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(1+)-complemented, (1+)-isomorphic copies of L_1 in dual Banach spaces

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(1+)-COMPLEMENTED, (1+)-ISOMORPHIC COPIES OF L_1 IN DUAL BANACH SPACES

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Abstract. The present paper contributes to the ongoing programme of quantification of isomorphic Banach space theory focusing on Pełczyński's classical work on dual Banach spaces containing L_1 ($= L_1[0,1]$) and the Hagler–Stegall characterisation of dual spaces containing complemented copies of L_1. We prove the following quantitative version of the Hagler–Stegall theorem asserting that for a Banach space X the following statements are equivalent:

- X contains almost isometric copies of $(\bigoplus_{n=1}^{\infty} \ell_\infty^n)_{\ell_1}$,
- for all $\varepsilon > 0$, X^* contains a $(1+\varepsilon)$-complemented, $(1+\varepsilon)$-isomorphic copy of L_1,
- for all $\varepsilon > 0$, X^* contains a $(1+\varepsilon)$-complemented, $(1+\varepsilon)$-isomorphic copy of $C[0,1]^*$.

Moreover, if X is separable, one may add the following assertion:

- for all $\varepsilon > 0$, there exists a $(1+\varepsilon)$-quotient map $T\colon X \to C(\Delta)$ so that $T^*[C(\Delta)^*]$ is $(1+\varepsilon)$-complemented in X^*,

where Δ is the Cantor set.

1. Introduction

In 1968, Pełczyński [15] showed that if a Banach space X contains an isomorphic copy of ℓ_1, then the dual space X^* contains an isomorphic copy of L_1 and proved that the converse holds as well subject to a mild technical condition that was later removed by Hagler [6]. More precisely, the result stated that the isomorphic containment of ℓ_1 is equivalent to the following assertions: X^* contains a subspace isomorphic to L_1, X^* contains a subspace isomorphic to $C[0,1]^*$. When X is separable, these are further equivalent to the assertions: X^* contains a subspace isomorphic to $\ell_1([0,1])$, $C[0,1]$ is a quotient of X.

Shortly after, Hagler and Stegall [8] obtained a 'complemented' version of aforementioned theorem, that is the following result.

Theorem 1.1 (Hagler–Stegall). *Let X be a Banach space. Then the following assertions are equivalent:*

(1) *X contains a subspace isomorphic to $(\bigoplus_{n=1}^{\infty} \ell_\infty^n)_{\ell_1}$;*

(2) *X^* contains a complemented subspace isomorphic to L_1;*

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(3) *X^* contains a complemented subspace isomorphic to $C[0,1]^*$;*

(4) *X^* contains an infinite set K such that K is equivalent to the usual basis of $\ell_1(\Gamma)$ for some Γ, $[K]$ is complemented in X^*, and K is dense-in-itself in the weak* topology on X^*;*

If, in addition, X is separable, then the assertions (1)–(4) *are equivalent to*

(5) *There exists a surjective operator $T\colon X \to C[0,1]$ such that $T^*[C[0,1]^*]$ is complemented in X^*.*

Subsequently, Dilworth, Girardi, and Hagler [5] proved the following isometric version of Pełczyński's result mentioned earlier by means of the notion of asymptotically isometric copies of ℓ_1.

Theorem 1.2. *Let X be a Banach space. Then the following are equivalent:*

(1) *X contains an asymptotically isometric copy of ℓ_1;*

(2) *X^* contains an isometric copy of L_1;*

(3) *X^* contains an isometric copy of $C[0,1]^*$.*

The result was refined further by Hagler [7] who provided the following quantitative characterisations of dual spaces containing complemented isometric copies of L_1.

Theorem 1.3 (Hagler)**.** *Let X be a Banach space and $\lambda \geqslant 1$. The following assertions are equivalent:*

(1) *X contains $(1,\lambda)$-asymptotic copies of $\ell_1 \oplus (\bigoplus_{n=1}^\infty \ell_\infty^n)_{\ell_1}$;*

(2) *X^* contains λ-complemented subspaces isometric to L_1;*

(3) *X^* contains λ-complemented subspaces isometric to $C[0,1]^*$;*

(4) *X^* contains an infinite set K such that K is isometrically equivalent to the usual basis of $\ell_1(\Gamma)$ for some Γ, $[K]$ is λ-complemented in X^*, and K is dense in itself in the weak* topology on X^*.*

If, in addition, X is separable, then the above assertions are equivalent to

(5) *There exists a quotient map $T\colon X \to C(\Delta)$ such that $T^*[C(\Delta)^*]$ is λ-complemented in X^*.*

The purpose of this note is to quantify the aforementioned results, especially Theorem 1.1, in the spirit of the recent research on quantitative versions of various theorems and properties of Banach spaces (see [2, 10, 11] and references therein).

In order to state the main results of the paper, we employ the following four quantities denoted by lower-case Greek letters and defined as suprema of certain sets; when the sets happen to be empty, we use the convention that then each of the quantities is 0. Hereinafter X and Y will stand for Banach spaces; $\mathcal{B}(X,Y)$ is the space of (bounded, linear) operators from X to Y.

- $\alpha_Y(X) = \sup\{\|T^{-1}\|^{-1} \colon T \in \mathcal{B}(Y, X) \text{ an isomorphism with } \|T\| \leqslant 1\}$.
  α_Y, being directly related to the Banach–Mazur distance, measures how far Y is from being isomorphically embeddable into X. Obviously, $\alpha_Y(X) = 1$ if and only if X contains almost isometric copies of Y, that is, for every $\varepsilon > 0$, X contains a subspace $(1+\varepsilon)$-isomorphic to Y.
- $\beta_Y(X) = \sup\{(\|A\|\|B\|)^{-1} \colon A \in \mathcal{B}(X, Y), B \in \mathcal{B}(Y, X),\ AB = I_Y\}$.
  $\beta_Y(X)$ measures how far Y is from being isomorphic to a complemented subspace of X: $\beta_Y(X) = 1$ if and only if for every $\varepsilon > 0$, there exists a subspace M of X so that M is $(1+\varepsilon)$-isomorphic to Y and $(1+\varepsilon)$-complemented in X.
- $\gamma_Y(X) = \sup\{\delta(T) \colon T \in \mathcal{B}(X, Y) \text{ is a surjective operator with } \|T\| \leqslant 1\}$,
  where $\delta(T) = \sup\{c > 0 \colon cB_Y \subseteq TB_X\}$. $\gamma_Y(X)$ measures how far Y is from being isomorphic to a quotient of X: $\gamma_Y(X) = 1$ if and only if Y is a $(1+\varepsilon)$-(linear) quotient of X for every $\varepsilon > 0$.
- $\theta_Y(X) = \sup\{(\|A\|\|S\|)^{-1} \colon A \in \mathcal{B}(X, Y), S \in \mathcal{B}(X^*, Y^*), SA^* = I_{Y^*}\}$.
  We have $\theta_Y(X) = 1$ if and only if, for every $\varepsilon > 0$, there exists a $(1+\varepsilon)$-quotient map $T \colon X \to Y$ so that $T^*[Y^*]$ is $(1+\varepsilon)$-complemented in X^*.

