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Mathematical analysis of fluids in motion

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Title: Mathematical analysis of fluids in motion

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Abstract: The aim of this work is to provide new results of global existence for different evolution equations of fluid mechanics. We are in general interested in finding weak solutions without restrictions on the size of initial data. The proofs of existence are based on several different approaches including energy methods, convergence analysis of finite numerical methods and convex integration. All these techniques significantly exploit results of mathematical analysis and other branches of mathematics.

Keywords: compressible Navier–Stokes equations, weak solutions, energy methods, convergent numerical schemes, convex integration

Dedicated to...

...my fiancée Aňa, because with her everything makes sense,

...my parents Renáta and Josef, for their eternal love and the endless freedom they gave me when I had decided to become a mathematician,

...my amazing brother Tomáš,

...the rest of my family.

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Contents

Introduction

The mathematical theory of the nonlinear evolutionary partial differential equations describing fluids in motion enjoys years of fruitful development. We restrict our attention to the question of the existence of solutions *in dimension three without restrictions on the time interval and size of the initial data*. It is well known that the problem of existence of smooth global solutions for Navier–Stokes–like models is far from being solved. For that reason, we provide only global existence results for weak solutions. The techniques which are used in this thesis are reflecting some recent trends in the field, particularly:

- existence results based on energy and weak convergence methods,
- convergence of numerical schemes to weak solutions and
- applications of convex integration on equations of fluid mechanics.

The core of this work is presented in Chapters 3–6 and consists of four articles [15], [35], [66] and [68] written by the author or with a contribution of the author. They contain new results, which are interesting[1] for the the recent development in the field. The mentioned articles are introduced in their original versions. The rest of the thesis plays only a supporting role.

The outline of the thesis is the following.

- The principal object of studies in Chapters 3–5 are two different models for compressible fluids, namely the *compressible Navier–Stokes–Fourier system* and the *compressible Navier–Stokes system with entropy transport*. In Chapter 1, we derive these models from the general system of balance equations of fluid thermodynamics. As a priori estimates provide a useful piece of heuristics in the theory of weak solutions weak theory, we also present their derivation for the former model.[2] The rest of Chapter 1 motivates the characteristic aspects of the theory of weak solutions for the compressible Navier–Stokes equations applied to the system from Chapter 5. These aspects are playing an inevitable role in Chapters 3–5; therefore, their condensed summary might be helpful in the forthcoming parts of the thesis.[3] A discussion of the role of the adiabatic coefficient is also included. It should be mentioned that the content of Chapter 1 is not original. The presentation was somewhat inspired by book [32] and this reference also covers the topic to the last detail.

- Chapter 2 contains comments to the presented articles. Here we mention also their significance and point out the most important steps leading to them.

- Chapters 3–6 contain the original articles. Each chapter can be read independently.

[1] They were already published in the first quartile mathematical journals.
[2] In fact, obtaining a priori estimates for the latter model is less demanding
[3] At least this was the only motivation of the author to include them.

- In Conclusion, we give some possible extensions of the content of the thesis.

We note that the terminology and notation we use are standard for the theory of partial differential equations.

1. Relevant fluid models and review of analytical techniques

There are a few approaches how to describe the evolution of fluids. The mathematical results of the thesis were obtained for models coming from continuum mechanics. We assume that the fluid at each time $t \in (0,T)$ occupies a fixed domain $\Omega \subseteq \mathbb{R}^3$ on which the physical quantities like the vector field describing the motion of the fluid, density, pressure or the temperature are defined pointwise. We parametrize $(0,T) \times \Omega$ by the Euclidean coordinates (t,x) (also known as the Eulerian coordinates). Let us consider that Ω has a Lipschitz boundary. We assume that the motion of the fluid is described by the *state variables* $\varrho\colon [0,T) \times \Omega \to [0,\infty)$ (*density*), $\boldsymbol{u}\colon [0,T) \times \Omega \to \mathbb{R}^3$ (*velocity field*) and $\vartheta\colon [0,T) \times \Omega \to [0,+\infty)$ (*temperature*).

As the starting point for our considerations, we take the balance equations of fluid thermodynamics.[1] These consist of:

- the balance of mass (or the *continuity equation*)

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0 \quad \text{in } (0,T) \times \Omega, \tag{1.1a}$$

  which corresponds to the principle of conservation of mass.

- the balance of linear momentum (or the *momentum equation*)

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) = \mathrm{div}_x(\mathbb{S} - p\mathbb{I}) + \varrho \mathbf{f} \quad \text{in } (0,T) \times \Omega \tag{1.1b}$$

  with a function $p = p(\varrho, \vartheta)$, a matrix valued function[2] $\mathbb{S} = \mathbb{S}(\boldsymbol{u})$ (*viscous stress tensor*) and a given vector valued function $\mathbf{f}\colon [0,T) \times \Omega \to \mathbb{R}^3$ (*external forces*).

- the balance of internal energy (or the *internal energy equation*)

$$\partial_t(\varrho e) + \mathrm{div}_x(\varrho e \mathbf{u}) + \mathrm{div}_x(\mathbf{q}) = (\mathbb{S} - p\mathbb{I}) : \nabla_x \mathbf{u} + \varrho Q \quad \text{in } (0,T) \times \Omega \tag{1.1c}$$

  with a scalar function $e = e(\varrho, \vartheta)$ (*internal energy*), a vector field $\mathbf{q} = \mathbf{q}(\vartheta)$ (*heat flux*) and a given vector valued function $Q\colon [0,T) \times \Omega \to \mathbb{R}^3$ (the *external heat sources*).

For the simplicity, we assume that $\mathbf{f} = (0,0,0)^T$ and $Q = 0$.

The evolution is prescribed for the quantities ϱ, $\varrho \boldsymbol{u}$ and ϱe, so it makes sense to consider the following initial data:

$$\varrho(0,x) = \varrho_0(x), \quad ,(\varrho \boldsymbol{u})(0,x) = \mathbf{m}_0(x) \quad (\varrho e)(0,x) = r_0(x) \quad \text{in } \Omega.$$

To describe the interaction of the fluid with its surroundings, we supplement the system by a set of boundary conditions. The particular choice corresponding to the content of the thesis is the following:

$$\boldsymbol{u} = 0 \quad \text{and} \quad \nabla_x \vartheta \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega$$

[1] They are presented in the "classical" pointwise forms. Another possibility is to used s.c. Reynold's formulation which is more reasonable in the context of weak solutions. We refer to [8, Appendix A] for this treatment.

[2] The dependence only on $\boldsymbol{u}$ will be commented later.

where $\mathbf{n}(x)$ denotes the outer normal vector to Ω at $x \in \partial\Omega$.

Observe also that (at least formally) by multiplying (1.1b) by $\boldsymbol{u}$ and combining the result with (1.1a), we receive the *balance of kinetic energy*[3]

$$\partial_t \left(\frac{1}{2} \varrho |\mathbf{u}|^2 \right) + \operatorname{div}_x \left(\frac{1}{2} \varrho |\mathbf{u}|^2 \mathbf{u} + p\boldsymbol{u} - \mathbb{S}\boldsymbol{u} \right) = -(\mathbb{S} - p\mathbb{I}) : \nabla_x \boldsymbol{u}. \tag{1.2}$$

Taking the sum of the kinetic and internal energy balance, we obtain *the total energy conservation*

$$\partial_t \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e \right) + \operatorname{div}_x \left\{ \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e + p \right) \mathbf{u} - \mathbb{S}\boldsymbol{u} + \mathbf{q} \right\} = 0. \tag{1.3}$$

There are many classical references on the derivation of the system with a more detailed presentation of the quantities and relations appearing in (1.1). We would like to mention e.g. [3], [18], [44] or [75].

1.1 Derivation of the models for Chapters 3–5

To close system (1.1), we consider the following *constitutive relations* for the rest of the quantities:

- *Newton's law of viscosity*:

$$\mathbb{S} = \mathbb{S}(\nabla_x \boldsymbol{u}) = \mu \left(\nabla_x \boldsymbol{u} + (\nabla_x \boldsymbol{u})^T - \frac{2}{3} \operatorname{div}_x \boldsymbol{u} \mathbb{I} \right) + \eta \operatorname{div}_x \boldsymbol{u} \mathbb{I}.$$

  with constants μ and η satisfying $\mu > 0$ and $\eta \geq 0$.[4] The following form is also sometimes used:

$$\mathbb{S}(\nabla_x \boldsymbol{u}) = \mu \left(\nabla_x \boldsymbol{u} + (\nabla_x \boldsymbol{u})^T \right) + \lambda \operatorname{div}_x \boldsymbol{u} \mathbb{I}$$

  along with $\lambda + \frac{2}{3}\mu \geq 0$.

- *Fourier's law*

$$\mathbf{q} = -\kappa(\vartheta) \nabla_x \vartheta$$

  where κ is a smooth function satisfying

$$k_1(1 + \vartheta^2) \leq \kappa(\vartheta) \leq k_2(1 + \vartheta^2) \quad \text{for all } \vartheta \in [0, \infty). \tag{1.4}$$

- *Gibbs's relation*: there exists a function $s = s(\varrho, \vartheta)$ (called *entropy*) such that

$$De = \vartheta Ds + p \frac{D\varrho}{\varrho^2}, \tag{1.5}$$

[3]It is well known that this *testing* by the velocity $\boldsymbol{u}$ is impossible for the weak solutions of the momentum equation when Ω is three–dimensional.

[4]It should be noted that the coefficients should also depend on the temperature. However, the situation when μ depends on ϑ remains an open in the theory of global weak solutions for compressible fluids (except some particular models - see e.g. [42])

where D stands for the differential with respect to variables ϱ and ϑ. Let us make some observations coming directly from the presented assumptions.

By assuming that e, p and s are smooth functions of ϱ and ϑ, we obtain from $\partial^2_{\varrho,\vartheta} e = \partial^2_{\vartheta,\varrho} e$ and (1.5) the relation

$$\partial_\varrho s + \frac{\partial_\vartheta p}{\varrho^2} = 0. \tag{1.6}$$

By the means of (1.5), one can derive from the internal energy *the balance of entropy*:

$$\partial_t(\varrho s) + \operatorname{div}(\varrho s \boldsymbol{u}) + \operatorname{div}_x\left(-\kappa(\vartheta)\frac{\nabla_x \vartheta}{\vartheta}\right) = \frac{\mathbb{S} : \nabla_x \boldsymbol{u}}{\vartheta} + \frac{\kappa(\vartheta)|\nabla_x \vartheta|^2}{\vartheta^2}. \tag{1.7}$$

Finally, we have to specify the pressure and the internal energy (or the entropy). This is the place where the two models considered in Chapters 3–5 use different assumptions and we present them separately in Subsection 1.1.1 and 1.1.2.

1.1.1 Compressible Navier–Stokes with entropy transport

Let us start with the well known model for the *perfect gas*. It is given by Boyle's law:

$$p = p(\varrho, \vartheta) = R\varrho\vartheta$$

where $R \in (0, \infty)$.In this case, a direct combination of (1.5) with (1.6) implies that

$$\partial_\varrho e = 0$$

and the internal energy is equal to the *thermal energy* $e = e(\vartheta) = Q(\vartheta)$. A physically reasonable quantity called *the specific thermal capacity* is then defined by the relation $c_v(\vartheta) = \partial_\vartheta Q$. By utilizing once again (1.5) and (1.6), we obtain

$$\partial_\vartheta s = \frac{c_v(\vartheta)}{\vartheta} \quad \text{and} \quad \partial_\varrho s = -\frac{R}{\varrho}.$$

Therefore up to a constant

$$s(\varrho, \vartheta) = \int_1^\vartheta \frac{c_v(z)}{z}\,\mathrm{d}z - R\log(\varrho).$$

A usual simplification, consistent with Boyle's law, is to take $c_v \in (0, \infty)$ In such case, we observe that

$$\vartheta = e^{s/c_v}\varrho^{R/c_v}. \tag{1.8}$$

Finally, we consider that the conduction of heat and its generation by viscous dissipation in (1.7) are neglected. Hence

$$\partial_t(\varrho s) + \operatorname{div}_x(\varrho s \boldsymbol{u}) = 0 \tag{1.9}$$

and we can consider $(\varrho, \boldsymbol{u}, s)$ as state variables. Moreover, at least for smooth solutions with $\varrho > 0$, equation (1.9) is equivalent to

$$\partial_t s + \boldsymbol{u}\nabla_x s = 0. \tag{1.10}$$

Equation (1.10) means that the motion of the fluid is adiabatic (see [3, 3.4, 3.5] or [58, Chapter 1]) but the entropy is not necessarily constant. Despite the simplification, this model has its place in the physics of the atmosphere (see the discussion in [39]).

Observe that (1.8) implies that

$$p(\varrho, s) = Re^{s/c_v} \varrho^{\gamma}$$

where $\gamma = \frac{R+c_v}{c_v} > 0$ is called the *adiabatic constant*. For the standard temperatures on the Earth and standard monomolecular gases, γ satisfies approximately

$$1 < \gamma = 1 + \frac{2}{\text{degrees of freedom of a molecule}}.$$

For example, one has $\gamma = \frac{5}{3}$ for monatomic gases, $\gamma = \frac{7}{5}$ for diatomic gases and lower constants for other physically reasonable models of gases.

We call (1.1a), (1.1b) with (1.10) the *compressible Navier–Stokes system with entropy transport*. In this case, the boundary condition on $\mathbf{q}$ does not appear and we supplement the system by the condition $s(0, x) = s_0(x)$ in Ω.

As the entropy is transported along the trajectories of the fluid, it stays constant whenever the initial entropy s_0 is constant. This leads to the well–known *isentropic compressible Navier–Stokes system*, which played the main role in the development of the recent theory of weak solutions for the compressible fluid. Unlike the case with the entropy transport, assuming constant entropy simplifies the model in such a way that some important aspects of the physics of the atmosphere are not captured (e.g. the internal gravity waves), see the discussion in [39].

1.1.2 Compressible Navier–Stokes–Fourier system

A more sophisticated choice of the pressure is the following one[5]

$$p(\varrho, \vartheta) = p_e(\varrho) + \vartheta p_\vartheta(\varrho)$$

with the *elastic pressure* p_e and p_ϑ are function of ϱ, so $\partial_\vartheta p = p_\vartheta$.

Directly from (1.5) and (1.6), we obtain

$$\partial_\varrho e = \vartheta \left(\partial_\varrho s + \frac{p_\vartheta}{\varrho^2} \right) + \frac{p_e(\varrho)}{\varrho^2} = \frac{p_e(\varrho)}{\varrho^2}.$$

Therefore the internal energy has the following form:

$$e(\varrho, \vartheta) = P_e(\varrho) + Q(\vartheta)$$

where the *elastic potential* P_e is given by

$$P_e(\varrho) = \int_1^\varrho \frac{p_e(z)}{z^2} \, \mathrm{d}z.$$

For the simplicity, we once again assume that $Q(\vartheta) = c_v \vartheta$. The evolution of the elastic part of the internal energy $\varrho P_e(\varrho)$ can be derived from the continuity

[5] which is considered in [32, Chapter 1]

equation. To this end, we observe that the smooth solutions of the continuity equation also satisfy the *renormalized continuity equation*

$$\partial_t B(\varrho) + \operatorname{div}(B(\varrho)\boldsymbol{u}) = (B(\varrho) - B'(\varrho)\varrho) \operatorname{div} \boldsymbol{u}$$

for any sufficiently regular function $B\colon \mathbb{R} \to \mathbb{R}$. Taking $B(\varrho) = \varrho P_e(\varrho)$ leads to

$$\partial_t(\varrho P_e(\varrho)) + \operatorname{div}(\varrho P_e(\varrho)\boldsymbol{u}) = -p_e(\varrho) \operatorname{div} \boldsymbol{u}.$$

Due to this fact, the internal energy equation simplifies to the *thermal energy equation*

$$c_v \partial_t(\varrho\vartheta) + c_v \operatorname{div}_x(\varrho\vartheta\boldsymbol{u}) + \operatorname{div}_x \mathbf{q}(\vartheta) = \mathbb{S}(\nabla_x \boldsymbol{u}) : \nabla_x \boldsymbol{u} - \vartheta p_\vartheta(\varrho) \operatorname{div}_x \boldsymbol{u}. \quad (1.11)$$

The system (1.1a), (1.1b) with (1.11) is called *the Navier–Stokes–Fourier system* (we also prescribe initial data $(\varrho\vartheta)(0,x) = \chi_0(x)$ in Ω).

Typically, $p_e(\varrho) \approx \varrho^\gamma$ and $p_\vartheta(\varrho) \approx \varrho^\Gamma$ with positive constants a, R, γ and Γ. We will consider only the following special from of the pressure which appears in Chapter 5:

$$p(\varrho, \vartheta) = a_1 \varrho^\gamma + a_2 \varrho + \varrho\vartheta \quad (1.12)$$

with $\gamma > 3$ and a_1, $a_2 > 0$. In this case,

$$P_e(\varrho) = \frac{a_1}{\gamma - 1}\varrho^{\gamma-1} + \varrho \log \varrho \quad \text{and} \quad s(\varrho, \vartheta) = c_v \log(\vartheta) - \log \varrho.$$

1.2 Remarks on the existence theory for weak solutions

As it has been already mentioned in Introduction, we present the standard a priori estimates for a typical example of compressible models. We also comment the important parts of the existence theory, which is based Lions theory of compactness [65], an extension by Feireisl [31] and the construction of an approximation scheme (in [40] and [41]).

We restrict our considerations on the compressible Navier–Stokes–Fourier system with the pressure given by (1.12), and the initial and boundary data described in the previous section.[6]

Having an evolutionary differential equation in the classical form, an important task is to specify a reasonable definition of weak solutions. The standard heuristics reads as follows - weak solutions should be limits of smooth solutions[7] whenever the latter exist. This leads to the question of finding suitable *a priori estimates*. As we are solely interested in global solutions for "large initial data", we would like to find for any T estimates which are independent of the size of the initial data.

We recall that the standard strategy of proving the existence of weak solution proceeds as follows - we construct a family of approximation problems which one can solve and their solutions are uniformly bounded in suitable Banach spaces

[6] A similar system is treated in Chapter 5.

[7] This is typically the case when we have a sequence of initial data which are "losing regularity".

similarly to the *a priori estimates*. If the spaces are reflexive, we can take a weakly convergent subsequence and try to show that it converges to the solution of the former system. For nonlinear problems with poor a priori estimates, we have to use so called *weak convergence methods* (see also [28]) and the structure of the equations to improve the notion of convergence.

In this chapter, it's not our motivation to present the proof of the existence (as we can refer the reader to [32]). We only recall a priori estimates for the Navier–Stokes–Fourier system (Subsection (1.2.1)) in order to motivate the notion of weak solution from Chapter 5.[8] After that, we will revise the important weak convergence methods leading to the existence result. In order to do that in a bit systematic manner, we consider the problem of *compactness* (or *stability*)[9] and emphasize the crucial parts in the proof (Subsection 1.2.2). It should be noted that that "perturbed versions" of these compactness techniques will appear in Chapters 3–4; therefore, this section might serve for motivational reasons.

To conclude the section, we recall (together with a few comments) the basic convergence scheme which has been shown to be useful in the isentropic case. This might be seen as a motivation for Chapter 5 where a similar scheme is constructed for the system with entropy transport.

1.2.1 Global a priori estimates and better integrability of the pressure and the temperature

Assume that the triplet $(\varrho, \boldsymbol{u}, \vartheta)$ of smooth functions solves the Navier–Stokes–Fourier system and (1.12) with smooth initial conditions $(\varrho_0, \varrho_0\boldsymbol{u}_0, \varrho_0\vartheta_0)$.

Moreover, we assume that ϱ, ϱ_0 and ϑ, ϑ_0 are positive functions and Ω is sufficiently smooth and $T \in (0, \infty)$.

Estimates from continuity equation

Let us denote by $X = X(t, \xi)$ the solutions of the differential equation for characteristics, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t} X(t, \xi) = \boldsymbol{u}(t, X(t, \xi)), \quad X(0, \xi) = \xi.$$

Solving the continuity equation by the method of characteristics then implies that

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\varrho(t, X(t, \xi))\right) = -\varrho(t, X(t, \xi))(\mathrm{div}_x\, \boldsymbol{u})(t, X(t, \xi)).$$

Hence

$$\varrho(t, X(t, \xi)) = \varrho_0(\xi) e^{-\int_0^t (\mathrm{div}_x\, \boldsymbol{u})(s, X(s,\xi))\,\mathrm{d}s}, \tag{1.13}$$

so the density is a priori bounded from below by 0. Regrettably, it's not known whether $\mathrm{div}_x\, \boldsymbol{u}$ is a priori bounded in $L^\infty((0,T)\times\Omega)$, which decreases the potential of (1.13) to derive a priori lower and upper bounds on ϱ. (This leads to possible occurrence of vacuums for weak solutions, i.e. regions where $\varrho = 0$ with positive Lebesgue's measure.)

[8] And also in Chapters 3–4, where the situation with a priori estimates is simpler.

[9] I.e. having a sequence of strong solutions uniformly bounded in the norms given by a priori estimates, is it true that there exists a subsequence converging to the weak solution?

Estimates from total energy equality

Integrating over Ω the continuity equation (and reflecting the boundary conditions) leads to the mass conservation

$$\int_\Omega \varrho(t,x)\,\mathrm{d}x = \int_\Omega \varrho(0,x)\,\mathrm{d}x = M_0 > 0.$$

The total energy equality (1.3) provides a substantial source of global estimates. If we integrate its both sides over $(0,t)\times\Omega$ and recall the boundary conditions we obtain

$$\int_\Omega c_v\varrho(t,x)\vartheta(t,x) + \frac{a_1}{\gamma-1}\varrho^\gamma + \varrho(t,x)\log(\varrho(t,x)) + \frac{1}{2}\varrho|\boldsymbol{u}|^2(t,x)\,\mathrm{d}x$$
$$= \int_\Omega c_v\varrho_0(x)\vartheta_0(x) + \frac{a_1}{\gamma-1}(\varrho_0(x))^\gamma + \varrho_0(x)\log(\varrho_0(x)) + \frac{1}{2}\varrho_0(x)|\boldsymbol{u}_0(x)|^2\,\mathrm{d}x =: E_0.$$

As $t\in(0,T)$ has been taken arbitrarily, the initial energy E_0 controls

- ϱ in $L^\infty(0,T;L^\gamma(\Omega))$,
- $\sqrt{\varrho}|\boldsymbol{u}|$ in $L^\infty(0,T;L^2(\Omega))$,
- $\varrho\vartheta$ in $L^\infty(0,T;L^1(\Omega))$ and
- $\varrho\boldsymbol{u} = \sqrt{\varrho}\sqrt{\varrho}\boldsymbol{u}$ in $L^\infty\left(0,T;L^{\frac{2\gamma}{\gamma+1}}(\Omega;\mathbb{R}^3)\right)$ by the means of Hölder's inequality.

Estimates from entropy balance

The control over the dissipative mechanisms follows from the entropy balance (1.7). Previously, (1.7) was derived from the internal energy balance. Equivalently, it follows also from the thermal energy balance. Indeed, as $\varrho s(\varrho,\vartheta) = c_v\varrho\log(\vartheta) - \varrho\log(\varrho)$, we renormalize[10] (1.11) (particularly by dividing the equation by ϑ^{-1}) and subtract the renormalized continuity equation with $B(\varrho)=\log(\varrho)$ to get (1.7). Integratingl over $(0,t)\times\Omega$ gives us

$$\int_0^T\int_\Omega \frac{\mathbb{S}(\nabla_x\boldsymbol{u}):\nabla_x\boldsymbol{u}}{\vartheta} + \frac{\kappa(\vartheta)|\nabla_x\vartheta|^2}{\vartheta^2}\,\mathrm{d}x\,\mathrm{d}s \tag{1.14}$$
$$= \int_\Omega c_v\varrho(T,x)\log(\vartheta(T,x)) - \varrho(T,x)\log(\varrho(T,x))\,\mathrm{d}x - S_0$$

where S_0 is defined as follows:

$$S_0 = \int_\Omega c_v\varrho_0(x)\log(\vartheta_0(x)) - \varrho_0(x)\log(\varrho(x))\,\mathrm{d}x.$$

[10] Observe that the entropy equation could be considered as a renolmalized form of the thermal energy balance.

We use the growth condition on κ from (1.4) and get

$$\begin{aligned}\int_0^t \int_\Omega \frac{\mathbb{S}(\nabla_x \boldsymbol{u}) : \nabla_x \boldsymbol{u}}{\vartheta} + k_1 \left(1 + \frac{1}{\vartheta^2}\right) |\nabla_x \vartheta|^2 \,\mathrm{d}x \,\mathrm{d}s \qquad (1.15)\\ \leq \int_\Omega c_v \varrho(t,x)(\log(\vartheta(t,x)))^+ \,\mathrm{d}x - S_0 \\ \leq \int_\Omega c_v \varrho(t,x)(\vartheta(t,x)) \,\mathrm{d}x - S_0 \leq E_0 - S_0.\end{aligned}$$

Coupling the previous estimate with the inequality

$$\|v\|^2_{L^2(\Omega)} \leq C(\|\varrho\|_{L^\infty(L^\gamma}, \|\varrho\|^{-1}_{L^1(\Omega)}) \left(\|\nabla_x v\|^2_{L^2(\Omega)} + \left(\int_\Omega \varrho |v| \right)^2 \right)$$

(see [32, Lemma 3.2]), we get

- $\log(\vartheta)$ and ϑ in $L^2(0,T;W^{1,2}(\Omega))$.

Estimates from thermal energy balance

To estimate the velocity field without the multiplicator $\frac{1}{\vartheta}$, we integrate the thermal energy balance and get

$$\begin{aligned}\int_0^T \int_\Omega \mathbb{S}(\nabla_x \boldsymbol{u}) : \nabla_x \boldsymbol{u} \,\mathrm{d}x \,\mathrm{d}t \leq E_0 + \int_0^T \int_\Omega \vartheta \varrho |\operatorname{div}_x \boldsymbol{u}| \,\mathrm{d}x \,\mathrm{d}t \\ \leq E_0 + \|\vartheta\|_{L^2(L^6)} \|\varrho\|_{L^\infty(L^3)} \|\nabla_x \boldsymbol{u}\|_{L^2(L^2)}.\end{aligned}$$

We observe that

$$\mathbb{S}(\nabla_x \boldsymbol{u}) : \nabla_x \boldsymbol{u} = \operatorname{div}_x(\mathbb{S}(\nabla_x \boldsymbol{u}) \nabla_x \boldsymbol{u}) - (\mu \triangle_x \boldsymbol{u} - (\mu + \lambda) \nabla_x \operatorname{div}_x) \cdot \boldsymbol{u};$$

therefore,

$$\int_\Omega \mathbb{S}(\nabla_x \boldsymbol{u}) : \nabla_x \boldsymbol{u} \,\mathrm{d}x \,\mathrm{d}t = \mu \int_\Omega |\nabla_x \boldsymbol{u}|^2 \,\mathrm{d}x \,\mathrm{d}t + (\mu + \lambda) \int_\Omega |\operatorname{div}_x \boldsymbol{u}|^2 \,\mathrm{d}x \,\mathrm{d}t.$$

Hence,

- $\boldsymbol{u}$ is a priori bounded in $L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3))$ by the means of Young's and Poincaré's inequality.[11]

Using Hölder's inequality, we conclude that

- $\varrho \boldsymbol{u} \in L^2\left(0,T; L^{\frac{6\gamma}{6+\gamma}}(\Omega;\mathbb{R}^3)\right)$,
- $\varrho \boldsymbol{u} \otimes \boldsymbol{u} = \sqrt{\varrho}(\sqrt{\varrho}\boldsymbol{u}) \otimes \boldsymbol{u} \in L^2\left(0,T; L^{\frac{6\gamma}{3+4\gamma}}(\Omega;\mathbb{R}^{3\times 3})\right)$ and
- $\varrho \vartheta \boldsymbol{u} = \sqrt{\varrho}(\sqrt{\varrho}\boldsymbol{u})\vartheta \in L^2\left(0,T; L^{\frac{6\gamma}{3+4\gamma}}(\Omega;\mathbb{R}^3)\right)$.

[11] This a priori estimate holds provided $\gamma \geq 3$.

Finally, the estimates for the temperature can be improved using the thermal energy equation. Quite inconveniently, the right–hand side of (1.11) is controlled only in $L^1((0,T)\times\Omega)$. On the other hand, the viscous dissipation $\mathbb{S}(\nabla_x \boldsymbol{u}) : \nabla_x \boldsymbol{u}$ does not change its sign. By multiplying the equation by $\vartheta^{-\omega}$ integrating over $(0,T)\times\Omega$, we receive

$$
\begin{aligned}
\omega \int_0^T \int_\Omega \frac{\kappa(\vartheta)|\nabla_x \vartheta|^2}{|\vartheta^{\omega+1}|} \,\mathrm{d}x\,\mathrm{d}s \leq \int_\Omega \varrho(T,x) \frac{1}{1-\omega} \vartheta^{1-\omega}(t,x)\,\mathrm{d}x \\
- \int_\Omega \varrho_0(x) \frac{1}{1-\omega} \vartheta_0^{1-\omega}(x)\,\mathrm{d}x - \int_0^T \int_\Omega \frac{1}{\vartheta^\omega} \mathbb{S} : \nabla_x \boldsymbol{u}\,\mathrm{d}x\,\mathrm{d}t \\
+ \int_0^T \int_\Omega \vartheta^{1-\omega} \varrho \operatorname{div}_x \boldsymbol{u}\,\mathrm{d}x\,\mathrm{d}s.
\end{aligned}
$$

Therefore $\nabla_x \vartheta^{\frac{3-\omega}{2}}$ is a priori bounded in $L^2(0,T;L^2(\Omega))$ for any fixed $\omega\in(0,1]$. As $\vartheta^{\frac{3-\omega}{2}} \in L^1((0,T)\times\Omega)$, a standard Poincaré–type argument guarantees a priori bounds on

– $\vartheta^{\frac{3-\omega}{2}}$ in $L^2(0,T;L^2(\Omega))$,

– ϑ in $L^p(0,T;L^q(\Omega))$ whenever $p\in[1,3)$ and $q\in[1,9)$.

The collected estimates are not yet sufficient. The terms p, $\mathbb{S}(\nabla_x \boldsymbol{u}) : \nabla_x \boldsymbol{u}$ and $\kappa(\vartheta)\nabla_x\vartheta$ aren't a priori bounded in a Banach space with weakly precompact bounded sets. Let us provide some comments how this inconvenience is overcome:

- The problem with the pressure can be overcome due to the structure of the momentum equation leading to better estimates, see in Subsection 1.2.1.

- Better a priori estimates for the velocity field are perhaps beyond the scope of the recent possibilities. On the other hand, the term is non–negative. Therefore, there is a possibility to replace (1.11) by the thermal energy inequality

$$
c_v \partial_t(\varrho\vartheta) + c_v \operatorname{div}_x(\varrho\vartheta\boldsymbol{u}) - \operatorname{div}_x(\kappa(\vartheta)\nabla_x\vartheta) \geq \mathbb{S} : \nabla_x \boldsymbol{u} - \vartheta p_\vartheta(\varrho) \operatorname{div}_x \boldsymbol{u} \quad (1.16)
$$

  together with the total energy inequality

$$
\begin{aligned}
\int_\Omega \varrho(\tau,x) \left(\frac{1}{2} |\boldsymbol{u}(\tau,x)|^2 + e(\varrho(\tau,x),\vartheta(\tau,x)) \right) \quad (1.17) \\
\leq \int_\Omega \varrho_0(x) \left(\frac{1}{2} |\boldsymbol{u}_0(x)|^2 + e(\varrho_0(x),\vartheta_0(x)) \right)
\end{aligned}
$$

  holding for almost every $\tau\in(0,T)$. Moreover, the initial condition for (1.16) has to be also in the form of inequality. Note that any smooth solution of this formulation with two inequalities also satisfies the kinetic energy equality. It is easy to check that then (1.16) holds with equality.

