or, in the case of a reduction of an output, from legal restrictions (mainly with respect to labour) on immediate reduction of used amount of an input. (Our qualitative results would not change if we assumed that only increases in an output are costly, or asymmetry between costs of increasing and reducing an output.) Thus, these costs are, by their economic substance,

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[5] A reader may suggest that it would be easier and the model would become more tractable if we assumed that $\gamma_i$ is linear for each $i \in I$, i.e., $\gamma_i(x) = k_ix$, where $k_i > 0$, for every $i \in I$. However, in this case payoff relevance of a previous period output vector would be blurred. It could effect firm i's preference over two sequences of price vectors only if i-th component of first element of one of them was from the interval $(c_i - k_i, c_i + k_i)$. Moreover, the assumption that $\gamma_i$ is strictly convex has a (realistic) implication that firm's competitive supply $y_i(p_i, z_i)$ (see below) is finite for all feasible $p_i$ and $z_i$.

different from investments into enlarging or maintaining a capacity.

Firm $i \in I$ discounts revenue and all costs with a discount factor $\beta_i \in (0, 1)$ and we set $\beta = (\beta_1, ..., \beta_n)$.

In a period $t \in \{1, 2, . . . \}$, each firm, taking into account its output in the previous period, names a price $p_i(t) \in P_i = [0, \rho]$. We let $P = X_{i\in I}P_i$. (Since reduction of an output to zero is costly, firm's supply can be positive even at prices below marginal costs of production.) For each $i \in I$, initial output level $q_i(0) \in [0, D(0)]$ is given. The interpretation of this can be that the analyzed oligopoly existed before the period one, but we started to observe and analyze it only in the period one.

A vector of prices in a period t, $p(t) = (p_1(t),..., p_n(t))$, together with a vector of outputs in a period t - 1, $q(t-1) = (q_1(t-1),..., q_n(t-1))$, uniquely determines a vector of outputs in a period t, $q(t)$. Let $y_i(p_i, z_i)$ be firm i's competitive supply at price $p_i$, provided that its output in the previous period was $z_i \geq 0$. That is, $y_i(p_i, z_i)$ is the output firm i would supply if it was not constrained by supply by other firms and by market demand. We have $y_i(p_i, z_i) = \text{argmax}\{(p_i - c_i)\lambda - \gamma_i(|\lambda - z_i|)|\lambda \in [0, \infty)\}$. Clearly, if $p_i - c_i > \gamma_i'(0)$ (where the apostrophe denotes first derivative) then $y_i(p_i, z_i) > z_i$ and it satisfies the first order condition $p_i - c_i = \gamma_i'[y_i(p_i, z_i) - z_i]$. If $-\gamma_i'(0) \leq p_i - c_i \leq \gamma_i'(0)$ then $y_i(p_i, z_i) = z_i$. If $p_i - c_i < -\gamma_i'(0)$ then $y_i(p_i, z_i) < z_i$ and it satisfies the first order conditions $p_i - c_i \leq -\gamma_i'[z_i - y_i(p_i, z_i)]$ and $(p_i - c_i + \gamma_i'[z_i - y_i(p_i, z_i)])y_i(p_i, z_i) = 0$.

Consider $p \in P$ and $z \in [0, D(0)]^n$. The vector of outputs sold when current price vector is p and current state vector is z, $q(p, z)$ is defined inductively. For price $\lambda$ and price vector p set $J_0(\lambda, p) = \{i \in I|\ p_i = \lambda\}$. Let $\lambda_0 = \min_{i\in I}p_i$. For firm $i = \min\{j \in J_0(\lambda_0, p)|\ y_j(p_j, z_j) \leq y_k(p_k, z_k)\ \forall\ k \in J_0(\lambda_0, p)\}$ the output sold is $q_i(p, z) = \min\{y_i(p_i, z_i),\ |J_0(\lambda_0,$

