Perturbations of surjective algebra homomorphisms between algebras of operators on Banach spaces

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Some notation & motivation

Lemma (Corollary of Carl Neumann series)

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then ϕ is an automorphism too.

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Let $\mathcal{H}$ be a separable Hilbert space, let $\phi, \psi \colon \mathscr{B}(\mathcal{H}) \to \mathscr{B}(\mathcal{H})$ be continuous algebra homomorphisms.

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Theorem (Molnár, PAMS, 1998)

Let $\mathcal{H}$ be a separable Hilbert space, let $\phi, \psi \colon \mathscr{B}(\mathcal{H}) \to \mathscr{B}(\mathcal{H})$ be continuous algebra homomorphisms. If ψ is surjective with

$$\|\psi(A) - \phi(A)\| < \|A\| \tag{2}$$

for all non-zero $A \in \mathscr{B}(\mathcal{H})$, then ϕ is surjective too.

Motivation, the question

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- *The condition* (2) *cannot be replaced with*

$$\|\psi(A) - \phi(A)\| \leqslant \|A\| \quad (\forall A \in \mathscr{B}(\mathcal{H}) \setminus \{0\}). \tag{3}$$

Indeed, take $\psi = id_{\mathscr{B}(\mathcal{H})}$ *and* $\phi = 0$ *for a counterexample.*

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Molnár's proof relies heavily on the C^*-algebra structure of $\mathscr{B}(\mathcal{H})$ and on the geometry of $\mathcal{H}$.

The main results, I.

Theorem A (H.–Tarcsay)

Let X and Y be non-zero Banach spaces such that Y is separable and reflexive. Assume X satisfies one of the following:

1. $X = L_p[0,1]$, *where* $1 < p < \infty$; *or*
2. X *is a reflexive Banach space with a subsymmetric Schauder basis.*

Let $\psi, \phi \colon \mathscr{B}(X) \to \mathscr{B}(Y)$ be algebra homomorphisms such that ψ is surjective. If

$$\|\psi(A) - \phi(A)\| < \|A\|$$

for each non-zero $A \in \mathscr{B}(X)$, then ϕ is an isomorphism.

The main results, II.

Theorem B (H.–Tarcsay)

Let X and Y be non-zero Banach spaces such that Y is separable and reflexive. Assume X satisfies one of the following:

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2. X *is a reflexive Banach space with a subsymmetric Schauder basis.*

Let $\phi\colon \mathscr{B}(X) \to \mathscr{B}(Y)$ *be a continuous, injective algebra homomorphism. If* $\mathrm{Ran}(\phi)$ *contains an operator with dense range, and* ϕ *maps rank one idempotents into rank one idempotents, then* ϕ *is an isomorphism.*

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The study of representations of $\mathscr{B}(X)$ on separable Banach spaces goes back to the work of Berkson and Porta (*Representations of* $\mathscr{B}(X)$, JFA, '69).

Examples and non-examples

Example

Each of the following spaces is reflexive and has a subsymmetric basis, hence satisfies the conditions of Theorems A and B:

(a) The sequence spaces ℓ_p, where $1 < p < \infty$;

(b) every reflexive Orlicz sequence space l_M with Orlicz function M satisfying the Δ_2-condition $\limsup_{t\to 0} M(2t)/M(t) < \infty$;

(c) every Lorentz sequence space $d(w,p)$, where $p > 1$, $w = (w_n)_{n\in\mathbb{N}}$ is non-increasing, $w_1 = 1$, $\lim_{n\to\infty} w_n = 0$ and $\sum_{n=1}^{\infty} w_n = \infty$.

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Proposition (H.–Tarcsay)

Let X be the p^{th} James space J_p (where $1 < p < \infty$) or the Semadeni space $C[0, \omega_1]$. There is a continuous, injective algebra homomorphism $\phi \colon \mathscr{B}(X) \to \mathscr{B}(X)$ with $\phi(I_X) = I_X$ which maps rank one operators into rank one operators but ϕ is not surjective.

The proof of Theorem A, assuming Theorem B

Drop all assumptions on X and Y for now, except:

In the following, let X and Y be arbitrary non-zero Banach spaces, and let $\psi, \phi \colon \mathscr{B}(X) \to \mathscr{B}(Y)$ be algebra homomorphisms such that

$$\|\psi(A) - \phi(A)\| < \|A\|$$

for each non-zero $A \in \mathscr{B}(X)$.