- To solve the problem with the heat flux, we observe that

$$
\kappa(\vartheta)\nabla_x\vartheta = \nabla_x(\mathcal{K}(\vartheta)).
$$

  As $\mathcal{K}(\vartheta) \approx \vartheta^3$, we cannot control this term a priori even in $L^1((0,T)\times\Omega)$. On the other hand, a technical estimate leads to an a priori bound of ϑ in $L^3(0,T;L^3(\Omega))$ (see [32, Section 5.2]) and it's not obvious whether it could be improved. This is a reason why all derivatives from $\triangle_x \mathcal{K}(\vartheta)$ have to be sent to the test functions in the weak formulation.

Improved pressure estimates

To avoid technical issues, we assume that $\Omega = \mathbb{T}^3$ (i.e. we consider the periodic boundary conditions). To get better estimates on the pressure, we apply div_x on the momentum equation, whence

$$-\triangle_x p = \operatorname{div}_x \operatorname{div}_x(\varrho \boldsymbol{u} \otimes \boldsymbol{u}) - \operatorname{div}_x \operatorname{div}_x \mathbb{S} + \operatorname{div}_x \partial_t(\varrho \boldsymbol{u}). \tag{1.18}$$

We will treat the differential operators as multiplier operators.[12] As the solutions are assumed to be smooth, this approach can be verified by the means of Fourier series (or basic elliptic regularity theory). Accordingly,

$$-p = \left(\frac{\nabla_x}{\sqrt{\triangle_x}} \otimes \frac{\nabla_x}{\sqrt{\triangle_x}} \right) : (\varrho \boldsymbol{u} \otimes \boldsymbol{u} + \mathbb{S}) + \frac{\operatorname{div}_x}{\triangle_x} \partial_t(\varrho \boldsymbol{u}). \tag{1.19}$$

Let us recall (see e.g. [47, Section 3.6]) that the (vector) multiplier operator $\frac{\nabla_x}{\sqrt{\triangle_x}}$ is bounded from $L^r(\mathbb{T}^3)$ to $L^r(\mathbb{T}^3; \mathbb{R}^3)$ for $r \in (1, \infty)$ because it corresponds to the symbol $i\frac{\mathbf{k}}{|\mathbf{k}|}$. As $\varrho \boldsymbol{u} \otimes \boldsymbol{u}$ and $\mathbb{S}$ belong to $L^q((0,T) \times \Omega)$ with $q > 1$, we are left with the task to estimate the term with the time derivative in (1.19). This cannot be done directly, so we multiply the equality (1.19) by a function σ which is compactly supported in time and satisfies

$$\partial_t \sigma + \operatorname{div}_x(\sigma \boldsymbol{u}) = f.$$

Next, we integrate the equation for $p\sigma$ over $(0,T) \times \mathbb{T}^3$, apply integration by parts with respect to time and use the duality between $i\frac{\mathbf{k}}{|\mathbf{k}|^2}$ and $-i\frac{\mathbf{k}\cdot}{|\mathbf{k}|^2}$. Hence

$$\begin{aligned} -\int_0^T \int_{\mathbb{T}^3} p\sigma \,\mathrm{d}x\,\mathrm{d}t &= I_1 + \int_0^T \int_{\mathbb{T}^3} (\varrho \boldsymbol{u}) \cdot \frac{\nabla_x}{\triangle_x^{-1}} \partial_t \sigma \,\mathrm{d}x\,\mathrm{d}t \\ &= I_1 - \int_0^T \int_{\mathbb{T}^3} (\varrho \boldsymbol{u}) \cdot \frac{\nabla_x}{\sqrt{\triangle_x}} \frac{\operatorname{div}_x}{\sqrt{\triangle_x}} (\sigma \boldsymbol{u}) \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{\mathbb{T}^3} (\varrho \boldsymbol{u}) \cdot \frac{\nabla_x}{\triangle_x} f \,\mathrm{d}x\,\mathrm{d}t \end{aligned} \tag{1.20}$$

with

$$I_1 = \int_0^T \int_{\mathbb{T}^3} \left[\left(\frac{\nabla_x}{\sqrt{\triangle_x}} \otimes \frac{\nabla_x}{\sqrt{\triangle_x}} \right) : (\varrho \boldsymbol{u} \otimes \boldsymbol{u} + \mathbb{S}) \right] \sigma \,\mathrm{d}x\,\mathrm{d}t.$$

At the first glance, a reasonable choice of σ would be $\varrho\eta$ where $\eta = \eta(t)$ is a smooth function compactly supported in $(0,T)$. However, we can obtain suitable estimates on the last term of (1.20) only if

$$\int_{\mathbb{T}^3} f \,\mathrm{d}x = 0.$$

To overcome this problem, we take

$$\sigma(t,x) = \varrho(t,x) - \int_{\mathbb{T}^3} \varrho(t,y) \,\mathrm{d}y.$$

[12]Particularly, $\nabla_x \approx i\mathbf{k}$, $\operatorname{div}_x \approx i\mathbf{k}\cdot$, $\triangle_x \approx |\mathbf{k}|^2$, $\sqrt{\triangle_x} \approx |\mathbf{k}|$, etc.

By the means of a priori estimates and standard properties of the multiplier operators (e.g. (1.23)), we conclude that[13]

$$\begin{aligned}\int_0^T \int_{\mathbb{T}^3} p(\varrho,\vartheta)\varrho\eta \,\mathrm{d}x\,\mathrm{d}t &\lesssim \|\mathbb{S}\|_{L^2(L^2)}\|\varrho\|_{L^2(L^2)} + \|\varrho\boldsymbol{u}\|^2_{L^2(L^2)}\|\eta\|_{L^\infty(0,T)} \\ &+ \left\|\frac{\nabla_x}{\triangle_x}\left(\varrho - \int_{\mathbb{T}^3}\varrho\right)\right\|_{L^\infty\left(L^{\frac{2\gamma}{\gamma-2}}\right)} \|\eta'\|_{L^1(0,T)}\|\varrho\|_{L^\infty(L^\gamma)}\|\sqrt{\varrho}\boldsymbol{u}\|_{L^\infty(L^2)} \\ &\lesssim C(E_0)(1+\|\eta'\|_{L^1(0,T)})\end{aligned}$$

as long as $\gamma \geq 3$. By taking a suitable sequence of cut–off functions η, we conclude a priori bounds on

- $p(\varrho,\vartheta)\varrho$ in $L^1((0,T)\times\mathbb{T}^3)$,
- ϱ in $L^{\gamma+1}((0,T)\times\mathbb{T})$.

The estimate also works on the level of weak solutions, where the pressure term is tested by test function $\phi \in \mathcal{D}((0,T)\times\mathbb{T}^3)$ and looks as follows:

$$\int_0^T \int_{\mathbb{T}^3} p(\varrho,\vartheta)\,\mathrm{div}_x\,\phi\,\mathrm{d}x\,\mathrm{d}t. \tag{1.21}$$

Taking inversion of the Laplace operator and multiplying (1.18) by σ corresponds to testing (1.21) by $\frac{\nabla_x}{\triangle_x}\sigma$. The operator $\mathcal{A}(f) = \frac{\nabla_x}{\triangle_x}f$ defined for functions with zero mean has the following properties

$$\mathrm{div}_x\,\mathcal{A}(f) = f, \tag{1.22}$$

$$\|\nabla_x\mathcal{A}(f)\|_p \leq C\|f\|_p. \tag{1.23}$$

The regularity of $\mathcal{A}(\varrho)$ with respect to time follows from the continuity equation and due to the basic Sobolev embeddings $\mathcal{A}(\varrho)\in C([0,T]\times\overline{\Omega};\mathbb{R}^3)$, which makes $\mathcal{A}(\varrho)$ a suitable test function for the momentum equation.

The control over the term $\|\varrho\boldsymbol{u}\|^2_{L^2(L^2)}$ is no longer available when $\gamma < 3$. In that case, one can take

$$\sigma(t,x) = \varrho^\omega(t,x) - \int_{\mathbb{T}^3}\varrho^\omega(t,y)\,\mathrm{d}y$$

with $\omega < 1$ and similar steps give a priori estimate for $\varrho^\delta p \in L^1((0,T)\times\mathbb{T}^3)$ for some $\delta > 0$ as long as $\gamma > \frac{3}{2}$.

When Ω is general, the treatment gets complicated. However, the first step consists in the construction of an operator $\mathcal{A} \approx \mathrm{div}_x^{-1}$ which satisfies (1.22). If Ω has Lipschitz boundary then there exists such operator $\mathcal{A}$ (called *Bogovskii operator*) having an additional property - $\mathcal{A}(f)$ has zero traces on $\partial\Omega$ (see Lemma 4.3.9).

[13] Here we use $a \lesssim b$ if $a \leq cb$ and the constant $c > 0$ is independent on the data.

1.2.2 Essential parts of the Lions theory of compactness

As we have already mentioned, we will demonstrate the essential steps leading to the compactness result. Once again, we restrict ourselves to the case of the compressible Navier–Stokes–Fourier system, with (1.12) (so $\gamma > 3$).

Let $(\varrho_n, \boldsymbol{u}_n, \vartheta_n)$ be a sequence of smooth solutions of the compressible Navier–Stokes–Fourier with (1.12) which are uniformly bounded by the estimates obtained in the previous subsection. For a reason which we will specify later, let us also assume that the initial data of ϱ_n are strongly convergent. We pick a subsequence[14] such that

$$\varrho_n \to \varrho,\ \boldsymbol{u}_n \to \boldsymbol{u},\ \vartheta_n \to \vartheta \quad \text{almost everywhere in } (0,T)\times\Omega$$

and the following terms converge weakly:[15]

- $\varrho_n \overset{*}{\rightharpoonup} \varrho$ in $L^\infty(L^\gamma(\Omega))$ and also in $L^{\gamma+1}(0,T;L^{\gamma+1}(\Omega))$,
- $\boldsymbol{u}_n \rightharpoonup \boldsymbol{u}$ in $L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3))$,
- $\vartheta_n \rightharpoonup \vartheta$ in $L^2(0,T;W^{1,2}(\Omega))$ and also in $L^p(0,T;L^q(\Omega))$ whenever $p\in[1,3)$ and $q \in [1,9)$,
- $\mathbb{S}(\nabla_x \boldsymbol{u}_n) \rightharpoonup \mathbb{S}(\nabla_x \boldsymbol{u})$ in $L^2(0,T;L^2(\Omega))$,
- $\varrho_n\boldsymbol{u}_n \overset{*}{\rightharpoonup} \overline{\varrho\boldsymbol{u}}$ in $L^\infty\left(0,T;L^{\frac{2\gamma}{\gamma+1}}(\Omega;\mathbb{R}^{3\times3})\right)$,
- $\varrho_n\boldsymbol{u}_n\otimes\boldsymbol{u}_n \rightharpoonup \overline{\varrho\boldsymbol{u}\otimes\boldsymbol{u}}$ in $L^2\left(0,T;L^{\frac{6\gamma}{3+4\gamma}}(\Omega;\mathbb{R}^3)\right)$,
- $\varrho_n\vartheta_n \rightharpoonup \overline{\varrho\vartheta}$ in $L^2\left(0,T;L^{\frac{6\gamma}{6+\gamma}}(\Omega)\right)$,
- $\varrho_n\vartheta_n\boldsymbol{u}_n \rightharpoonup \overline{\varrho\vartheta\boldsymbol{u}}$ in $L^2\left(0,T;L^{\frac{6\gamma}{3+4\gamma}}(\Omega;\mathbb{R}^3)\right)$,
- $p(\varrho_n,\vartheta_n) \rightharpoonup \overline{p(\varrho,\vartheta)}$ in $L^r((0,T)\times\Omega)$ for some $r>1$,
- $\varrho_n\vartheta_n \operatorname{div}_x \boldsymbol{u}_n \rightharpoonup \overline{\varrho_n\vartheta_n \operatorname{div}_x \boldsymbol{u}_n}$ in $L^r((0,T)\times\Omega)$ for some $r>1$,

and also

- for all non–negative functions $\phi \in L^\infty((0,T)\times\Omega)$

$$\int_0^T\int_\Omega \mathbb{S}(\nabla_x\boldsymbol{u}) : \nabla_x\boldsymbol{u}\phi \,\mathrm{d}x\,\mathrm{d}t \le \liminf_{n\to\infty}\int_0^T\int_\Omega \mathbb{S}(\nabla_x\boldsymbol{u}_n) : \nabla_x\boldsymbol{u}_n\phi\,\mathrm{d}x\,\mathrm{d}t.$$

To deal with the majority of the multilinear terms, we observe that the structure of the balance laws gives the convergence of ϱ_n to ϱ in $C([0,T];L^\gamma_\omega(\Omega))$ (at least for a subsequence).[16] Moreover, the embedding $L^\gamma(\Omega)\hookrightarrow (W^{1,2}(\Omega))^*$ is compact. This, together with the convergence of $\boldsymbol{u}_n$ in $L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3))$, implies that $\overline{\varrho\boldsymbol{u}} = \varrho\boldsymbol{u}$. By the same token, $\overline{\varrho\boldsymbol{u}\otimes\boldsymbol{u}} = \varrho\boldsymbol{u}\otimes\boldsymbol{u}$, $\overline{\varrho\vartheta}=\varrho\vartheta$ and $\overline{\varrho\vartheta\boldsymbol{u}} = \varrho\vartheta\boldsymbol{u}$. See also Section 4.6 for more details.

[14] Later, we won't mention this explicitly.

[15] Here we adopt a common notation. Namely, we denote the weak limits of nonlinear terms by pointwise limits under the bar sign.

[16] For a Banach space X, we denote by $C(I;X_\omega)$ the set of continuous functions from I to X supplemented with the standard weak topology.

Convergence of the density

The following observation plays a major role in showing the strong convergence of ϱ_n:

Lemma 1.2.1. *Consider $U \subseteq \mathbb{R}^d$. Let $\varrho_n \rightharpoonup \varrho$ in $L^1(U)$ and $B\colon \mathbb{R} \to \mathbb{R}$ be a convex function such that $B(\varrho_n) \rightharpoonup \overline{B(\varrho)}$ in $L^1(\mathbb{R}^d)$. Then*

$$B(\varrho) \leq \overline{B(\varrho)} \quad \text{a.e. in } U.$$

Moreover, if B is strictly convex and $B(\varrho) = \overline{B(\varrho)}$ then $\varrho_n \to \varrho$ a.e. in U.

Hence for B strictly convex,

$$\int_\Omega \overline{B(\varrho)} - B(\varrho)\,\mathrm{d}x$$

can be considered as a kind of measure of oscillations of the sequence ϱ_n.

To exploit this observation, we renormalize the continuity equations using $B(z) = z\log(z)$:

$$\partial_t B(\varrho_n) + \operatorname{div}(B(\varrho_n)\boldsymbol{u}_n) = -\varrho_n \operatorname{div}_x \boldsymbol{u}_n \tag{1.24}$$

and pass to the limit

$$\partial_t \overline{B(\varrho)} + \operatorname{div}_x(\overline{B(\varrho)\boldsymbol{u}}) = -\overline{\varrho \operatorname{div}_x \boldsymbol{u}}. \tag{1.25}$$

We would also like to get a similar equation for $B(\varrho)$, i.e. we want ϱ to be renormalized solution of the continuity equation.[17] It turns out that this is possible at least for $\gamma \geq \frac{9}{5}$ based on the DiPerna–Lions theory from [26]. Since we consider $\gamma > 3$, we conclude

$$\partial_t B(\varrho) + \operatorname{div}_x(B(\varrho)\boldsymbol{u}) = -\varrho \operatorname{div}_x \boldsymbol{u}. \tag{1.26}$$

By subtracting (1.25) from (1.26) and integrating over Ω,[18] we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \overline{B(\varrho)} - B(\varrho)\,\mathrm{d}x = \int_\Omega \overline{-\varrho \operatorname{div}_x \boldsymbol{u}} + \varrho \operatorname{div}_x \boldsymbol{u}\,\mathrm{d}x. \tag{1.27}$$

What is left is to show that the right–hand side of (1.27) is non–positive. Based on that knowledge, Lemma 1.2.1 provides us almost everywhere convergence of the densities (thus $\overline{p(\varrho,\vartheta)} = p(\varrho,\vartheta)$) *as soon as the initial data for the density are strongly convergent.*

In order to demonstrate non–positivity of (1.27), properties of so called *effective viscous flux* are used. The effective viscous flux is a quantity

$$p - (2\mu + \lambda)\operatorname{div}_x \boldsymbol{u} \tag{1.28}$$

[17] The definition of renormalized weak solutions can be found in Chapter 4 (Definition 4.2.4). At the same place, the basic sufficient condition for a weak solution to be renormalized is recalled (Lemma 4.3.5 and Remark 4.3.6).

[18] It is also a consequence of renormalization that $\phi \equiv 1$ is a suitable test function for the weak formulation of the continuity equation.

which be formally computed from the momentum equation by the means of the operator $\triangle_x^{-1} \operatorname{div}_x$.[19] Most importantly, compensated compactness techniques lead to

$$\overline{(p-(2\mu+\lambda)\operatorname{div}_x \boldsymbol{u})B(\varrho)} = (\bar{p}-(2\mu+\lambda)\operatorname{div}_x \boldsymbol{u})\overline{B(\varrho)}$$

for suitable functions B (see e.g. Lemma 4.7.1).

Equipped with the this knowledge, we can estimate (1.27) as follows

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \overline{B(\varrho)} - B(\varrho)\,\mathrm{d}x &= \frac{1}{2\mu+\lambda}\int_\Omega \overline{p(\varrho,\vartheta)}\varrho - \overline{p(\varrho,\vartheta)\varrho}\,\mathrm{d}x \\
&= \frac{1}{2\mu+\lambda}\int_\Omega \overline{a_1\varrho^\gamma + a_2\varrho}\varrho - \overline{a_1\varrho^{\gamma+1}+a_2\varrho^2}\,\mathrm{d}x + \frac{1}{2\mu+\lambda}\int_\Omega \varrho^2\vartheta - \overline{\varrho^2\vartheta}\,\mathrm{d}x \\
&\le \frac{1}{2\mu+\lambda}\int_\Omega \varrho^2\vartheta - \overline{\varrho^2\vartheta}\,\mathrm{d}x,
\end{aligned}$$

where the last step made use of the monotonicity of the elastic part of the pressure p. According to Lemma 1.2.1, $\varrho^2 \le \overline{\varrho^2}$; therefore the only point remaining concerns showing

$$\overline{\varrho^2\vartheta} = \overline{\varrho^2}\vartheta. \tag{1.29}$$

To this end, we renormalize the continuity equations for ϱ_n obtaining $\varrho_n^2 \to \overline{\varrho_n^2}$ in $C([0,T]; L^2_\omega(\Omega))$ and (1.29) follows from the similar argument which was used for the limit passage in $\varrho\boldsymbol{u}$.

The case $\gamma < \frac{9}{5}$

The complexity of the approach substantially increases in the case $\gamma < \frac{9}{5}$. It is true that the limit ϱ also satisfies the renormalized version of the continuity equation but the standard argument of DiPerna–Lions fails. To overcome this problem, the concept of the *oscillation defect measure* introduced in [31] comes into play. It is defined as follows:

$$\operatorname{osc}[\varrho_n \to \varrho] = \sup_{k\in\mathbb{N}}\left(\limsup_{n\to\infty}\int_{(0,T)\times\Omega} |T_k(\varrho) - T_k(\varrho_n)|^{\gamma+1}\,\mathrm{d}x\,\mathrm{d}t\right)$$

where the truncation functions T_k are specified e.g. by (4.41).

It can be shown that $\operatorname{osc}[\varrho_n \to \varrho]$ is bounded by the means of uniform estimates on $(\varrho_n, \boldsymbol{u}_n, \vartheta_n)$. Moreover, boundedness of the oscillation defect measure is another condition ensuring that the weak limit of ϱ_n is a renormalized solution of the continuity equation. Using this knowledge, a refined argument based on Lemma 1.2.1 yields $\overline{p(\varrho,\vartheta)} = p(\varrho,\vartheta)$ (see e.g. [32]).

Thermal energy balance

The last step is to pass to the limit in $\mathcal{K}(\vartheta)$ and $\varrho\vartheta\operatorname{div}_x \boldsymbol{u}$. Both passages are non–trivial and it is not our intention to give a complete presentation of them. Let us at least point out some important parts (see [32] for the details).

[19] There is a similarity to (1.18) as

$$\operatorname{div}_x \operatorname{div}_x \mathbb{S}(\nabla_x \boldsymbol{u}) = \triangle_x (2\mu+\lambda)\operatorname{div}_x \boldsymbol{u}.$$

The first main problem lies in the fact that we don't know whether the sequence of temperatures converges pointwise. On the other hand, one can show that $\vartheta_n \to \vartheta$ a.e. on $\{\varrho > 0\}$. "Bad" a priori estimates for $\mathcal{K}(\vartheta)$ brings another obstruction. However, it is possible to pass to the limit in this term at least in the biting sense. Due to the better a priori estimates on $\varrho\mathcal{K}(\vartheta)$, the biting limit $\overline{\mathcal{K}(\vartheta)}$ can be identified with $\mathcal{K}(\vartheta)$ out of a possible vacuum $\{\varrho = 0\}$, more precisely,

$$\varrho\overline{\mathcal{K}(\vartheta)} = \varrho\mathcal{K}(\vartheta).$$

Finally, it should be mentioned that we have skipped the question of attaining the initial conditions. To this end, a similar treatment to the one coming from the theory of balance laws (see e.g. [18]) with integrable fluxes can be applied here (a careful approach is needed for the thermal energy balance, where the initial condition is satisfied only as an inequality).

1.2.3 Approximation schemes

As we have mentioned above, a complete proof of the existence of weak solutions is based on a construction of approximation problems.

The first scheme which was shown to be convergent for the isentropic compressible Navier–Stokes with $\gamma > \frac{9}{5}$ was presented in [40][20] and had the following form

$$\partial_t\varrho + \mathrm{div}_x(\varrho\boldsymbol{u}) = \varepsilon\triangle_x\varrho,$$
$$\partial_t(\varrho\boldsymbol{u}) + \mathrm{div}_x(\varrho\boldsymbol{u}\otimes\boldsymbol{u}) + \delta\nabla_x\varrho^\beta + \nabla_x\varrho^\gamma = -\varepsilon\nabla_x\boldsymbol{u}\nabla_x\varrho + \mathrm{div}_x(\mathbb{S}(\nabla_x\boldsymbol{u}))$$

with parameters ε, $\delta > 0$ and sufficiently large $\beta > 0$. The existence of solutions to (1.30) follows from the Galerkin method. Then, ε in (1.30) is sent to zero eliminating the *artificial viscosity* $\varepsilon\triangle_x\varrho$ and also the term $\varepsilon\nabla_x\boldsymbol{u}\nabla_x\varrho$ which compensates the artificial viscosity in uniform estimates. The last step consist of taking $\delta \to 0$ and making the *artificial pressure* term ϱ^β disappear. It should be mentioned that the ideas mentioned in Subsection 1.2.2 are used during the limiting processes. A similar scheme and its convergence is a topic of Chapter 4.

It should be noted that at least when $\gamma < 3$, the scheme is canonical in the existence theory of weak solutions as the majority of the known results uses a perturbed version of (1.30).

Under condition $\gamma > 3$, it is possible to introduce finite–dimensional numerical schemes converging to a weak solution. The first such scheme was constructed in [55] for the isentropic case and another one appears in Chapter 5. The main reason for this constraint on γ emanates from the presence of a term reminiscent of $\varepsilon\triangle_x\varrho$, which has to be controlled in the pressure estimates.

[20] In fact, a few schemes were proposed in [65]; however, none of them has been ever used.

2. Discussion on the results of the thesis

2.1 Compressible Navier–Stokes system with entropy transport

Papers [68] (Chapter 3) and [66] (Chapter 4)[1] are linked to the model with the transport equation for the entropy (4.1). *Their main outcome is the existence of global weak solutions for large initial data under the assumption* $\gamma > \frac{3}{2}$. This substantially extends the previously known compactness result of P. L. Lions from [65, Chapter 8] for $\gamma \geq \frac{9}{5}$.

It might be of interest to briefly recall the strategy leading to the compactness result in [65]. To this end, assume that $(\varrho_n, \boldsymbol{u}_n, s_n)$ is a sequence of weak solutions to (4.1) converging weakly[2] to $(\varrho, \boldsymbol{u}, s)$. The main advantage of the scenario $\gamma \geq \frac{9}{5}$ lies in the fact that $\varrho\beta(s)$ is a renormalized weak solution of the continuity equation for any continuous function β. Particularly, the choice $\beta \equiv 1$ yields the convergence of the densities almost everywhere. Consequently, one can show the convergence of the pressure term and the rest of the compactness argument follows directly from this fact.

The standard renormalization argument of DiPerna and Lions is not directly applicable in the case $\gamma \in (3/2, 9/5)$. The alternative method of the defect measures developed in [31] uses the structure of the momentum equation; for our system, it provides that $\varrho\beta(s)$ is a renormalized solution only for $\beta(s) = e^{s/\gamma}$. Most importantly, we don't know if ϱ is a renormalized solution, which is the main source of difficulties when $\gamma < \frac{9}{5}$.

2.1.1 Compactness of solutions

Chapter 3 deals successfully with these difficulties and provides the existence of weak solutions for $\gamma > \frac{3}{2}$ under the assumption that there exists a suitable sequence $(\varrho_n, \boldsymbol{u}_n, s_n)$ of weak solutions to the problem (see Theorem 3.3.1). The result does not directly imply the compactness of the set of weak solutions as we assume that each ϱ_n belongs to $L^2(0, T; L^2(\Omega))$ (but not uniformly, which is important). On the other hand, the result of Chapter 3 forms a reasonable starting point for the rest of the existence theory.

Main contributions of [66]

There are a few crucial ideas leading to the main result. Let us emphasize the most interesting one. To this end, we recall that for smooth solutions we have

$$\boldsymbol{u} \cdot \nabla_x s = \operatorname{div}_x(s\boldsymbol{u}) - s \operatorname{div}_x \boldsymbol{u} \tag{2.1}$$

and the right–hand side is used in the definition of weak solutions of the transport equation (see Subsection 3.1.1). In order to prove Theorem 3.3.1 we have to pass

[1] The results came historically in this order.
[2] Suitable spaces for the convergence are presented in Proposition 3.2.1.

to the limit in the term $s_n \operatorname{div}_x \boldsymbol{u}_n$. However, sequences s_n and $\operatorname{div}_x \boldsymbol{u}_n$ converge only weakly, so an argument based on the compensated compactness is needed. To establish

$$\overline{s \operatorname{div}_x \boldsymbol{u}} = s \operatorname{div}_x \boldsymbol{u}, \tag{2.2}$$

one has to revisit the effective viscous flux lemma

$$\overline{(p - c \operatorname{div}_x(\boldsymbol{u}))\,\sigma} = (\bar{p} - c \operatorname{div}_x(\boldsymbol{u}))\,\bar{\sigma}. \tag{2.3}$$

We can ask what are some sufficient conditions for a sequence σ_n which would guarantee (2.3). It turns out that (2.3) holds for σ_n satisfying the continuity equation with the velocity field $\boldsymbol{u}_n$ and with sufficiently integrable right–hand side (see Lemma 3.4.2). The sequence s_n fulfils this condition, whence (2.2) follows from (2.3).

2.1.2 Approximation scheme

The main objective of Chapter 4 is to supplement the result of Chapter 3 by a suitable approximation scheme and to complete the proof of the existence of weak solutions. Due to the hyperbolic character of the transport equation, some new ideas are needed to resolve this problem.

The main result of Chapter 4 is Theorem 4.2.7, which gives the existence of global weak solutions in the case $\gamma > \frac{3}{2}$. We also consider the system with an equation

$$\partial_t(\varrho s) + \operatorname{div}_x(\varrho s \boldsymbol{u}) = 0, \tag{2.4}$$

which is formally equivalent to the transport equation for the entropy. Surprisingly, we get only a weaker result in this case, namely the existence of weak solutions for $\gamma \geq \frac{9}{5}$ (Theorem 4.2.2).[3] This asymmetry emerges from the already mentioned lack of renormalization techniques.

Main contributions of [66]

Although the scheme (4.53) is a reminiscence of the standard "scheme" mentioned in (1.2.3), we believe it wasn't a trivial task to derive it from the initial system.

It occurs that (among infinitely many others) there is an equation formally equivalent to (2.4) which is quite preferable for the construction of the approximation scheme. It is the continuity equation

$$\partial_t(Z) + \operatorname{div}_x(Z \boldsymbol{u}) = 0 \tag{2.5}$$

for $Z = \varrho e^{s/\gamma}$. Let us observe that in variables $(\varrho, \boldsymbol{u}, Z)$, the pressure term has the familiar form Z^γ; therefore, it seems reasonable to construct an approximation scheme in these variables. Theorem 4.2.5 encompasses the existence result for our system coupled with (2.5) for $\gamma > \frac{3}{2}$. Moreover, the relation

$$c_* \varrho \leq Z \leq c^* \varrho, \quad 0 < c_* < c^* < \infty \tag{2.6}$$

[3] It should be noted that even this result reasonably extends the one of Lions as we have also shown the existence of weak solution.

holds a.e. in $(0,T) \times \Omega$ as long as it is satisfied for ϱ_0 and Z_0.[4]

The next non–trivial task is to return from the artificial state variable Z to s. Recall that Z represents $\varrho e^{s/\gamma}$; therefore, in order to "compute s from Z" the first step would be to "divide" (2.5) by ϱ. This encompasses quite engaged renormalisation techniques because we cannot prevent the density ϱ from vanishing on a set of positive measure (see Section (4.8)). The relation (2.6) plays also an important role in this part.[5] The condition $\gamma \geq \frac{9}{5}$ is necessary to enforce the renormalization techniques. As a conclusion, we obtain the existence of weak solutions for the systems with (2.4) and the original transport of entropy. Finally, we observe that Theorem 4.2.7 for $\gamma \in (\frac{3}{2}, \frac{9}{5})$ follows from Theorem 3.3.1.

2.2 Convergent numerical schemes for the Navier–Stokes–Fourier system

By means of mathematical analysis, one can rigorously study the properties of numerical schemes and show their convergence to the exact solutions. This task seems to be quite ambitious for the compressible Navier–Stokes system. The main quest is then to reduce a nonlinear infinite dimensional problem on a sequence of finite dimensional ones converging to the exact solutions of the primal problem.

Perhaps surprisingly, such schemes can be constructed even in the case of three dimensions. The first complete result of this kind for the isentropic compressible Navier–Stokes system was established by Karper in [55]. Later on, Feireisl et. al. in [37] obtained a comparable outcome for the Navier–Stokes–Fourier system. The conclusions of [37] and [55] hold for domains Ω which admit a sequence of conformal shape regular sequence tetrahedral meshes (see Subsection 5.2.2 for the definition). Particularly, Ω itself has to be a polyhedron, and the existence of such meshes is only assumed.