$p)|^{-1}D(\lambda_0)\}$[6] and we set $J_1(\lambda_0, p) = J_0(\lambda_0, p)\setminus\{i\}$ and $D_1(\lambda_0) = D(\lambda_0) - q_i(p, z)$. If $J_1(\lambda_0, p) \neq \emptyset$, we find firm $i = \min\{j \in J_1(\lambda_0, p) |\ y_j(p_j, z_j) \leq y_k(p_k, z_k)\ \forall\ k \in J_1(\lambda_0, p)\}$, the output sold by this firm is $q_i(p, z) = \min\{y_i(p_i, z_i), |J_1(\lambda_0, p)|^{-1}D_1(\lambda_0)\}$, and we set $J_2(\lambda_0, p) = J_1(\lambda_0, p)\setminus\{i\}$ and $D_2(\lambda_0) = D_1(\lambda_0) - q_i(p, z)$. We continue in this way with $J_k(\lambda_0, p)$ and $D_k(\lambda_0)$ for k = 3, ..., $|J_0(\lambda_0, p)| - 1$. Then, unless $J_0(\lambda_0, p) = I$, we set $\lambda_1 = \min\{p_i |\ i \in I\setminus J_0(\lambda_0, p)\}$ and, using the residual demand $D_r(\lambda_1)$, equal to $D(\lambda_1)$ minus sum of outputs sold by firms in $J_0(\lambda_0, p)$, instead of $D(\lambda_0)$, we repeat the above procedure for firms in $J_0(\lambda_1, p)$. We continue in this way until we determine $q_i(p, z)$ for all $i \in I$. We let $q(p, z) = (q_i(p, z))_{i\in I}$.

Since changes in an output in a period t in comparison with an output in a period t - 1 are costly and $\gamma_i$ is strictly convex for all $i \in I$, an output vector q(t - 1) is payoff relevant in a period t. Therefore, we can associate it with a state vector in a period t and the set $Q = [0, D(0)]^n$ with the state space.[7]

Firm i's ($i \in I$) profit in a period t, gross of costs of changing an output in comparison with a period t - 1, is

(1) $$\pi_i[p(t), q(t - 1)] = (p_i(t) - c_i)\ q_i[p(t), q(t - 1)].$$

Firm i's net profit in a period t is $\pi_i[p(t), q(t - 1)] - \alpha_i[q_i(t - 1), q_i(p(t), q(t - 1))]$.

We assume that binding contracts between firms are not possible.

A Markov strategy of firm i, $s_i$, is a function that assigns to each element of the state space a price charged by firm i, i.e., $s_i:Q \rightarrow P_i$. Thus, a Markov strategy is a special case of

[6] For a finite set A the symbol $|A|$ denotes its cardinality.

[7] This definition of the state space does not reflect the requirement that (since period 2) for any feasible state vector z we have $z = q(p, z')$ for some $p \in P$ and some feasible state vector z'. The narrower definition of the state space, taking this into account, would impose additional restrictions on the initial state, would be cumbersome, and would not change the results of our analysis.

a closed loop strategy.[8] The set of all feasible Markov strategies of firm i is denoted by $S_i$. We let $S = X_{i\in I}S_i$, $s = (s_1, ..., s_n) \in S$, and write $s_{-i}$ for $(s_1, ..., s_{i-1}, s_{i+1}, ..., s_n)$.[9]

For each firm $i \in I$, average discounted net profit is its payoff function. Average discounted net profit of firm i in a subgame with an initial state q, when firms follow a Markov strategy profile s, is

$$(2) \qquad \Pi_i(s, q) = (1 - \beta_i) \sum_{t=1}^{\infty} \beta_i^{t-1}\{\pi_i[s(q(t-1)), q(t-1)] - \alpha_i[q_i(t-1), q_i(t)]\}, \quad \forall\, i \in I,$$

where $q(0) = q$ and $q(t) = q[s(q(t-1)), q(t-1)]$ for each positive integer t. (Without loss of generality, we can number the first period of a subgame by one.) We set $\Pi(s, p) = (\Pi_1(s, p), ..., \Pi_n(s, p))$. In what follows, "$G_{DO}$," or "the game $G_{DO}$," refers to the analyzed dynamic oligopoly.

A Markov perfect equilibrium is a profile of Markov strategies that yields a Nash equilibrium in every subgame of $G_{DO}$. The following definition expresses this more formally.