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The triangle inequality yields

$$\|\psi(A)\| \leqslant \|\psi(A) - \phi(A)\| + \|\phi(A)\| < \|A\| + \|\phi(A)\|. \tag{4}$$

Similarly, $\|\phi(A)\| < \|A\| + \|\psi(A)\|$. In particular, ϕ is continuous if and only if ψ is continuous.

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Lemma (Injectivity Lemma)

Let $P \in \mathscr{B}(X)$ be a norm one idempotent. Then $P \in \mathrm{Ker}(\phi)$ if and only if $P \in \mathrm{Ker}(\psi)$. Consequently, ψ is injective if and only if ϕ is injective.

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Proof of Lemma

Assume $P \in \mathrm{Ker}(\phi)$. Then it follows from (4) that $\|\psi(P)\| < \|P\| = 1$. As $\psi(P) \in \mathscr{B}(Y)$ is an idempotent, this is equivalent to saying $\psi(P) = 0$. The other direction is analogous.

Proof of Lemma

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Proposition (A preserver result)

Let $P \in \mathscr{B}(X)$ be a norm one idempotent. Then $\mathrm{Ran}(\psi(P)) \cong \mathrm{Ran}(\phi(P))$. If ψ is surjective, then $\phi(I_X) = I_Y$. Moreover, if ψ is an isomorphism, then $\mathrm{Ran}(\phi(P)) \cong \mathrm{Ran}(P)$.

The proof of the preserver result

Fact (corollary of a result of Zemánek)

If X is a Banach space and $P, Q \in \mathscr{B}(X)$ are idempotents with $\|P - Q\| < 1$, then $\mathrm{Ran}(P) \cong \mathrm{Ran}(Q)$.

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As $\|P\| = 1$, the estimate $\|\psi(P) - \phi(P)\| < 1$ and Fact imply $\mathrm{Ran}(\psi(P)) \cong \mathrm{Ran}(\phi(P))$.

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$$\|I_Y - \phi(I_X)\| = \|\psi(I_X) - \phi(I_X)\| < 1,$$

which by the Carl Neumann series implies that $\phi(I_X)$ is invertible in $\mathscr{B}(Y)$. As $\phi(I_X)$ is an idempotent, $\phi(I_X) = I_Y$ must hold.

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which by the Carl Neumann series implies that $\phi(I_X)$ is invertible in $\mathscr{B}(Y)$. As $\phi(I_X)$ is an idempotent, $\phi(I_X) = I_Y$ must hold. Suppose ψ is an isomorphism. By Eidelheit's Theorem there is an isomorphism $S \in \mathscr{B}(X, Y)$ such that $\psi(A) = SAS^{-1}$ for each $A \in \mathscr{B}(X)$. In particular, $(SP)(PS^{-1}) = SPS^{-1} = \psi(P)$ and $(PS^{-1})(SP) = P$ imply that $\mathrm{Ran}(P) \simeq \mathrm{Ran}(\psi(P))$. By the first part of the proposition $\mathrm{Ran}(\phi(P)) \simeq \mathrm{Ran}(P)$ follows. $\square$

The proof of Theorem A, assuming Theorem B

Proposition (Johnson–Phillips–Schechtman, H.–Tarcsay)

Let X be a Banach space such that one of the following two conditions is satisfied.

1. *X has a subsymmetric Schauder basis; or*
2. *$X = L_p[0,1]$ where $1 \leqslant p < \infty$.*

Then $\mathscr{B}(X)$ admits a bounded set $\mathcal{Q}$ of commuting idempotents such that $\mathcal{Q}$ has cardinality $\mathfrak{c}$ and $\mathrm{Ran}(P) \cong X$ for every $P \in \mathcal{Q}$ and PQ is finite-rank for each distinct $P, Q \in \mathcal{Q}$.