One can ask if a similar numerical scheme can be constructed when the domain is "physical" (e.g. bounded with a smooth boundary). Affirmative answer in the case of the isentropic model was offered in [36] and for the Navier–Stokes–Fourier system in [35]. The latter is presented in Chapter 5. *Therein, the convergence of numerical solutions to the weak solutions of the Navier–Stokes–Fourier system is provided by Theorem 5.3.1.*

It should be noted that Theorem 5.3.1 gives only the existence of a "convergent subsequence" (*conditional convergence*). However, when Ω has regular boundary (at least $C^{2+\nu}$ with $\nu > 0$), the situation is more pleasant. This issue is discussed in Section 5.7, which also an important contribution of the paper.[6]

The rest of this section is devoted to the main contributions of Chapter 5.

[4] Although it might seem surprising, it is a consequence of the fact that the entropy s is a priori bounded in $L^\infty((0,T) \times \Omega)$ by the initial data and $Z = \varrho e^{s/\gamma}$.

[5] We believe that this part of the proof might find some applications for similarly degenerate evolutionary differential equations.

[6] This result is not available in [37] as the domain Ω is polygonal (i.e. without adequate regularity).

2.2.1 Approximation of the physical domain

As it has been mentioned, the previous result for the Navier–Stokes–Fourier system has been known only for a fixed boundary Ω. Hence, the first issue covered by Chapter 5 is that of convergence of polyhedral domains Ω_h to Ω.

The assumptions on Ω sufficient for the proof of Theorem 5.3.1 to hold are the following:

- $\partial\Omega \in C^1$,
- there exists a family of polyhedral domains $\{\Omega_h\}_h$ such that each Ω_h admits a conformal shape regular tetrahedral mesh,
- Ω_n approximates Ω in the following sense

$$\Omega \subset \overline{\Omega} \subset \Omega_h \subset \left\{ x \in \mathbb{R}^3 \;\middle|\; \mathrm{dist}[x, \Omega] < h \right\}. \tag{2.7}$$

Let us mention that the properties of the tetrahedral meshes are very strict. However, it turns out that any bounded domain Ω with C^1 boundary complies with the presented assumptions (see [51]), which makes our principal result very general.[7] Let us point out that (2.7) is rather weak notion of convergence and gives us no information about a possible convergence of the normal vector fields of Ω_n. This causes a delicate problem in connection with the homogeneous Neumann boundary conditions for the temperature.

Having discretizations with respect to space (i.e. meshes on Ω_n) specified, we can turn our attention to the construction of the numerical scheme (e.g. by discretizing some spaces of functions and differential operators). As this is not the original part of the paper, we postpone the discussion concerning the numerical schemes to Chapter 5. We can also refer to [37] or [55] for more detail about numerical methods.

2.2.2 The main theorem and important parts of the proof

Firstly, let us sketch the common strategy used in the numerical analysis to obtain a convergence result.

Having a numerical scheme (with a discretization parameter h) for a differential equation formulated, it is important to check whether it is *stable* and *consistent* with the given differential equation. Broadly speaking, "stability" consists of showing a uniform control over the approximate solutions in suitable discrete spaces of functions (this part should remind us a priori estimates for the Navier–Stokes–Fourier system). The consistency means that the numerical scheme is consistent with the primal differential equation. To this end, we plug the numerical solution into the weak formulation. If the error goes to zero as $h \to 0$ we call the scheme consistent. The consistency formulation is also suitable if we want to adopt the compactness techniques mentioned in Subsection 1.2.2.

Many proofs in Chapter 5 are skipped as they are the same as in [37] with a slight difference - the meshes are *unfitted*, i. e. $\cup E_n \neq \overline{\Omega}$. Due to the boundary

[7] We note that a suitable sequence of domains Ω_n might be constructed based on a special periodic tessellations of $\mathbb{R}^3$.

approximation (2.7), this causes no problem except one case - the consistency of the discrete thermal method. This part is original and perhaps a bit surprising. The weak formulation of the thermal energy balance is tested by smooth functions ϕ satisfying $\nabla_x \phi \cdot \mathbf{n}$. But to control the approximation error in the proof of consistency, one has to control

$$\int_{\partial\Omega_h} \mathcal{K}(\vartheta) \nabla_x \phi \cdot \mathbf{n} \, \mathrm{d}S_x. \tag{2.8}$$

The problem is that the normal vector fields of Ω_n doesn't necessarily converge to those of Ω.[8] Despite this problem, the convergence is established based on Lemma 5.6.1.

2.2.3 Additional remarks

Finally, we remark that all results on the convergence of numerical solutions to the weak solutions of the Compressible Navier–Stokes assume $\gamma > 3$ (see also the discussion in Subsestion 1.2.3).

2.3 Application of convex integration on different models in fluid mechanics

The last article, which is presented in the thesis (in Chapter 6), considers inviscid versions of models used in meteorology on the scales typical for oceans or the atmosphere (see e.g. [81, Chapter II, 7]). There are at least two reasons why the topic has its place in the thesis. Firstly, the models considered in Chapter 6 fall into the category of differential equations of fluid mechanics. Secondly, it provides the existence of global weak solutions for quite general initial data independent of their size. On the other hand, the mathematical techniques leading to the global existence cannot be comparable with the week compactness (or energy) methods presented in Chapter 3–5.

We consider two models of *inviscid incompressible fluid*, namely - inviscid Boussinesq equation (6.1) and inviscid primitive equations (6.2) (their viscous counterparts appear e.g. in [61] and [62]). *The main propositions - Theorem 6.2.1, Theorem 6.2.2 and Corollary 6.2.4 - establish the global existence of weak solutions for both models.* It should be noted that the conclusion might be surprising at least in the case of the primitive equations. Due to the structure of the primitive equations, not too much is known about the energy estimates. We are only aware of the local existence of strong solutions *in two dimensions* for the so–called *homogeneous hydrostatic equations* (see e.g. [4], [48]) which is equivalent to the inviscid primitive equations with constant temperature.[9] It is also known that for a special choice of initial data, smooth solutions of the three–dimensional inviscid primitive equations develop singularities in finite time (see [10]). The three–dimensional scenario is quite open and even existence of local strong solutions

[8]This situation occurs canonically in the case when Ω_n are constructed using periodic tessellations, which is the case for construction of Ω_n in [51].

[9]The results of Chapter 6 also apply on the homogeneous hydrostatic equations.

emanating from "general" initial data is not known.[10] In this sense, Chapter 6 contains so far the first existence result in three spacial dimensions for general initial data. The usual drawback of the method of convex integration is that the solutions from Chapter 6 are far from being unique.

2.3.1 Used method

The method of convex integration was developed to solve problems emanating from differential geometry. Quite recently, the first occurrence is in [22] for the incompressible Euler system, it was shown to be applicable on differential equations of fluid mechanics. Canonically, the convex integration provides the existence of infinitely many weak solutions of a single initial–value problem. In other words, some inviscid models of fluid mechanics are extremely underdetermined on the level of weak solutions.[11] The proofs of our main results use the technique known from [22] and extended in [33] on the systems similar to the incompressible Euler equations.

However, there is a qualitative difference between the incompressible Euler equations and the primitive equations. The latter contain degenerate equation (6.2c) so it seems that the systematic treatment known for the former cannot be applied. The contrary is true. Quite surprisingly, if we formally overdetermine the primitive equations by adding a suitable evolution equation, we can adapt the approach of [33].

Additional discussion and relevant references are presented in Chapter 6.

[10] The technique of [4] cannot be directly extended as they depend on the form of $\mathrm{div}_x\, \boldsymbol{u} = 0$ in two dimensions.

[11] It is still a challenging open problem to find a criterion which would ensure uniqueness of weak solutions in these cases.

3. Stability result for Navier-Stokes Equations with Entropy Transport

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Abstract

A stability result for the compressible Navier-Stokes system with transport equation for entropy s is shown. The proof comes as an outcome of the isentropic case and additional properties of the effective viscous flux. We deal with the pressure term in the form $\varrho^\gamma e^s$ with adiabatic index $\gamma > 3/2$; therefore the crucial renormalization method is restricted.

3.1 Introduction

Our aim is to show a stability result for global solutions of the compressible Navier-Stokes system supplemented by the transport equation for a scalar quantity (Theorem 3.3.1 and Corollary 3.3.3). Influence of this quantity on the pressure term is also considered. Systems of this kind are limit models for the Navier-Stokes-Fourier system when the thermal conduction coefficient is taken zero and the heating from viscous dissipation can be neglected. Such models arise e.g. in meteorology, see [56].

The considered system reads

$$\partial_t \varrho + \operatorname{div}(\varrho \mathbf{u}) = 0 \tag{3.1}$$

$$\partial_t(\varrho \mathbf{u}) + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) - \mu \triangle \mathbf{u} - (\mu + \lambda)\nabla \operatorname{div} \mathbf{u} + \nabla p(\varrho, s) = \varrho \mathbf{f} \tag{3.2}$$

$$\partial_t s + \nabla s \cdot \mathbf{u} = 0, \tag{3.3}$$

where ϱ, s, are scalar unknown functions on $\Omega \times (0,T)$ and $\mathbf{u}\colon \Omega \times (0,T) \to \mathbb{R}^3$.[1] We suppose $\Omega \subseteq \mathbb{R}^3$ to be a bounded domain with Lipschitz boundary. We also suppose homogeneous Dirichlet condition for $\mathbf{u}$.[2]

We assume that $\mu > 0$ and $\lambda + 2/3\mu > 0$ (which is the widely used assumption) and add the following constitutive relation for the pressure term

$$p(\varrho, s) = \varrho^\gamma \mathcal{T}(s), \tag{3.4}$$

where $\mathcal{T}$ is a continuous and positive function. We also consider initial data ϱ_0, $(\varrho \mathbf{u})_0$ and s_0.

First result on stability of the system (3.1), (3.2) with the transport equation was published by P.-L. Lions under rather non-physical assumption $\gamma > 9/5$,

[1] We use the classical terminology for unknown functions - *density function* for ϱ, *velocity vector field* for $\mathbf{u}$ and *momentum vector field* for $\varrho \mathbf{u}$ and *entropy* for s.

[2] In cases when Ω is the whole space or torus (with periodic boundary conditions on $\mathbf{u}$) we can adapt analogous techniques and obtain the same result.

see Chapter 5 and Chapter 8 of [65]. The result (for $\gamma > 9/5$) was then used by Bresch et al. in [5] where is shown that the low Mach number limit for the considered system is the compressible isentropic Navier-Stokes equation.

Existence of solutions for the compressible Navier-Stokes system with equation for temperature of parabolic type and $\gamma > 3/2$ was provided by Feireisl, see e. g. [32]. For $\gamma < 9/5$ no results have been published if the parabolic equation for temperature is replaced by less regular transport equation for entropy.

We show a kind of stability result for solutions under mild assumtions on the sequence of densities. We apply schemes from [65] and [32]. The lack of space regularity for density in case $\gamma < 9/5$ unables us to renormalize the continuity equation (3.1) using renormalization techniques including defect measures provided by [31]. The main reason is that in the polytropic case (i. e. with non-constant entropy) the pressure is not a monotone function of density but rather of $\tilde{\varrho} = \varrho \mathcal{T}(s)^{1/\gamma}$. We use invariance of the transport equation (Lemma 3.3.2) with respect to renormalization. This gives two consequences - one can work with a more suitable form of the pressure term, namely $\mathcal{T}(s) = 1/s$, and one can combine the continuity equation for density and the transport equation for entropy to conclude thee continuity equation for $\tilde{\varrho}$. Then it is possible to use techniques from [32] to show convergence of the pressure term. However, we cannot provide strong convergence of either ϱ_n or s_n (only of $\tilde{\varrho_n}$). The main problem then lies in convergence of $s_n \operatorname{div} u_n$, which can be treated due to a generalized form (Lemma 3.4.2) of so called effective viscous flux identity.

We specify the difference between this result and the result of Lions. In the case $\gamma > 9/5$ it is possible to improve integrability of the limit density, namely $\varrho \in L^2\big((0,T);L^2(\Omega)\big)$. Under this condition one can renormalize the continuity equation for ϱ without additional assumptions. If γ is only greater then 3/2, the structure of the momentum equation is needed[3] to show that the continuity equation for ϱ can be renormalized. But as was already mentioned, this structure works for $\tilde{\varrho}$ and not for ϱ.

3.1.1 Weak formulation

We call a triplet

$$(\varrho, s, \mathbf{u}) \in L^\infty\big((0,T);L^\gamma(\Omega)\big) \times \cap_{q\geq 1} L^\infty\big((0,T);L^q(\Omega)\big) \times L^2\big((0,T);W_0^{1,2}(\Omega)\big)$$

a weak solution to (3.1), (3.2) and (3.3) satisfying homogeneous Dirichlet boundary condition and initial conditions ϱ_0, $(\varrho u)_0$ and s_0 if

[3] At least, it is not known if the continuity equation with such low integrability of the density can be renormalized without some additional structure (e. g. given by the momentum equation).

- equalities (3.1) and (3.3) are satisfied in the sense of distributions, i.e.[4]

$$\int_0^T\int_\Omega \varrho\partial_t\varphi + \int_0^T\int_\Omega \varrho\mathbf{u}\cdot\nabla\varphi = 0 \tag{3.5}$$

$$\left.\begin{aligned} \int_0^T\int_\Omega \varrho\mathbf{u}\cdot\partial_t\eta + \int_0^T\int_\Omega \varrho\mathbf{u}\otimes\mathbf{u} : \nabla\eta + \int_0^T\int_\Omega p(\varrho,s)\operatorname{div}\eta \\ -\mu\int_0^T\int_\Omega \nabla\mathbf{u} : \nabla\eta - \int_0^T\int_\Omega (\lambda+\mu)\operatorname{div}\mathbf{u}\operatorname{div}\eta = \int_{(0,T)\times\Omega} \varrho\mathbf{f}\cdot\eta \end{aligned}\right\} \tag{3.6}$$

$$\int_0^T\int_\Omega s\partial_t\varphi + \int_0^T\int_\Omega s\mathbf{u}\cdot\nabla\varphi - \int_0^T\int_\Omega s\operatorname{div}\mathbf{u}\varphi = 0 \tag{3.7}$$

for any $\varphi\in\mathcal{D}\big((0,T)\times\Omega\big)$ and $\eta\in\mathcal{D}\big((0,T)\times\Omega\big)^3$. Where $\mathcal{D}\big((0,T)\times\Omega\big)$ is the space of C^∞ functions with compact support in $(0,T)\times\Omega$.

- the quantities are in the following sense continuous with respect to time

$$(\varrho,\varrho\mathbf{u},s)\in C\big([0,T];L^\gamma_\omega(\Omega)\big)\times C\big([0,T];L^{m_\infty}_\omega(\Omega)\big)\times\cap_{q\geq1}C\big([0,T];L^q_\omega(\Omega)\big)$$

and $\varrho(0)=\varrho_0$, $(\varrho\mathbf{u})(0)=(\varrho\mathbf{u})_0$, $s(0)=s_0$.

We note that $C\big([0,T];X_\omega\big)$ is the space of continuous functions from $[0,T]$ to Banach space X endowed with the weak topology.

3.2 A priori estimates

We assume in this section $(\varrho,s,\mathbf{u})$ to be a sufficiently smooth solution to (3.1), (3.2) and (3.3) with smooth initial data. Then the entropy is transported along characteristics given by the flow

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{X}(t,x)=\mathbf{u}(t,\mathbf{X}(t,x)). \tag{3.8}$$

As

$$\frac{\mathrm{d}}{\mathrm{d}t}s(t,\mathbf{X}(t,x))=0,$$

the entropy stays bounded by the initial condition for all $t\in[0,T]$. By the same method one can derive a priori non-negativity for the density ϱ (when ϱ_0 is non-negative).

Next, we multiply the momentum equation by $\mathbf{u}$ and integrate both sides over Ω. We obtain (thanks to the continuity equation for ϱ and the boundary condition for u)

$$\begin{aligned}\partial_t\int_\Omega\frac{1}{2}\varrho|\mathbf{u}|^2+\mu\int_\Omega|\nabla\mathbf{u}|^2+(\lambda+\mu)\int_\Omega(\operatorname{div}(\mathbf{u}))^2 \\ -\int_\Omega\mathcal{T}(s)\varrho^\gamma\operatorname{div}(\mathbf{u})=\int_\Omega\varrho\mathbf{f}\cdot\mathbf{u}.\end{aligned} \tag{3.9}$$

[4] When there is no ambiguity, we omit the symbols for variables with respect to which we integrate.

We multiply (3.3) by $\varrho B'(s)$ and use (3.1), where B is a smooth function. We then obtain the renormalized version of the equation, namely

$$\partial_t(\varrho B(s)) + \operatorname{div}(\varrho B(s)\mathbf{u}) = 0. \tag{3.10}$$

Put $B(s) = \mathcal{T}(s)^{1/\gamma}$ and denote $\tilde{\varrho} = B(s)\varrho$. We then derive an estimates similar to the isentropic case $p = p(\varrho)$, instead we deal with the pressure in the form $p = p(\tilde{\varrho})$. We test (3.10) by $C'(\tilde{\varrho})$ and obtain

$$\partial_t(C(\tilde{\varrho})) + \operatorname{div} C(\tilde{\varrho})\mathbf{u} + (C'(\tilde{\varrho})\tilde{\varrho} - C(\tilde{\varrho}))\operatorname{div}\mathbf{u} = 0. \tag{3.11}$$

We then put $C(\tilde{\varrho}) = \tilde{\varrho}P(\tilde{\varrho})$ for

$$P(z) = \int_1^z \frac{q^\gamma}{q^2}\,\mathrm{d}q = \frac{1}{\gamma-1}z^{\gamma-1} - 1 \tag{3.12}$$

and realize that

$$(C'(\tilde{\varrho})\tilde{\varrho} - C(\tilde{\varrho}))\operatorname{div}\mathbf{u} = \tilde{\varrho}^\gamma \operatorname{div}\mathbf{u}.$$

Applying this equality to (3.9) we end with energy equality in form

$$\partial_t \int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \tilde{\varrho}P(\tilde{\varrho})\right) + \mu\int_\Omega |\nabla\mathbf{u}|^2 + (\mu+\lambda)\int_\Omega (\operatorname{div}(\mathbf{u}))^2 = \int_\Omega \varrho\mathbf{f}\cdot\mathbf{u} \tag{3.13}$$

from which can be deduced the following global in time estimates.

Proposition 3.2.1. *Let* $(\varrho, s, \mathbf{u})$ *be a smooth solution to* (3.1)-(3.3) *then*

- s *is bounded in* $L^\infty\big((0,T)\times\Omega\big)$,
- $\tilde{\varrho}$ *and* ϱ *are bounded in* $L^\infty\big((0,T); L^\gamma(\Omega)\big)$ *and nonnegative,*
- $\mathbf{u}$ *is bounded in* $L^2\big((0,T); W_0^{1,2}(\Omega)\big)$,
- $\varrho\mathbf{u}$ *is bounded in* $L^\infty\big((0,T); L^{m_\infty}(\Omega)\big)$,
- $\varrho\mathbf{u}$ *is bounded in* $L^2\big((0,T); L^{m_2}(\Omega)\big)$,

where the bounds depend on the triplet $(\varrho, s, \mathbf{u})$ *only through the initial data. The exponents* m_2 *and* m_∞ *are given by*

$$m_\infty = \frac{2\gamma}{\gamma+1},$$
$$m_2 = \frac{6\gamma}{6+\gamma}.$$

3.3 Weak sequential stability and global existence

We state the main result on the stability of weak solutions. First, observe that if s is a solution of the transport equation and B a differentiable function then (at least formally) $B(s)$ is a solution of the same equation with initial condition

$B(s_0)$. This invariance with respect to renormalization gives us flexibility in the form of the pressure term. We set $\zeta = (\mathcal{T}^{-1}(s))^{1/\gamma}$ and observe that

$$p = \left(\frac{\varrho}{\zeta}\right)^{\gamma}, \quad \tilde{\varrho} = \frac{\varrho}{\zeta}. \tag{3.14}$$

As $\mathcal{T}$ is positive, ζ has values in $(1/C, C)$ for some $C > 0$ if and only if s is bounded. As we will see later, the quantity ϱ/ζ has more suitable form when passing to limit than $\varrho\zeta$.

Theorem 3.3.1. *Let* $(\varrho_n, \mathbf{u}_n, \zeta_n)$ *be a sequence of weak solutions to* (3.1) - (3.3) *with initial data*

$$(\varrho_{n,0}, (\varrho\mathbf{u})_{n,0}, \zeta_{n,0}) \to (\varrho_0, (\varrho\mathbf{u})_0, \zeta_0) \quad \textit{strongly in } L^{\gamma} \times L^{m_\infty} \times L^{\infty}$$

satisfying energy inequality

$$\left[\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \tilde{\varrho}P(\tilde{\varrho})\right)\right]_0^T + \mu\int_\Omega |\nabla\mathbf{u}|^2 + (\mu+\lambda)\int_\Omega (\operatorname{div}(\mathbf{u}))^2 \le \int_0^T\int_\Omega \varrho\mathbf{f}\cdot\mathbf{u} \tag{3.15}$$

for P *given by* (3.12) *and*

$$1/C \le \operatorname{ess\,inf} \zeta_n \le \operatorname{ess\,sup} \zeta_n \le C.$$

Let also $\varrho_n \in L^2\big(0, T; L^2(\Omega)\big)$. *Then there exists a subsequence* $(\varrho_{n_k}, \mathbf{u}_{n_k}, \zeta_{n_k})$ *convergent weakly to a solution to* (3.1)-(3.3) *with initial data* $(\varrho_0, (\varrho\mathbf{u})_0, \zeta_0)$ *and* p *given by* (3.14).

Remark. We emphasize that we do not suppose ϱ_n to be equibounded in $L^2\big((0,T)\times(\Omega)\big)$ because this bound is not given a priori (unless $\gamma \ge 2$).[5] This assumption is essential for renormalization techniques (see Lemma 3.4.1). We note that the commonly used approximation scheme (see [41]) gives such regularity for ϱ_n in the final approximative step.

Proof. (Theorem 3.3.1). Step 1 *- strong convergence of the makeshift density.* We put $\tilde{\varrho}_n = \varrho_n/\zeta_n$ and observe that $p(\varrho,\zeta) = \tilde{\varrho}^{\gamma}$. The function $\tilde{\varrho}$ also satisfies the continuity equation (see Lemma 3.4.1 - recall also that $(\varrho_n, \mathbf{u}_n)$ can be extended from Ω to the whole space by zero). Hence we use the well-known results for the isentropic case (see [32]) and obtain

$$\tilde{\varrho}_n \to \tilde{\varrho} \quad \text{a. e. and also in } C\big([0,T]; L^{\gamma}(\Omega)\big). \tag{3.16}$$

Step 2 *- passing to the limit in the transport equation.* From (3.16) we derive the weak convergence of

$$\varrho_n = \tilde{\varrho}_n\zeta_n \rightharpoonup \tilde{\varrho}\zeta,$$

therefore $\varrho/\zeta = \tilde{\varrho}$ and $\varrho^{\gamma}/\zeta^{\gamma} = \tilde{\varrho}^{\gamma}$. Hence we satisfied the momentum equation.

[5] In the case $\gamma > 9/5$ one can improve the a priori regularity using appropriate test function (e.g. by the Bogovskiĭ operator of the density) to obtain $L^2((0,T); L^2(\Omega))$ bound.

The pair $(\zeta_n, \mathbf{u}_n)$ solves the transport equation in the weak sense, so

$$\int_0^T \int_\Omega \zeta_n \partial_t \phi + \int_0^T \int_\Omega \zeta_n \mathbf{u}_n \cdot \nabla \phi - \zeta_n \operatorname{div} \mathbf{u}_n \phi = 0 \tag{3.17}$$

for any $\phi \in \mathcal{D}(\Omega)$. Passing to the limit in (3.17) we conclude that

$$\int_0^T \int_\Omega \zeta \partial_t \phi + \int_0^T \int_\Omega \zeta \mathbf{u} \cdot \nabla \phi - \overline{\zeta \operatorname{div} \mathbf{u}} \phi = 0.$$

Next we use properties of the effective viscous flux (Lemma 3.4.2) and realize that for any $\phi \in \mathcal{D}([0, T])$ and $\eta \in \mathcal{D}(\Omega)$:

$$\begin{aligned} \lim_{n \to \infty} \int_0^T \int_{\mathbb{R}^3} \phi \eta \left(\tilde{\varrho}_n^\gamma - (2\mu + \lambda) \operatorname{div} \mathbf{u}_n \right) \zeta_n \\ = \int_0^T \int_{\mathbb{R}^3} \phi \eta \left(\overline{\tilde{\varrho}^\gamma} - (2\mu + \lambda) \operatorname{div} \mathbf{u} \right) \zeta. \end{aligned} \tag{3.18}$$

As the sequence $\{\tilde{\varrho}_n\}$ converges strongly, one realizes that

$$\overline{\zeta \operatorname{div} \mathbf{u}} = \zeta \operatorname{div} \mathbf{u}$$

and so $(\zeta, \mathbf{u})$ solves the transport equation in the weak sense. Weak continuity in time of ϱ, $\varrho\mathbf{u}$, ζ and satisfaction of the initial conditions can be shown by standard techniques for evolution equations. □

Remark. The proof did not provide strong (or pointwise) convergence of ζ_n or ϱ_n. We sketch the main obstructions which we cannot avoid. For any continuous B we can renormalize equations for ζ_n and ζ. Then due to Lemma 3.4.2 we deduce that

$$\partial_t (\overline{B(\zeta)} - B(\zeta)) + \mathbf{u} \cdot \nabla (\overline{B(\zeta)} - B(\zeta)) = 0.$$

in the weak sense. Therefore

$$\begin{aligned} \left[\int_\Omega \overline{B(\zeta)}(s, x) - B(\zeta)(s, x) \, \mathrm{d}x \right]_0^t \\ = - \int_0^t \int_\Omega \operatorname{div} \mathbf{u}(s, x) \left(\overline{B(\zeta)}(s, x) - B(\zeta)(s, x) \right) \mathrm{d}x \, \mathrm{d}s \end{aligned}$$

but we cannot utilise Gronwall's lemma, unless $\operatorname{div} \mathbf{u} \in L^\infty\big((0, T) \times \Omega\big)$. However, this a priori bound is not known. One may also try to derive almost everywhere convergence of densities. However, for $\gamma < 9/5$ it is more complex to renormalize the equation of continuity. Approach using defect measures developed in [31] demands compatible structure of the pressure term with the continuity equation for density. The method is straightforwardly applicable only in the case of slight perturbations of the isentropic case $p = p(\varrho)$, namely in the case when $p(\varrho, s) \approx C\varrho^\gamma + r(\varrho, s)$, where r is a term of lower order.

The following claim is a corollary of renormalization techniques. For proof see e.g. Chapter 4 of [32].

Lemma 3.3.2. *Let*

$$(\zeta, \mathbf{u}) \in \Big(L^\infty\big((0,T)\times\Omega\big) \cap C\big([0,T]; L^q_\omega(\Omega)\big)\Big) \times L^2\big((0,T); W^{1,2}(\Omega)\big)$$

be a weak solution to (3.3) *with* $\zeta(0) = \zeta_0 \in L^\infty$. *Then for every* $B \in C(\mathbb{R})$ *is* $(B(\zeta), \mathbf{u})$ *a weak solution to* (3.3) *with* $B(\zeta) \in C\big([0,T]; L^q(\Omega)\big)$ *and* $(B(\zeta))(0) = B(\zeta_0)$.

The next theorem is a straigtforward corollary of Theorem 3.3.1 and Lemma 3.3.2.

Corollary 3.3.3. *Let* $\mathcal{T} \in C(\mathbb{R})$ *be a positive invertible function. Let* $(\varrho_n, s_n, \mathbf{u}_n)$ *be a sequence of weak solutions to* (3.1) - (3.3) *with initial data*

$$(\varrho_{n,0}, (\varrho\mathbf{u})_{n,0}, s_{n,0}) \to (\varrho_0, (\varrho\mathbf{u})_0, s_0) \quad \text{strongly in } L^\gamma \times L^{m_\infty} \times L^\infty$$

and $p = \varrho^\gamma \mathcal{T}(s)$ *satisfying inequality* (3.15) *for* P *given by* (3.12), $\tilde{\varrho} = \varrho \mathcal{T}^{1/\gamma}(s)$ *and* s_n *uniformly bounded in* $L^\infty\big((0,T)\times\Omega\big)$. *Let also* $\varrho_n \in L^2\big(0,T; L^2(\Omega)\big)$. *Then there exists a weak solution to* (3.1) - (3.3) *with the limit initial data and* $p = \varrho^\gamma \mathcal{T}(s)$.

3.4 Auxiliary lemmas

In this section we summarize additional claims which were used in the previous parts. The first one is based on renormalization techniques presented in [26].

Lemma 3.4.1. *Let*

$$(\varrho, \mathbf{u}) \in L^2((0,T); L^2(\mathbb{R}^d)) \times L^2((0,T); W^{1,2}(\mathbb{R}^d))$$

be a weak solution to the continuity equation and

$$(\zeta, \mathbf{u}) \in L^\infty((0,T)\times\mathbb{R}^d) \times L^2((0,T); W^{1,2}(\mathbb{R}^d))$$

a weak solution to a transport equation. Then $(\varrho\zeta, \mathbf{u})$ *is a weak solution to the continuity equation.*

Proof. Let $\eta \in \mathcal{D}(\mathbb{R}^d)$ be a non-negative function with $\|\eta\|_{L^1(\mathbb{R}^d)} = 1$ and denote $\eta_\varepsilon = 1/\varepsilon^n \eta(\cdot/\varepsilon)$. We mollify both equations with respect to the space variables by testing the weak formulation for any $y \in \mathbb{R}^d$ by functions $\eta_\varepsilon(\cdot - y)$. We obtain equations

$$\partial_t[\varrho]_\varepsilon + \operatorname{div}([\varrho]_\varepsilon\mathbf{u}) = \operatorname{div}([\varrho]_\varepsilon\mathbf{u}) - \operatorname{div}([\varrho\mathbf{u}]_\varepsilon), \tag{3.19}$$

$$\partial_t[\zeta]_\varepsilon + \mathbf{u}\cdot\nabla[\zeta]_\varepsilon = \mathbf{u}\cdot\nabla[\zeta]_\varepsilon - [\mathbf{u}\cdot\nabla\zeta]_\varepsilon, \tag{3.20}$$

where $[g]_\varepsilon = g * \eta_\varepsilon$. Then, we multiply (3.19) by $[\zeta]_\varepsilon$ and according to (3.20) we get

$$\begin{aligned} &\partial_t\left([\varrho]_\varepsilon[\zeta]_\varepsilon\right) + \operatorname{div}([\varrho]_\varepsilon[\zeta]_\varepsilon\mathbf{u}) \\ &\quad = \left(\operatorname{div}([\varrho]_\varepsilon\mathbf{u}) - \operatorname{div}([\varrho\mathbf{u}]_\varepsilon)\right)[\zeta]_\varepsilon + \left(\mathbf{u}\cdot\nabla[\zeta]_\varepsilon - [\mathbf{u}\cdot\nabla\zeta]_\varepsilon\right)[\varrho]_\varepsilon. \end{aligned} \tag{3.21}$$

The right hand side converges to zero in $L^1((0,T)\times\mathbb{R}^d)$ due to well-known Friedrich's commutator lemma. The weak convergence of derivatives on the left-hand side is assured by the strong convergence of the mollified functions. □

We recall the celebrated effective viscous flux identity, which can be postulated in a slightly generalized form.