A straightforward argument shows that

$$\beta_Y(X) \leqslant \theta_Y(X) \leqslant \beta_{Y^*}(X^*). \tag{1.1}$$

By using the aforementioned four quantities, we quantify the aforementioned results as follows.

Theorem A. *Let X be a Banach space. Then*

$$\alpha_{\ell_1}(X) = \alpha_{L_1}(X^*) = \alpha_{C[0,1]^*}(X^*).$$

If, in addition, X is separable, then

$$\alpha_{\ell_1}(X) = \alpha_{\ell_1([0,1])}(X^*) = \gamma_{C[0,1]}(X).$$

Theorem B. *Let X be a Banach space. Then*

$$\alpha_{(\oplus_{n=1}^\infty \ell_\infty^n)_{l_1}}(X) = \beta_{C[0,1]^*}(X^*) = \beta_{L_1}(X^*).$$

If, in addition, X is separable, then

$$\theta_{C(\Delta)}(X) = \beta_{L_1}(X^*).$$

The following two corollaries follows immediately from Theorem A:

Corollary A. *Let X be a Banach space. The following are equivalent:*

1. *X contains a subspace isomorphic to ℓ_1;*
2. *X^* contains almost isometric copies of L_1;*
3. *X^* contains almost isometric copies of $C[0,1]^*$;*

If, in addition, X is separable, then the assertions (1)–(3) *are equivalent to:*

(4) X^* *contains almost isometric copies of* $\ell_1([0,1])$*;*
(5) $C[0,1]$ *is a* $(1+\varepsilon)$*-quotient of* X *for every* $\varepsilon > 0$.

Corollary B. *Let X be a Banach space and let Y be one of the spaces: L_1 or $C[0,1]^*$. If Y is isomorphic to a subspace of X^*, then X^* contains an almost isometric copy of Y. If, in addition, X is separable, then an analogous assertion is valid for $Y = \ell_1([0,1])$.*

The following $(1+\varepsilon)$-version of Theorem 1.1 follows from Theorem B.

Corollary C. *Let X be a Banach space. Then the following assertions are equivalent:*

(1) *X contains almost isometric copies of* $(\bigoplus_{n=1}^{\infty} \ell_\infty^n)_{l_1}$*;*
(2) *X^* contains a $(1+\varepsilon)$-complemented subspace that is $(1+\varepsilon)$-isomorphic to L_1 for every $\varepsilon > 0$;*
(3) *X^* contains a $(1+\varepsilon)$-complemented subspace that is $(1+\varepsilon)$-isomorphic to $C[0,1]^*$ for every $\varepsilon > 0$.*

If, in addition, X is separable, then

(4) *For every $\varepsilon > 0$, there exists a $(1+\varepsilon)$-quotient map $T\colon X \to C(\Delta)$ so that $T^*[C(\Delta)^*]$ is $(1+\varepsilon)$-complemented in X^*.*

2. Preliminaries

Our notation and terminology are standard and mostly in-line with [1] and [14]. Throughout the paper, all Banach spaces are infinite-dimensional. We work with real scalar but the result can be easily amended to the complex too. By a *subspace* we understand a closed, linear subspace and by an *operator* we understand a bounded, linear map. Let X be a Banach space. We denote by B_X the closed unit ball of X. I_X stands for the identity operator on X and $J_X\colon X \to X^{**}$ is the canonical embedding. For a subset $K \subseteq X$, $[K]$ stands for the closed linear span of K. For a subspace $M \subseteq X$, we denote by i_M the inclusion map from M into X. For $\lambda \geqslant 1$, we say that a surjective operator $T\colon X \to Y$ is a λ-*quotient map* if $\|T\| \operatorname{co}(T) \leqslant \lambda$, where

$$\operatorname{co}(T) = \inf\{c > 0\colon B_Y \subseteq c \cdot TB_X\}.$$

Quotient maps are 1-quotient maps according to the above terminology. A norm-one surjective operator $T\colon X \to Y$ is a quotient map if and only if T is a (1+)-quotient map, that is, $(1+\varepsilon)$-quotient map for every $\varepsilon > 0$.

The *Banach–Mazur distance* $\mathrm{d}(X,Y)$ between two isomorphic Banach spaces X and Y is defined by $\inf \|T^{-1}\|$, where the infimum is taken over all norm-one isomorphisms T from X onto Y. As defined by Lindenstrauss and Rosenthal [13], for $1 \leqslant p \leqslant \infty$ and $\lambda \geqslant 1$, a Banach space X is said to be a $\mathcal{L}_{p,\lambda}$-*space* whenever for every finite-dimensional subspace E of X there is a finite-dimensional subspace F of X such that $F \supseteq E$ and $\mathrm{d}(F, l_p^{\dim F}) \leqslant \lambda$. We say that a Banach space X is an $\mathcal{L}_{p,\lambda+}$-*space* if it is an $\mathcal{L}_{p,\lambda+\varepsilon}$-*space* for all $\varepsilon > 0$.

Following the notation from [8], we denote

$$\mathcal{F} = \{(n,i)\colon n = 0,1,\ldots; i = 0,1,\ldots,2^n - 1\}$$

and, for $(n,i),(m,j) \in \mathcal{F}$ we write $(n,i) \geqslant (m,j)$ whenever

- $n \geqslant m$,
- $2^{n-m} j \leqslant i \leqslant 2^{n-m}(j+1) - 1$.