DEFINITION 1. A profile of Markov strategies $s \in S$ is a Markov perfect equilibrium of $G_{DO}$ if, for each state $q \in [0, D(0)]^n$, for every firm $i \in I$, and for each strategy $s_i' \in S_i$, $\Pi_i(s, q) \geq \Pi_i((s_i', s_{-i}), q)$.

[8] We restrict attention to pure strategies. Due to costs of changing an output, randomized behavioural strategies do not seem to be appealing, unless differences between firms' marginal costs of production are large. (In this case a continuous strategy Markov perfect equilibrium in behavioural objectively correlated strategies could be Pareto superior to the equilibria selected in this paper.) Moreover, allowing for behavioural randomized strategies would not increase realism of the model.

[9] Other symbols with the subscript "−i" have analogous meaning.

If n - 1 firms use Markov strategies then there is a best response of the remaining firm against them (chosen from the whole set of its closed loop strategies) that is also a Markov strategy. Therefore, a Markov perfect equilibrium is still a subgame perfect equilibrium when the Markov restriction is not imposed.

A firm i's Markov strategy $s_i$ is continuous if it is a continuous function from Q to $P_i$. The set of all feasible continuous Markov strategies of firm i is denoted by $S^*_i$. We let $S^* = X_{i\in I}S^*_i$.

A continuous strategy Markov perfect equilibrium is a Markov perfect equilibrium strategy profile in which all strategies are continuous functions. For the sake of completeness, we give the formal definition.

DEFINITION 2. A continuous strategy Markov perfect equilibrium of $G_{DO}$ is a profile of continuous Markov strategies $s^* \in S^*$ such that for each state $q \in [0, D(0)]^n$, for every firm $i \in I$, and for each strategy $s'_i \in S_i$, $\Pi_i(s^*, q) \geq \Pi_i((s'_i, s^*_{-i}), q)$.

Note that in Definition 2 we (have to) explicitly require that a strategy profile that is a continuous strategy Markov perfect equilibrium is immune to all unilateral deviations to all Markov strategies, including those that are not continuous.

## 3. CONTINUOUS STRATEGY MARKOV PERFECT EQUILIBRIA

In this Section we give first sufficient (in Proposition 1) and then necessary (in

Proposition 2) condition of an existence of a continuous strategy Markov perfect equilibrium.

PROPOSITION 1. Let a price vector $p^* \in P$ be symmetric with $p_1^* \le p_1^{mon}$. Assume that there exists a symmetric price vector $p^0 \in P$ satisfying

(3) $y_i(p_1^0, 0) \ge (n - 1)^{-1}D(p_1^0)$ for all $i \in I$,

and

(4) $(p_1^0 - c_i)\, D(p_1^0) < \pi_i(p^*, q^*)$ for all $i \in I$,

where $q_i^* = n^{-1}D(p_1^*)$ for all $i \in I$. Then there exists a vector of discount factors $\beta^* \in (0, 1)^n$ such that for each $\beta \in X_{i\in I}[\beta_i^*, 1)$ there is a continuous strategy Markov perfect equilibrium of $G_{DO}$ in which, in each continuation equilibrium, the play converges to the price vector $p^*$ and the output vector $q^*$.

PROOF. *Description of the equilibrium strategy profile* $s^*$. Set $q_i^0 = n^{-1}D(p_1^0)$ for each $i \in I$. Let $r \in (0, 1)$ and define a function $\xi$:Q $\rightarrow$ [0, 1] by

$$(5) \qquad \xi(q) = r\frac{q_1^0 - \min(\max(\max_{i\in I} q_i,\ q_1^*),\ q_1^0)}{q_1^0 - q_1^*} + 1 - r.$$

Then the strategy profile $s^*$ is defined by

(6) $s^*(q) = \xi(q)p^* + [1 - \xi(q)]p^0$.

*The strategy profile* $s^*$ *forms a Markov perfect equilibrium.* Strategies are functions of a current payoff relevant state only, so they are Markov. Since $G_{DO}$ is continuous at infinity it is enough to examine unilateral single period deviations.