Lemma 3.4.2. *Let* $(\varrho_n, \mathbf{u}_n, s_n)$ *be weak solutions to* (3.1), (3.2) *and* (3.3) *uniformly bounded by a priori estimates and weakly convergent to* $(\varrho, \mathbf{u}, s)$*. Let*

- p_n *be uniformly bounded in* $L^r((0,T)\times\Omega)$ *for some* $r > 1$ *and weakly convergent to* p*,*
- $\sigma_n \rightharpoonup^* \sigma$ *in* $L^\infty((0,T)\times\Omega)$ *with* $\partial_t\sigma_n + \operatorname{div}(\sigma_n u_n) = \kappa_n$ *for* κ_n *boudned in* $L^2((0,T);L^2(\Omega))$*.*

Then after passing to a subsequence, if needed, we obtain

$$\begin{aligned}\lim_{n\to\infty}\int_0^T\int_{\mathbb{R}^3}\phi\eta\big(p_n-(2\mu+\lambda)\operatorname{div}\mathbf{u}_n\big)\sigma_n \\ =\int_0^T\int_{\mathbb{R}^3}\phi\eta\big(p-(2\mu+\lambda)\operatorname{div}\mathbf{u}\big)\sigma\end{aligned}\tag{3.22}$$

for any $\eta \in \mathcal{D}(\Omega)$ *and* $\phi \in \mathcal{D}((0,T))$*.*[6]

Remark. Broadly speaking, the sequence $\{p_n - (2\mu+\lambda)\operatorname{div}\mathbf{u}_n\}$ behaves like L^1 strongly convergent if tested by a bounded solutions of (nonhomogeneous) continuity equation with streamlines induced by $\mathbf{u}_n$.

Remark. The proof of Lemma 3.4.2 follows from the proof for the known special case $\sigma_n = B(\varrho_n)$ and continuity equation in the form

$$\partial_t B(\varrho_n) + \operatorname{div}(B(\varrho_n)\mathbf{u}_n) = (B(\varrho) - B'(\varrho)\varrho)\operatorname{div} u$$

for a B bounded C^1 function with compactly supported $B'(t)$. The only difference is the presence of κ_n, which does not have to be connected with the left hand side of the continuity equation. However, if we take $\kappa_n \rightharpoonup \kappa$ in $L^2\big((0,T);L^2(\Omega)\big)$, then

$$\int_{\mathbb{R}^3}\phi\eta\varrho_n\mathbf{u}_n\cdot\nabla\triangle^{-1}\kappa_n \to \int_{\mathbb{R}^3}\phi\eta\varrho\mathbf{u}\cdot\nabla\triangle^{-1}\kappa$$

as $\varrho_n\mathbf{u}_n$ converges in $L^\infty([0,T];L^{2\gamma/(\gamma+1)}_\omega(\Omega)) \hookrightarrow L^2([0,T];W^{-1,2}(\Omega))$ and

$$\nabla\triangle^{-1}\kappa_n \rightharpoonup \nabla\triangle^{-1}\kappa \quad \text{in } L^2\big((0,T);W^{1,2}(\mathbb{R}^3)\big)$$

because of linearity and degree of the singular operator $\nabla\triangle^{-1}$. For the details (how to deal with other terms) see [65] or [32]. A version of this theorem can be also found in [71].

[6] Compactly supported functions may be extended by zero if needed.

4. Existence of weak solutions for compressible Navier-Stokes equations with entropy transport

Corresponds to the article:

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Abstract

We consider the compressible Navier-Stokes system with variable entropy. The pressure is a nonlinear function of the density and the entropy/potential temperature which, unlike in the Navier-Stokes-Fourier system, satisfies only the transport equation. We provide existence results within three alternative weak formulations of the corresponding classical problem. Our constructions hold for the optimal range of the adiabatic coefficients from the point of view of the nowadays existence theory.

4.1 Introduction

The purpose of this paper is to analyze the model of flow of compressible viscous fluid with variable entropy. Such flow can be described by the compressible Navier-Stokes equations coupled with an additional equation describing the evolution of the entropy. In case when the conductivity is neglected, the changes of the entropy are solely due to the transport and the whole system can be written as:

$$\partial_t \varrho + \operatorname{div}(\varrho \boldsymbol{u}) = 0 \text{ in } (0,T)\times\Omega, \tag{4.1a}$$

$$\partial_t(\varrho s) + \operatorname{div}(\varrho s \boldsymbol{u}) = 0 \text{ in } (0,T)\times\Omega, \tag{4.1b}$$

$$\partial_t(\varrho \boldsymbol{u}) + \operatorname{div}(\varrho \boldsymbol{u}\otimes\boldsymbol{u}) + \nabla p = \operatorname{div}\mathbb{S} \text{ in } (0,T)\times\Omega, \tag{4.1c}$$

where the unknowns are the density $\varrho\colon (0,T)\times\Omega \to \mathbb{R}_+\cup\{0\}$, the entropy $s\colon (0,T)\times\Omega\to\mathbb{R}_+$ and the velocity of fluid $\boldsymbol{u}\colon (0,T)\times\Omega\to\mathbb{R}^3$, and where Ω is a three dimensional domain with a smooth boundary $\partial\Omega$.

The momentum, the continuity and the entropy equations are additionally coupled by the form of the pressure p, we assume that

$$p(\varrho, s) = \varrho^\gamma \mathcal{T}(s), \quad \gamma > 1, \tag{4.2}$$

where $\mathcal{T}(\cdot)$ is a given smooth and strictly monotone function from $\mathbb{R}_+$ to $\mathbb{R}_+$, in particular $\mathcal{T}(s) > 0$ for $s > 0$.

We assume that the fluid is Newtonian and that the viscous part of the stress tensor is of the following form

$$\mathbb{S} = \mathbb{S}(\nabla\boldsymbol{u}) = 2\mu\Big(\mathbb{D}(\boldsymbol{u}) - \frac{1}{3}\operatorname{div}\boldsymbol{u}\mathbb{I}\Big) + \eta \operatorname{div}_x \boldsymbol{u}\mathbb{I}$$

with $\mathbb{D}(\boldsymbol{u}) = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T)$. Viscosity coefficients μ and η are assumed to be constant, hence we can write

$$\operatorname{div} \mathbb{S}(\nabla \boldsymbol{u}) = \mu \Delta \boldsymbol{u} + (\mu + \lambda) \nabla \operatorname{div} \boldsymbol{u}$$

with $\lambda = \eta - \frac{2}{3}\mu$. To keep the ellipticity of the Lamé operator we require that

$$\mu > 0, \quad 3\lambda + 2\mu > 0. \tag{4.3}$$

The system is supplemented by the initial and the boundary conditions:

$$\varrho(0,x) = \varrho_0(x), \ (\varrho s)(0,x) = S_0(x), \ (\varrho \boldsymbol{u})(0,x) = \boldsymbol{q}_0(x), \tag{4.4}$$

$$\boldsymbol{u}_{|(0,T)\times\partial\Omega} = \boldsymbol{0}. \tag{4.5}$$

System (4.1) is a model of motion of compressible viscous gas with variable entropy transported by the flow. The quantity $\theta = [\mathcal{T}(s)]^{1/\gamma}$ can be also interpreted as a potential temperature in which case the pressure (4.2) takes the form $(\varrho\theta)^\gamma$ and has been studied in [39, 60].

We aim at proving the existence of global in time weak solutions to system (4.1). Note that at least for smooth solution the continuity equation (4.1a) allows us to reformulate (4.1b) as a pure transport equation for s, we have

$$\partial_t \varrho + \operatorname{div}(\varrho \boldsymbol{u}) = 0 \text{ in } (0,T) \times \Omega, \tag{4.6a}$$

$$\partial_t s + \boldsymbol{u} \cdot \nabla s = 0 \text{ in } (0,T) \times \Omega, \tag{4.6b}$$

$$\partial_t(\varrho \boldsymbol{u}) + \operatorname{div}(\varrho \boldsymbol{u} \otimes \boldsymbol{u}) + \nabla p = \operatorname{div} \mathbb{S} \text{ in } (0,T) \times \Omega. \tag{4.6c}$$

In contrast to entropy equation in system (4.1) the above form is insensitive to appearance of vacuum states; in fact it is completely decoupled from the continuity equation. The regularity of the density in the compressible Navier-Stokes-type systems is in general rather delicate matter. Therefore, one can expect that proving the existence of solutions to system (4.1) requires more severe assumptions than to get a relevant solution to (4.6). This observation will be reflected in the range of parameter γ which determines the quality of a priori estimates for the argument of the pressure – $Z = \varrho[\mathcal{T}(s)]^{1/\gamma}$ according to the notation from above.

In order to clarify this issue a little more let us introduce a third formulation of system (4.1) describing the evolution of the pressure argument $Z = \varrho[\mathcal{T}(s)]^{\frac{1}{\gamma}}$ instead of the entropy itself. We have:

$$\partial_t \varrho + \operatorname{div}(\varrho \boldsymbol{u}) = 0 \text{ in } (0,T) \times \Omega, \tag{4.7a}$$

$$\partial_t Z + \operatorname{div}(Z \boldsymbol{u}) = 0 \text{ in } (0,T) \times \Omega, \tag{4.7b}$$

$$\partial_t(\varrho \boldsymbol{u}) + \operatorname{div}(\varrho \boldsymbol{u} \otimes \boldsymbol{u}) + \nabla Z^\gamma = \operatorname{div} \mathbb{S} \text{ in } (0,T) \times \Omega. \tag{4.7c}$$

Again, the above formulation is equivalent with the previous ones provided the solution is regular enough, which, however, may not be true in case of weak solutions.

The above discussion motivates distinction between the cases when the evolution of the entropy is described by the continuity, the transport or the renormalized transport equation. Indeed, the form of the entropy equation, although used to describe the same phenomena, is a diagnostic marker indicating the notion of plausible solution to the whole system. Our paper contains an existence analysis for all three systems: (4.1), (4.6) and (4.7) within suitably adjusted definitions of weak solutions. Such an approach allows us to emphasise the implications between the solutions and to better understand the restrictions of renormalization technique. These issues, absent in the analysis of the standard single density systems, are of great importance for more complex multi-component or multi-phase flows. Our results show possible applications of nowadays classical tools in the analysis of the Navier-Stokes system to challenging problems, e.g. constitutive equation involving nonlinear combinations of hyperbolic quantities: densities, concentrations, etc.

The outline of the paper is the following. We first consider system (4.7), for which we are able to show the existence of a weak solution using standard technique available for the compressible Navier–Stokes system, see [41]. Next, using a special form of renormalization, and division of equation (4.7b) by ϱ, we show that we may replace (4.7b) by (4.6b) and finally by (4.1b). We are able to handle (4.6b) as well as (4.7b) for the optimal range of γ's (i.e. $\gamma > \frac{3}{2}$), while getting equation (4.1b) requires the assumption $\gamma \geq \frac{9}{5}$. This is a restriction under which the renormalization theory of DiPerna–Lions [26] can be applied.

In Section 4.2, we introduce the definition of the weak solutions to all three systems mentioned above and present our main existence theorems. Then, in Section 4.3 we recall some specific classical results which are then used in the proof. Further, in Sections 4.4 and 4.5 we prove the existence of weak solutions to system (4.7); we introduce several levels of approximations and prove the existence of solutions at each step by performing relevant limit passages in Sections 4.6 and 4.7. Finally, in Section 4.8 we prove the existence of weak solution to systems (4.1) and (4.6).

4.2 Weak solutions, existence results

Throughout our analysis we naturally distinguish two different situations. They are associated to the magnitude of the adiabatic exponent γ. From the point of view of theory of global in time weak solutions, it is reasonable to assume that

$$\gamma > \frac{3}{2}. \tag{4.8}$$

This assumption provides L^1 bound of the convective term and is necessary for application of nowadays techniques. Under this condition we will first prove the existence of a weak solution to system (4.7), see Theorem 4.2.5. Then we shall deduce from this result existence of weak solutions for the formulation (4.6) still under assumption (4.8), see Theorem 4.2.5. This result is not equivalent to the existence of weak solutions to system (4.1) though. The latter can be proved solely under the restriction

$$\gamma \geq \frac{9}{5}. \tag{4.9}$$

Indeed, the latter more restricted range of γ's enables to obtain L^2 estimate of the density and, as mentioned in the introduction, makes it possible to apply the DiPerna-Lions theory of the renormalized solutions to the transport equation (4.6b) and to multiply it by ϱ within the class of weak solutions.

4.2.1 Weak solutions to system (4.1)

Let us first introduce the definition of a weak solution to our original system (4.1). We assume that the initial data (4.4) satisfy:

$$
\begin{aligned}
&\varrho_0 : \Omega \to \mathbb{R}_+, \;\; s_0 : \Omega \to \mathbb{R}_+, \;\; \boldsymbol{u}_0 : \Omega \to \mathbb{R}^3, \\
&\varrho_0 \in L^\gamma(\Omega), \;\; \int_\Omega \varrho_0 \mathrm{d}x > 0, \\
&S_0 = \varrho_0 s_0, \;\; s_0 \in L^\infty(\Omega), \;\; \boldsymbol{q}_0 = \varrho_0 \boldsymbol{u}_0 \in L^{\frac{2\gamma}{\gamma+1}}(\Omega, \mathbb{R}^3).
\end{aligned}
\tag{4.10}
$$

The choice of nontrivial initial condition for s on the set $\{\varrho_0 = 0\}$ will play an important role in the last section. Indeed, there is a certain difference in the proof of the case $s_0 = const$, and s_0 non-constant on this set. We consider

Definition 4.2.1. *Suppose the initial conditions satisfy (4.10). We say that the triplet $(\varrho, s, \boldsymbol{u})$ is a weak solution of problem (4.1)–(4.5) if:*

$$
(\varrho, s, \boldsymbol{u}) \in L^\infty(0,T; L^\gamma(\Omega)) \times L^\infty((0,T)\times\Omega) \times L^2(0,T; W^{1,2}_0(\Omega, \mathbb{R}^3)), \tag{4.11}
$$

and for any $t \in [0,T]$ we have:

(i) $\varrho \in C([0,T]; L^\gamma_\omega(\Omega))$ and the continuity equation (4.1a) is satisfied in the weak sense

$$
\begin{aligned}
&\int_\Omega \varrho(t,\cdot)\varphi(t,\cdot)\;\mathrm{d}x - \int_\Omega \varrho_0\varphi(0,\cdot)\;\mathrm{d}x \\
&\quad = \int_0^t \int_\Omega \Big(\varrho\partial_t\varphi + \varrho\boldsymbol{u}\cdot\nabla\varphi \Big)\;\mathrm{d}x\;\mathrm{d}\tau, \forall \varphi \in C^1([0,T]\times\overline{\Omega});
\end{aligned}
\tag{4.12}
$$

(ii) $\varrho s \in C([0,T]; L^\gamma_\omega(\Omega))$ and equation (4.1b) is satisfied in the weak sense

$$
\begin{aligned}
&\int_\Omega (\varrho s)(t,\cdot)\varphi(t,\cdot)\;\mathrm{d}x - \int_\Omega S_0\varphi(0,\cdot)\;\mathrm{d}x \\
&\quad = \int_0^t \int_\Omega \Big(\varrho s\partial_t\varphi + \varrho s\boldsymbol{u}\cdot\nabla\varphi \Big)\;\mathrm{d}x\;\mathrm{d}\tau, \forall \varphi \in C^1([0,T]\times\overline{\Omega});
\end{aligned}
\tag{4.13}
$$

(iii) $\varrho\boldsymbol{u} \in C([0,T]; L^{\frac{2\gamma}{\gamma+1}}_\omega(\Omega,\mathbb{R}^3))$ and the momentum equation (4.1c) is satisfied in the weak sense

$$
\begin{aligned}
&\int_\Omega (\varrho\boldsymbol{u})(t,\cdot)\cdot\boldsymbol{\psi}(t,\cdot)\;\mathrm{d}x - \int_\Omega \boldsymbol{q}_0\cdot\boldsymbol{\psi}(0,\cdot)\;\mathrm{d}x = \int_0^t \int_\Omega \Big(\varrho\boldsymbol{u}\cdot\partial_t\boldsymbol{\psi} + \varrho\boldsymbol{u}\otimes\boldsymbol{u} : \nabla\boldsymbol{\psi} \\
&\quad + \varrho^\gamma\mathcal{T}(s)\operatorname{div}\boldsymbol{\psi} - \mathbb{S}(\nabla\boldsymbol{u}) : \nabla\boldsymbol{\psi} \Big)\;\mathrm{d}x\;\mathrm{d}\tau, \forall \boldsymbol{\psi} \in C^1_c([0,T]\times\Omega, \mathbb{R}^3);
\end{aligned}
\tag{4.14}
$$

(iv) the energy inequality

$$
\begin{aligned}
&\mathcal{E}^1(\varrho, s, \boldsymbol{u})(t) + \int_0^t \int_\Omega \Big(\mu|\nabla\boldsymbol{u}|^2 + (\mu+\lambda)(\operatorname{div}\boldsymbol{u})^2 \Big)\;\mathrm{d}x\;\mathrm{d}\tau \\
&\quad \le \mathcal{E}^1(\varrho_0, s_0, \boldsymbol{u}_0)
\end{aligned}
\tag{4.15}
$$

holds for a.a $t \in (0,T)$, where

$$
\mathcal{E}^1(\varrho, s, \boldsymbol{u}) = \int_\Omega \Big(\frac{1}{2}\varrho|\boldsymbol{u}|^2 + \frac{\varrho^\gamma\mathcal{T}(s)}{\gamma-1} \Big)\;\mathrm{d}x.
$$

The first main result concerning solutions meant by Definition 4.2.1 reads.

Theorem 4.2.2. *Let μ, λ satisfy (4.3), $\gamma \geq \frac{9}{5}$ and the initial data $(\varrho_0, S_0, \boldsymbol{q}_0)$ satisfy (4.10). Then there exists a weak solution $(\varrho, s, \boldsymbol{u})$ to problem (4.1)–(4.5) in the sense of Definition 4.2.1.*

4.2.2 Weak solution to system (4.7)

The restriction on γ in Theorem 4.2.2 is obviously not satisfactory as all the physically reasonable values of γ are less or equal that $\frac{5}{3}$. We are able to relax this constraint for system (4.7). Formally, taking $Z = \varrho(\mathcal{T}(s))^{\frac{1}{\gamma}}$ in (4.7) one can recover our original system (4.1). However, for the weak solution this formal argument cannot be made rigorous unless we assume that $\gamma \geq \frac{9}{5}$. Nevertheless, system (4.7) is a good starting point for our considerations. Indeed, for reasonable initial and boundary conditions it can be shown that it possesses a weak solution for $\gamma > \frac{3}{2}$, using more or less standard approach. Proving existence of solutions directly for system (4.1) seems not to be so simple.

We assume that the initial data for system (4.7) are

$$\varrho_0 : \Omega \to \mathbb{R}_+, \quad s_0 : \Omega \to \mathbb{R}_+, \quad \boldsymbol{u}_0 : \Omega \to \mathbb{R}^3,$$

$$\varrho(0, x) = \varrho_0(x), \ Z(0, x) = Z_0(x), \ (\varrho\boldsymbol{u})(0, x) = \boldsymbol{q}_0(x) = \varrho_0\boldsymbol{u}_0(x), \tag{4.16}$$

and they satisfy

$$\left.\begin{array}{r}(\varrho_0, Z_0) \in L^\gamma(\Omega)^2, \quad \varrho_0, Z_0 \geq 0 \text{ a.e. in } \Omega, \quad \displaystyle\int_\Omega \varrho_0 \,\mathrm{d}x > 0, \\ 0 \leq c_\star\varrho_0 \leq Z_0 \leq c^\star\varrho_0 \text{ a.e. in } \Omega, \quad 0 < c_\star \leq c^\star < \infty, \boldsymbol{q}_0 \in L^{\frac{2\gamma}{\gamma+1}}(\Omega, \mathbb{R}^3).\end{array}\right\} \tag{4.17}$$

Then we have

Definition 4.2.3. *Suppose that the initial conditions satisfy (4.17). We say that the triplet $(\varrho, Z, \boldsymbol{u})$ is a weak solution of problem (4.7) with the initial and boundary conditions (4.5), (4.16) if*

$$(\varrho, Z, \boldsymbol{u}) \in L^\infty(0, T; L^\gamma(\Omega)) \times L^\infty(0, T; L^\gamma(\Omega)) \times L^2(0, T; W_0^{1,2}(\Omega, \mathbb{R}^3)), \tag{4.18}$$

and for any $t \in (0, T]$ we have:

(i) $\varrho \in C([0, T]; L^\gamma_\omega(\Omega))$ and the continuity equation (4.7a) is satisfied in the weak sense

$$\begin{aligned}&\int_\Omega \varrho(t, \cdot)\varphi(t, \cdot) \,\mathrm{d}x - \int_\Omega \varrho_0\varphi(0, \cdot) \,\mathrm{d}x \\ &= \int_0^t \int_\Omega \Big(\varrho\partial_t\varphi + \varrho\boldsymbol{u} \cdot \nabla\varphi\Big) \,\mathrm{d}x \,\mathrm{d}\tau, \forall\varphi \in C^1([0, T] \times \overline{\Omega});\end{aligned} \tag{4.19}$$

(ii) $Z \in C([0, T]; L^\gamma_\omega(\Omega))$ and equation (4.7b) is satisfied in the weak sense

$$\begin{aligned}&\int_\Omega Z(t, \cdot)\varphi(t, \cdot) \,\mathrm{d}x - \int_\Omega Z_0\varphi(0, \cdot) \,\mathrm{d}x \\ &= \int_0^t \int_\Omega \Big(Z\partial_t\varphi + Z\boldsymbol{u} \cdot \nabla\varphi\Big) \,\mathrm{d}x \,\mathrm{d}\tau, \forall\varphi \in C^1([0, T] \times \overline{\Omega});\end{aligned} \tag{4.20}$$

(iii) $\varrho\boldsymbol{u} \in C([0,T]; L_\omega^{\frac{2\gamma}{\gamma+1}}(\Omega,\mathbb{R}^3))$ *and the momentum equation (4.1c) is satisfied in the weak sense*

$$\int_\Omega (\varrho\boldsymbol{u})(t,\cdot)\cdot\boldsymbol{\psi}(t,\cdot)\,\mathrm{d}x - \int_\Omega \boldsymbol{q}_0\cdot\boldsymbol{\psi}(0,\cdot)\,\mathrm{d}x = \int_0^t\int_\Omega \Big(\varrho\boldsymbol{u}\cdot\partial_t\boldsymbol{\psi} + \varrho\boldsymbol{u}\otimes\boldsymbol{u} : \nabla\boldsymbol{\psi}$$
$$+ Z^\gamma \operatorname{div}\boldsymbol{\psi} - \mathbb{S}(\nabla\boldsymbol{u}) : \nabla\boldsymbol{\psi}\Big)\,\mathrm{d}x\,\mathrm{d}\tau, \forall\boldsymbol{\psi} \in C_c^1([0,T]\times\Omega,\mathbb{R}^3); \quad (4.21)$$

(iv) the energy inequality

$$\mathcal{E}^2(\varrho, Z, \boldsymbol{u})(t) + \int_0^t\int_\Omega \Big(\mu|\nabla\boldsymbol{u}|^2 + (\mu+\lambda)(\operatorname{div}\boldsymbol{u})^2\Big)\,\mathrm{d}x\,\mathrm{d}\tau$$
$$\le \mathcal{E}^2(\varrho_0, Z_0, \boldsymbol{u}_0) \quad (4.22)$$

holds for a.a $t \in (0,T)$*, where*

$$\mathcal{E}^2(\varrho, Z, \boldsymbol{u}) = \int_\Omega \Big(\frac{1}{2}\varrho|\boldsymbol{u}|^2 + \frac{Z^\gamma}{\gamma-1}\Big)\,\mathrm{d}x. \quad (4.23)$$

Before presenting the existence result for the auxiliary problem, let us recall the definition of a renormalized solution to equation (4.7b):

Definition 4.2.4. *We say that equation (4.7b) holds in the sense of renormalized solutions, provided* $(Z,\boldsymbol{u})$*, extended by zero outside of* Ω*, satisfy*

$$\partial_t b(Z) + \operatorname{div}(b(Z)\boldsymbol{u}) + \big(b'(Z)Z - b(Z)\big)\operatorname{div}\boldsymbol{u} = 0 \text{ in } \mathcal{D}'((0,T)\times\mathbb{R}^3), \quad (4.24)$$

where

$$b \in C^1(\mathbb{R}), \quad b'(z) = 0, \quad \forall z \in \mathbb{R} \text{ large enough.} \quad (4.25)$$

We have the following existence result for solutions defined by Definition 4.2.3

Theorem 4.2.5. *Let* μ, λ *satisfy (4.3),* $\gamma > \frac{3}{2}$*, and the initial data* $(\varrho_0, Z_0, \boldsymbol{q}_0)$ *satisfy (4.17).*

Then there exists a weak solution $(\varrho, Z, \boldsymbol{u})$ *to problem (4.7) with boundary conditions (4.5), in the sense of Definition 4.2.3. Moreover,* $(Z,\boldsymbol{u})$ *solves (4.7b) in the renormalized sense and*

$$0 \le c_\star\varrho \le Z \le c^\star\varrho$$

a.e. in $(0,T)\times\Omega$.

4.2.3 Weak solution to system (4.6)

If we replace (4.1b) by (4.6b) (using also the renormalization of the latter), the result is also much better than in Theorem 4.2.2, in fact optimal from the point of view of nowadays theory of compressible Navier–Stokes equations. In order to formulate the result precisely, we first rewrite system (4.6) in a slightly different way. We look for a triplet $(\varrho, \zeta, \boldsymbol{u})$ solving the system of equations

$$\partial_t\varrho + \operatorname{div}(\varrho\boldsymbol{u}) = 0, \quad (4.26a)$$

$$\partial_t \zeta + \boldsymbol{u} \cdot \nabla \zeta = 0, \tag{4.26b}$$

$$\partial_t(\varrho \boldsymbol{u}) + \operatorname{div}(\varrho \boldsymbol{u} \otimes \boldsymbol{u}) + \nabla \left(\frac{\varrho}{\zeta}\right)^\gamma = \operatorname{div} \mathbb{S}(\nabla \boldsymbol{u}), \tag{4.26c}$$

with initial conditions

$$\varrho(0, x) = \varrho_0(x),\ \zeta(0, x) = \zeta_0(x),\ (\varrho \boldsymbol{u})(0, x) = \boldsymbol{q}_0(x), \tag{4.27}$$

such that $\zeta_0 = \frac{\varrho_0}{Z_0}$ and satisfying assumptions (4.17), in particular

$$\zeta_0 \in \Big((c^\star)^{-1}, (c_\star)^{-1}\Big). \tag{4.28}$$

Then the weak solution is defined as follows.

Definition 4.2.6. *Suppose the initial conditions* $(\varrho_0, \zeta_0, \boldsymbol{q}_0)$ *satisfy (4.28) and (4.17) (for* ϱ_0 *and* $\boldsymbol{q}_0$*), We say that the triplet* $(\varrho, \zeta, \boldsymbol{u})$ *is a weak solution of problem (4.26) emanating from the initial data* $(\varrho_0, \zeta_0, \boldsymbol{q}_0)$ *if*

$$(\varrho, \zeta, \boldsymbol{u}) \in L^\infty(0, T; L^\gamma(\Omega)) \times L^\infty((0, T) \times \Omega) \times L^2(0, T; W_0^{1,2}(\Omega, \mathbb{R}^3)), \tag{4.29}$$

and for any $t \in (0, T]$ *we have:*

(i) $\varrho \in C([0, T]; L^\gamma_\omega(\Omega))$ *and the continuity equation (4.26a) is satisfied in the weak sense*

$$\int_\Omega \varrho(t, \cdot)\varphi(t, \cdot)\ \mathrm{d}x - \int_\Omega \varrho_0 \varphi(0, \cdot)\ \mathrm{d}x = \int_0^t \int_\Omega \Big(\varrho \partial_t \varphi + \varrho \boldsymbol{u} \cdot \nabla \varphi\Big)\ \mathrm{d}x\ \mathrm{d}\tau, \forall \varphi \in C^1([0, T] \times \overline{\Omega}); \tag{4.30}$$

(ii) $\zeta \in C([0, T]; L^\infty_\omega(\Omega))$ *and equation (4.26b) is satisfied in the weak sense*

$$\int_\Omega \zeta(T, \cdot)\varphi(T, \cdot)\ \mathrm{d}x - \int_\Omega \zeta_0 \varphi(0, \cdot)\ \mathrm{d}x = \int_0^t \int_\Omega \Big(\zeta \partial_t \varphi + \zeta \operatorname{div}(\boldsymbol{u}\varphi)\Big)\ \mathrm{d}x\ \mathrm{d}\tau, \forall \varphi \in C^1([0, T] \times \overline{\Omega}); \tag{4.31}$$

(iii) $\varrho \boldsymbol{u} \in C([0, T]; L_\omega^{\frac{2\gamma}{\gamma+1}}(\Omega, \mathbb{R}^3))$ *and the momentum equation (4.26c) is satisfied in the weak sense*

$$\int_\Omega (\varrho \boldsymbol{u})(t, \cdot) \cdot \boldsymbol{\psi}(t, \cdot)\ \mathrm{d}x - \int_\Omega \boldsymbol{q}_0 \cdot \boldsymbol{\psi}(0, \cdot)\ \mathrm{d}x = \int_0^t \int_\Omega \Big(\varrho \boldsymbol{u} \cdot \partial_t \boldsymbol{\psi} + \varrho \boldsymbol{u} \otimes \boldsymbol{u} : \nabla \boldsymbol{\psi} + \left(\frac{\varrho}{\zeta}\right)^\gamma \operatorname{div} \boldsymbol{\psi} - \mathbb{S}(\nabla \boldsymbol{u}) : \nabla \boldsymbol{\psi}\Big)\ \mathrm{d}x\ \mathrm{d}\tau, \forall \boldsymbol{\psi} \in C^1_c([0, T] \times \Omega, \mathbb{R}^3); \tag{4.32}$$

(iv) the energy inequality

$$\mathcal{E}^2(\varrho, \varrho/\zeta, \boldsymbol{u})(t) + \int_0^t \int_\Omega \Big(\mu |\nabla \boldsymbol{u}|^2 + (\mu + \lambda)(\operatorname{div} \boldsymbol{u})^2\Big)\ \mathrm{d}x\ \mathrm{d}\tau \leq \mathcal{E}^2(\varrho_0, \varrho_0/\zeta_0, \boldsymbol{u}_0) \tag{4.33}$$

holds for a.a $t \in (0, T)$*, where* $\mathcal{E}^2$ *is defined through (4.23).*

The last result concerns the existence of solutions meant by Definition 4.2.6.

Theorem 4.2.7. *Let μ, λ satisfy (4.3), $\gamma > \frac{3}{2}$, and the initial data $(\varrho_0, \zeta_0, \boldsymbol{q}_0)$ satisfy (4.28) and (4.17) (for ϱ_0 and $\boldsymbol{q}_0$).*

Then there exists a weak solution $(\varrho, \zeta, \boldsymbol{u})$ to problem (4.26) with boundary conditions (4.5), in the sense of Definition 4.2.6. Moreover, $(\varrho, \boldsymbol{u})$ solves (4.26a) and $(\zeta, \boldsymbol{u})$ solves (4.26b) in the renormalized sense.