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In particular, there is a family of subspaces $(X_i)_{i \in \Gamma}$ of X such that

- *there is $K > 0$ such that X_i is K-complemented in X $(\forall i \in \Gamma)$;*
- *$X_i \cong X$ for each $i \in \Gamma$;*
- *$X_i \cap X_j$ is finite-dimensional for all distinct $i, j \in \Gamma$;*
- *I has cardinality $\mathfrak{c}$.*

Proof idea for $X = L_p[0,1]$

Recall that $L_p[0,1]$ is isometrically isomorphic to $L_p(\{0,1\}^{\mathbb{N}}, \Lambda, \nu)$, where

$$(\{0,1\}^{\mathbb{N}}, \Lambda, \nu) := (\{0,1\}, \mathcal{P}(\{0,1\}), \mu)^{\mathbb{N}}, \quad \mu(\{0\}) = 1/2 = \mu(\{1\}).$$

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For any $S \subseteq \mathbb{N}$ let us define

$$\pi_S \colon \{0,1\}^{\mathbb{N}} \to \{0,1\}^S; \quad (x_n)_{n\in\mathbb{N}} \mapsto (x_n)_{n\in S}$$

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If S is an infinite, $L_p(\{0,1\}^{\mathbb{N}}, \Lambda_S, \nu|_{\Lambda_S})$ and $L_p[0,1]$ are isometrically isomorphic.

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$$\mathcal{Q} := \{\mathbb{E}(\cdot|\Lambda_N) \colon N \in \mathcal{D}\} \qquad \square$$

The proof of Theorem A, assuming Theorem B

Lemma (Folklore)

Let X be a Banach space and let $\mathcal{Q}$ be a bounded family of non-zero, mutually orthogonal idempotents in $\mathscr{B}(X)$. Then the density of X is at least the cardinality of $\mathcal{Q}$.

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As a corollary of the previous Proposition and Lemma, we obtain:

Corollary (Dichotomy result)

Let X be a Banach space such that one of the following two conditions is satisfied.

1. *X has a subsymmetric Schauder basis; or*
2. *$X = L_p[0,1]$ where $1 \leqslant p < \infty$.*

Let Y be a separable Banach space. Let $\theta \colon \mathscr{B}(X) \to \mathscr{B}(Y)$ be a continuous algebra homomorphism. Then θ is either injective or $\theta = 0$.

From this point on, we assume that the properties prescribed by the conditions of Theorem A hold for the Banach spaces X and Y, and $\psi \colon \mathscr{B}(X) \to \mathscr{B}(Y)$ is assumed to be surjective. We recall that due to the deep automatic continuity result of B. E. Johnson, any surjective homomorphism between algebras of operators of Banach spaces is automatically continuous.

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Proof of Theorem A.

We first observe that ψ is automatically injective. Indeed, Y is non-zero, hence ψ is non-zero, since it is surjective. By the "Dichotomy result" it follows that ψ is injective.
Thus by "Injectivity Lemma" ϕ is injective too. Continuity of ψ implies that ϕ is continuous. Furthermore, from the "Preserver result" we conclude that $\phi(I_X) = I_Y$ (which witnesses that $\mathrm{Ran}(\phi)$ contains an operator with dense range), and ϕ preserves rank one idempotents. Hence Theorem B applies. □

Ingredients for the proof of Theorem B

Recall:

Theorem B (H.–Tarcsay)

Let X and Y be non-zero Banach spaces such that Y is separable and reflexive. Assume X satisfies one of the following:

1. $X = L_p[0,1]$, *where* $1 < p < \infty$; *or*
2. X *is a reflexive Banach space with a subsymmetric Schauder basis.*

Let $\phi\colon \mathscr{B}(X) \to \mathscr{B}(Y)$ be a continuous, injective algebra homomorphism. If $\mathrm{Ran}(\phi)$ contains an operator with dense range, and ϕ maps rank one idempotents into rank one idempotents, then ϕ is an isomorphism.

Ingredients for the proof of Theorem B

Strategy

By Eidelheit's Theorem we know that if $\phi\colon \mathscr{B}(X) \to \mathscr{B}(Y)$ is a (ring) isomorphism, then there is a (Banach space) isomorphism $S\colon X \to Y$ such that

$$\phi(A) = SAS^{-1} \qquad (\forall A \in \mathscr{B}(X)).$$

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$$\phi(A) = SAS^{-1} \qquad (\forall A \in \mathscr{B}(X)).$$

In the setup of Theorem A, we will see that the operator S is of the form

$$S\colon X \to Y; \quad x \mapsto \phi(x \otimes f_0)y_0$$

for some $f_0 \in X^*$ and $y_0 \in Y$.

The proof of Theorem A

If X has a subsymmetric basis, let this be denoted by (b_n). If $X = L_p[0,1]$, where $1 < p < \infty$, then (b_n) denotes the Haar basis. In both cases (P_n) stands for the sequence of coordinate projections associated to (b_n). As X is reflexive, (P_n) is a b.a.i. for the compact operators $\mathscr{K}(X)$.