Let $\Delta$ be a small positive real number such that (4) still holds when we replace $p^0$ by a symmetric price vector $p' \in P$ with each component equal to $p_1^0 + \Delta$. A deviation by firm i at a state $q \in Q$ with $s^*(q) \notin [p^0, p']$ causes[10] the play to switch along the line segment $[p^0, p^*]$ towards $p^0$. (A common price prescribed by $s^*$ after a deviation is lower than it would be if a deviation did not take place.) For the limit case $\beta_i = r = 1$ this strictly reduces firm i's continuation average discounted net profit. Due to continuity of the latter in $\beta_i$ and r the same holds also for $\beta_i < 1$ and $r < 1$ close enough to one. There is no danger that the minimum value of r (at which a deviation does not increase a deviator's continuation average discounted net profit) tends to one as a magnitude of a deviation (a difference between a deviating firm's output resulting from a deviation and its output in the case without a deviation) tends to zero. The reason is that a very small deviation in output (which, with respect to (3), must be an upward one) results from a downward deviation in price that reduces a deviator's single period net profit.

A deviation at a state $q \in Q$ with $s^*(q) \in [p^0, p']$ restarts the movement along the line segment $[p^0, p^*]$, starting with a vector from $[p^0, p']\backslash\{p'\}$. That is, it triggers a continuation equilibrium that gives a deviating firm a continuation equilibrium average discounted net profit no higher than the one it earns in a subgame in the first period of which a deviation took place. The conditions (4) and (3) imply that, for a deviator's discount factor close enough to one, a single period net profit from such deviation is strictly lower than a deviator's continuation equilibrium average discounted net profit in a subgame in the first period of which a deviation took place. Therefore, such a deviation decreases a deviator's overall (starting form the period in which it took place) continuation average discounted net profit.

*Continuity of* $s^*$ follows from continuity of functions maximum, minimum, and $\xi$. Q.E.D.

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[10] For two n-dimensional vectors a and b the symbol [a, b] denotes the set of all their convex combinations.

The condition (3) ensures that, when the equilibrium strategy profile $s^*$ prescribes the price vector $p^0$, firms reach the output vector $q^0$ in one step and profitable upward unilateral deviations in price are not possible. We included this condition in the above form in Proposition 1 in order to simplify the latter. We could relax it, by replacing $(n - 1)^{-1}$ by $n^{-1}$ (so that, when $s^*$ prescribes $p^0$, firms still reach $q^0$ in one step) and specifying conditions analogous to (4) for a maximum of single period net profit of each firm from a unilateral upward deviation in price from $p_1^0$.

It is worth noting that, for each specific form of demand function D and every set of specific forms of functions $\gamma_i$, $i \in I$, Proposition 1 can be expressed by imposing restrictions only on discount factors, parameters of functions $\gamma_i$, and firms' marginal costs of production, i.e., by imposing restrictions only on primitives of the model.

PROPOSITION 2. Assume that the profile of continuous Markov strategies $s^* \in S^*$ is a continuous strategy Markov perfect equilibrium of $G_{DO}$ for all vectors of discount factors $\beta \in X_{i\in I}[\beta_i^*, 1)$, where $\beta^* \in (0, 1)^n$. Assume also that the equilibrium average discounted net profit of each firm is bounded away from zero (as its discount factor tends to one). Then each continuation equilibrium path converges (in P) to a symmetric price vector $p^* \in P$ with $p_1^* \leq p_1^{mon}$.

PROOF. First, a sequence of price vectors generated by a continuation equilibrium cannot approach a cycle between different price vectors. It would have to approach it by switching between several paths - its subsequences, each of them converging to one component of the cycle. After some profitable unilateral deviations it would not be possible to determine, only on the basis of a current output vector, from which of them a deviation took place. Thus, either for at least one of them a set of states from which the play switches to it would be open, which contradicts continuity of strategies, or the set of paths along which the play can approach (after a profitable unilateral deviation) the cycle would have to be uncountable. In the latter case, the set of paths cannot be partitioned among components of the cycle without violating the continuity property of strategies.