Using the result of Theorem 4.2.7, we may easily obtain a solution to system (4.6). Indeed, we may define

$$s = \mathcal{T}^{-1}(\zeta^{-\gamma})$$

and use the fact that equation (4.26b) holds in the renormalized sense.

Remark 4.2.8. *Note that in two space dimensions, all results hold for any $\gamma > 1$. In both two and three space dimensions, we can also include a non-zero external force on the right-hand side of the momentum equation, i.e. we have additionally the term $\varrho\boldsymbol{f}$ on the right-hand side of (4.1c), (4.6c) and (4.7c). For $\boldsymbol{f} \in L^\infty((0,T)\times\Omega, \mathbb{R}^3)$ we would get the same results as in Theorems 4.2.2, 4.2.5 and 4.2.7.*

4.3 Auxiliary results

Before proving our main theorems, we recall several auxiliary results used in this paper. These are mostly standard results and we include them them only for the sake of clarity of presentation.

Lemma 4.3.1. *Let $\mu > 0$, $\lambda + 2\mu > 0$. Then there exists a positive constant c such that*

$$\mu\|\nabla\boldsymbol{u}\|^2_{L^2(\Omega,\mathbb{R}^{3\times3})} + (\lambda+\mu)\|\operatorname{div}\boldsymbol{u}\|_{L^2(\Omega)} \geq c\|\nabla\boldsymbol{u}\|_{L^2(\Omega,\mathbb{R}^{3\times3})}. \tag{4.34}$$

Lemma 4.3.2. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$. If $g_n \to g$ in $C([0,T]; L^q_\omega(\Omega))$, $1 < q < \infty$ then $g_n \to g$ strongly in $L^p(0,T; W^{-1,r}(\Omega))$ provided $L^q(\Omega) \hookrightarrow\hookrightarrow W^{-1,r}(\Omega)$.*

Note that $L^q(\Omega) \hookrightarrow\hookrightarrow W^{-1,r}(\Omega)$ holds for Ω a Lipschitz domain in $\mathbb{R}^3$ for $1 \leq r \leq \frac{3}{2}$ if $q > 1$ arbitrary or for $\frac{3}{2} < r < \infty$ provided $q > \frac{3r}{3+r}$.

Lemma 4.3.3. *Let $1 \leq q < \infty$. Let the sequence $g_n \in C([0,T], L^q_\omega(\Omega))$ be bounded in $L^\infty(0,T; L^q(\Omega))$. Then it is uniformly bounded on $[0,T]$. More precisely, we have*

$$\operatorname{ess\,sup}_{t\in(0,T)} \|g_n(t)\|_{L^q(\Omega)} \leq C \Rightarrow \sup_{t\in[0,T]} \|g_n(t)\|_{L^q(\Omega)} \leq C, \tag{4.35}$$

where c is a positive constant independent of n.

Lemma 4.3.4. *Let $1 < p, q < \infty$ and let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$. Let $\{g_n\}_{n\in\mathbb{N}}$ be a sequence of functions defined on $[0,T]$ with values in $L^q(\Omega)$ such that*

$$g_n \in C([0,T], L^q_\omega(\Omega)), \quad \begin{cases} g_n \text{ is uniformly continuous in } W^{-1,p}(\Omega) \\ g_n \text{ and uniformly bounded in } L^q(\Omega). \end{cases} \tag{4.36}$$

Then, at least for a chosen subsequence

$$g_n \to g \text{ in } C([0,T], L^q_\omega(\Omega)). \tag{4.37}$$

If, moreover, $L^q(\Omega) \hookrightarrow\hookrightarrow W^{-1,p}(\Omega)$*, then*

$$g_n \to g \qquad \text{in } C([0,T]; W^{-1,p}(\Omega)). \tag{4.38}$$

Next, let us consider weak solutions to the continuity equation

$$\partial_t Z + \operatorname{div}(Z\boldsymbol{u}) = 0, \quad Z(0,\cdot) = Z_0(\cdot). \tag{4.39}$$

As a result of the DiPerna–Lions [26] theory we have

Lemma 4.3.5. *Assume* $Z \in L^q((0,T)\times\Omega)$ *and* $\boldsymbol{u} \in L^2(0,T;W^{1,2}_0(\Omega))$*, where* $\Omega \subset \mathbb{R}^3$ *is a domain with Lipschitz boundary. Let* $(Z,\boldsymbol{u})$ *be a weak solution to (4.39) and* $q \geq 2$*. Then* $(Z,\boldsymbol{u})$ *is also a renormalized solution to (4.39), i.e. it solves (4.24) in the sense of distributions on* $(0,T)\times\mathbb{R}^3$ *provided* Z*,* $\boldsymbol{u}$ *are extended by zero outside of* Ω*.*

Remark 4.3.6. *By density argument and standard approximation technique, we may extend the validity of (4.24) to functions* $b \in C([0,\infty) \cap C^1(0,\infty))$ *such that*

$$|b'(t)| \leq Ct^{-\lambda_0}, \qquad \lambda_0 < -1, \quad t \in (0,1],$$

$$|b'(t)| \leq Ct^{\lambda_1}, \qquad -1 < \lambda_1 \leq \frac{q}{2} - 1, \quad t \geq 1.$$

Lemma 4.3.7. *Let*

$$(s,\mathbf{u}) \in \Big(L^\infty((0,T)\times\Omega) \cap C([0,T]; L^q_\omega(\Omega))\Big) \times L^2(0,T; W^{1,2}(\Omega,\mathbb{R}^3))$$

be a weak solution to (4.6b) with $s(0,\cdot) = s_0 \in L^\infty(\Omega)$*. Then for every* $B \in C(\mathbb{R})$*,* $(B(s),\boldsymbol{u})$ *is a distributional solution to (4.6b), i.e.*

$$\partial_t B(s) + \boldsymbol{u}\cdot\nabla B(s) = 0$$

in $\mathcal{D}'((0,T)\times\Omega)$*. Moreover,* s *and* $B(s) \in C([0,T]; L^r(\Omega))$ *for all* $r < \infty$ *and* $B(s)(0,\cdot) = B(s_0)$*.*

In some situations when the DiPerna–Lions theory is not applicable, i.e. when $q < 2$ in Lemma 4.3.5, we can still prove that the solution is in fact a renormalized one using the approach from [31]. To this purpose one has to consider the oscillation defect measure of the sequence Z_δ approximating Z, i.e.

$$\operatorname{osc}_q(Z_\delta - Z) = \sup_{k\in\mathbb{N}} \limsup_{\delta\to 0^+} \|T_k(Z_\delta) - T_k(Z)\|_{L^q((0,T)\times\Omega)}, \tag{4.40}$$

where

$$T_k(z) = kT\Big(\frac{z}{k}\Big), \quad z \in \mathbb{R}, \quad k \geq 1, \tag{4.41}$$

with $T \in C^\infty(\mathbb{R})$ such that

$$T(z) = z \text{ for } z \leq 1, \quad T(z) = 2 \text{ for } z \geq 3, \quad T \text{ concave, non-decreasing.} \tag{4.42}$$

We have

Lemma 4.3.8. *Let $\Omega \subset \mathbb{R}^3$ a domain with Lipschitz boundary. Assume that $(Z_\delta, \boldsymbol{u}_\delta)$ is a sequence of renormalized solutions to the continuity equation such that*

$$Z_\delta \to Z \qquad \text{weakly in } L^1((0,T) \times \Omega),$$

$$\boldsymbol{u}_\delta \to \boldsymbol{u} \qquad \text{weakly in } L^2(0,T; W_0^{1,2}(\Omega, \mathbb{R}^3))$$

such that $\operatorname{osc}_q(Z_\delta - Z) < \infty$ for some $q > 2$. Then $(Z, \boldsymbol{u})$ is a renormalized solution to the continuity equation.

We further need the following well-known result [20,69] concerning the solution operator to the problem

$$\begin{aligned} \operatorname{div} \boldsymbol{v} &= f, \\ \boldsymbol{v}|_{\partial\Omega} &= \mathbf{0}. \end{aligned} \tag{4.43}$$

Lemma 4.3.9. *Let Ω be a Lipschitz domain in $\mathbb{R}^3$. For any $1 < p < \infty$ there exists a solution operator $\mathcal{B}\colon \{f \in L^p(\Omega); \int_\Omega f \,\mathrm{d}x = 0\} \to W_0^{1,p}(\Omega, \mathbb{R}^3)$ to (4.43) such that for $\boldsymbol{v} = \mathcal{B}f$ it holds*

$$\|\boldsymbol{v}\|_{W^{1,p}(\Omega)} \le C(\Omega, p) \|f\|_{L^p(\Omega)}.$$

Next, we report the following general result concerning the compensated compactness (see [67] or [74])

Lemma 4.3.10. *Let $\boldsymbol{U}_n$, $\boldsymbol{V}_n$ be two sequences such that*

$$\boldsymbol{U}_n \to \boldsymbol{U} \qquad \text{weakly in } L^p(\Omega, \mathbb{R}^3),$$

$$\boldsymbol{V}_n \to \boldsymbol{V} \qquad \text{weakly in } L^q(\Omega, \mathbb{R}^3),$$

where $\frac{1}{s} = \frac{1}{p} + \frac{1}{q} < 1$, and

$$\operatorname{div} \boldsymbol{U}_n \quad \text{is precompact in } W^{-1,r}(\Omega),$$

$$\operatorname{curl} \boldsymbol{V}_n \quad \text{is precompact in } W^{-1,r}(\Omega, \mathbb{R}^{3\times 3})$$

for a certain $r > 0$. Then

$$\boldsymbol{U}_n \cdot \boldsymbol{V}_n \to \boldsymbol{U} \cdot \boldsymbol{V} \qquad \text{weakly in } L^s(\Omega).$$

We will further need the following operators

$$\boldsymbol{\mathcal{A}}[\cdot] = \{\mathcal{A}_i\}_{i=1,2,3}[\cdot] = \nabla \Delta^{-1}[\cdot], \tag{4.44}$$

where Δ^{-1} stands for the inverse of the Laplace operator on $\mathbb{R}^3$. To be more specific, the Fourier symbol of $\mathcal{A}_j$ is

$$\mathcal{F}(\mathcal{A}_j)(\xi) = \frac{-\mathrm{i}\xi_j}{|\xi|^2}. \tag{4.45}$$

Note that for a sufficiently smooth v

$$\sum_{i=1}^{3} \partial_i \, \mathcal{A}_i[v] = v \tag{4.46}$$

and, by virtue of the classical Marcinkiewicz multiplier theorem,

$$\|\nabla\boldsymbol{\mathcal{A}}[v]\|_{L^s(\Omega,\mathbb{R}^3)} \le C(s,\Omega)\|v\|_{L^s(\Omega)}, 1 < s < \infty. \tag{4.47}$$

Note that (see [42]) if $v, \partial_t v \in L^p((0,T)\times\mathbb{R}^3)$, then

$$\partial_t\boldsymbol{\mathcal{A}}[v(t,\cdot)](x) = \boldsymbol{\mathcal{A}}[\partial_t v(t,\cdot)](x) \text{ for a.a } (t,x)\in(0,T)\times\mathbb{R}^3. \tag{4.48}$$

Next, let us also introduce the so-called *Riesz operators*

$$\mathcal{R}_{ij}[\cdot] = \partial_j\,\mathcal{A}_i[\cdot] = \partial_j\partial_i\Delta^{-1}[\cdot], \tag{4.49}$$

or, in terms of Fourier symbols, $\mathcal{F}(\mathcal{R}_{ij})(\xi) = \frac{\xi_i\xi_j}{|\xi|^2}$. We recall some of its evident properties needed in the sequel. We have

$$\sum_{i=1}^{3}\mathcal{R}_{ii}[g] = g, \quad g\in L^r(\mathbb{R}^3), \quad 1<r<\infty, \tag{4.50}$$

$$\int_{\mathbb{R}^3}\mathcal{R}_{ij}[u]v\ \mathrm{d}x = \int_{\mathbb{R}^3} u\,\mathcal{R}_{ij}[v]\,\mathrm{d}x, \quad u\in L^r(\mathbb{R}^3), v\in L^{r'}(\mathbb{R}^3), \quad 1<r<\infty, \tag{4.51}$$

and

$$\|\,\mathcal{R}_{ij}[u]\|_{L^p(\mathbb{R}^3)} \le c(p)\|u\|_{L^p(\mathbb{R}^3)}, \qquad 1<p<\infty. \tag{4.52}$$

4.4 Approximation

We first focus on the proof of the auxiliary result, i.e. on Theorem 4.2.5. The problem can be viewed as compressible Navier–Stokes system with two densities, where one is connected with inertia of the fluid and the other one with the pressure. The proof of Theorem 4.2.5 is hence very similar to the construction of solutions to the usual barotropic Navier–Stokes equations.

The purpose of this section is to introduce subsequent levels of approximation and to formulate relevant existence theorems for each of them. The proofs of these theorems are presented afterwards by performing several limit passages when corresponding approximation parameters vanish. We first regularize the pressure in order to get higher integrability of Z (and also of ϱ) in order to obtain the renormalized continuity equations using the DiPerna–Lions technique [26]. Next we regularize the continuity equations (for both ϱ and Z). The construction of a solution is done at another level of approximation, the Galerkin approximation for the velocity.

4.4.1 First approximation level

A weak solution of problem (4.7) (4.5) is obtained as a limit when $\delta\to 0^+$ of the solutions to following problem

$$\partial_t\varrho + \operatorname{div}(\varrho\boldsymbol{u}) = 0, \tag{4.53a}$$

$$\partial_t Z + \operatorname{div}(Z\boldsymbol{u}) = 0, \tag{4.53b}$$

$$\partial_t(\varrho \boldsymbol{u}) + \operatorname{div}(\varrho \boldsymbol{u} \otimes \boldsymbol{u}) + \nabla Z^\gamma + \delta \nabla Z^\beta = \operatorname{div} \mathbb{S}(\nabla \boldsymbol{u}) \tag{4.53c}$$

with the boundary conditions

$$\boldsymbol{u}_{|(0,T)\times\partial\Omega} = \boldsymbol{0}, \tag{4.54}$$

and modified initial data

$$\left.\begin{aligned} (\varrho(0,\cdot), Z(0,\cdot)) = (\varrho_{0,\delta}(\cdot), Z_{0,\delta}(\cdot)) \in C^\infty(\overline{\Omega}, \mathbb{R}^2), \\ 0 < c_\star \varrho_{0,\delta} \le Z_{0,\delta} \le c^\star \varrho_{0,\delta} \text{ in } \overline{\Omega}, \end{aligned}\right\} \tag{4.55a}$$

$$\nabla \varrho_{0,\delta} \cdot \boldsymbol{n}_{|(0,T)\times\partial\Omega} = 0, \quad \nabla Z_{0,\delta} \cdot \boldsymbol{n}_{|(0,T)\times\partial\Omega} = 0, \tag{4.55b}$$

$$(\varrho \boldsymbol{u})(0,\cdot) = \boldsymbol{q}_{0,\delta}(\cdot) \in C^\infty(\overline{\Omega}, \mathbb{R}^3). \tag{4.55c}$$

The specific assumption on the initial data (4.55b) is not needed here, at this approximation level we would be satisfied with less regular approximation without this condition. However, more regular approximation with the above mentioned compatibility condition is needed at another approximation level and we prefer to regularize the initial condition just once.

Note that we require $\boldsymbol{q}_{0,\delta} \to \boldsymbol{q}_0$ in $L^{(2\gamma)/(\gamma+1)}(\Omega;\mathbb{R}^3)$ and $\varrho_{0,\delta} \to \varrho_0$, $Z_{0,\delta} \to Z_0$, both in $L^\gamma(\Omega)$. While the first part, i.e. the initial condition for the linear momentum, is easy to ensure by standard mollification, the regularization of the initial condition for Z and ϱ is more complex. However, we may multiply Z_0 by a suitable cut-off function (to set the function to be zero near the boundary), then add a small constant to this function and finally mollify it; i.e.

$$Z_{0,\delta} = (\varphi_\delta Z_0 + \delta) * \omega_\delta.$$

It is not difficult to see that for suitably chosen cut-off function φ_δ[1] all properties connected with $Z_{0,\delta}$ in (4.55a)–(4.55b) will be fulfilled as well as $Z_{0,\delta} \to Z_0$ in $L^\gamma(\Omega)$ for $\delta \to 0^+$. Similarly we proceed for ϱ_0. By a suitable regularization of the initial linear momentum we may also ensure that

$$\frac{|\boldsymbol{q}_{0,\delta}|^2}{\varrho_{0,\delta}} 1_{\{\varrho_0>0\}} \to \frac{|\boldsymbol{q}_0|^2}{\varrho_0} 1_{\{\varrho_0>0\}}$$

in $L^1(\Omega)$.

4.4.2 Second approximation level

We prove the existence of a solution to problem (4.53)–(4.55) by letting $\epsilon \to 0^+$ in the following approximate system. Given $\epsilon, \delta > 0$, we consider

$$\partial_t \varrho + \operatorname{div}(\varrho \boldsymbol{u}) = \epsilon \Delta \varrho, \tag{4.56a}$$

$$\partial_t Z + \operatorname{div}(Z \boldsymbol{u}) = \epsilon \Delta Z, \tag{4.56b}$$

$$\partial_t(\varrho \boldsymbol{u}) + \operatorname{div}(\varrho \boldsymbol{u} \otimes \boldsymbol{u}) + \nabla Z^\gamma + \delta \nabla Z^\beta + \epsilon \nabla \boldsymbol{u} \cdot \nabla \varrho = \operatorname{div}(\mathbb{S}(\nabla \boldsymbol{u})), \tag{4.56c}$$

supplemented with the boundary conditions

$$\nabla_x \varrho \cdot \boldsymbol{n}_{|(0,T)\times\partial\Omega} = 0, \quad \nabla_x Z \cdot \boldsymbol{n}_{|(0,T)\times\partial\Omega} = 0, \tag{4.57}$$

$$\boldsymbol{u}_{|(0,T)\times\partial\Omega} = \boldsymbol{0}, \tag{4.58}$$

and modified initial data (4.55) (see the comments above).

[1] We may take $\varphi_\delta \in C_c^\infty(\Omega)$ such that $0 \le \varphi_\delta \le 1$ in Ω with $\varphi_\delta(x) = 1$ if (for $x \in \Omega$) $\operatorname{dist}\{x, \partial\Omega\} \ge \frac{\delta}{2}$ and $\varphi_\delta(x) = 0$ if $\operatorname{dist}\{x, \partial\Omega\} \le \frac{\delta}{4}$.

4.4.3 Existence results for the approximate systems

Let us present now the existence result for the first approximation level

Proposition 4.4.1. *Let $\beta \geq \max(\gamma, 4)$, $\delta > 0$. Then, given initial data $(\varrho_{0,\delta}, Z_{0,\delta}, \boldsymbol{u}_{0,\delta})$ as in (4.55), there exists a finite energy weak solution $(\varrho, Z, \boldsymbol{u})$ to problem (4.53)–(4.55) such that*

$$(\varrho, Z, \boldsymbol{u}) \in [L^\infty(0,T; L^\beta(\Omega))]^2 \times L^2(0,T; W_0^{1,2}(\Omega, \mathbb{R}^3)), \tag{4.59}$$

$$0 \leq c_\star \varrho \leq Z \leq c^\star \varrho \ \textit{a.e in } (0,T) \times \Omega, \tag{4.60}$$

and for any $t \in (0,T)$ we have:

(i) $\varrho \in C([0,T]; L^\beta_\omega(\Omega))$ and the continuity equation (4.53a) is satisfied in the weak sense

$$\begin{aligned} &\int_\Omega \varrho(t,\cdot)\varphi(t,\cdot)\ \mathrm{d}x - \int_\Omega \varrho_{0,\delta}\varphi(0,\cdot)\ \mathrm{d}x \\ &= \int_0^t \int_\Omega \big(\varrho\partial_t\varphi + \varrho\boldsymbol{u}\cdot\nabla\varphi\big)\ \mathrm{d}x\ \mathrm{d}\tau, \forall \varphi \in C^1([0,T]\times\overline{\Omega}); \end{aligned} \tag{4.61}$$

(ii) $Z \in C([0,T]; L^\beta_\omega(\Omega))$ and equation (4.53b) is satisfied in the weak sense

$$\begin{aligned} &\int_\Omega Z(t,\cdot)\varphi(t,\cdot)\ \mathrm{d}x - \int_\Omega Z_{0,\delta}\varphi(0,\cdot)\ \mathrm{d}x \\ &= \int_0^t \int_\Omega \big(Z\partial_t\varphi + Z\boldsymbol{u}\cdot\nabla\varphi\big)\ \mathrm{d}x\ \mathrm{d}\tau, \forall \varphi \in C^1([0,T]\times\overline{\Omega}); \end{aligned} \tag{4.62}$$

(iii) $\varrho\boldsymbol{u} \in C([0,T]; L^{(2\beta)/(\beta+1)}_\omega(\Omega, \mathbb{R}^3))$ and the momentum equation (4.53c) is satisfied in the weak sense

$$\begin{aligned} &\int_\Omega \varrho\boldsymbol{u}(t,\cdot)\cdot\boldsymbol{\psi}(t,\cdot)\ \mathrm{d}x - \int_\Omega \boldsymbol{q}_{0,\delta}\cdot\boldsymbol{\psi}(0,\cdot)\ \mathrm{d}x = \int_0^t\int_\Omega \Big(\varrho\boldsymbol{u}\cdot\partial_t\boldsymbol{\psi} + \varrho\boldsymbol{u}\otimes\boldsymbol{u} : \nabla\boldsymbol{\psi} \\ &+ Z^\gamma \operatorname{div}\boldsymbol{\psi} + \delta Z^\beta \operatorname{div}\boldsymbol{\psi} - \mathbb{S}(\nabla\boldsymbol{u}) : \nabla\boldsymbol{\psi}\Big)\ \mathrm{d}x\ \mathrm{d}\tau, \forall \boldsymbol{\psi} \in C^1_c([0,T]\times\Omega, \mathbb{R}^3); \end{aligned} \tag{4.63}$$

(iv) the energy inequality

$$\mathcal{E}_\delta(\varrho, \boldsymbol{u}, Z)(t) + \int_0^t\int_\Omega \mathbb{S}(\nabla\boldsymbol{u}) : \nabla\boldsymbol{u}\ \mathrm{d}x\ \mathrm{d}\tau \leq \mathcal{E}_\delta(\varrho_{0,\delta}, Z_{0,\delta}, \boldsymbol{u}_{0,\delta}) \tag{4.64}$$

holds for a.a $t \in (0,T)$, where $\mathcal{E}_\delta(\varrho, \boldsymbol{u}, Z) = \int_\Omega \left(\frac{1}{2}\varrho|\boldsymbol{u}|^2 + \frac{\delta}{\beta-1}Z^\beta + \frac{1}{\gamma-1}Z^\gamma\right)\ \mathrm{d}x$;

(v) the following estimates hold with constants independent of δ

$$\sup_{t\in[0,T]} \|\varrho(t)\|^\gamma_{L^\gamma(\Omega)} + \sup_{t\in[0,T]} \|Z(t)\|^\gamma_{L^\gamma(\Omega)} \leq C(\gamma, c_\star)\, \mathcal{E}_\delta(\varrho_{0,\delta}, Z_{0,\delta}, \boldsymbol{u}_{0,\delta}), \tag{4.65}$$

$$\delta \sup_{t\in[0,T]} \|\varrho(t)\|^\beta_{L^\beta(\Omega)} + \delta \sup_{t\in[0,T]} \|Z(t)\|^\beta_{L^\beta(\Omega)} \leq C(\beta, c_\star)\, \mathcal{E}_\delta(\varrho_{0,\delta}, Z_{0,\delta}, \boldsymbol{u}_{0,\delta}), \tag{4.66}$$

$$\|\boldsymbol{u}\|_{L^2(0,T;W_0^{1,2}(\Omega,\mathbb{R}^3))} \leq C\,\mathcal{E}_\delta(\varrho_{0,\delta}, Z_{0,\delta}, \boldsymbol{u}_{0,\delta}), \tag{4.67}$$

$$\sup_{t\in[0,T]} \|\varrho\boldsymbol{u}\|_{L^{\frac{2\gamma}{\gamma+1}}(\Omega,\mathbb{R}^3))} + \sup_{t\in[0,T]} \|Z\boldsymbol{u}\|_{L^{\frac{2\gamma}{\gamma+1}}(\Omega,\mathbb{R}^3)} \leq C(\gamma, c^\star, \mathcal{E}_\delta(\varrho_{0,\delta}, Z_{0,\delta}, \boldsymbol{u}_{0,\delta})), \tag{4.68}$$

$$\|\varrho\boldsymbol{u}\|_{L^2\left(0,T;L^{\frac{6\gamma}{\gamma+6}}(\Omega,\mathbb{R}^3)\right)} + \|Z\boldsymbol{u}\|_{L^2\left(0,T;L^{\frac{6\gamma}{\gamma+6}}(\Omega,\mathbb{R}^3)\right)} \leq C(\gamma, c_\star, \mathcal{E}_\delta(\varrho_{0,\delta}, Z_{0,\delta}, \boldsymbol{u}_{0,\delta})), \tag{4.69}$$

$$\|\varrho|\boldsymbol{u}|^2\|_{L^1\left(0,T;L^{\frac{3\gamma}{\gamma+3}}(\Omega)\right)} + \|Z|\boldsymbol{u}|^2\|_{L^1\left(0,T;L^{\frac{3\gamma}{\gamma+3}}(\Omega)\right)} \leq C(\gamma, c_\star, \mathcal{E}_\delta(\varrho_{0,\delta}, Z_{0,\delta}, \boldsymbol{u}_{0,\delta})), \tag{4.70}$$

$$\|\varrho|\boldsymbol{u}|^2\|_{L^2\left(0,T;L^{\frac{6\gamma}{4\gamma+3}}(\Omega)\right)} + \|Z|\boldsymbol{u}|^2\|_{L^2\left(0,T;L^{\frac{6\gamma}{4\gamma+3}}(\Omega)\right)} \leq C(\gamma, c_\star, \mathcal{E}_\delta(\varrho_{0,\delta}, Z_{0,\delta}, \boldsymbol{u}_{0,\delta})), \tag{4.71}$$

$$\begin{aligned} &\|\varrho\|^{\gamma+\theta}_{L^{\gamma+\theta}((0,T)\times\Omega)} + \delta\|\varrho\|^{\beta+\theta}_{L^{\beta+\theta}((0,T)\times\Omega)} + \|Z\|^{\gamma+\theta}_{L^{\gamma+\theta}((0,T)\times\Omega)} + \delta\|Z\|^{\beta+\theta}_{L^{\beta+\theta}((0,T)\times\Omega)} \\ &\quad \leq C(\gamma, c_\star, \mathcal{E}_\delta(\varrho_{0,\delta}, Z_{0,\delta}, \boldsymbol{u}_{0,\delta})), \end{aligned} \tag{4.72}$$

where $\theta = \min\{\frac{2}{3}\gamma - 1, \frac{\gamma}{2}\}$*. Moreover, equations (4.53a), (4.53b) hold in the sense of renormalized solutions in* $\mathcal{D}'((0,T)\times\Omega)$ *and* $\mathcal{D}'((0,T)\times\mathbb{R}^3)$ *provided* $\varrho, Z, \boldsymbol{u}$ *are prolonged by zero outside* Ω*.*

We have for the second approximation level

Proposition 4.4.2. *Suppose* $\beta \geq \max(4,\gamma)$*. Let* ϵ*,* $\delta > 0$*. Assume the initial data* $(\varrho_{0,\delta}, Z_{0,\delta}, \boldsymbol{u}_{0,\delta})$ *satisfy (4.55). Then there exists a weak solution* $(\varrho, Z, \boldsymbol{u})$ *to problem (4.55)–(4.58) such that*

$$(\varrho, Z, \boldsymbol{u}) \in [L^\infty(0,T;L^\beta(\Omega)) \cap L^2(0,T;W^{1,2}(\Omega))]^2 \times L^2(0,T;W_0^{1,2}(\Omega,\mathbb{R}^3)), \tag{4.73}$$

$$0 \leq c_\star\varrho \leq Z \leq c^\star\varrho \text{ a.e in } (0,T)\times\Omega, \tag{4.74}$$

and for any $t \in (0,T)$ *we have:*

(i) $\varrho \in C([0,T];L^\beta_\omega(\Omega))$ *and the continuity equation (4.56a) is satisfied in the weak sense*

$$\begin{aligned} &\int_\Omega \varrho(t,\cdot)\varphi(t,\cdot)\,\mathrm{d}x - \int_\Omega \varrho_{0,\delta}\varphi(0,\cdot)\,\mathrm{d}x \\ &= \int_0^t\int_\Omega \Big(\varrho\partial_t\varphi + \varrho\boldsymbol{u}\cdot\nabla\varphi - \epsilon\nabla\varrho\cdot\nabla\varphi\Big)\,\mathrm{d}x\,\mathrm{d}\tau, \quad \forall\varphi\in C^1([0,T]\times\overline{\Omega}); \end{aligned} \tag{4.75}$$

(ii) $Z \in C([0,T];L^\beta_\omega(\Omega))$ *and equation (4.56b) is satisfied in the weak sense*

$$\begin{aligned} &\int_\Omega Z(t,\cdot)\varphi(t,\cdot)\,\mathrm{d}x - \int_\Omega Z_{0,\delta}\varphi(0,\cdot)\,\mathrm{d}x \\ &= \int_0^t\int_\Omega \Big(Z\partial_t\varphi + Z\boldsymbol{u}\cdot\nabla\varphi - \epsilon\nabla Z\cdot\nabla\varphi\Big)\,\mathrm{d}x\,\mathrm{d}\tau, \quad \forall\varphi\in C^1([0,T]\times\overline{\Omega}); \end{aligned} \tag{4.76}$$

(iii) $\varrho \boldsymbol{u} \in C\left([0,T]; L_\omega^{\frac{2\beta}{\beta+1}}(\Omega,\mathbb{R}^3)\right)$ *and the momentum equation (4.56c) is satisfied in the weak sense*

$$\int_\Omega \varrho\boldsymbol{u}(t,\cdot)\cdot\boldsymbol{\psi}(t,\cdot)\,\mathrm{d}x - \int_\Omega \boldsymbol{q}_{0,\delta}\cdot\boldsymbol{\psi}(0,\cdot)\,\mathrm{d}x = \int_0^t\int_\Omega \Big(\varrho\boldsymbol{u}\cdot\partial_t\boldsymbol{\psi} + \varrho\boldsymbol{u}\otimes\boldsymbol{u} : \nabla\boldsymbol{\psi}$$
$$+ Z^\gamma \operatorname{div}\boldsymbol{\psi} + \delta Z^\beta \operatorname{div}\boldsymbol{\psi} - \mathbb{S}(\nabla\boldsymbol{u}) : \nabla\boldsymbol{\psi} + \epsilon\nabla\varrho\cdot\nabla\boldsymbol{u}\cdot\boldsymbol{\psi}\Big)\,\mathrm{d}x\,\mathrm{d}\tau,$$
$$\forall \boldsymbol{\psi}\in C_c^1([0,T]\times\Omega,\mathbb{R}^3); \tag{4.77}$$

(iv) the energy inequality

$$\mathcal{E}_\delta(\varrho,\boldsymbol{u},Z)(t) + \int_0^t\int_\Omega \Big(\mathbb{S}(\nabla\boldsymbol{u}) : \nabla\boldsymbol{u} + \frac{\epsilon\gamma}{\gamma-1} Z^{\gamma-2}|\nabla Z|^2$$
$$+ \frac{\epsilon\delta\beta}{\beta-1} Z^{\beta-2}|\nabla Z|^2\Big)\,\mathrm{d}x\,\mathrm{d}\tau \le \mathcal{E}_\delta(\varrho_{0,\delta}, Z_{0,\delta}, \boldsymbol{u}_{0,\delta}) \tag{4.78}$$

holds for a.a $t\in(0,T)$*, where* $\mathcal{E}_\delta(\varrho,\boldsymbol{u},Z)$ *is the same as in Proposition 4.4.1;*

(v) the following estimates hold with constants independent of ϵ

$$\sup_{t\in[0,T]} \|\varrho(t)\|^\beta_{L^\beta(\Omega)} + \sup_{t\in[0,T]} \|Z(t)\|^\beta_{L^\beta(\Omega)} \le C(\beta, c_\star)\,\mathcal{E}_\delta(\varrho_{0,\delta}, Z_{0,\delta}, \boldsymbol{u}_{0,\delta}), \tag{4.79}$$
$$\|\boldsymbol{u}\|_{L^2(0,T;W_0^{1,2}(\Omega,\mathbb{R}^3))} \le C\,\mathcal{E}_\delta(\varrho_{0,\delta}, Z_{0,\delta}, \boldsymbol{u}_{0,\delta}), \tag{4.80}$$
$$\sup_{t\in[0,T]} \|\varrho\boldsymbol{u}\|_{L^{\frac{2\beta}{\beta+1}}(\Omega,\mathbb{R}^3))} + \sup_{t\in[0,T]} \|Z\boldsymbol{u}\|_{L^{\frac{2\beta}{\beta+1}}(\Omega,\mathbb{R}^3)} \le C(\beta, c_\star, \mathcal{E}_\delta(\varrho_{0,\delta}, Z_{0,\delta}, \boldsymbol{u}_{0,\delta})), \tag{4.81}$$
$$\epsilon\Big(\|\nabla\varrho\|^2_{L^2((0,T)\times\Omega,\mathbb{R}^3)} + \|\nabla Z\|^2_{L^2((0,T)\times\Omega,\mathbb{R}^3)}\Big) \le C(\beta, c_\star, \mathcal{E}_\delta(\varrho_{0,\delta}, Z_{0,\delta}, \boldsymbol{u}_{0,\delta})), \tag{4.82}$$
$$\|\varrho|\boldsymbol{u}|^2\|_{L^2\left(0,T;L^{\frac{6\beta}{4\beta+3}}(\Omega)\right)} + \|Z|\boldsymbol{u}|^2\|_{L^2\left(0,T;L^{\frac{6\beta}{4\beta+3}}(\Omega)\right)} \le C(\beta, c_\star, \mathcal{E}_\delta(\varrho_{0,\delta}, Z_{0,\delta}, \boldsymbol{u}_{0,\delta})), \tag{4.83}$$
$$\|\varrho\|_{L^{\beta+1}((0,T)\times\Omega)} + \|Z\|_{L^{\beta+1}((0,T)\times\Omega)} \le C(\beta, c_\star, \mathcal{E}_\delta(\varrho_{0,\delta}, Z_{0,\delta}, \boldsymbol{u}_{0,\delta})). \tag{4.84}$$

4.5 Existence for the second approximation level

We are not going to present detailed proof of Proposition 4.4.2, as it is similar to the corresponding step in the existence proof for the barotropic Navier–Stokes equations, cf. [69]. In what follows we only explain main ideas as well as how to obtain the crucial estimate (4.74).