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Lemma (H.–Tarcsay, folklore(?))

Let Y be a reflexive Banach space and let (Q_n) be a bounded, monotone increasing sequence ($Q_nQ_m = Q_m = Q_mQ_n$ for $m \leqslant n$) of idempotents in $\mathscr{B}(Y)$. There exists and idempotent $Q \in \mathscr{B}(Y)$ such that (Q_n) converges to Q in the strong operator topology.

The proof of Theorem A

If X has a subsymmetric basis, let this be denoted by (b_n). If $X = L_p[0,1]$, where $1 < p < \infty$, then (b_n) denotes the Haar basis. In both cases (P_n) stands for the sequence of coordinate projections associated to (b_n). As X is reflexive, (P_n) is a b.a.i. for the compact operators $\mathscr{K}(X)$.

Lemma (H.–Tarcsay, folklore(?))

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Proof sketch

- $\mathscr{B}(Y)$ is a dual Banach algebra with predual $Y \widehat{\otimes}_\pi Y^*$;
- standard convex combination trick;
- Mazur's Theorem.

The proof of Theorem A

Since $(\phi(P_n))$ is a bounded, monotone increasing sequence of idempotents in $\mathscr{B}(Y)$ it follows from the Lemma that there exists an idempotent $P \in \mathcal{B}(Y)$ such that $(\phi(P_n))$ converges to P in the strong operator topology.

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We show that $P = I_Y$. To this end we consider the map

$$\theta \colon \mathscr{B}(X) \to \mathscr{B}(Y); \quad A \mapsto (I_Y - P)\phi(A)(I_Y - P).$$

It can be shown that the map θ is a continuous algebra homomorphism with $\theta|_{\mathscr{K}(X)} = 0$.

Back to the proof of Theorem A

Clearly θ is not injective. As Y is separable, the "Dichotomy result" implies $\theta = 0$. By the assumption, we can take $T \in \mathscr{B}(X)$ such that $\phi(T)$ has dense range. Consequently

$$0 = \theta(T) = (I_Y - P)\phi(T)(I_Y - P) = (I_Y - P)\phi(T)$$

So $(I_Y - P)|_{\mathrm{Ran}(\phi(T))} = 0$ and $\mathrm{Ran}(\phi(T))$ is dense in Y, hence $P = I_Y$.

Back to the proof of Theorem A

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So $(I_Y - P)|_{\mathrm{Ran}(\phi(T))} = 0$ and $\mathrm{Ran}(\phi(T))$ is dense in Y, hence $P = I_Y$.

Let $x_0 \in X$ be such that $\|x_0\| = 1$, and choose $f_0 \in X^*$ such that $\langle x_0, f_0 \rangle = 1 = \|f_0\|$. As ϕ is injective, we can pick $y_0 \in Y^*$ with $\|y_0\| = 1$ such that $\phi(x_0 \otimes f_0)y_0 \neq 0$. Thus we can define the non-zero map

$$S \colon X \to Y; \quad x \mapsto \phi(x \otimes f_0)y_0$$

which is easily seen to be linear and bounded. It can be shown that

$$SA = \phi(A)S \qquad (\forall A \in \mathscr{B}(X)). \tag{5}$$

It remains to show that S is an isomorphism.

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   - the injectivity of S; and
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    - that ϕ maps rank one idempotents to rank one idempotents;
    - that $(\phi(P_n))$ converges to I_Y in the strong operator topology; and
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   - the injectivity of S.

Thus S is invertible, hence

$$\phi(A) = SAS^{-1} \qquad (\forall A \in \mathscr{B}(X)). \tag{6}$$

as claimed. □

Almost done...

Something that's not in the paper:

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- $X = \mathcal{X}_p$, *($2 < p < \infty$) the complemented subspace of $L_p[0,1]$ which is not isomorphic to ℓ_2, ℓ_p, $\ell_2 \oplus \ell_p$ or $L_p[0,1]$. Proof uses recent results of Johnson–Phillips–Schechtman;*

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Remark

The conclusion of Theorems A and B holds for the following Banach spaces X:

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- $X = T$, *the Tsirelson space. (Health warning: details need to be checked. Joint with N. J. Laustsen.)*

OK, the very last slide, really

Thank you for your attention :)

Sources

- L. Molnár, "Stability of the surjectivity of endomorphisms and isometries of $\mathscr{B}(\mathcal{H})$", PAMS (1998);
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