Let $\Delta = \{0,1\}^{\mathbb{N}}$ be the Cantor set endowed with the metric

$$\mathrm{d}\big((a_n)_{n=1}^\infty, (b_n)_{n=1}^\infty\big) = \sum_{n=1}^\infty \frac{1}{2^n}|a_n - b_n| \quad \big((a_n)_n, (b_n)_n \in \Delta\big).$$

By Miljutin's Theorem ([1, Lemma 4.4.7]), $C[0,1]$ is isomorphic (but not isometric) to $C(\Delta)$. It is well-known that $C(\Delta)^*$ and $C[0,1]^*$ are linearly isometric, though.

3. Proof of Theorem A

The present section is devoted to the proof of Theorem A and is conveniently split into more digestible parts.

Proof. Step 1. $\alpha_{\ell_1}(X) \leqslant \alpha_{L_1}(X^*)$.

Let $0 < c < \alpha_{\ell_1}(X)$. Then there exists an operator $T\colon \ell_1 \to X$ such that $\|T\| \leqslant 1$ and $c\|z\| \leqslant \|Tz\|$ $(z \in \ell_1)$. Setting $Y = T\ell_1$ yields an operator $A\colon L_1 \to Y^*$ so that

$$\|f\| \leqslant \|Af\| \leqslant c^{-1}\|f\| \quad (f \in L_1). \tag{3.1}$$

Indeed, we may define $S\colon \ell_1 \to Y$ by $Sz = Tz$ $(z \in \ell_1)$. Take an isometric embedding $U : L_1 \to l_\infty$ (l_∞ is universal for all separable Banach spaces and their conjugate spaces). Then $A = (S^*)^{-1}U$ is the required operator. By the 1-injectivity of L_∞, we obtain an operator $B\colon X^{**} \to \ell_\infty$ so that $Bi_Y^{**} = A^*$ and $\|B\| = \|A^*\| = \|A\|$:

$$\begin{array}{ccc} X^{**} & & \\ \uparrow{\scriptstyle i_Y^{**}} & \searrow^{\exists B} & \\ Y^{**} & \xrightarrow{A^*} & L_\infty \end{array}$$

Passing to the adjoints, we get a commutative diagram:

$$\begin{array}{ccccccc} L_1 & \xrightarrow{J_{L_1}} & L_1^{**} & \xrightarrow{A^{**}} & Y^{***} & \xrightarrow{J_Y^*} & Y^* \\ & & & \searrow^{B^*} & \uparrow{\scriptstyle i_Y^{***}} & & \uparrow{\scriptstyle i_Y^*} \\ & & & & X^{***} & \xrightarrow{J_X^*} & X^* \end{array}$$

Let $R = J_X^* B^* J_{L_1}\colon L_1 \to X^*$. Clearly, $\|R\| \leqslant \|B\| = \|A\| \leqslant c^{-1}$. Moreover, it follows from chasing the above diagram as well as from (3.1) that

$$\|Rf\| \geqslant \|i_Y^* J_X^* B^* J_{L_1} f\| = \|J_Y^* i_Y^{***} B^* J_{L_1} f\| = \|J_Y^* A^{**} J_{L_1} f\| \geqslant \|f\| \quad (f \in L_1).$$

Consequently, $\alpha_{L_1}(X^*) \geqslant c$. Since c was arbitrary, the proof of the inequality is complete.

Step 2. $\alpha_{L_1}(X^*) \leqslant \alpha_{\ell_1}(X)$.

Let $0 < c < \alpha_{L_1}(X^*)$ and $\varepsilon > 0$. Then there is an operator $T\colon L_1 \to X^*$ with $\|T\| \leqslant 1$ so that $\|Tf\| \geqslant c\|f\|$ $(f \in L_1)$. We set $Y = TL_1$. We may then take a sequence $(y_n)_{n=1}^\infty$ that is dense in Y and choose $z_n \in B_X$ so that $|\langle y_n, z_n\rangle| \geqslant (1-\varepsilon)\|y_n\|$ $(n \in \mathbb{N})$. Letting $Z = [z_n \colon n \in \mathbb{N}]$, one may observe that the restriction map $J\colon Y \to Z^*, y \mapsto y|_Z$ satisfies $\|Jy\| \geqslant (1-\varepsilon)\|y\|$ $(y \in Y)$. The composite operator $S = JT\colon L_1 \to Z^*$ satisfies

$$\|f\| \geqslant \|Sf\| \geqslant (1-\varepsilon)\|Tf\| \geqslant c(1-\varepsilon)\|f\| \quad (f \in L_1).$$

This means that $\alpha_{L_1}(Z^*) \geqslant c(1-\varepsilon)$. It follows from [4, Theorem 1.1] that

$$\gamma_{C(\Delta)}(Z) = \alpha_{L_1}(Z^*) \geqslant c(1-\varepsilon).$$

We take an operator $R\colon Z \to C(\Delta)$ with $\|R\| \leqslant 1$ so that $RB_Z \supseteq c(1-\varepsilon)B_{C(\Delta)}$ and an isometric embedding $U\colon \ell_1 \to C(\Delta)$. For each $n \in \mathbb{N}$, pick $x_n \in B_Z$ so that $Rx_n = c(1-\varepsilon)Ue_n^*$, where $(e_n^*)_{n=1}^\infty$ is the unit vector basis of ℓ_1. It is easy to check that

$$c(1-\varepsilon)\sum_{k=1}^n |a_k| \leqslant \|\sum_{k=1}^n a_k x_k\| \qquad (n \in \mathbb{N}, a_1, \ldots, a_n \in \mathbb{R}).$$

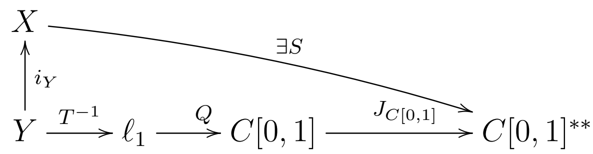
Finally, if we define $A\colon \ell_1 \to X$ by assigning $e_n \mapsto x_n$ $(n \in \mathbb{N})$ and extend linearly to the linear span and then, by density, to the whole of ℓ_1, then

$$c(1-\varepsilon)\|z\| \leqslant \|Az\| \leqslant \|z\| \quad (z \in \ell_1).$$

Consequently, $\alpha_{\ell_1}(X) \geqslant \|A^{-1}\|^{-1} \geqslant c(1-\varepsilon)$. Letting $\varepsilon \to 0$, we arrive at $\alpha_{\ell_1}(X) \geqslant c$. As c is arbitrary, the proof of Step 2 is complete.

Step 3. $\alpha_{\ell_1}(X) \leqslant \alpha_{C[0,1]^*}(X^*)$.