Second, a continuation equilibrium cannot prescribe a non-convergent sequence of price vectors that does not approach a cycle between different price vectors. Firms' single period net profits along it would not be monotonic. Therefore the method of punishment described in the proof of Proposition 1 could not be used. After some profitable unilateral deviations it would not be possible to determine, only on the basis of a current output vector, from which price vector prescribed by $s^*$ a deviation took place. Therefore it would not be possible to punish all of them in a way compatible with continuity of strategies. (There would be pairs of different profitable unilateral deviations, leading to the same state, requiring switching to different continuation continuous Markov strategy profiles in order to punish them.)

Third, continuation equilibrium paths cannot converge to an asymmetric price vector $p' \in P$. Since each firm's equilibrium average discounted net profit is bounded away from zero (as its discount factor tends to one), when the play approached p' the firms in $J_0(\min_{j\in I}p'_j, p')$ could not produce outputs whose sum approached $D(\min_{j\in I}p'_j)$. This would imply that $-\gamma'_i(0) \leq p'_i - c_i \leq \gamma'_i(0)$ for all $i \in J_0(\min_{j\in I}p'_j, p')$. Therefore $s^*_i(0) > p'_i$ for all $i \in J_0(\min_{j\in I}p'_j, p')$. This, in turn, would imply that, in the subgame with the initial state 0, the play could only converge to p', passing through $s^*(0)$, without reaching it in any finite time. (Otherwise profitable unilateral deviations taking place when $s^*$ prescribes $s^*(0)$ could not be punished in a way compatible with continuity of strategies.) In some small neighbourhood of p' the play would have to pass only through price vectors with all coordinates above

$\min_{j \in I} p'_j$. This would make a punishment of some profitable unilateral deviations, taking place when $s^*$ prescribed a price vector from that neighbourhood, in a way compatible with continuity of strategies impossible.

Fourth, we show that continuation equilibrium paths cannot converge to a symmetric price vector $p^* \in P$ with $p_1^* > p_1^{mon}$. Suppose that they do converge to such price vector. Consider a unilateral deviation by firm one at a state in a neighbourhood of $q^* = (n^{-1}D(p_1^*), ..., n^{-1}D(p_1^*))$, giving it output slightly higher than $q_1^*$ but lower than $n^{-1}D(p_1^{mon})$. Due to continuity of strategies such deviation (despite the fact that it is not profitable) cannot be ignored. If a magnitude of this deviation is small enough then a continuation equilibrium triggered by it prescribes a symmetric price vector p with $p_1 \in [p_1^{mon}, p_1^*)$ in the period following the one in which a deviation took place and then the play proceeds along the line segment $[p, p^*]$. For $\beta_1$ close enough to one this increases the overall (starting from the period when a deviation took place) continuation average discounted net profit, so such a deviation is not deterred by $s^*$. (In equilibrium, movement along the line segment in the state space corresponding to the movement along the line segment $[p, p^*]$ in P can be described by the function $q(t + 1) = q[s^*(q(t)), q(t)]$, relating current and future state. If parameters of this function[11] are in a small neighbourhood of those for which the movement along $[p, p^*]$ stops the claim holds. For a sufficiently small magnitude of firm one's deviation the sequence of price vectors generated by $s^*$ in the subgame with the initial state p' must be close, on an element-wise basis, to the one generated by the latter values of parameters of the function $q[s^*(q(t)), q(t)]$.) Q.E.D.

[11] In the case of a continuous strategy Markov perfect equilibrium described in the proof of Proposition 1 "r" is the only parameter of this function.

From the point of view of net profits, the price vector $p^* = (p_1^{mon}, ..., p_1^{mon})$, leading to the output vector $(n^{-1}D(p_1^{mon}), ..., n^{-1}D(p_1^{mon}))$, strictly Pareto dominates each symmetric price vector p with every component below $p_1^{mon}$, leading to an output vector $(n^{-1}D(p_1), ..., n^{-1}D(p_1))$. Thus, when all discount factors are close enough to one (and the assumptions of Proposition 2 hold), the continuous strategy Markov perfect equilibria based on convergence to $p^*$ are obvious candidates for selection. In the following Section we justify this intuition on the basis of the weakest possible concept of renegotiation-proofness.