We introduce another approximation level, the Galerkin approximation for the velocity. We take a suitable basis $\{\boldsymbol{\Phi}_j\}_{j=1}^\infty$ in $W_0^{1,2}(\Omega,\mathbb{R}^3)$, orthonormal in $L^2(\Omega,\mathbb{R}^3)$, and replace (4.77) by

$$\int_\Omega \partial_t(\varrho\boldsymbol{u}^n)\cdot\boldsymbol{\Phi}_j\,\mathrm{d}x = \int_\Omega \Big(\varrho\boldsymbol{u}^n\otimes\boldsymbol{u}^n : \nabla\boldsymbol{\Phi}_j + Z^\gamma\operatorname{div}\boldsymbol{\Phi}_j + \delta Z^\beta \operatorname{div}\boldsymbol{\Phi}_j$$
$$- \mathbb{S}(\nabla\boldsymbol{u}^n) : \nabla\boldsymbol{\Phi}_j + \epsilon\nabla\varrho\cdot\nabla\boldsymbol{u}^n\cdot\boldsymbol{\Phi}_j\Big)\,\mathrm{d}x, \quad \forall j = 1,2,\ldots,n, \tag{4.85}$$

where ϱ and Z solves (4.56a) and (4.56b), respectively, with $\boldsymbol{u}$ replaced by $\boldsymbol{u}^n$, and

$$\boldsymbol{u}^n(t,x) = \sum_{j=1}^{n} a_j^n(t)\boldsymbol{\Phi}_j(x).$$

The initial condition for the momentum equation reads

$$\varrho(0,\cdot)\boldsymbol{u}^n(0,\cdot) = P^n(\boldsymbol{q}_{0,\delta})(\cdot)$$

with P^n the corresponding orthogonal projection on the space spanned by $\{\boldsymbol{\Phi}_j\}_{j=1}^n$. We construct the solutions to the n-th Galerkin approximation by means of a version of the Schauder fixed point theorem. The fundamental step in this procedure is derivation of the a priori estimates. They can be obtained by using the solution $\boldsymbol{u}^n$ as a test function in (4.85) and combining it with (4.56b) as well as with (4.56a). We then deduce

$$\mathcal{E}_\delta(\varrho, Z, \boldsymbol{u}^n)(t) + \int_0^t \int_\Omega \Big(\mathbb{S}(\nabla \boldsymbol{u}^n) : \nabla \boldsymbol{u}^n + \epsilon\gamma Z^{\gamma-2}|\nabla Z|^2 + \epsilon\delta\beta Z^{\beta-2}|\nabla Z|^2\Big)\, \mathrm{d}x\, \mathrm{d}\tau$$
$$\leq \mathcal{E}_\delta(\varrho_{0,\delta}, Z_{0,\delta}, P^n(\boldsymbol{q}_{0,\delta})/\varrho_{0,\delta}) \leq C \quad (4.86)$$

with C independent of n (also of ϵ and δ). Next, testing equations (4.56a) and (4.56b) by ϱ and Z, respectively, we also have

$$\|\varrho\|^2_{L^2(\Omega)}(t) + \|Z\|^2_{L^2(\Omega)}(t) + \epsilon \int_0^t \Big(\|\nabla\varrho\|^2_{L^2(\Omega;\mathbb{R}^3)} + \|\nabla Z\|^2_{L^2(\Omega;\mathbb{R}^3)}\Big)\, \mathrm{d}\tau \leq C \quad (4.87)$$

provided $\beta \geq 4$. Note also that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \varrho\, \mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega Z\, \mathrm{d}x = 0.$$

To prove inequalities (4.74) we use a simple comparison principle between ϱ and Z. Taking $c_\star, c^\star$ as in (4.55a) we may write

$$\partial_t(Z - c_\star\varrho) + \mathrm{div}\,\big(\boldsymbol{u}^n(Z - c_\star\varrho)\big) - \epsilon\Delta(Z - c_\star\varrho) = 0$$

and

$$\partial_t(c^\star\varrho - Z) + \mathrm{div}\,\big(\boldsymbol{u}^n(c^\star\varrho - Z)\big) - \epsilon\Delta(c^\star\varrho - Z) = 0.$$

As both equations have non-negative initial conditions, it is easy to see that also the solutions are non-negative and due to the uniqueness of solutions we deduce that

$$0 < c_\star\varrho \leq Z \leq c^\star\varrho < \infty \quad (4.88)$$

a.e. in $(0,T)\times\Omega$. Combining (4.88) with (4.86) we also have

$$\|\varrho\|_{L^\infty(0,T;L^\beta(\Omega))} \leq C \quad (4.89)$$

with $C = C(c_\star, \delta, \mathcal{E}_\delta)$. The regularity of solutions to parabolic problems allows us to deduce that we have independently of n

$$\|\partial_t\varrho\|_{L^q(0,T;L^q(\Omega))} + \|\partial_t Z\|_{L^q(0,T;L^q(\Omega))} + \|\varrho\|_{L^q(0,T;W^{2,q}(\Omega))} + \|Z\|_{L^q(0,T;W^{2,q}(\Omega))} \leq C(\epsilon) \quad (4.90)$$

for all $q \in (1,\infty)$. These estimates are sufficient to apply the fixed point argument, but also to pass to the limit $n \to \infty$. To this aim, recall also that ϱ and Z belong to $C\left([0,T]; L^\beta_\omega(\Omega)\right)$ and $\varrho \boldsymbol{u}^n$ to $C([0,T]; L^{(2\beta)/(\beta+1)}_\omega(\Omega,\mathbb{R}^3))$. Hence, using several general results from Section 3 (see Lemmas 4.3.2–4.3.4) we may pass to the limit with $n \to \infty$ to recover system (4.55)–(4.58) as stated in Proposition 4.4.2. To finish the proof of this proposition, we have to show estimate (4.84). To this aim, we use as test function in (4.77) $\boldsymbol{\psi}$, solution to (cf. Lemma 4.3.9 in Section 3)

$$\operatorname{div} \boldsymbol{\psi} = Z - \frac{1}{|\Omega|}\int_\Omega Z \, \mathrm{d}x$$

with homogeneous Dirichlet boundary conditions. Due to properties of the Bogovskii operator we may prove

$$\|Z\|_{L^{\beta+1}((0,T)\times\Omega)} \le C$$

which, together with (4.88), finishes the proof of Proposition 4.4.2.

4.6 Vanishing viscosity limit: proof of Proposition 4.4.1

4.6.1 Limit passage based on the a priori estimates

At this stage, we are ready to pass to the limit for $\epsilon \to 0^+$ to get rid of the diffusion term in the equations (4.56a), (4.56b) as well as of the ϵ-dependent term in (4.56c). Note that the parameter δ is kept fixed throughout this procedure so that we may use the estimates derived above, except (4.90). Accordingly, the solution of problem (4.55)–(4.58) obtained in Proposition 4.4.2 above will be denoted $(\varrho_\epsilon, Z_\epsilon, \boldsymbol{u}_\epsilon)$.

First of all, by virtue of (4.80) and (4.82), we obtain

$$\epsilon \nabla \varrho_\epsilon \cdot \nabla \boldsymbol{u}_\epsilon \to 0 \text{ in } L^1((0,T)\times\Omega),$$

and, analogously,

$$\epsilon \nabla Z_\epsilon,\ \epsilon \nabla \varrho_\epsilon \to 0 \text{ in } L^2((0,T)\times\Omega).$$

From estimates (4.79)–(4.84) we further deduce

$$\varrho_\epsilon \to \varrho \text{ weakly-} \star \text{ in } L^\infty(0,T;L^\beta(\Omega)) \text{ and weakly in } L^{\beta+1}((0,T)\times\Omega), \quad (4.91a)$$

$$Z_\epsilon \to Z \text{ weakly-} \star \text{ in } L^\infty(0,T;L^\beta(\Omega)) \text{ and weakly in } L^{\beta+1}((0,T)\times\Omega), \quad (4.91b)$$

$$\boldsymbol{u}_\epsilon \to \boldsymbol{u} \ \text{ weakly in } L^2(0,T;W^{1,2}_0(\Omega,\mathbb{R}^3)), \quad (4.91c)$$

passing to subsequences if necessary.

By virtue of (4.74) and the weak $L^{\beta+1}$-convergence derived above we obtain

$$0 \le c_\star \varrho \le Z \le c^\star \varrho \text{ a.e. in } (0,T)\times\Omega. \quad (4.92)$$

Due to (4.75), (4.76), (4.82) and (4.84), ϱ_ϵ and Z_ϵ are uniformly continuous in $W^{-1,\frac{2\beta}{\beta+1}}(\Omega)$. Since they belong to $C([0,T];L^\beta_\omega(\Omega))$ and they are uniformly bounded in $L^\beta(\Omega)$ (by virtue of (4.79)), we use Lemma 4.3.4, in order to get at least for a chosen subsequence

$$\varrho_\epsilon \to \varrho, \quad Z_\epsilon \to Z \text{ in } C([0,T]; L^\beta_\omega(\Omega)). \tag{4.93}$$

Once we realize that the imbedding $L^s(\Omega) \hookrightarrow W^{-1,2}(\Omega)$ is compact for $s > \frac{6}{5}$, we apply Lemma 4.3.2 to ϱ_ϵ and Z_ϵ, and obtain

$$\varrho_\epsilon \to \varrho, \quad Z_\epsilon \to Z \text{ in } L^p(0,T; W^{-1,2}(\Omega)), \quad 1 \le p < \infty. \tag{4.94}$$

Consequently, by virtue of the previous formula, (4.81) and (4.91c) we obtain

$$\varrho_\epsilon \boldsymbol{u}_\epsilon \to \varrho \boldsymbol{u}, \quad Z_\epsilon \boldsymbol{u}_\epsilon \to Z\boldsymbol{u} \text{ weakly-}\star \text{ in } L^\infty\left(0,T; L^{\frac{2\beta}{\beta+1}}(\Omega,\mathbb{R}^3)\right). \tag{4.95}$$

Taking into account (4.77) and (4.79)–(4.84) we conclude that $\varrho_\epsilon \boldsymbol{u}_\epsilon$ is uniformly continuous in $W^{-1,s}(\Omega,\mathbb{R}^3)$, where $s = \frac{\beta+1}{\beta}$. Since

$$\varrho_\epsilon \boldsymbol{u}_\epsilon \in C\left([0,T]; L_\omega^{\frac{2\beta}{\beta+1}}(\Omega,\mathbb{R}^3)\right)$$

and since it is uniformly bounded in $L^{\frac{2\beta}{\beta+1}}(\Omega,\mathbb{R}^3)$ (see (4.81)), Lemma 4.3.4 yields

$$\varrho_\epsilon \boldsymbol{u}_\epsilon \to \varrho \boldsymbol{u} \text{ in } C\left([0,T]; L_\omega^{\frac{2\beta}{\beta+1}}(\Omega,\mathbb{R}^3)\right). \tag{4.96}$$

The imbedding $L^{\frac{2\beta}{\beta+1}}(\Omega) \hookrightarrow W^{-1,2}(\Omega)$ is compact, hence we deduce from Lemma 4.3.2

$$\varrho_\epsilon \boldsymbol{u}_\epsilon \to \varrho \boldsymbol{u} \text{ strongly in } L^p(0,T; W^{-1,2}(\Omega,\mathbb{R}^3)). \tag{4.97}$$

It implies, together with (4.91c) that

$$\varrho_\epsilon \boldsymbol{u}_\epsilon \otimes \boldsymbol{u}_\epsilon \to \varrho \boldsymbol{u} \otimes \boldsymbol{u} \text{ in } L^q((0,T)\times\Omega; \mathbb{R}^{3\times 3}) \tag{4.98}$$

for some $q > 1$.

We have proven that the limits ϱ, Z and $\boldsymbol{u}$ satisfy for any $t \in [0,T]$ the following system of equations

$$\begin{aligned}
&\int_\Omega \varrho(t,\cdot)\varphi(t,\cdot)\ \mathrm{d}x - \int_\Omega \varrho_{0,\delta}\varphi(0,\cdot)\ \mathrm{d}x \\
&\quad = \int_0^t \int_\Omega \Big(\varrho\partial_t\varphi + \varrho\boldsymbol{u}\cdot\nabla\varphi\Big)\ \mathrm{d}x\ \mathrm{d}\tau, \forall \varphi \in C^1([0,T]\times\overline{\Omega});
\end{aligned} \tag{4.99}$$

$$\begin{aligned}
&\int_\Omega Z(t,\cdot)\varphi(t,\cdot)\ \mathrm{d}x - \int_\Omega Z_{0,\delta}\varphi(0,\cdot)\ \mathrm{d}x \\
&\quad = \int_0^t \int_\Omega \Big(Z\partial_t\varphi + Z\boldsymbol{u}\cdot\nabla\varphi\Big)\ \mathrm{d}x\ \mathrm{d}\tau, \forall \varphi \in C^1([0,T]\times\overline{\Omega});
\end{aligned} \tag{4.100}$$

$$\begin{aligned}
&\int_\Omega \varrho\boldsymbol{u}(t.\cdot)\cdot\boldsymbol{\psi}(t,\cdot)\ \mathrm{d}x\ \mathrm{d}t - \int_\Omega \boldsymbol{q}_{0,\delta}\cdot\boldsymbol{\psi}(0,\cdot)\ \mathrm{d}x\ \mathrm{d}t = \int_0^t \int_\Omega \Big(\varrho\boldsymbol{u}\cdot\partial_t\boldsymbol{\psi} + \varrho\boldsymbol{u}\otimes\boldsymbol{u} : \nabla\boldsymbol{\psi} \\
&\quad + \overline{p}\operatorname{div}\boldsymbol{\psi} - \mathbb{S}(\nabla\boldsymbol{u}) : \nabla\boldsymbol{\psi}\Big)\ \mathrm{d}x\ \mathrm{d}\tau, \forall \boldsymbol{\psi} \in C^1_c([0,T]\times\Omega,\mathbb{R}^3),
\end{aligned} \tag{4.101}$$

where, by virtue of (4.84),

$$Z_\epsilon^\gamma + \delta Z_\epsilon^\beta \to \overline{p} \text{ weakly in } L^{\frac{\beta+1}{\beta}}((0,T)\times\Omega). \tag{4.102}$$

In particular, equations (4.56a), (4.56b) and (4.56c) (with $\overline{p}$ instead of $Z^\gamma + \delta Z^\beta$) are satisfied in the sense of distributions and the limit functions satisfy the initial condition

$$\varrho(0,\cdot) = \varrho_{0,\delta}(\cdot), \quad Z(0,\cdot) = Z_{0,\delta}(\cdot), \quad (\varrho\boldsymbol{u})(0,\cdot) = \boldsymbol{q}_{0,\delta}(\cdot), \tag{4.103}$$

where $(\varrho_{0,\delta}, Z_{0,\delta}, \boldsymbol{q}_{0,\delta})$ are defined in (4.55).

Thus our ultimate goal is to show that

$$\overline{p} = Z^\gamma + \delta Z^\beta \tag{4.104}$$

which is equivalent to the strong convergence of Z_ϵ in $L^1((0,T)\times\Omega)$.

4.6.2 Effective viscous flux

We introduce the quantity $Z^\gamma + \delta Z^\beta - (\lambda + 2\mu)\operatorname{div}\boldsymbol{u}$ called usually the effective viscous flux. This quantity enjoys remarkable properties for which we refer to Hoff [50], Lions [65], or Serre [73]. We have the following crucial result.

Lemma 4.6.1. *Let $\varrho_\epsilon, Z_\epsilon, \boldsymbol{u}_\epsilon$ be the sequence of approximate solutions, the existence of which is guaranteed by Proposition 4.4.2, and let $\varrho, Z, \boldsymbol{u}$ and $\overline{p}$ be the limits appearing in (4.91a), (4.91b), (4.91c) and (4.102) respectively. Then*

$$\lim_{\epsilon\to 0^+} \int_0^T \psi \int_\Omega \phi\Big(Z_\epsilon^\gamma + \delta Z_\epsilon^\beta - (\lambda+2\mu)\operatorname{div}\boldsymbol{u}_\epsilon\Big) Z_\epsilon \,\mathrm{d}x\,\mathrm{d}t$$
$$= \int_0^T \psi \int_\Omega \phi\Big(\overline{p} - (\lambda+2\mu)\operatorname{div}\boldsymbol{u}\Big) Z \,\mathrm{d}x\,\mathrm{d}t$$

for any $\psi \in C_c^\infty((0,T))$ and $\phi \in C_c^\infty(\Omega)$, passing to subsequences, if necessary.

The proof of Lemma 4.6.1 is based on the Div-Curl Lemma of compensated compactness, see Lemma 4.3.10. We will not present it here, as it is a relatively standard result in the theory of weak solutions to the compressible Navier-Stokes equations; see e.g. [69] for more details. The basic tools for the proof can be found in Section 3. We shall give more details to the proof of a similar result used in the limit passage $\delta \to 0$, where, moreover, several arguments are more subtle than here.

We conclude this section by showing (4.104) and, consequently, strong convergence of the sequence Z_ϵ in $L^1((0,T)\times\Omega)$.

Recall that Z solves (4.100) in the sense of renormalized equations, see Lemma 4.3.5. Thus, we take $b(Z) = Z\ln Z$ (see Remark 4.3.6) to get

$$\int_0^T \int_\Omega Z \operatorname{div}_x \boldsymbol{u} \,\mathrm{d}x\,\mathrm{d}t = \int_\Omega Z_{0,\delta}\ln(Z_{0,\delta})\,\mathrm{d}x - \int_\Omega Z(T)\ln\big(Z(T)\big)\,\mathrm{d}x. \tag{4.105}$$

On the other hand, Z_ϵ solves (4.56b) a.e on $(0,T)\times\Omega$, in particular,

$$\partial_t b(Z_\epsilon) + \operatorname{div}_x(b(Z_\epsilon)\boldsymbol{u}_\epsilon) + \big(b'(Z_\epsilon)Z_\epsilon - b(Z_\epsilon)\big)\operatorname{div}\boldsymbol{u}_\epsilon - \epsilon\Delta b(Z_\epsilon) \le 0$$

for any b convex and globally Lipschitz on $\mathbb{R}^+$; whence

$$\int_0^T \int_\Omega \Big(b'(Z_\epsilon)Z_\epsilon - b(Z_\epsilon)\Big) \operatorname{div} \boldsymbol{u}_\epsilon \ \mathrm{d}x \ \mathrm{d}t \leq \int_\Omega b(Z_{0,\delta}) \ \mathrm{d}x - \int_\Omega b\big(Z_\epsilon(T)\big) \ \mathrm{d}x$$

from which we easily deduce

$$\int_0^T \int_\Omega Z_\epsilon \operatorname{div} \boldsymbol{u}_\epsilon \ \mathrm{d}x \ \mathrm{d}t \leq \int_\Omega Z_{0,\delta} \ln(Z_{0,\delta}) \ \mathrm{d}x - \int_\Omega Z_\epsilon(T) \ln\big(Z_\epsilon(T)\big) \ \mathrm{d}x. \quad (4.106)$$

Note that

$$\int_\Omega Z(T)\ln(Z(T)) \ \mathrm{d}x \leq \liminf_{\epsilon\to 0^+} \int_\Omega Z_\epsilon(T) \ln\big(Z_\epsilon(T)\big) \ \mathrm{d}x.$$

Take two non-decreasing sequences ψ_n, ϕ_n of non-negative functions such that

$$\psi_n \in C_c^\infty(0,T), \psi_n \to 1, \phi_n \in C_c^\infty(\Omega), \phi_n \to 1. \quad (4.107)$$

Lemma 4.6.1 implies that

$$\begin{aligned}
&\limsup_{\epsilon\to 0^+} \int_0^T \psi_m \int_\Omega \phi_m (Z_\epsilon^\gamma + \delta Z_\epsilon^\beta) Z_\epsilon \ \mathrm{d}x \ \mathrm{d}t \\
&\quad \leq \limsup_{\epsilon\to 0^+} \int_0^T \psi_n \int_\Omega \phi_n (Z_\epsilon^\gamma + \delta Z_\epsilon^\beta) Z_\epsilon \ \mathrm{d}x \ \mathrm{d}t \\
&\quad \leq \lim_{\epsilon\to 0^+} \int_0^T \psi_n \int_\Omega \phi_n \big(Z_\epsilon^\gamma + \delta Z_\epsilon^\beta - (\lambda+2\mu) \operatorname{div} \boldsymbol{u}_\epsilon\big) Z_\epsilon \ \mathrm{d}x \ \mathrm{d}t \\
&\qquad + (\lambda+2\mu) \limsup_{\epsilon\to 0^+} \int_0^T \psi_n \int_\Omega \phi_n Z_\epsilon \operatorname{div} \boldsymbol{u}_\epsilon \ \mathrm{d}x \ \mathrm{d}t \\
&\quad \leq \int_0^T \psi_n \int_\Omega \phi_n (\overline{p} - (\lambda+2\mu) \operatorname{div}_x \boldsymbol{u}) Z \ \mathrm{d}x \ \mathrm{d}t \\
&\qquad + (\lambda+2\mu) \limsup_{\epsilon\to 0^+} \int_0^T \int_\Omega Z_\epsilon |1-\psi_n\phi_n| |\operatorname{div} \boldsymbol{u}_\epsilon| \ \mathrm{d}x \ \mathrm{d}t \\
&\qquad + (\lambda+2\mu) \limsup_{\epsilon\to 0^+} \int_0^T \int_\Omega Z_\epsilon \operatorname{div} \boldsymbol{u}_\epsilon \ \mathrm{d}x \ \mathrm{d}t.
\end{aligned}$$

Using also (4.105) and (4.106), we observe that

$$\begin{aligned}
&\limsup_{\epsilon\to 0^+} \int_0^T \psi_m \int_\Omega \phi_m (Z_\epsilon^\gamma + \delta Z_\epsilon^\beta) Z_\epsilon \ \mathrm{d}x \ \mathrm{d}t \leq \int_0^T \int_\Omega \overline{p} Z \ \mathrm{d}x \ \mathrm{d}t \\
&\quad + \eta(n) + (\lambda+2\mu) \Big[\int_\Omega Z(T)\ln(Z(T)) \ \mathrm{d}x - \limsup_{\epsilon\to 0^+} \int_\Omega Z_\epsilon(T) \ln(Z_\epsilon(T)) \ \mathrm{d}x \Big]
\end{aligned}$$

for all $m \leq n$, where

$$\eta(n) \to 0 \text{ for } n \to \infty.$$

Thus we have proved

$$\limsup_{\epsilon\to 0^+} \int_0^T \psi_m \int_\Omega \phi_m (Z_\epsilon^\gamma + \delta Z_\epsilon^\beta) Z_\epsilon \ \mathrm{d}x \ \mathrm{d}t \leq \int_0^T \int_\Omega \overline{p} Z \ \mathrm{d}x \ \mathrm{d}t, \quad \forall m \geq 1.$$

To conclude the proof of (4.104), we make use of a (slightly modified) Minty's trick. Since the nonlinearity $P(Z) = Z^\gamma + \delta Z^\beta$ is monotone, we have for any $v \in L^{\beta+1}((0,T)\times\Omega)$

$$\int_0^T \psi_m \int_\Omega \phi_m (P(Z_\epsilon) - P(v))(Z_\epsilon - v) \;\mathrm{d}x\;\mathrm{d}t \geq 0$$

and, consequently,

$$\begin{aligned}\int_0^T \int_\Omega \overline{p} Z \;\mathrm{d}x\;\mathrm{d}t + \int_0^T \psi_m \int_\Omega \phi_m P(v) v \;\mathrm{d}x\;\mathrm{d}t \\ - \int_0^T \psi_m \int_\Omega \phi_m (\overline{p} v + P(v) Z) \;\mathrm{d}x\;\mathrm{d}t \geq 0.\end{aligned}$$

Now, letting $m \to \infty$, we get

$$\int_0^T \int_\Omega (\overline{p} - P(v))(Z - v) \;\mathrm{d}x\;\mathrm{d}t \geq 0$$

and the choice $v = Z + \eta\varphi$, $\eta \to 0$, $\varphi \in C_c^\infty((0,T)\times\Omega)$ arbitrary, yields the desired conclusion

$$\overline{p} = Z^\gamma + \delta Z^\beta.$$

To finish the proof of Proposition 4.4.1 we have to show (4.72). To this aim, we use as test function in (4.53c) solution to (cf. Lemma 4.3.9 in Section 3)

$$\operatorname{div} \boldsymbol{\psi} = Z^\theta - \frac{1}{|\Omega|}\int_\Omega Z^\theta \,\mathrm{d}x$$

with homogeneous Dirichlet boundary conditions, where $\theta > 0$ is a constant. Due to properties of the Bogovskii operator we may show (the proof is similar to the case of compressible Navier–Stokes equations, see e.g. [69])

$$\|Z\|^{\gamma+\theta}_{L^{\gamma+\theta}((0,T)\times\Omega)} + \delta \|Z\|^{\beta+\theta}_{L^{\beta+\theta}((0,T)\times\Omega)} \leq C$$

with $\theta \leq \min\{\frac{\gamma}{2}, \frac{2}{3}\gamma - 1\}$. Other estimates can be obtained easily. The proof of Proposition 4.4.1 is finished.

4.7 Passing to the limit in the artificial pressure term. Proof of Theorem 4.2.5

Our next goal is to let $\delta \to 0^+$. We will relax the assumptions on the growth of the pressure and on the regularity of the initial data. We are again confronted with a missing estimate for the sequence of densities which would guarantee the strong convergence. Additional problems will arise from the fact that the a priori bounds for the density do not allow us to apply the DiPerna–Lions transport theory, see Lemma 4.3.5. To overcome these difficulties, we will apply to system (4.53) Feireisl's approach. Accordingly, the solution of problem (4.53) obtained in Proposition 4.4.1 above will be denoted $\varrho_\delta, Z_\delta, \boldsymbol{u}_\delta$.

4.7.1 Limit passage based on a priori estimates

Using estimates independent of the parameter δ, i.e. (4.65)–(4.72), as well as the procedure at the beginning of the previous section we show (see also [69])

$$\varrho_\delta \to \varrho \text{ in } C([0,T]; L^\gamma_\omega(\Omega)), \tag{4.108a}$$

$$Z_\delta \to Z \text{ in } C([0,T]; L^\gamma_\omega(\Omega)), \tag{4.108b}$$

$$\boldsymbol{u}_\delta \to \boldsymbol{u} \text{ weakly in } L^2(0,T; W^{1,2}_0(\Omega, \mathbb{R}^3)), \tag{4.108c}$$

$$\varrho_\delta \boldsymbol{u}_\delta \to \varrho \boldsymbol{u} \text{ in } C([0,T]; L^{\frac{2\gamma}{\gamma+1}}_\omega(\Omega, \mathbb{R}^3)), \tag{4.108d}$$

$$\varrho_\delta \boldsymbol{u}_\delta \otimes \boldsymbol{u}_\delta \to \varrho \boldsymbol{u} \otimes \boldsymbol{u} \text{ weakly in } L^q((0,T) \times \Omega, \mathbb{R}^{3\times 3}) \quad \text{for some } q > 1, \tag{4.108e}$$

$$\varrho^\gamma_\delta \to \overline{\varrho^\gamma} \text{ weakly in } L^{\frac{\gamma+\theta}{\gamma}}((0,T) \times \Omega), \tag{4.108f}$$

$$Z^\gamma_\delta \to \overline{Z^\gamma} \text{ weakly in } L^{\frac{\gamma+\theta}{\gamma}}((0,T) \times \Omega), \tag{4.108g}$$

$$\delta Z^\beta_\delta \to 0 \text{ weakly in } L^q((0,T) \times \Omega), \quad \text{for some } q > 1, \tag{4.108h}$$

passing to subsequences as the case may be.