Let $0 < c < \alpha_{\ell_1}(X)$. There exists a contractive operator $T\colon \ell_1 \to X$ that is bounded below by c, that is, $c\|z\| \leqslant \|Tz\|$ $(z \in \ell_1)$. Set $Y = T[\ell_1]$. Take a quotient map $Q\colon \ell_1 \to C[0,1]$. Since $C[0,1]^{**}$ is 1-injective, we get an operator $S\colon X \to C[0,1]^{**}$ so that $\|S\| = \|J_{C[0,1]}QT^{-1}\|$ and $S|_Y = J_{C[0,1]}QT^{-1}$. Let us summarise this in the diagram:



Let us consider the composite map $A = S^*J_{C[0,1]^*}\colon C[0,1]^* \to X^*$ and fix $\varepsilon > 0$. For each $\mu \in C[0,1]^*$, we get

$$\|A\mu\| \geqslant \sup_{y \in B_Y} |\langle J_{C[0,1]}QT^{-1}y, \mu\rangle| \geqslant \sup_{z \in B_{\ell_1}} |\langle \mu, Qz\rangle| \geqslant \frac{1}{1+\varepsilon} \sup_{f \in B_{C[0,1]}} |\langle \mu, f\rangle| = \frac{\|\mu\|}{1+\varepsilon}.$$

Moreover, it is easy to see that $\|A\| \leqslant c^{-1}$. Hence we arrive at

$$\frac{1}{1+\varepsilon}\|\mu\| \leqslant \|A\mu\| \leqslant c^{-1}\|\mu\| \quad (\mu \in C[0,1]^*).$$

This implies $\alpha_{C[0,1]^*}(X^*) \geqslant c/(1+\varepsilon)$. The arbitrariness of c and ε completes the proof of Step 3.

Step 4. $\alpha_{C[0,1]^*}(X^*) \leqslant \alpha_{L_1}(X^*)$.

This is trivial since L_1 can be isometrically embedded into $C[0,1]^*$.

Step 5. $\alpha_{\ell_1([0,1])}(X^*) \leqslant \alpha_{\ell_1}(X)$ if X is separable.

Let $0 < c < \alpha_{\ell_1([0,1])}(X^*)$. Similarly as before, we may take a contractive operator $T\colon \ell_1([0,1]) \to X^*$ so that $\|Tf\| \geqslant c\|f\|$ $(f \in \ell_1([0,1]))$. Set $K = (T^*J_X)[B_X]$. Then K is separable, bounded, convex, and $\overline{K}^{w^*} \supseteq cB_{\ell_\infty([0,1])}$, *i.e.*, the weak* closure of $c^{-1}K$ contains the unit ball of $\ell_\infty([0,1])$. In order to complete the proof of Step 5, we require to make two claims that are slight modifications of Hagler's results from [6].

Claim 1. Let C be a separable, bounded and convex subset of $\ell_\infty([0,1])$ whose weak* closure contains $B_{\ell_\infty([0,1])}$. Let $\Gamma_1, \ldots, \Gamma_n, \Gamma_{n+1}, \ldots, \Gamma_{n+m}$ be pairwise disjoint subsets of $[0,1]$ with cardinality $\mathfrak{c}$, the continuum. Then, for every $0 < \varepsilon < 1$, there exists $f \in C$ such that for all $i = 1, 2, \ldots, n$ one has

$$\operatorname{card}\{\gamma \in \Gamma_i\colon f(\gamma) \geqslant 1-\varepsilon\} = \mathfrak{c},$$

whereas for all $i = n+1, \ldots, n+m$

$$\operatorname{card}\{\gamma \in \Gamma_i\colon f(\gamma) \leqslant \varepsilon - 1\} = \mathfrak{c}.$$

Claim 2. Let C be a separable, bounded, and convex subset of $\ell_\infty([0,1])$ whose weak* closure contains $B_{\ell_\infty([0,1])}$. Given $m \geqslant 1$ and a finite collection $\Gamma_0, \Gamma_1, \ldots, \Gamma_{2^m-1}$ of pairwise disjoint subsets of $[0,1]$ each having the cardinality $\mathfrak{c}$, for every $0 < \varepsilon < 1$, there exists $f \in C$ so that for every $i = 0, \ldots, 2^m - 1$

$$\operatorname{card}\{\gamma \in \Gamma_i\colon (-1)^i f(\gamma) \geqslant 1-\varepsilon\} = \mathfrak{c}.$$

Indeed, one may define

$$\begin{aligned} \Gamma'_i &= \Gamma_{2i}, & 0 \leqslant i \leqslant 2^{m-1} - 1 \\ \Gamma'_{2^{m-1}+i} &= \Gamma_{2i+1}, & 0 \leqslant i \leqslant 2^{m-1} - 1. \end{aligned}$$

Then it readily follows from Claim 1 that there is $f \in C$ so that for all $0 \leqslant i \leqslant 2^{m-1} - 1$

$$\begin{aligned} &\operatorname{card}\{\gamma \in \Gamma'_i\colon f(\gamma) \geqslant 1-\varepsilon\} = \mathfrak{c} \\ &\operatorname{card}\{\gamma \in \Gamma'_{2^{m-1}+i}\colon f(\gamma) \leqslant \varepsilon - 1\} = \mathfrak{c}. \end{aligned}$$

In other words, $\operatorname{card}\{\gamma \in \Gamma_k : (-1)^k f(\gamma) \geqslant 1-\varepsilon\} = \mathfrak{c}$ for all $0 \leqslant k \leqslant 2^m - 1$, so the claim is justified.

Let $0 < \varepsilon < 1$. By [15, Propositon 2.2], there exists a sequence $(f_n)_{n=1}^{\infty}$ in $c^{-1}K$ so that

$$(1-\varepsilon)\sum_{n=1}^{m}|a_n| \leqslant \|\sum_{n=1}^{m}a_nf_n\| \leqslant \frac{1}{c}\sum_{n=1}^{m}|a_n| \tag{3.2}$$

for all m and all scalars $a_1, a_2, \ldots, a_m$. For each n, pick $x_n \in B_X$ so that $cf_n = T^*J_Xx_n$. It follows from (3.2) that

$$c(1-\varepsilon)\sum_{n=1}^{m}|a_n| \leqslant \|\sum_{n=1}^{m}a_nx_n\| \leqslant \sum_{n=1}^{m}|a_n| \tag{3.3}$$

for all m and all scalars $a_1, a_2, \ldots, a_m$.