## 4. EQUILIBRIUM SELECTION

In this Section we deal with the selection from the set of all continuous strategy Markov perfect equilibria of $G_{DO}$, in which all firms are active (their equilibrium average discounted net profit is bounded away from zero as their discount factors tend to one). We first briefly comment on Markov perfect equilibria in which some firm(s) is (are) inactive. Then we define the concept of renegotiation-quasiproofness. Finally, we show that all continuous strategy Markov perfect equilibria of $G_{DO}$, which are renegotiation-quasiproof for discount factors arbitrarily close to one and in which equilibrium average discounted net profit of each firm is bounded away from zero, prescribe the play converging to the symmetric price vector $p^* = (p_1^{mon}, ..., p_1^{mon})$.

Equilibria in which some firm(s) is (are) not active arise if some coalition $C \subset I$, $C \neq \emptyset$, can make all its members better off, in comparison with the continuous strategy Markov perfect equilibria with all firms active that are renegotiation-quasiproof for discount factors arbitrarily close to one, by forcing the firms in I\C out of the market forever. These can be of two types. First, a coalition C, (acting as if) ignoring the existence of the firms in

I\C, chooses an equilibrium (of the game derived from $G_{DO}$ by replacing the set of firms I by C), in which the firms in I\C are forced out of the market. This case can be handled by the approach based on renegotiation-quasiproofness. Second, a coalition C can make all its members better off (in comparison with the continuous strategy Markov perfect equilibria with all firms active that are renegotiation-quasiproof for discounts factors arbitrarily close to one) by driving (in a finite number of periods) the firms in I\C out from the market (i.e., all firms in C in each continuation equilibrium, after a finite number of periods bounded from above, charge a price that is higher than $c_j + \gamma_j'(0)$ for all $j \in C$ and lower than $c_i - \gamma_i'(0)$ for each $i \in I\backslash C$), but this outcome would not be selected if the existence of the firms in I\C was ignored. The analysis of this case would require an explicit examination of entry and exit, which is beyond the scope of this paper. (Once the firms in I\C exit, a coalition C has a clear incentive to increase price. Thus, a thorough analysis of conditions under which an entry takes place would be crucial.)

DEFINITION 3. A continuous strategy Markov perfect equilibrium $s^* \in S^*$ of $G_{DO}$ is renegotiation-quasiproof if there does not exist another continuous strategy Markov perfect equilibrium of $G_{DO}$, $s' \in S^*$, such that $\Pi_i(s', q) > \Pi_i(s^*, q)$ for each $i \in I$ and all $q \in Q$.

Thus, a continuous strategy Markov perfect equilibrium $s^* \in S^*$ is renegotiation-quasiproof if firms, by (collectively) switching to another continuous strategy Markov perfect equilibrium, cannot increase continuation equilibrium average discounted net profit of every of them in each subgame. This is the weakest possible concept of renegotiation-proofness. A switching to another equilibrium is assumed to take place only if it increases continuation average discounted payoff of each player in each subgame. (It is assumed not to take place if it will increase continuation average discounted payoff of each player only in a subgame in which it takes place.) Since it is so weak concept, we do not find it appropriate to call it simply "renegotiation-proofness." On the other hand, we cannot call it "weak renegotiation-proofness," because this term was already used by Farrell and Maskin (1989) for a different concept.

As it is usual in the literature on renegotiation-proofness, the set of (subgame perfect) equilibria to which firms are allowed to renegotiate is restricted here. Since we assume (for reasons explained in the Introduction) that, at the beginning of the game, firms will coordinate on a continuous strategy Markov perfect equilibrium, there is no reason to assume that they will renegotiate to some other type of (subgame perfect) equilibrium.

When all discount factors are close to one, the concept of renegotiation-quasiproofness coincides with the concept of renegotiation-proofness used in Maskin’s and Tirole’s (1988) paper on dynamic price competition. In their paper a renegotiation is based on a change of a price vector that is infinitely repeated (after a finite number of periods) in each continuation equilibrium. In our case it is based on a change of a limit of all continuation equilibrium paths.

PROPOSITION 3. There exists a vector of discount factors $\beta^* \in (0, 1)^n$ such that a continuous strategy Markov perfect equilibrium $s^*$ of $G_{DO}$, in which each firm’s equilibrium average discounted net profit is bounded away from zero (as its discount factor tends to one), is renegotiation-quasiproof for all $\beta \in X_{i\in I}[\beta^*_i, 1)$ if and only if it prescribes in each continuation equilibrium the play converging to the price vector $p^* = (p^{mon}_1, p^{mon}_1, ..., p^{mon}_1)$.