Consequently, $\varrho, Z, \boldsymbol{u}$ satisfy

$$\partial_t \varrho + \operatorname{div}(\varrho \boldsymbol{u}) = 0 \text{ in } \mathcal{D}'((0,T) \times \mathbb{R}^3), \tag{4.109}$$

$$\partial_t Z + \operatorname{div}(Z \boldsymbol{u}) = 0 \text{ in } \mathcal{D}'((0,T) \times \mathbb{R}^3), \tag{4.110}$$

$$\partial_t(\varrho \boldsymbol{u}) + \operatorname{div}(\varrho \boldsymbol{u} \otimes \boldsymbol{u}) + \nabla \overline{Z^\gamma} = \mu \Delta \boldsymbol{u} + (\mu + \lambda) \nabla \operatorname{div} \boldsymbol{u} \text{ in } \mathcal{D}'((0,T) \times \Omega, \mathbb{R}^3). \tag{4.111}$$

Thus the only thing to complete the proof of Theorem 4.2.5 is to show the strong convergence of Z_δ in $L^1((0,T) \times \Omega)$ which is actually equivalent to identifying $\overline{Z^\gamma} = Z^\gamma$.

4.7.2 Strong convergence of Z_δ

Recall that the cut-off functions T and T_k were introduced in (4.41)–(4.42).

Effective viscous flux

As in Section 6, we need the following auxiliary result:

Lemma 4.7.1. *Let $\varrho_\delta, Z_\delta, \boldsymbol{u}_\delta$ be the sequence of approximate solutions constructed by means of Proposition 4.4.1. Then*

$$\lim_{\delta \to 0^+} \int_0^T \psi \int_\Omega \phi \big(Z^\gamma_\delta - (\lambda + 2\mu) \operatorname{div} \boldsymbol{u}_\delta \big) T_k(Z_\delta) \ \mathrm{d}x \ \mathrm{d}t = \int_0^T \psi \int_\Omega \phi \big(\overline{Z^\gamma} - (\lambda + 2\mu) \operatorname{div} \boldsymbol{u} \big) \overline{T_k(Z_\delta)} \ \mathrm{d}x \ \mathrm{d}t \tag{4.112}$$

for any $\psi \in C^\infty_c((0,T))$ and $\phi \in C^\infty_c(\Omega)$, passing to subsequences, if necessary.

Proof. Recall that we have for $\delta > 0$ the renormalized form of equation (4.53b)

$$\partial_t(T_k(Z_\delta)) + \operatorname{div}(T_k(Z_\delta)\boldsymbol{u}_\delta) + (Z_\delta T_k'(Z_\delta) - T_k(Z_\delta))\operatorname{div}\boldsymbol{u}_\delta = 0, \tag{4.113}$$

however, for the limit we only have

$$\partial_t(\overline{T_k(Z)}) + \operatorname{div}(\overline{T_k(Z)}\boldsymbol{u}) + \overline{(ZT_k'(Z) - T_k(Z))\operatorname{div}\boldsymbol{u}} = 0, \tag{4.114}$$

both in the sense of distributions.

We use as the test function in the approximated momentum equation (4.53c) the function

$$\boldsymbol{\varphi}_\delta = \psi\phi\nabla\Delta^{-1}[1_\Omega T_k(Z_\delta)] = \psi\phi\boldsymbol{\mathcal{A}}[1_\Omega T_k(Z_\delta)],\ k\in\mathbb{N},$$

and for the limit equation (4.111) the test function

$$\boldsymbol{\varphi} = \psi\phi\nabla\Delta^{-1}[1_\Omega\overline{T_k(Z)}] = \psi\phi\boldsymbol{\mathcal{A}}[1_\Omega\overline{T_k(Z)}],\ k\in\mathbb{N}.$$

Here, $\psi \in C_c^\infty(0,\infty)$ and $\phi\in C_c^\infty(\Omega)$, for the definition of $\boldsymbol{\mathcal{A}}$ see Section 3. Note that thanks to properties of ψ and ϕ we indeed extend our domain from Ω onto the whole space $\mathbb{R}^3$. It allows then to work with $\boldsymbol{\mathcal{A}}$ defined in terms of Fourier multipliers.

We get

$$\begin{aligned}
&\lim_{\delta\to0^+}\int_0^T\psi\int_\Omega\Big(\phi Z_\delta^\gamma T_k(Z_\delta) + Z_\delta^\gamma\nabla\phi\cdot\boldsymbol{\mathcal{A}}[1_\Omega T_k(\varrho_\delta)]\Big)\ \mathrm{d}x\ \mathrm{d}t \qquad (4.115)\\
&-\lim_{\delta\to0^+}\int_0^T\psi\int_\Omega\phi\Big(\mu\nabla\boldsymbol{u}_\delta:\boldsymbol{\mathcal{R}}[1_\Omega T_k(Z_\delta)] + (\lambda+\mu)\operatorname{div}\boldsymbol{u}_\delta T_k(Z_\delta)\Big)\ \mathrm{d}x\ \mathrm{d}t\\
&-\lim_{\delta\to0^+}\int_0^T\psi\int_\Omega\Big(\mu\nabla\boldsymbol{u}_\delta\cdot\nabla\phi\cdot\boldsymbol{\mathcal{A}}[1_\Omega T_k(Z_\delta)] + (\lambda+\mu)\operatorname{div}\boldsymbol{u}_\delta\nabla\phi\cdot\boldsymbol{\mathcal{A}}[1_\Omega T_k(Z_\delta)]\Big)\ \mathrm{d}x\ \mathrm{d}t\\
&=\int_0^T\psi\int_\Omega\Big(\phi\overline{Z^\gamma}\ \overline{T_k(Z)} - \overline{Z^\gamma}\nabla\phi\cdot\boldsymbol{\mathcal{A}}[1_\Omega\overline{T_k(Z)}]\Big)\ \mathrm{d}x\ \mathrm{d}t\\
&-\int_0^T\psi\int_\Omega\phi\Big(\mu\nabla\boldsymbol{u}:\boldsymbol{\mathcal{R}}[1_\Omega\overline{T_k(Z)}] + (\lambda+\mu)\operatorname{div}\boldsymbol{u}\overline{T_k(Z)}\Big)\ \mathrm{d}x\ \mathrm{d}t\\
&-\int_0^T\psi\int_\Omega\Big(\mu\nabla\boldsymbol{u}\cdot\nabla\phi\cdot\boldsymbol{\mathcal{A}}[1_\Omega\overline{T_k(Z)}] + (\lambda+\mu)\operatorname{div}\boldsymbol{u}\nabla\phi\cdot\boldsymbol{\mathcal{A}}[1_\Omega\overline{T_k(Z)}]\Big)\ \mathrm{d}x\ \mathrm{d}t\\
&+\lim_{\delta\to0^+}\int_0^T\psi\int_\Omega\Big(\phi\varrho_\delta\boldsymbol{u}_\delta\cdot\boldsymbol{\mathcal{A}}[\operatorname{div}(T_k(Z_\delta)\boldsymbol{u}_\delta) + (Z_\delta T_k'(Z_\delta) - T_k(Z_\delta))\operatorname{div}\boldsymbol{u}_\delta]\\
&\qquad -\varrho_\delta(\boldsymbol{u}_\delta\otimes\boldsymbol{u}_\delta):\nabla\left(\phi\boldsymbol{\mathcal{A}}[1_\Omega T_k(Z_\delta)]\right)\Big)\ \mathrm{d}x\ \mathrm{d}t\\
&-\int_0^T\psi\int_\Omega\Big(\phi\varrho\boldsymbol{u}\cdot\boldsymbol{\mathcal{A}}[\operatorname{div}(\overline{T_k(Z)}\boldsymbol{u}) + \overline{(ZT_k'(Z) - T_k(Z))\operatorname{div}\boldsymbol{u}}]\\
&\qquad -\varrho(\boldsymbol{u}\otimes\boldsymbol{u}):\nabla\left(\phi\boldsymbol{\mathcal{A}}[1_\Omega\overline{T_k(Z)}]\right)\Big)\ \mathrm{d}x\ \mathrm{d}t\\
&-\lim_{\delta\to0^+}\int_0^T\partial_t\psi\int_\Omega\phi\varrho_\delta\boldsymbol{u}_\delta\cdot\boldsymbol{\mathcal{A}}(T_k(Z_\delta))\ \mathrm{d}x\ \mathrm{d}t + \int_0^T\partial_t\psi\int_\Omega\phi\varrho\boldsymbol{u}\cdot\boldsymbol{\mathcal{A}}[\overline{T_k(Z)}]\ \mathrm{d}x\ \mathrm{d}t.
\end{aligned}$$

We have

$$\int_\Omega \phi \nabla \boldsymbol{u}_\delta : \boldsymbol{\mathcal{R}}[1_\Omega T_k(Z_\delta)] \ \mathrm{d}x = \int_\Omega \phi \sum_{i,j=1}^{3} \left(\partial_{x_j} u_\delta^i \mathcal{R}_{ij}[1_\Omega T_k(Z_\delta)] \right) \ \mathrm{d}x$$

$$= \int_\Omega \sum_{i,j=1}^{3} \left(\partial_{x_j} (\phi u_\delta^i) \mathcal{R}_{ij}[1_\Omega T_k(Z_\delta)] \right) \ \mathrm{d}x - \int_\Omega \sum_{i,j=1}^{3} \left(\partial_{x_j} \phi u_\delta^i \mathcal{R}_{ij}[1_\Omega T_k(Z_\delta)] \right) \ \mathrm{d}x$$

$$= \int_\Omega \phi \operatorname{div} \boldsymbol{u}_\delta T_k(Z_\delta) \ \mathrm{d}x + \int_\Omega \nabla\phi \cdot \boldsymbol{u}_\delta T_k(Z_\delta) \, \mathrm{d}x - \int_\Omega \sum_{i,j=1}^{3} \left(\partial_{x_j} \phi u_\delta^i \mathcal{R}_{ij}[1_\Omega T_k(Z_\delta)] \right) \ \mathrm{d}x.$$

Consequently, going back to (4.115) and dropping the compact terms, where we use

$$\mathcal{A}[1_\Omega T_k(\varrho_\delta)] \to \mathcal{A}[1_\Omega \overline{T_k(\varrho)}] \text{ in } C([0,T] \times \overline{\Omega}),$$

we obtain

$$\lim_{\delta\to 0^+} \int_0^T \psi \int_\Omega \phi \Big(Z_\delta^\gamma T_k(Z_\delta) - (\lambda + 2\mu) \operatorname{div} \boldsymbol{u}_\delta T_k(Z_\delta) \Big) \ \mathrm{d}x \ \mathrm{d}t \tag{4.116}$$

$$- \int_0^T \psi \int_\Omega \phi \Big(\overline{Z^\gamma} \ \overline{T_k(Z)} - (\lambda + 2\mu) \operatorname{div} \boldsymbol{u} \overline{T_k(Z)} \Big) \ \mathrm{d}x \ \mathrm{d}t$$

$$= \lim_{\delta\to 0^+} \int_0^T \psi \int_\Omega \Big(\varrho_\delta \boldsymbol{u}_\delta \cdot \boldsymbol{\mathcal{A}}[\operatorname{div}(T_k(Z_\delta)\boldsymbol{u}_\delta)] - \varrho_\delta (\boldsymbol{u}_\delta \otimes \boldsymbol{u}_\delta) : \boldsymbol{\mathcal{R}}[1_\Omega T_k(Z_\delta)] \Big) \ \mathrm{d}x \ \mathrm{d}t$$

$$- \int_0^T \psi \int_\Omega \Big(\phi \varrho \boldsymbol{u} \cdot \boldsymbol{\mathcal{A}}[\operatorname{div}(\overline{T_k(Z)}\boldsymbol{u})] - \varrho (\boldsymbol{u} \otimes \boldsymbol{u}) : \boldsymbol{\mathcal{R}}[1_\Omega \overline{T_k(Z)}] \Big) \ \mathrm{d}x \ \mathrm{d}t.$$

Our goal is to show that the right-hand side of (4.116) vanishes. We write

$$\int_\Omega \phi \Big[\varrho_\delta \boldsymbol{u}_\delta \cdot \boldsymbol{\mathcal{A}}[1_\Omega \operatorname{div}(T_k(Z_\delta)\boldsymbol{u}_\delta)] - \varrho_\delta (\boldsymbol{u}_\delta \otimes \boldsymbol{u}_\delta) : \boldsymbol{\mathcal{R}}[1_\Omega T_k(Z_\delta)] \Big] \, \mathrm{d}x$$

$$= \int_\Omega \phi \boldsymbol{u}_\delta \cdot \Big[T_k(Z_\delta) \boldsymbol{\mathcal{A}}[\operatorname{div}(1_\Omega \varrho_\delta \boldsymbol{u}_\delta)] - \varrho_\delta \boldsymbol{u}_\delta \cdot \boldsymbol{\mathcal{R}}[1_\Omega T_k(Z_\delta)] \Big] \ \mathrm{d}x + l.o.t.,$$

where l.o.t. denotes lower order terms (with derivatives on ϕ) and appear due to the integration by parts in the first term on the left-hand side. We consider the bilinear form

$$[\boldsymbol{v}, \boldsymbol{w}] = \sum_{i,j=1}^{3} \Big(v^i \mathcal{R}_{ij}[w^j] - w^i \mathcal{R}_{ij}[v^j] \Big),$$

where

$$\boldsymbol{v} = \boldsymbol{v}(Z) = (T_k(Z), T_k(Z), T_k(Z)), \qquad \boldsymbol{w} = \boldsymbol{w}(\varrho, \boldsymbol{u}) = \varrho \boldsymbol{u}.$$

We may write

$$\sum_{i,j=1}^{3} \Big(v^i \mathcal{R}_{ij}[w^j] - w^i \mathcal{R}_{ij}[v^j] \Big)$$

$$= \sum_{i,j=1}^{3} \Big((v^i - \mathcal{R}_{ij}[v^j]) \mathcal{R}_{ij}[w^j] - (w^i - \mathcal{R}_{ij}[w^j]) \mathcal{R}_{ij}[v^j] \Big) = \boldsymbol{U} \cdot \boldsymbol{V} - \boldsymbol{W} \cdot \boldsymbol{Z},$$

where

$$U^i = \sum_{j=1}^{3}(v^i - \mathcal{R}_{ij}[v^j]),\ W^i = \sum_{j=1}^{3}(w^i - \mathcal{R}_{ij}[w^j]),\ \operatorname{div}\boldsymbol{U} = \operatorname{div}\boldsymbol{W} = 0,$$

and

$$V^i = \partial_{x_i}\left(\sum_{j=1}^{3}\Delta^{-1}\partial_{x^j}w^j\right),\ Z^i = \partial_{x_i}\left(\sum_{j=1}^{3}\Delta^{-1}\partial_{x^j}v^j\right),\ i = 1, 2, 3.$$

Therefore we may apply the Div-Curl lemma (Lemma 4.3.10) and using

$$T_k(Z_\delta) \to \overline{T_k(Z)} \text{ in } C_{\text{weak}}([0, T]; L^q(\Omega)),\ 1 \le q < \infty,$$

$$\varrho_\delta \boldsymbol{u}_\delta \to \varrho\boldsymbol{u} \text{ in } C_{\text{weak}}([0, T]; L^{2\gamma/(\gamma+1)}(\Omega; \mathbb{R}^3)),$$

we conclude that

$$T_k(Z_\delta)(t, \cdot)\boldsymbol{\mathcal{A}}[1_\Omega \operatorname{div}(\varrho_\delta \boldsymbol{u}_\delta)(t, \cdot)] - (\varrho_\delta \boldsymbol{u}_\delta)(t, \cdot) \cdot \boldsymbol{\mathcal{R}}[1_\Omega T_k(Z_\delta)(t, \cdot)] \tag{4.117}$$

$$\to$$

$$\overline{T_k(Z)}(t, \cdot)\boldsymbol{\mathcal{A}}[1_\Omega \operatorname{div}(\varrho\boldsymbol{u})(t, \cdot)] - (\varrho\boldsymbol{u})(t, \cdot) \cdot \boldsymbol{\mathcal{R}}[1_\Omega \overline{T_k(Z)}(t, \cdot)]$$

weakly in $L^s(\Omega; \mathbb{R}^3)$ for all $t \in [0, T]$,

with

$$s < \frac{2\gamma}{\gamma + 1}.$$

Note that $s > \frac{6}{5}$ since $\gamma > \frac{3}{2}$ and thus the convergence in (4.117) takes place in the space

$$L^q(0, T; W^{-1,2}(\Omega)) \text{ for any } 1 \le q < \infty;$$

going back to (4.116), we have

$$\lim_{\delta \to 0^+} \int_0^T \psi \int_\Omega \phi\Big(Z_\delta^\gamma T_k(Z_\delta) - (\lambda + 2\mu)\operatorname{div}\boldsymbol{u}_\delta T_k(Z_\delta)\Big)\ \mathrm{d}x\ \mathrm{d}t \tag{4.118}$$

$$= \int_0^T \psi \int_\Omega \phi\Big(\overline{Z^\gamma}\ \overline{T_k(Z)} - (\lambda + 2\mu)\operatorname{div}\boldsymbol{u}\overline{T_k(Z)}\Big)\ \mathrm{d}x\ \mathrm{d}t.$$

□

Remark 4.7.2. *Observe that an analogue of equality (4.113) holds also when we consider σ_δ instead of $T_k(Z_\delta)$, where σ_δ are uniformly essentially bounded and satisfy*

$$\partial_t \sigma_\delta + \operatorname{div}(\sigma_\delta \boldsymbol{u}_\delta) = f_\delta$$

where f_δ are bounded in $L^2((0, T) \times \Omega)$ (see [68] and [71]). This generalization will be necessary in Section 4.8.

Oscillation defect measure and renormalized solutions

The main results of this part are essentially taken over from [31]:

Lemma 4.7.3. *There exists a constant c independent of k such that*

$$\limsup_{\delta\to 0^+} \|T_k(Z_\delta) - T_k(Z)\|_{L^{\gamma+1}((0,T)\times\Omega)} \le c \tag{4.119}$$

with c independent of $k \ge 1$.

Proof. One has

$$\begin{aligned}
&\limsup_{\delta\to 0^+} \int_0^T \int_\Omega \left(Z_\delta^\gamma T_k(Z_\delta) - \overline{Z^\gamma}\ \overline{T_k(Z)} \right) \, \mathrm{d}x \, \mathrm{d}t \\
&\quad = \limsup_{\delta\to 0^+} \int_0^T \int_\Omega (Z_\delta^\gamma - Z^\gamma)(T_k(Z_\delta) - T_k(Z)) \, \mathrm{d}x \, \mathrm{d}t \\
&\qquad + \int_0^T \int_\Omega (\overline{Z^\gamma} - Z^\gamma)(T_k(Z) - \overline{T_k(Z)}) \, \mathrm{d}x \, \mathrm{d}t \\
&\quad \ge \limsup_{\delta\to 0^+} \int_0^T \int_\Omega (Z_\delta^\gamma - Z^\gamma)(T_k(Z_\delta) - T_k(Z)) \, \mathrm{d}x \, \mathrm{d}t \\
&\quad \ge \limsup_{\delta\to 0^+} \int_0^T \int_\Omega |T_k(Z_\delta) - T_k(Z)|^{\gamma+1} \, \mathrm{d}x \, \mathrm{d}t,
\end{aligned} \tag{4.120}$$

as $Z \mapsto Z^\gamma$ is convex, T_k concave on $\mathbb{R}_+$, and

$$(z^\gamma - y^\gamma)(T_k(z) - T_k(y)) \ge |T_k(z) - T_k(y)|^{\gamma+1} \tag{4.121}$$

for all $z, y \ge 0$. Hence,

$$\limsup_{\delta\to 0^+} \int_0^T \int_\Omega |T_k(Z_\delta) - T_k(Z)|^{\gamma+1} \, \mathrm{d}x \, \mathrm{d}t \le \int_0^T \int_\Omega (\overline{Z^\gamma T_k(Z)} - \overline{Z^\gamma}\ \overline{T_k(Z)}) \, \mathrm{d}x \, \mathrm{d}t. \tag{4.122}$$

On the other hand,

$$\begin{aligned}
&\limsup_{\delta\to 0^+} \int_0^T \int_\Omega \operatorname{div} \boldsymbol{u}_\delta \big(T_k(Z_\delta) - \overline{T_k(Z)}\big) \, \mathrm{d}x \, \mathrm{d}t \\
&\quad = \limsup_{\delta\to 0^+} \int_0^T \int_\Omega \big(T_k(Z_\delta) - T_k(Z) + T_k(Z) - \overline{T_k(Z)}\big) \operatorname{div} \boldsymbol{u}_\delta \, \mathrm{d}x \, \mathrm{d}t \\
&\quad \le 2 \sup_{\delta > 0} \| \operatorname{div} \boldsymbol{u}_\delta \|_{L^2((0,T)\times\Omega)} \limsup_{\delta\to 0^+} \|T_k(Z_\delta) - T_k(Z)\|_{L^2((0,T)\times\Omega)}.
\end{aligned} \tag{4.123}$$

Relations (4.121), (4.122) combined with Lemma 4.7.1 yield the desired conclusion. □

Using the result of Lemma 4.7.3 one has the following crucial assertion (see Lemma 4.3.8):

Lemma 4.7.4. *The limit functions $(Z, \boldsymbol{u})$ solve (4.7b) in the sense of renormalized solutions, i.e.,*

$$\partial_t b(Z) + \operatorname{div}(b(Z)\boldsymbol{u}) + ((b'(Z)Z - b(Z)) \operatorname{div} \boldsymbol{u} = 0 \tag{4.124}$$

holds in $\mathcal{D}'((0,T) \times \mathbb{R}^3)$ for any $b \in C^1(\mathbb{R})$ satisfying (4.25) provided $(Z, \boldsymbol{u})$ are extended by zero outside Ω.

Strong convergence of the density

We are going to complete the proof of Theorem 4.2.5. To this end, we introduce a family of functions $(L_k)_{k\geq 1}$:

$$L_k(z) = z\int_1^z \frac{T_k(s)}{s^2}\ \mathrm{d}s.$$

Note that L_k is convex for any $k \geq 1$ and

$$ZL_k'(Z) - L_k(Z) = T_k(Z). \tag{4.125}$$

We can use the fact that $(Z_\delta, \boldsymbol{u}_\delta)$ are renormalized solutions of (4.53b) to deduce

$$\partial_t L_k(Z_\delta) + \operatorname{div}\big(L_k(Z_\delta)\boldsymbol{u}_\delta\big) + T_k(Z_\delta)\operatorname{div}\boldsymbol{u}_\delta = 0 \tag{4.126}$$

in $\mathcal{D}'((0,T)\times\mathbb{R}^3)$ with Z_δ, $\boldsymbol{u}_\delta$ extended by zero outside of Ω. Similarly, by virtue of (4.110) and Lemma 4.7.4 (as above, we may justify the use of $L_k(\cdot)$ by density argument)

$$\partial_t L_k(Z) + \operatorname{div}\big(L_k(Z)\boldsymbol{u}\big) + T_k(Z)\operatorname{div}\boldsymbol{u} = 0 \tag{4.127}$$

in $\mathcal{D}'((0,T)\times\mathbb{R}^3)$.

In view of (4.126), we have

$$L_k(Z_\delta) \to \overline{L_k(Z)} \text{ in } C([0,T]; L^q_\omega(\Omega)) \tag{4.128}$$

for all $1 \leq q < \infty$. Hence (4.126) yields

$$\partial_t\overline{L_k(Z)} + \operatorname{div}(\overline{L_k(Z)}\boldsymbol{u}) + \overline{T_k(Z)\operatorname{div}\boldsymbol{u}} = 0 \tag{4.129}$$

in $\mathcal{D}'((0,T)\times\mathbb{R}^3)$. Therefore, (4.127) and (4.129) imply

$$\int_\Omega\Big(\overline{L_k(Z(T))} - L_k(Z(T))\Big)\ \mathrm{d}x = \int_0^T\int_\Omega\Big(T_k(Z)\operatorname{div}\boldsymbol{u} - \overline{T_k(Z)\operatorname{div}\boldsymbol{u}}\Big)\ \mathrm{d}x\ \mathrm{d}t.$$

Due to convexity of $L_k(\cdot)$ we have

$$\begin{aligned}0 &\leq \int_0^T\int_\Omega\Big(T_k(Z)\operatorname{div}\boldsymbol{u} - \overline{T_k(Z)\operatorname{div}\boldsymbol{u}}\Big)\ \mathrm{d}x\ \mathrm{d}t\\ &\leq \int_0^T\int_\Omega\Big(T_k(Z) - \overline{T_k(Z)}\Big)\operatorname{div}\boldsymbol{u}\ \ \mathrm{d}x\ \mathrm{d}t\\ &\quad + \int_0^T\int_\Omega\Big(\overline{T_k(Z)}\operatorname{div}\boldsymbol{u} - \overline{T_k(Z)\operatorname{div}\boldsymbol{u}}\Big)\ \mathrm{d}x\ \mathrm{d}t.\end{aligned}$$

Now, the effective viscous flux equality (4.112) and (4.122) implies

$$\begin{aligned}&\frac{1}{2\mu+\lambda}\limsup_{\delta\to 0^+}\int_0^T\int_\Omega |T_k(Z_\delta) - T_k(Z)|^{\gamma+1}\ \mathrm{d}x\ \mathrm{d}t\\ &\quad\leq \int_0^T\int_\Omega\Big(\overline{T_k(Z)\operatorname{div}\boldsymbol{u}} - \overline{T_k(Z)}\operatorname{div}\boldsymbol{u}\Big)\ \mathrm{d}x\ \mathrm{d}t;\end{aligned}$$

whence

$$\frac{1}{2\mu+\lambda}\limsup_{\delta\to 0^+}\int_0^T\int_\Omega |T_k(Z_\delta)-T_k(Z)|^{\gamma+1}\ \mathrm{d}x\ \mathrm{d}t$$
$$\leq \int_0^T\int_\Omega |T_k(Z)-\overline{T_k(Z)}||\operatorname{div}\boldsymbol{u}|\ \mathrm{d}x\ \mathrm{d}t$$
$$\leq C\|T_k(Z)-\overline{T_k(Z)}\|_{L^1((0,T)\times\Omega)}^{\frac{\gamma-1}{2\gamma}}\|T_k(Z)-\overline{T_k(Z)}\|_{L^{\gamma+1}((0,T)\times\Omega)}^{\frac{\gamma+1}{2\gamma}}.$$

Recall that

$$\|T_k(Z)-\overline{T_k(Z)}\|_{L^1((0,T)\times\Omega)}\leq \|T_k(Z)-Z\|_{L^1((0,T)\times\Omega)}+\|\overline{T_k(Z)}-Z\|_{L^1((0,T)\times\Omega)},$$

yielding

$$\lim_{k\to\infty}\|T_k(Z)-\overline{T_k(Z)}\|_{L^1((0,T)\times\Omega)}=0.$$

As

$$\sup_{k\geq 1}\limsup_{\delta\to 0^+}\|T_k(Z_\delta)-T_k(Z)\|_{L^{\gamma+1}((0,T)\times\Omega)}<+\infty,$$

we also have

$$\lim_{k\to\infty}\limsup_{\delta\to 0^+}\|T_k(Z_\delta)-T_k(Z)\|_{L^{\gamma+1}((0,T)\times\Omega)}=0.$$

Therefore, one verifies that

$$Z_\delta\to Z \text{ strongly in } L^q((0,T)\times\Omega)$$

for any $q<\gamma+\theta$. The proof of Theorem 4.2.5 is finished.

4.8 Proof of equivalent formulations

From Theorem 4.2.5 it follows that for any $\gamma>\frac{3}{2}$ there exists a triple of functions

$$(\varrho,Z,\boldsymbol{u})\in L^\infty(0,T;L^\gamma(\Omega))\times L^\infty(0,T;L^\gamma(\Omega))\times L^2(0,T;W_0^{1,2}(\Omega,\mathbb{R}^3)) \quad (4.130)$$

satisfying equations (4.7) in the sense specified in Definition 4.2.3. However, in what follows, we will use the result only for $\gamma\geq\frac{9}{5}$.

Our aim will be to deduce from this the existence of $s\in L^\infty((0,T)\times\Omega)$ such that the pressure in the momentum equation equals $p=\varrho^\gamma\mathcal{T}(s)$ satisfying either equality (4.20) or the distributional formulation of (4.6b) with corresponding initial data in a similar way as suggested in Feireisl et al. [39].

4.8.1 The case $\gamma\geq\frac{9}{5}$

We first present the main ideas of the proof which corresponds to the situation

$$\frac{Z_0}{\varrho_0}1_{\{\varrho_0=0\}}=T(s_0)^{\frac{1}{\gamma}}1_{\{\varrho_0=0\}}=1.$$

Due to the construction we know that functions $(\varrho,Z,\boldsymbol{u})$ extended by zero outside of Ω fulfill equations (4.7a), (4.7b) in the sense of distributions on the whole

$(0,T)\times\mathbb{R}^3$. Therefore, we may test both of these equations by $\xi_\eta(x-\cdot)$, where ξ_η is a standard mollifier. We obtain the following equations

$$\partial_t\varrho_\eta + \operatorname{div}(\varrho_\eta \boldsymbol{u}) = r^1_\eta, \tag{4.131}$$

$$\partial_t Z_\eta + \operatorname{div}(Z_\eta \boldsymbol{u}) = r^2_\eta, \tag{4.132}$$

satisfied a.e. in $(0,T)\times\mathbb{R}^3$, where by a_η we denoted $a*\xi_\eta$. From the Friedrichs lemma (see e.g. [69]) we know that r^1_η, r^2_η converge to 0 strongly in $L^1((0,T)\times\mathbb{R}^3)$ as $\eta\to 0^+$ (the strong convergence of r^1_η requires the stronger assumption on γ). Now we multiply the first equation by $-\frac{(Z_\eta+\lambda)}{(\varrho_\eta+\lambda)^2}$ and the second by $\frac{1}{\varrho_\eta+\lambda}$ with $\lambda>0$, respectively. Note that for η fixed $\partial_t\varrho_\eta$ and $\partial_t Z_\eta$ belong to $L^\infty(0,T;C^\infty_c(\mathbb{R}^3))$, so these are sufficiently regular test functions. After some manipulations, we obtain the following equation

$$\partial_t\left(\frac{Z_\eta+\lambda}{\varrho_\eta+\lambda}\right) + \operatorname{div}\left[\left(\frac{Z_\eta+\lambda}{\varrho_\eta+\lambda}\right)\boldsymbol{u}\right] - \left[\frac{(Z_\eta+\lambda)\varrho_\eta}{(\varrho_\eta+\lambda)^2} + \frac{\lambda}{\varrho_\eta+\lambda}\right]\operatorname{div}\boldsymbol{u}$$
$$= -r^1_\eta\frac{Z_\eta+\lambda}{(\varrho_\eta+\lambda)^2} + r^2_\eta\frac{1}{\varrho_\eta+\lambda}$$

satisfied a.e. in $(0,T)\times\mathbb{R}^3$. Note that

$$Z_\eta(t,x) = \int_{\mathbb{R}^3} Z(t,y)\xi_\eta(x-y)\mathrm{d}y \le c^\star\int_{\mathbb{R}^3}\varrho(t,y)\xi_\eta(x-y)\mathrm{d}y = c^\star\varrho_\eta,$$

and therefore

$$\frac{Z_\eta+\lambda}{\varrho_\eta+\lambda} \le \frac{c^\star\varrho_\eta+\lambda}{\varrho_\eta+\lambda} \le \max\{1,c^\star\}, \qquad \frac{1}{\varrho_\eta+\lambda}\le\frac{1}{\lambda}.$$

So, for λ fixed, we may use the strong convergence of $\varrho_\eta\to\varrho$, $Z_\eta\to Z$ and the dominated convergence theorem to let $\eta\to 0$ and to obtain the following equation

$$\partial_t\left(\frac{Z+\lambda}{\varrho+\lambda}\right) + \operatorname{div}\left[\left(\frac{Z+\lambda}{\varrho+\lambda}\right)\boldsymbol{u}\right] - \left[\frac{(Z+\lambda)\varrho}{(\varrho+\lambda)^2} + \frac{\lambda}{\varrho+\lambda}\right]\operatorname{div}\boldsymbol{u} = 0$$

which is satisfied in the sense of distributions on $(0,T)\times\mathbb{R}^3$. Before we pass to the limit with $\lambda\to 0^+$ note that we may distinguish two situations

- for $\varrho=0$ we have $Z=0$ and therefore $\frac{Z+\lambda}{\varrho+\lambda}=1$, while $\frac{(Z+\lambda)\varrho}{(\varrho+\lambda)^2}+\frac{\lambda}{\varrho+\lambda}=1$,
- for $\varrho>0$ we have $\frac{Z+\lambda}{\varrho+\lambda}\le\max\{1,c^\star\}$, $\varrho+\lambda\to\varrho$, $Z+\lambda\to Z$ strongly in $L^\infty(0;T;L^2_{\mathrm{loc}}(\mathbb{R}^3))$, therefore $\frac{Z+\lambda}{\varrho+\lambda}\to\frac{Z}{\varrho}$ strongly in $L^\infty((0,T)\times\mathbb{R}^3)$ and so $\frac{(Z+\lambda)\varrho}{(\varrho+\lambda)^2}+\frac{\lambda}{\varrho+\lambda}\to\frac{Z}{\varrho}$.