We are now in a position to define an operator $S\colon \ell_1 \to X$ by the assignment $e_n^* \mapsto x_n$. By (3.3), we have

$$\alpha_{\ell_1}(X) \geqslant \|S^{-1}\|^{-1} \geqslant c(1-\varepsilon).$$

As $0 < c < \alpha_{\ell_1}(X)$ and $0 < \varepsilon < 1$ were arbitrary, we proved that $\alpha_{\ell_1}(X) \geqslant \alpha_{\ell_1([0,1])}(X^*)$.

Step 6. $\alpha_{C[0,1]^*}(X^*) \leqslant \alpha_{\ell_1([0,1])}(X^*)$.

This inequality follows immediately from the elementary fact that $\ell_1([0,1])$ can be isometrically embedded into $C[0,1]^*$ via Dirac delta functionals.

Step 7. $\alpha_{L_1}(X^*) = \gamma_{C[0,1]}(X)$, whenever X is separable.

This is [4, Theorem 1.1 (b)]. □

4. Proof of Theorem B

We proceed as in the proof of Theorem A by splitting it into a number of independent steps.

Proof of Theorem B. Step 1. $\alpha_{(\bigoplus_{n=1}^{\infty}\ell_\infty^n)_{\ell_1}}(X) \leqslant \beta_{C(\Delta)^*}(X^*)$.

Since $Z = (\bigoplus_{n=1}^{\infty}\ell_\infty^{2^n})_{\ell_1}$ embeds isometrically into $(\bigoplus_{n=1}^{\infty}\ell_\infty^n)_{\ell_1}$, it suffices to prove that $\alpha_Z(X) \leqslant \beta_{C(\Delta)^*}(X^*)$. For this, let us fix $0 < c < \alpha_Z(X)$. Then there exists a contractive operator $R\colon Z \to X$ that is bounded below by c.

Let us consider a double-indexed family $(\Delta_{n,i})_{n=0,i=0}^{\infty,2^n-1}$ of clopen subsets of the Cantor set such that

1. $\Delta_{0,0} = \Delta$, $\Delta_{n,i} = \Delta_{n+1,2i} \cup \Delta_{n+1,2i+1}$ $((n,i) \in \mathcal{F})$ and $\Delta_{n,i} \cap \Delta_{n,j} = \varnothing$ if $i \neq j$;
2. the diameter of $\Delta_{n,i}$ is $1/2^n$ $(0 \leqslant i \leqslant 2^n - 1)$.

We set $g_{n,i} = \mathbb{1}_{\Delta_{n,i}}$, which is a continuous function, $[g_{n,i}]_{i=0}^{2^n-1} \subseteq [g_{n+1,i}]_{i=0}^{2^{n+1}-1}$, $(g_{n,i})_{i=0}^{2^n-1}$ is isometrically equivalent to the unit vector basis of $\ell_\infty^{2^n}$ for all n and $\bigcup_{n=0}^{\infty}[g_{n,i}]_{i=0}^{2^n-1}$ is dense in $C(\Delta)$. We may then define an operator $T\colon Z \to C(\Delta)$ by the assignment $Te_{n,i} = g_{n,i}$. Clearly, $\|T\| = 1$.

Claim 1. If W is a finite-dimensional Banach space and $S\colon W \to C(\Delta)$ is an operator, then for every $\varepsilon > 0$, there exists an operator $\widehat{S}\colon W \to Z$ so that $\|\widehat{S}\| \leqslant (1+\varepsilon)\|S\|$ and $\|S - T\widehat{S}\| \leqslant \varepsilon$.

Proof of Claim 1. Let us fix an Auerbach basis $(w_k, w_k^*)_{k=1}^N$ for W ($\dim W = N$). Let $\delta > 0$ be such that $\delta N \leqslant \varepsilon\|S\|$ and $\delta N \leqslant \varepsilon$. Then, there exist a positive integer n and $(f_k)_{k=1}^N$ in $[g_{n,i}]_{i=0}^{2^n-1}$ so that $\|Sw_k - f_k\| < \delta$ $(k = 1, 2, \ldots, N)$. Write

$$f_k = \sum_{i=0}^{2^n-1} t_{k,i} g_{n,i}, \quad (k = 1, 2, \ldots, N).$$

Define an operator $\widehat{S}\colon W \to Z$ by $\widehat{S}w_k = \sum_{i=0}^{2^n-1} t_{k,i} e_{n,i}$. We *claim* that $\|\widehat{S}\| \leqslant (1+\varepsilon)\|S\|$ and $\|S - T\widehat{S}\| \leqslant \varepsilon$. Indeed, for $w = \sum_{k=1}^N a_k w_k \in W$, we have

$$\begin{aligned} \|\widehat{S}w\| &= \Big\|\sum_{k=1}^N a_k f_k\Big\| \\ &\leqslant \Big\|\sum_{k=1}^N a_k (f_k - Sw_k)\Big\| + \Big\|\sum_{k=1}^N a_k Sw_k\Big\| \\ &\leqslant (\delta N + \|S\|)\|w\| \\ &\leqslant (1+\varepsilon)\|S\|\|w\|. \end{aligned}$$

Furthermore,

$$\begin{aligned} \|Sw - T\widehat{S}w\| &= \Big\|\sum_{k=1}^N a_k \Big(Sw_k - \sum_{i=0}^{2^n-1} t_{k,i} g_{n,i}\Big)\Big\| \\ &= \Big\|\sum_{k=1}^N a_k (Sw_k - f_k)\Big\| \\ &\leqslant \delta N\|w\| \\ &\leqslant \varepsilon\|w\|. \end{aligned}$$

□

Let $\varepsilon > 0$. Since $C(\Delta)$ has the metric approximation property, there exists a net $(T_\alpha)_\alpha$ of finite-rank operators on $C(\Delta)$ such that

- $\limsup_\alpha \|T_\alpha\| \leqslant 1 + \varepsilon$,
- $\dim T_\alpha(C(\Delta)) \to \infty$,
- $T_\alpha \to I_{C(\Delta)}$ strongly.