PROOF. *Sufficiency.* Any continuous strategy Markov perfect equilibrium $s^*$ prescribing in each continuation equilibrium the play converging to the price vector $p^*$ is renegotiation-quasiproof for all discount factors close enough to one, because (by Proposition 2) there is no other continuous strategy Markov perfect equilibrium with the limit of single period net profit vectors weakly Pareto dominating their limit in $s^*$ (so a renegotiation to another continuous strategy Markov perfect equilibrium, prescribing the play converging to another limit, would not increase continuation average discounted net profit of each firm in any subgame) and $s^*[n^{-1}D(p_1^{mon}), n^{-1}D(p_1^{mon}), ..., n^{-1}D(p_1^{mon})] = p^*$ (so a renegotiation to another continuous strategy Markov perfect equilibrium, prescribing the play converging to $p^*$, cannot increase continuation average discounted net profit of any firm in the subgame with the initial state $(n^{-1}D(p_1^{mon}), n^{-1}D(p_1^{mon}), ..., n^{-1}D(p_1^{mon}))$.

*Necessity*. Consider a continuous strategy Markov perfect equilibrium s' prescribing in each continuation equilibrium the play converging to a symmetric price vector $p' \in P$ with $p_1' < p_1^{mon}$. For all discount factors close enough to one, a renegotiation to a continuous strategy Markov perfect equilibrium $s^*$ prescribing in each continuation equilibrium the play converging to the price vector $p^*$ increases continuation average discounted net profit of every firm in each subgame. Q.E.D.

Thus, the requirement of renegotiation-quasiproofness for discount factors arbitrarily close to one (although it does not identify the unique equilibrium) leads to the unique limit of all continuation equilibrium paths of all continuous strategy Markov perfect equilibria satisfying it. This limit is the symmetric price vector with each component equal to the monopoly price of the firm(s) with the lowest marginal costs of production.

## 5. CONCLUSIONS

In this paper we presented a method for selection of the unique limit of all continuation equilibrium paths in a dynamic Bertrand oligopoly with production of a homogeneous good to order and costly changes of an output. It utilizes a special feature of standard model of Bertrand oligopoly producing a homogeneous good - discontinuity of firm's demand function. This rules out continuous strategy Markov perfect equilibria (with all firms active) with all continuation equilibrium paths converging to an asymmetric price vector. Thus, the method described in this paper cannot be (directly) generalized for other dynamic games. Despite this we believe that the analysis of the game examined here - with respect to an extensive attention paid by economists to the underlying model of a static Bertrand oligopoly - is worthwhile.

The method described in this paper can be (despite the title of the paper) interpreted also as a method of selection of a (limit of continuation equilibrium paths of a pure strategy) subgame perfect equilibrium in the analyzed dynamic game. It is based on imposing three requirements on firms' strategies. First, they have to be Markov. This is an application of Harsanyi's and Selten's (1992) principle of invariance of a selected equilibrium with respect to isomorphism of games, implying that strategically equivalent subgames should have identical solution. Second, strategies have to be continuous (functions). This is a strengthening of the previous requirement - subgames that are strategically close to each other should have solutions that are close to each other (in a sense explained in the Introduction). Third, selected equilibrium strategy profiles have to be renegotiation-quasiproof for discount factors arbitrarily close to one - there should be no other equilibrium satisfying the previous two conditions that makes all firms better off in each subgame for some vector of discount factors less than but close to one.

It is worth noting that in the special case of a symmetric duopoly the (common) limit of continuation equilibrium paths (in the price space) of the continuous strategy Markov perfect equilibria identified in this paper as renegotiation-quasiproof for discount factors arbitrarily close to one - the symmetric price vector with both components equal to the monopoly price - coincides with the stationary part of the unique renegotiation-proof Markov perfect equilibrium identified in Maskin’s and Tirole’s paper (1988) on dynamic price competition (despite the different structure of the two models). We view this as an argument in favour of our result.

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