For each α, we may apply Claim 1 to the inclusion map $I_\alpha\colon T_\alpha[C(\Delta)] \to C(\Delta)$ to get an operator $\widehat{I_\alpha}\colon T_\alpha[C(\Delta)] \to Z$ so that $\|\widehat{I_\alpha}\| \leqslant 1+\varepsilon$ and $\|I_\alpha - T\widehat{I_\alpha}\| \leqslant (1+\dim T_\alpha[C(\Delta)])^{-2}$. Let S be a $\sigma(\mathcal{B}(Z^*, C(\Delta)^*), Z^*\widehat{\otimes}_\pi C(\Delta))$-cluster point of the net $((\widehat{I_\alpha}T_\alpha)^*)_\alpha$. A standard argument shows that $ST^* = I_{C(\Delta)^*}$.

Claim 2. There exists an operator $\widetilde{T}\colon C(\Delta)^* \to X^*$ so that $R^*\widetilde{T} = T^*$ and $\|\widetilde{T}\| \leqslant (1+\varepsilon)/c$.

The proof of the claim is a variation of the Lindenstrauss' compactness argument (see [9, Proposition 1] and [12, Lemma 2]). Since some amendments are required, we present the full reasoning.

Proof of Claim 2. We use the fact that $C(\Delta)^*$ is isometric to $L_1(\mu)$ for some infinite measure μ, and as such, it is a $\mathcal{L}_{1,1+}$-space. Let Λ be the collection of all finite-dimensional subspaces of $C(\Delta)^*$. Then, for each $\gamma \in \Lambda$ there exist $E_\gamma \in \Lambda$ with $\gamma \subseteq E_\gamma$ together with an isomorphism $U_\gamma \colon \ell_1^{\dim E_\gamma} \to E_\gamma$ so that $\|U_\gamma\|\|U_\gamma^{-1}\| \leqslant 1+\varepsilon$. Let $S_\gamma \colon Z \to E_\gamma^*$ be an operator such that $S_\gamma^* = T^*|_{E_\gamma}$ ($\gamma \in \Lambda$). By the 1-injectivity of $\ell_\infty^{\dim E_\gamma}$, there is an operator $R_\gamma \colon X \to \ell_\infty^{\dim E_\gamma}$ so that $R_\gamma R = U_\gamma^* S_\gamma$ and $\|R_\gamma\| \leqslant \|U_\gamma^* S_\gamma\|\|R^{-1}\| \leqslant \|U_\gamma\|\|T\|\|R^{-1}\|$. Let $T_\gamma = R_\gamma^* U_\gamma^{-1} \colon E_\gamma \to X^*$. Then $R^* T_\gamma = T^*|_{E_\gamma}$ and $\|T_\gamma\| \leqslant \frac{1+\varepsilon}{c}\|T\|$. For each γ, we define a non-linear, discontinuous function from $C(\Delta)^*$ to X^* by

$$\widetilde{T_\gamma} f = \begin{cases} T_\gamma f, & f \in E_\gamma \\ 0, & \text{otherwise.} \end{cases}$$

Then $(\widetilde{T_\gamma})_\gamma$ is a net in the compact space

$$\prod_{f \in C(\Delta)^*} \frac{1+\varepsilon}{c}\|T\|\|f\| B_{X^*}.$$

and as such, it has a cluster point $\widetilde{T}$. Standard arguments show that $\widetilde{T}$ is linear, $R^*\widetilde{T} = T^*$ and $\|\widetilde{T}\| \leqslant \frac{1+\varepsilon}{c}\|T\| = \frac{1+\varepsilon}{c}$. □

Finally, we get $SR^*\widetilde{T} = ST^* = I_{C(\Delta)^*}$ and hence

$$\beta_{C(\Delta)^*}(X^*) \geqslant (\|\widetilde{T}\|\|SR^*\|)^{-1} \geqslant \frac{c}{(1+\varepsilon)^3}.$$

Letting $\varepsilon \to 0$, we get $\beta_{C(\Delta)^*}(X^*) \geqslant c$. As c is arbitrary, we get Step 1.

Step 2. $\beta_{C[0,1]^*}(X^*) \leqslant \beta_{L_1}(X^*)$.

It is well known that L_1 is isometric to a 1-complemented subspace of $C[0,1]^*$ (see, *e.g.*, [1, p. 85]), which implies Step 2.

Step 3. $\beta_{L_1}(X^*) \leqslant \alpha_{(\bigoplus_{n=1}^\infty \ell_\infty^n)_{\ell_1}}(X)$.

Let $0 < c < \beta_{L_1}(X^*)$. Then there exist operators $A \colon L_1 \to X^*, B \colon X^* \to L_1$ so that $BA = I_{L_1}$, $\|A\| = 1$ and $\|B\| < 1/c$. Let $0 < \varepsilon < 1$ and $\varepsilon_n = \varepsilon/2^{2n+3}$ ($n = 0, 1, \ldots$).

By [8, Lemma 3], we get $(f_{n,i})_{(n,i)\in\mathcal{F}}$ in L_∞ and $(x_{n,i})_{(n,i)\in\mathcal{F}}$ in X satisfying

(1) $\|f_{n,i}\|_1 = 1$ and $f_{n,i} \geqslant 0$ everywhere for all $(n,i) \in \mathcal{F}$;

(2) For each n and $i \neq j$, $f_{n,i}(t)$ and $f_{n,j}(t)$ cannot be both non-zero for the same $t \in [0,1]$;

(3)
$$\langle Af_{n,i}, x_{m,j}\rangle = \begin{cases} 1, & (n,i) \geqslant (m,j), \\ 0, & \text{otherwise;} \end{cases}$$

(4) $\max_{0\leqslant i\leqslant 2^n-1} |t_i| \leqslant \|\sum_{i=0}^{2^n-1} t_i x_{n,i}\| \leqslant (1+\varepsilon_n)\cdot c^{-1} \max_{0\leqslant i\leqslant 2^n-1} |t_i|$ ($n = 0, 1, \cdots$; $t_0, \ldots, t_{2^n-1} \in \mathbb{R}$).

We may now define recursively a sequence $(W_{n,i})_{(n,i)\in\mathcal{F}}$ of non-empty weak*-closed subsets of B_{X^*} as follows:

- $W_{0,0} = \{x^* \in B_{X^*} \colon |\langle x^*, x_{0,0}\rangle - 1| \leqslant \varepsilon_0\}$,
- $W_{1,0} = W_{0,0} \cap \{x^* \in B_{X^*} \colon |\langle x^*, x_{1,0}\rangle - 1| \leqslant \varepsilon_1, |\langle x^*, x_{1,1}\rangle| \leqslant \varepsilon_1\}$,
- $W_{1,1} = W_{0,0} \cap \{x^* \in B_{X^*} \colon |\langle x^*, x_{1,1}\rangle - 1| \leqslant \varepsilon_1, |\langle x^*, x_{1,0}\rangle| \leqslant \varepsilon_1\}$,
- $W_{2,0} = W_{1,0} \cap \{x^* \in B_{X^*} \colon |\langle x^*, x_{2,0}\rangle - 1| \leqslant \varepsilon_2, |\langle x^*, x_{2,j}\rangle| \leqslant \varepsilon_2, j = 1, 2, 3\}$,
- $W_{2,1} = W_{1,0} \cap \{x^* \in B_{X^*} \colon |\langle x^*, x_{2,1}\rangle - 1| \leqslant \varepsilon_2, |\langle x^*, x_{2,j}\rangle| \leqslant \varepsilon_2, j = 0, 2, 3\}$,
- $W_{2,2} = W_{1,1} \cap \{x^* \in B_{X^*} \colon |\langle x^*, x_{2,2}\rangle - 1| \leqslant \varepsilon_2, |\langle x^*, x_{2,j}\rangle| \leqslant \varepsilon_2, j = 0, 1, 3\}$,
- $W_{2,3} = W_{1,1} \cap \{x^* \in B_{X^*} \colon |\langle x^*, x_{2,3}\rangle - 1| \leqslant \varepsilon_2, |\langle x^*, x_{2,j}\rangle| \leqslant \varepsilon_2, j = 0, 1, 2\}$,

and so on. By (3), each $W_{n,i}$ is non-empty. By the choice of ε_n, the sets $W_{n,i}, W_{n,j}$ are disjoint as long as $i \neq j$. Let

$$K = \bigcap_{n=0}^{\infty} \Big(\bigcup_{i=0}^{2^n-1} W_{n,i} \Big) \quad \text{and} \quad K_{n,i} = W_{n,i} \cap K \ \ \big((n,i) \in \mathcal{F}\big).$$

By (3), $Af_{n,i} \in W_{m,j}$ if $(n,i) \geqslant (m,j)$, which implies that each $K_{n,i}$ is non-empty. By the construction of the sequence $(W_{n,i})$, we see that $K_{0,0} = K, K_{n+1,2i} \cup K_{n+1,2i+1} = K_{n,i}$ and $K_{n,i} \cap K_{n,j} = \varnothing$ if $i \neq j$.

Let us define an operator $T \colon X \to C(K)$ by $\langle Tx, x^*\rangle = \langle x^*, x\rangle$ $(x \in X, x^* \in K)$. Then $|\langle Tx_{n,i}, x^*\rangle - 1| \leqslant \varepsilon_n$ if $x^* \in K_{n,i}$, and $|\langle Tx_{n,i}, x^*\rangle| \leqslant \varepsilon_n$ if $x^* \in \bigcup_{j\neq i} K_{n,j}$. Set $g_{n,i} = \mathbb{1}_{K_{n,i}}$, which is continuous as $K_{n,i}$ is clopen. Then $\|Tx_{n,i} - g_{n,i}\| \leqslant \varepsilon_n$. Moreover, $[g_{n,i}]_{i=0}^{2^n-1} \subseteq [g_{n+1,i}]_{i=0}^{2^{n+1}-1}$, $(g_{n,i})_{i=0}^{2^n-1}$ is isometrically equivalent to the unit vector basis of $\ell_\infty^{2^n}$ for all n, and

$$[g_{n,i} \colon (n,i) \in \mathcal{F}] = \overline{\bigcup_{n=0}^{\infty} [g_{n,i}]_{i=0}^{2^n-1}}$$

is isometric to $C(\Delta)$. Let Z be a subspace of $C(\Delta)$ isometric to $(\bigoplus_{n=1}^\infty \ell_\infty^n)_{\ell_1}$ and let $(z_{n,j})_{n=1,j=0}^{\infty,n-1}$ be a basis of Z isometrically equivalent to the unit vector basis of $(\bigoplus_{n=1}^\infty \ell_\infty^n)_{\ell_1}$. Fix $n \geqslant 1$. Then there exist $m > n$ and unit vecors $h_{n,j} \in [g_{m,i}]_{i=0}^{2^m-1}$ so that $\|z_{n,j} - h_{n,j}\| \leqslant \varepsilon/2^{n+3}$ $(j = 0, 1, \ldots, n-1)$. We write $h_{n,j} = \sum_{i=0}^{2^m-1} a_{i,j} g_{m,i}$ and define $y_{n,j} = \sum_{i=0}^{2^m-1} a_{i,j} x_{m,i} \in X$.

Claim. For all $(t_{n,j})_{n=1,j=0}^{\infty,n-1} \in (\bigoplus_{n=1}^\infty \ell_\infty^n)_{\ell_1}$ we have

$$(1 - \frac{\varepsilon}{2}) \sum_{n=1}^{\infty} \max_{0\leqslant j\leqslant n-1} |t_{n,j}| \leqslant \| \sum_{n=1}^{\infty} \sum_{j=0}^{n-1} t_{n,j} y_{n,j} \| \leqslant \frac{(1+\varepsilon)^2}{c} \sum_{n=1}^{\infty} \max_{0\leqslant j\leqslant n-1} |t_{n,j}|.$$

Indeed, by (4) we get

$$\begin{aligned} \| \sum_{j=0}^{n-1} t_{n,j} y_{n,j} \| &= \| \sum_{i=0}^{2^m-1} \Big(\sum_{j=0}^{n-1} a_{i,j} t_{n,j} \Big) x_{m,i} \| \\ &\leqslant \frac{1+\varepsilon_m}{c} \max_{0\leqslant i\leqslant 2^m-1} | \sum_{j=0}^{n-1} a_{i,j} t_{n,j} | \end{aligned}$$

$$
\begin{aligned}
&= \frac{1+\varepsilon_m}{c}\|\sum_{j=0}^{n-1} t_{n,j} h_{n,j}\| \\
&\leqslant \frac{1+\varepsilon_m}{c}\Big(\|\sum_{j=0}^{n-1} t_{n,j} z_{n,j}\| + \sum_{j=0}^{n-1} t_{n,j}(h_{n,j}-z_{n,j})\|\Big) \\
&\leqslant \frac{1+\varepsilon_m}{c}\Big(\max_{0\leqslant j\leqslant n-1}|t_{n,j}| + n\varepsilon/2^{n+3}\max_{0\leqslant j\leqslant n-1}|t_{n,j}|\Big) \\
&\leqslant \frac{(1+\varepsilon)^2}{c}\max_{0\leqslant j\leqslant n-1}|t_{n,j}|.
\end{aligned}
$$

Consequently,

$$
\|\sum_{n=1}^{\infty}\sum_{j=0}^{n-1} t_{n,j} y_{n,j}\| \leqslant \sum_{n=1}^{\infty}\|\sum_{j=0}^{n-1} t_{n,j} y_{n,j}\| \leqslant \frac{(1+\varepsilon)^2}{c}\sum_{n=1}^{\infty}\max_{0\leqslant j\leqslant n-1}|t_{n,j}|.
$$

On the other hand, by the choice of m and $h_{n,j}$, we arrive at

$$
\begin{aligned}
\|Ty_{n,j}-z_{n,j}\| &\leqslant \|Ty_{n,j}-h_{n,j}\| + \|h_{n,j}-z_{n,j}\| \\
&= \|\sum_{i=0}^{2^m-1} a_{i,j}(Tx_{m,i}-g_{m,i})\| + \varepsilon/2^{n+3} \\
&\leqslant \varepsilon_m 2^m \max_{0\leqslant i\leqslant 2^m-1}|a_{i,j}| + \varepsilon/2^{n+3} \\
&\leqslant \varepsilon/2^{n+3} + \varepsilon/2^{n+3} = \varepsilon/2^{n+2}.
\end{aligned}
$$

This implies

$$
\begin{aligned}
\|\sum_{n=1}^{\infty}\sum_{j=0}^{n-1} t_{n,j} y_{n,j}\| &\geqslant \|\sum_{n=1}^{\infty}\sum_{j=0}^{n-1} t_{n,j} Ty_{n,j}\| \\
&\geqslant \|\sum_{n=1}^{\infty}\sum_{j=0}^{n-1} t_{n,j} z_{n,j}\| - \|\sum_{n=1}^{\infty}\sum_{j=0}^{n-1} t_{n,j}(Ty_{n,j}-z_{n,j})\| \\
&\geqslant \sum_{n=1}^{\infty}\max_{0\leqslant j\leqslant n-1}|t_{n,j}| - \sum_{n=1}^{\infty} n\max_{0\leqslant j\leqslant n-1}|t_{n,j}|\frac{\varepsilon}{2^{n+2}} \\
&\geqslant (1-\frac{\varepsilon}{2})\sum_{n=1}^{\infty}\max_{0\leqslant j\leqslant n-1}|t_{n,j}|.
\end{aligned}
$$

Finally, by Claim, we get

$$
\alpha_{(\bigoplus_{n=1}^{\infty}\ell_\infty^n)_{\ell_1}}(X) \geqslant (1-\frac{\varepsilon}{2})\frac{c}{(1+\varepsilon)^2}.
$$

Letting $\varepsilon \to 0$ yields $\alpha_{(\bigoplus_{n=1}^{\infty}\ell_\infty^n)_{\ell_1}}(X) \geqslant c$; since c was arbitrary the proof of Step 3 is complete.

Step 4. $\theta_{C(\Delta)}(X) \leqslant \beta_{L_1}(X^*)$.

This step follows from (1.1) together with Step 2. We are now ready to establish the final step of the proof.

Step 5. Suppose that X is separable. Then $\beta_{L_1}(X^*) \leqslant \theta_{C(\Delta)}(X)$.

Let $0 < c < \beta_{L_1}(X^*)$. Then there exist operators $A\colon L_1 \to X^*, B\colon X^* \to L_1$ so that $BA = I_{L_1}, \|A\| = 1$, and $\|B\| < 1/c$.

Let $(f_{n,i})_{(n,i)\in\mathcal{F}}$ be a family of functions in L_∞, $(x_{n,i})_{(n,i)\in\mathcal{F}}$ in X, and $(W_{n,i})_{(n,i)\in\mathcal{F}}$ associated to $\varepsilon_n = 1/2^{2n+2}$ $(n = 0, 1, \dots)$ as described in Step 3. Since X is separable, we may assume that the d-diameter of $W_{n,i} \leqslant 2^{-n}$ for each i, where d is a metric giving the relative $\sigma(X^*, X)$-topology on B_{X^*}. Let

$$K = \bigcap_{n=0}^{\infty}\Big(\bigcup_{i=0}^{2^n-1} W_{n,i}\Big) \quad \text{and} \quad K_{n,i} = W_{n,i} \cap K \quad \big((n,i) \in \mathcal{F}\big).$$

Then K is a compact, totally disconnected metric space without isolated points, hence homeomorphic to Δ. Moreover, $K_{0,0} = K, K_{n+1,2i} \cup K_{n+1,2i+1} = K_{n,i}$ and $K_{n,i} \cap K_{n,j} = \varnothing$ if $i \neq j$. Hence $K = \bigcup_{i=0}^{2^n-1} K_{n,i}$ for all n. As seen in Step 3, the operator $T\colon X \to C(K)$, defined by $\langle Tx, x^* \rangle = \langle x^*, x \rangle$ $(x \in X, x^* \in K)$, satisfies $\|Tx_{n,i} - g_{n,i}\| \leqslant \varepsilon_n$, where $g_{n,i} = \mathbb{1}_{K_{n,i}} \in C(K)$.

An argument analogous to Step 1 yields that, if W is a finite-dimensional Banach space and $S\colon W \to C(K)$ is an operator, then, for every $\varepsilon > 0$, there exists an operator $\widehat{S}\colon W \to X$ so that $\|\widehat{S}\| \leqslant \frac{1+\varepsilon}{c}\|S\|$ and $\|S - T\widehat{S}\| \leqslant \varepsilon$.

Fix $\varepsilon > 0$. By an argument analogous to the one from Step 1, we get an operator $S\colon X^* \to C(K)^*$ with $\|S\| \leqslant \frac{(1+\varepsilon)^2}{c}$ so that $ST^* = I_{C(K)^*}$. This means that

$$\theta_{C(\Delta)}(X) = \theta_{C(K)}(X) \geqslant \frac{c}{(1+\varepsilon)^2}.$$

Letting $\varepsilon \to 0$, we get arrive at $\theta_{C(\Delta)}(X) \geqslant c$. As c is arbitrary, the proof is complete. □

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