Recall that this construction corresponds to the choice $\zeta_0=1$ in (4.28). In the more general case, for $\mathcal{T}(s_0)^{\frac{1}{\gamma}}=A_0$ we would have to replace λ in the numerator by $A_0\lambda$.

The case of non-constant $A_0 1_{\varrho_0=0}=\mathcal{T}(s_0)^{\frac{1}{\gamma}}1_{\varrho_0=0}\in L^\infty(\Omega)$ demands a bit more technical treatment. Let A be any solution of the transport equation (4.6b) with the initial data A_0. Such solution can be found using smoothing of $\boldsymbol{u}$ and solving the transport equation by the method of trajectories. Along with (4.131) we also test the transport equation for A by the same family of mollifiers obtaining

$$\partial_t A_\eta + \boldsymbol{u}\nabla A_\eta = r_\eta^3$$

with $r_\eta^3 \to 0$ in $L^1((0,T)\times\mathbb{R}^3)$ as $\eta \to 0^+$. In combination with the continuity equation we obtain

$$\partial_t(Z_\eta + \lambda A_\eta) + \operatorname{div}((Z_\eta + \lambda A_\eta)\boldsymbol{u}) = r_\eta^2 + \lambda(r_\eta^3 + A_\eta \operatorname{div}\boldsymbol{u}). \tag{4.133}$$

We multiply the last equality by $\frac{1}{\varrho_\eta+\lambda}$ and mimicking the previous approach we obtain

$$\partial_t\left(\frac{Z_\eta+\lambda A_\eta}{\varrho_\eta+\lambda}\right) + \operatorname{div}\left[\left(\frac{Z_\eta+\lambda A_\eta}{\varrho_\eta+\lambda}\right)\boldsymbol{u}\right] - \left[\frac{(Z_\eta+\lambda A_\eta)\varrho_\eta}{(\varrho_\eta+\lambda)^2} + \frac{\lambda A_\eta}{\varrho_\eta+\lambda}\right]\operatorname{div}\boldsymbol{u}$$
$$= -r_\eta^1\frac{Z_\eta+\lambda A_\eta}{(\varrho_\eta+\lambda)^2} + r_\eta^2\frac{1}{\varrho_\eta+\lambda} + r_\eta^3\frac{\lambda}{\varrho_\eta+\lambda}.$$

Next, we let $\eta \to 0^+$ and get

$$\partial_t\left(\frac{Z+\lambda A}{\varrho+\lambda}\right) + \operatorname{div}\left[\left(\frac{Z+\lambda A}{\varrho+\lambda}\right)\boldsymbol{u}\right] - \left[\frac{(Z+\lambda A)\varrho}{(\varrho+\lambda)^2} + \frac{\lambda A}{\varrho+\lambda}\right]\operatorname{div}\boldsymbol{u} = 0.$$

Let us denote $\theta = \frac{Z}{\varrho}$ for $\varrho > 0$ and $\theta = A$ for $\varrho = 0$. Observe that $c_* \le \theta \le c^*$ almost everywhere in $(0,T)\times\Omega$. Once again, we use the uniform boundedness of $\frac{Z+A\lambda}{\varrho+\lambda}$ and send $\lambda \to 0^+$ obtaining

$$\partial_t\theta + \operatorname{div}(\theta\boldsymbol{u}) - \theta\operatorname{div}\boldsymbol{u} = 0$$

or, equivalently,

$$\partial_t\theta + \boldsymbol{u}\cdot\nabla\theta = 0 \tag{4.134}$$

in the sense of distributions on $(0,T)\times\mathbb{R}^3$. The initial condition A_0 is attained in the sense of weak solutions for the transport equation. In addition, we can renormalize this equation, using any $G \in C^1(\mathbb{R})$ and deduce that

$$\partial_t G(\theta) + \boldsymbol{u}\cdot\nabla G(\theta) = 0 \tag{4.135}$$

is also satisfied in the sense of distributions on $(0,T)\times\mathbb{R}^3$. Taking for example $G(\theta) = \mathcal{T}^{-1}(\theta^\gamma)$, we obtain equation for s

$$\partial_t s + \boldsymbol{u}\cdot\nabla s = 0, \tag{4.136}$$

and, for $G(\theta) = B(\mathcal{T}^{-1}(\theta^\gamma))$, also its renormalized version

$$\partial_t B(s) + \boldsymbol{u}\cdot\nabla B(s) = 0 \tag{4.137}$$

satisfied in the sense of distributions on $(0,T)\times\mathbb{R}^3$ for any $B \in C^1(\mathbb{R})$.

In order to obtain the weak solution to problem (4.1b) we need to test equation (4.136) by ϱ. This is, however, not allowed due to low regularity of ϱ. Instead we will use $\varphi\varrho_\eta$, where $\varphi \in C_c^\infty((0,T)\times\mathbb{R}^3)$ and ϱ_η satisfies (4.131). Here we essentially use the fact that $\varrho \in L^2((0,T)\times\Omega)$, hence this step cannot be repeated for γ less then $\frac{9}{5}$. Then we also multiply (4.131) by φs and sum up the obtained expressions to deduce

$$\int_0^T \int_{\mathbb{R}^3} \varrho_\eta s \partial_t \varphi \, \mathrm{d}x \, \mathrm{d}t + \int_0^T \int_{\mathbb{R}^3} (\varrho_\eta s \boldsymbol{u}) \cdot \nabla \varphi \, \mathrm{d}x \, \mathrm{d}t = - \int_0^T \int_\Omega r^1_\eta s \varphi \, \mathrm{d}x \, \mathrm{d}t.$$

Having this formulation we pass to the limit with $\eta \to 0^+$, note that the term on the r.h.s. vanishes and therefore we obtain

$$\int_0^T \int_{\mathbb{R}^3} \varrho s \partial_t \varphi \, \mathrm{d}x \, \mathrm{d}t + \int_0^T \int_{\mathbb{R}^3} (\varrho s \boldsymbol{u}) \cdot \nabla \varphi \, \mathrm{d}x \, \mathrm{d}t = 0. \tag{4.138}$$

Note that if we start from (4.137), we can also get

$$\int_0^T \int_{\mathbb{R}^3} \varrho B(s) \partial_t \varphi \, \mathrm{d}x \, \mathrm{d}t + \int_0^T \int_{\mathbb{R}^3} (\varrho B(s) \boldsymbol{u}) \cdot \nabla \varphi \, \mathrm{d}x \, \mathrm{d}t = 0 \tag{4.139}$$

for any $B \in C^1(\mathbb{R})$.

Thus we have almost our formulation from Definition 4.2.1, except the initial condition. Indeed, for the moment we only know that equation $\partial_t(\varrho s) + \mathrm{div}(\varrho s \boldsymbol{u}) = 0$ is satisfied in the sense of distributions on $(0,T) \times \mathbb{R}^3$. Moreover, from the $L^\infty((0,T) \times \mathbb{R}^3)$ bound on s and the above equation, we deduce using Arzelà–Ascoli theorem that $\varrho s \in C([0,T]; L^\gamma_\omega(\Omega))$.

To recover the initial and the terminal condition, we need to use a test function φ from the space $C^1([0,T] \times \overline{\Omega})$ instead of $C^\infty_c((0,T) \times \Omega)$. To this purpose we define the following function

$$\varphi_\tau(t,x) = \begin{cases} \frac{t}{\tau} \varphi(\tau, x) & \text{for} \quad t \le \tau \\ \varphi(t,x) & \text{for} \quad \tau \le t \le T - \tau, \\ \frac{T-t}{\tau} \varphi(T-\tau, x) & \text{for} \quad T - \tau \le t \end{cases}$$

for $\varphi \in C^1([0,T] \times \overline{\Omega})$. Note that φ_τ is an admissible test function for (4.139), we can write

$$\begin{aligned} &\int_\tau^T \int_\Omega \varrho s \partial_t \varphi \, \mathrm{d}x \, \mathrm{d}t + \int_0^T \int_\Omega (\varrho s \boldsymbol{u}) \cdot \nabla \varphi \, \mathrm{d}x \, \mathrm{d}t \\ &\quad = -\frac{1}{\tau} \int_0^\tau \int_\Omega \varrho s \varphi(\tau, x) \, \mathrm{d}x \, \mathrm{d}t + \frac{1}{\tau} \int_{T-\tau}^T \int_\Omega \varrho s \varphi(T-\tau, x) \, \mathrm{d}x \, \mathrm{d}t. \end{aligned} \tag{4.140}$$

We represent function $\varphi(t,x)$ as $\varphi(t,x) = \psi(t)\zeta(x)$ (or approximate by such sums), where $\psi \in C^\infty_c((0,T))$, $\zeta \in C^\infty_c(\overline{\Omega})$, then the r.h.s. of (4.140) equals

$$\begin{aligned} &-\frac{1}{\tau} \int_0^\tau \int_\Omega \varrho s \varphi(\tau, x) \, \mathrm{d}x \, \mathrm{d}t + \frac{1}{\tau} \int_{T-\tau}^T \int_\Omega \varrho s \varphi(T-\tau, x) \, \mathrm{d}x \, \mathrm{d}t \\ &= -\frac{\psi(\tau)}{\tau} \int_0^\tau \int_\Omega \varrho s \zeta(x) \, \mathrm{d}x \, \mathrm{d}t + \frac{\psi(T-\tau)}{\tau} \int_{T-\tau}^T \int_\Omega \varrho s \zeta(x) \, \mathrm{d}x \, \mathrm{d}t, \end{aligned}$$

and by the weak continuity of ϱs, letting $\tau \to 0$, we conclude that

$$\begin{aligned} &\int_0^T \int_\Omega \varrho s \partial_t \varphi \, \mathrm{d}x \, \mathrm{d}t + \int_0^T \int_\Omega (\varrho s \boldsymbol{u}) \cdot \nabla \varphi \, \mathrm{d}x \, \mathrm{d}t \\ &\quad = - \int_\Omega (\varrho s)(0, \cdot) \varphi(0, \cdot) \, \mathrm{d}x + \int_\Omega (\varrho s)(T, \cdot) \varphi(T, \cdot) \, \mathrm{d}x \\ &\quad = - \int_\Omega S_0(\cdot) \varphi(0, \cdot) \, \mathrm{d}x + \int_\Omega (\varrho s)(T, \cdot) \varphi(T, \cdot) \, \mathrm{d}x \end{aligned} \tag{4.141}$$

and so the statement of Theorem 4.2.2 is proven. Similarly we may get the initial condition for $s(t,\cdot)$.

4.8.2 The case $\gamma > \frac{3}{2}$

The case of general γ has to be treated differently, due to the lack of $L^2((0,T)\times\Omega)$ estimate on ϱ and Z. The latter is necessary to apply the DiPerna-Lions technique of renormalization of the transport equation [26]. In the general case, we have to use more subtle technique developed by Feireisl, see e.g. [32] and used recently in [68] to study stability of solutions to system (4.6). In this section we will extend the stability result and prove existence of solutions to system (4.6) by giving a suitable sequence of approximative problems.

As a starting point for the further analysis we take system (4.53) with $\beta > \max\{\gamma, 4\}$ and initial data $Z_{0,\delta} = \frac{\varrho_{0,\delta}}{\zeta_{0,\delta}}$, with $\zeta_{0,\delta}$ satisfying (4.28). At this stage, we are able to repeat the procedure described in the previous section in order to recover equation (4.134) for θ_δ and its renormalized version (4.135) in the sense of distributions on $(0,T)\times\Omega$. Moreover, $\theta_\delta, \theta_\delta^{-1}$ are bounded in $L^\infty((0,T)\times\Omega)$ uniformly with respect to δ. Thus, our system

$$\partial_t \varrho_\delta + \operatorname{div}(\varrho_\delta \boldsymbol{u}_\delta) = 0, \tag{4.142a}$$

$$\partial_t B(\theta_\delta) + \boldsymbol{u}_\delta \cdot \nabla B(\theta_\delta) = 0, \tag{4.142b}$$

$$\partial_t(\varrho_\delta \boldsymbol{u}_\delta) + \operatorname{div}(\varrho_\delta \boldsymbol{u}_\delta \otimes \boldsymbol{u}_\delta) + \nabla(\varrho_\delta\theta_\delta)^\gamma + \delta\nabla(\varrho_\delta\theta_\delta)^\beta = \operatorname{div}\mathbb{S}(\nabla \boldsymbol{u}_\delta), \tag{4.142c}$$

where $\theta_\delta = \frac{Z_\delta}{\varrho_\delta}$, is satisfied in the sense of distributions on $(0,T)\times\Omega$. Observe that ϱ_δ belongs (not necessarily uniformly with respect to δ) to $L^\beta((0,T)\times\Omega)$ for each $\delta > 0$. At this stage we can use the stability result given by Theorem 3.1 in [68] and finish the proof of Theorem 4.2.7.

For the sake of completeness, we will provide the limit process $\delta \to 0^+$ following the arguments from [68]. We take $\zeta_\delta = \theta_\delta^{-1}$ and denote ζ the weak$-\star$ limit of ζ_δ (or its subsequence) in $L^\infty((0,T)\times\Omega)$. For any $\delta > 0$ the pair $(\zeta_\delta, \boldsymbol{u}_\delta)$ satisfies the transport equation in the weak sense (see Definition 4.2.6) along with the initial data $\zeta_{0,\delta} = \frac{Z_{0,\delta}}{\varrho_{0,\delta}}$. As we know from Section 4.7, sequence $Z_\delta = \frac{\varrho_\delta}{\zeta_\delta}$ (or its subsequence) converges strongly in $L^q((0,T)\times\Omega)$ to Z for any $q < \gamma + \theta$. Hence for the same q we have

$$\varrho_\delta = Z_\delta\zeta_\delta \to Z\zeta \text{ weakly in } L^q((0,T)\times\Omega).$$

Therefore ζ, ϱ and $\boldsymbol{u}$ satisfy in the weak sense

$$\partial_t \varrho + \operatorname{div}(\varrho \boldsymbol{u}) = 0, \tag{4.143a}$$

$$\partial_t(\varrho \boldsymbol{u}) + \operatorname{div}(\varrho \boldsymbol{u} \otimes \boldsymbol{u}) + \nabla\left(\frac{\varrho}{\zeta}\right)^\gamma = \operatorname{div}\mathbb{S}(\nabla \boldsymbol{u}). \tag{4.143b}$$

The next step is to show that the pair $(\zeta, \boldsymbol{u})$ satisfies the transport equation

$$\partial_t \zeta - \boldsymbol{u}\cdot\nabla\zeta = 0 \tag{4.143c}$$

in the weak sense. We apply the Div-Curl lemma (Lemma 4.3.10) with

$$\boldsymbol{U}_\delta = (\zeta_\delta, \zeta_\delta \boldsymbol{u}_\delta), \quad \boldsymbol{V}_\delta = (\boldsymbol{u}_\delta^j, 0, 0, 0),$$

where $j \in \{1,2,3\}$. We know that $\operatorname{div} \boldsymbol{U}_\delta$ and $\operatorname{curl} \boldsymbol{V}_\delta$ are bounded in $L^2((0,T)\times\Omega)$, hence precompact in $W^{-1,2}((0,T)\times\Omega)$. Therefore we obtain $\zeta_\delta \boldsymbol{u}_\delta \to \zeta \boldsymbol{u}$ weakly in $L^2((0,T)\times\Omega,\mathbb{R}^3)$. Due to the strong convergence of the pressure terms Z_δ^γ we get by the means of Lemma 4.7.1 (and Remark 4.7.2)

$$\zeta_\delta \operatorname{div} \boldsymbol{u}_\delta \to \zeta \operatorname{div} \boldsymbol{u} \text{ weakly in } L^2((0,T)\times\Omega).$$

Therefore (4.143c) is satisfied in the weak sense and due to the boundedness of ζ it is also a renormalized solution. The proof of Theorem 4.2.7 is complete.

5. A convergent numerical method for the full Navier-Stokes-Fourier system in smooth physical domains

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Abstract

We propose a mixed finite volume - finite element numerical method for solving the full Navier-Stokes-Fourier system describing the motion of a compressible, viscous, and heat conducting fluid. The physical domain occupied by the fluid has a smooth boundary and it is approximated by a family of polyhedral numerical domains. Convergence and stability of the numerical scheme is studied. The numerical solutions are shown to converge, up to a subsequence, to a weak solution of the problem posed on the limit domain.

5.1 Introduction

Numerical methods based on finite elements approximation use a mesh over the physical domain Ω. If the boundary of the latter is curved, meshes built up by means of *polygonal* elements can only approximate the kinematic boundary $\partial\Omega$. On the other hand, rigorous *error* estimates of the numerical methods usually require the exact solution of the problem to be smooth. Smooth solutions, however, can exist only on *regular* physical domains. It is therefore of interest to study the convergence of a numerical scheme in the situation when a family of numerical polyhedral domains Ω_h approaches, in a certain sense, the limit physical domain Ω. To avoid technicalities and since we are primarily interested in *smooth* solutions of the problem, only bounded domains with a sufficiently smooth boundary $\partial\Omega \in C^1$ will be considered although the principal results of this paper can be easily extended to less regular geometries, say $\partial\Omega$ Lipschitz.

5.1.1 Navier-Stokes-Fourier system

The motion of a compressible, viscous, and heat conducting fluid in the framework of continuum mechanics is characterized by three basic macroscopic (observable) quantities: the mass density $\varrho = \varrho(t, x)$, the absolute temperature $\vartheta = \vartheta(t, x)$, and the velocity field $\mathbf{u} = \mathbf{u}(t, x)$, depending on the time $t \in (0, T)$ and the reference (Eulerian) spatial position $x \in \Omega$. The time evolution of the fluid is governed by the *Navier-Stokes-Fourier system* of equations for Newtonian fluids:

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{5.1}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) = \mu \Delta \mathbf{u} + (\mu + \lambda) \nabla_x \mathrm{div}_x \mathbf{u}, \tag{5.2}$$

$$c_v \left[\partial_t(\varrho\vartheta) + \mathrm{div}_x(\varrho\vartheta\mathbf{u})\right] - \mathrm{div}_x(\kappa(\vartheta)\nabla_x\vartheta) = \mu|\nabla_x\mathbf{u}|^2 + \lambda|\mathrm{div}_x\mathbf{u}|^2 - \vartheta\frac{\partial p(\varrho,\vartheta)}{\partial\vartheta}\mathrm{div}_x\mathbf{u}, \tag{5.3}$$

with the pressure

$$p(\varrho,\vartheta) = a_1\varrho^\gamma + a_2\varrho + \varrho\vartheta,\ a_1, a_2 > 0, \tag{5.4}$$

$\mu, c_v > 0$, $\lambda \geq -\frac{2}{3}\mu$ and $\gamma > 3$. The heat conductivity $\kappa = \kappa(\vartheta)$ is continuously differentiable, satisfying

$$\underline{\kappa}(1+\vartheta^2) \leq \kappa(\vartheta) \leq \overline{\kappa}(1+\vartheta^2),\ \underline{\kappa} > 0. \tag{5.5}$$

Then we denote the primitive function $K(\vartheta) = \int_0^\vartheta \kappa(z)\ \mathrm{d}z$, i.e. $\kappa(\vartheta)\nabla_x\vartheta = \nabla_x K(\vartheta)$. For the sake of simplicity, the effect of external mechanical and heat sources is omitted in (5.2) and (5.3), respectively. The specific form of the constitutive relations is inspired by similar assumptions introduced in [32]. In particular, the problem (5.1 – 5.5), supplemented with suitable boundary conditions, admits a global-in-time weak solution for any finite energy initial data, see [32, Chapter 7, Theorem 7.1]. In the context of the existence theory developed in [32], the assumption $\gamma > 3$ is optimal.

The system of equations (5.1 – 5.3) is supplemented with the *no-slip* and *no-flux* boundary conditions

$$\mathbf{u}|_{\partial\Omega} = 0, \qquad -\kappa(\vartheta)\nabla_x\vartheta\cdot\mathbf{n}|_{\partial\Omega} = 0, \tag{5.6}$$

the initial state of the fluid is given by

$$\varrho(0,\cdot) = \varrho_0 > 0,\ \vartheta(0,\cdot) = \vartheta_0 > 0,\ \mathbf{u}(0,\cdot) = \mathbf{u}_0. \tag{5.7}$$

5.1.2 Numerical analysis

We propose a modification of the numerical method for the Navier-Stokes-Fourier system developed in [37] adapted to the physical domain with a smooth boundary, where the target domain Ω is approximated by a family of polyhedral (numerical) domains $\{\Omega_h\}_{h>0}$. A similar problem has been treated in [36] in the context of *barotropic fluids*, where the original numerical method of Karlsen and Karper [54], [55] has been adapted to the smooth domain setting. In contrast with [36], the presence of the heat equation (5.3), together with the Neumann type boundary condition $(5.6)_2$, create new difficulties addressed in the present paper.

Motivated by Karper [55], we use a mixed finite-element finite-volume method, where the convective terms are approximated by the standard upwind operator, while the diffusive term in the momentum equation is handled by means of the discontinuous Galerkin method based on the non-conformal finite elements of Crouzeix-Raviart type. Accordingly, we consider an *unfitted tetrahedral mesh* generating a family of numerical domains $\{\Omega_h\}_{h>0}$ such that (see Section 5.2.2 for details)

$$\Omega \subset \overline{\Omega} \subset \Omega_h \subset \mathcal{U}_h[\Omega] \equiv \left\{ x \in R^3 \ \middle|\ \mathrm{dist}[x,\Omega] < h \right\}. \tag{5.8}$$

Since the diffusion coefficient in the heat equation (5.3) is nonlinear, it seems more convenient to use the finite-volume scheme for the discretization of the heat flux as well. In order to prove stability and, more importantly, *consistency* of the resulting numerical method, the underlying mesh should be shape regular in the sense of Eymard et al. [29] and satisfy (5.8) at the same time. Examples of tetrahedral meshes complying with this stipulation were constructed in [51]. As a byproduct of our analysis, the theory developed here probably includes a treatment of variational crimes for the convection-diffusion equation with Neumann boundary conditions for finite volumes that can be of independent interest.

The paper is organized as follows. In Section 5.2, we introduce the concept of *weak solution* to the Navier-Stokes-Fourier system, together with the necessary numerical framework including the basic notation and properties of the underlying function spaces. In Section 5.3, we define the numerical method and state our main result concerning convergence towards a weak solution of the Navier-Stokes-Fourier system. Having exhausted the preliminary material, we report certain relations and estimates already obtained in [37]. Section 5.4 deals with numerical analogues of the renormalized version of the continuity and thermal energy balance as well as discrete version of the total energy balance. Section 5.5 addresses the issue of stability of the scheme, recalling the uniform bounds necessary for the limit passage. The material in these two sections is presented without proofs, with the references to the relevant parts of [37]. Section 5.6 is devoted to the problem of consistency and convergence of the scheme mimicking certain steps of the existence theory developed in [32, Chapter 7]. We conclude the paper in Section 5.7 by showing *unconditional convergence* of the scheme on condition that the numerical solutions remain bounded independently of the step parameter h.

5.2 Preliminaries, weak solutions, numerical framework

In this section, we collect the preliminary material concerning solvability of the Navier-Stokes-Fourier system and the apparatus of numerical analysis used in the paper.

5.2.1 Weak solutions

We use the concept of *weak formulation* of the problem (5.1 – 5.7) introduced in [32, Chapter 4]:

Definition 5.2.1. *A triple of functions* $[\varrho, \vartheta, \mathbf{u}]$ *is a* weak solution *to the problem (5.1 – 5.7) in the space-time cylinder* $(0,T) \times \Omega$ *if:*

$$\varrho \in L^\infty(0,T;L^\gamma(\Omega)),\ \vartheta \in L^2(0,T;L^6(\Omega)),\ \mathbf{u} \in L^2(0,T;W_0^{1,2}(\Omega;R^3)), \tag{5.9}$$

$$\varrho\mathbf{u} \in L^\infty(0,T;L^{\frac{2\gamma}{\gamma+1}}(\Omega;R^3)),\ \varrho\vartheta \in L^\infty(0,T;L^1(\Omega)); \tag{5.10}$$

$$\varrho \geq 0,\ \vartheta > 0\ a.a.\ in\ (0,T)\times\Omega; \tag{5.11}$$

$$\int_0^T \int_\Omega \Big[\varrho\partial_t\varphi + \varrho\mathbf{u}\cdot\nabla_x\varphi\Big]\ \mathrm{d}x\ \mathrm{d}t = -\int_\Omega \varrho_0\varphi(0,\cdot)\ \mathrm{d}x \tag{5.12}$$

for any $\varphi \in C_c^\infty([0,T) \times \overline{\Omega})$;

$$\int_0^T \int_\Omega \Big[\varrho \mathbf{u} \cdot \partial_t \varphi + \varrho \mathbf{u} \otimes \mathbf{u} : \nabla_x \varphi + p(\varrho, \vartheta) \mathrm{div}_x \varphi \Big] \ \mathrm{d}x \ \mathrm{d}t \tag{5.13}$$

$$= \int_0^T \int_\Omega \Big[\mu \nabla_x \mathbf{u} : \nabla_x \varphi + \lambda \mathrm{div}_x \mathbf{u} \ \mathrm{div}_x \varphi \Big] \ \mathrm{d}x \ \mathrm{d}t - \int_\Omega \varrho_0 \mathbf{u}_0 \cdot \varphi(0, \cdot) \ \mathrm{d}x$$

for any $\varphi \in C_c^\infty([0,T) \times \Omega; R^3)$;

$$\begin{aligned} &\int_0^T \int_\Omega \Big[c_v \Big(\varrho \vartheta \partial_t \varphi + \varrho \vartheta \mathbf{u} \cdot \nabla_x \varphi \Big) - \overline{K(\vartheta)} \Delta \varphi \Big] \ \mathrm{d}x \ \mathrm{d}t \\ &\quad + \int_0^T \int_\Omega \Big[\mu |\nabla_x \mathbf{u}|^2 + \lambda |\mathrm{div}_x \mathbf{u}|^2 \Big] \varphi \ \mathrm{d}x \ \mathrm{d}t - \int_0^T \int_\Omega \varrho \vartheta \mathrm{div}_x \mathbf{u} \varphi \ \mathrm{d}x \ \mathrm{d}t \\ &\quad \leq \int_\Omega c_v \varrho_0 \vartheta_0 \varphi(0, \cdot) \ \mathrm{d}x \end{aligned} \tag{5.14}$$

for any $\varphi \in C_c^\infty([0,T) \times \overline{\Omega})$, $\varphi \geq 0$, $\nabla_x \varphi \cdot \mathbf{n}|_{\partial\Omega} = 0$, *where*

$$\varrho \overline{K(\vartheta)} = \varrho K(\vartheta); \tag{5.15}$$

the energy inequality

$$\int_\Omega \left[\frac{1}{2} \varrho |\mathbf{u}|^2 + c_v \varrho \vartheta + \frac{a}{\gamma - 1} \varrho^\gamma + b \varrho \log(\varrho) \right] (\tau, \cdot) \ \mathrm{d}x \tag{5.16}$$

$$\leq \int_\Omega \left[\frac{1}{2} \varrho_0 |\mathbf{u}_0|^2 + c_v \varrho_0 \vartheta_0 + \frac{a}{\gamma - 1} \varrho_0^\gamma + b \varrho_0 \log(\varrho_0) \right] \ \mathrm{d}x \ \textit{for a.a.}\ \tau \in (0,T).$$

The *existence* of weak solutions to the Navier-Stokes-Fourier system on an arbitrary time interval $(0, T)$ was proved in [32, Chapter 7, Theorem 7.1]. The interested reader may consult [32] for a thorough discussion concerning the inequalities in (5.14), (5.16) as well as the interpretation of (5.15). Further properties of weak solutions, and, in particular, the problem of weak-strong uniqueness and conditional regularity are discussed in Section 5.7.

5.2.2 Mesh, finite elements

In what follows, we make systematically use of the following notation:

$$a \stackrel{<}{\sim} b \text{ if } a \leq cb,\ c > 0 \text{ a constant},\ a \approx b \text{ if } a \stackrel{<}{\sim} b \text{ and } b \stackrel{<}{\sim} a.$$

Here, "constant" means a generic quantity independent of the size of the mesh and the time step used in the numerical scheme.

Mesh

Our numerical scheme is constructed over a family of *polyhedral* domains Ω_h approximating Ω in the sense specified in (5.8). Furthermore, we suppose that each Ω_h admits a *conformal shape regular tetrahedral mesh* consisting of a set of compact elements $E \in E_h$, a set of faces $\Gamma \in \Gamma_h$, along with the associated normals $\mathbf{n}$, and a family of control points $x_E \in \mathrm{int}[E]$, enjoying the following property, cf. Eymard et al. [29, Chapter 3]:

1. The intersection $E \cap F$ of two elements $E, F \in E_h$, $E \neq F$ is either empty or their common face, edge, or vertex.

2. For any $E \in E_h$, $\mathrm{diam}[E] \approx h$, $r[E] \approx h$, where r denotes the radius of the largest sphere contained in E.

3. If E and F are two neighboring elements sharing a common face Γ, then the segment $[x_E, x_F]$ is perpendicular to Γ. We denote $d_\Gamma = |x_E - x_F| > 0$.

Remark 5.2.1. *If the mesh is well-centered (cf. VanderZee et al. [79]), the point x_E can be taken the center of the circumsphere of the element E. A well centered mesh satisfying (5.8) for a given domain Ω was constructed in [51].*

Remark 5.2.2. *Since our method is based on finite elements of first order, the expected rate of convergence should be the same even if the polygonal approximation of the physical domain is replaced by more sophisticated "curved" elements, cf. Lenoir [59].*

Each face $\Gamma \in \Gamma_h$ is associated with a normal vector $\mathbf{n}$. We shall write Γ_E whenever a face $\Gamma_E \subset \partial E$ is considered as a part of the boundary of the element E. In such a case, the normal vector to Γ_E is always the *outer* normal vector with respect to E. Moreover, for any function g continuous on each element E, we set

$$g^{\mathrm{out}}|_\Gamma = \lim_{\delta \to 0+} g(\cdot + \delta \mathbf{n}),\ g^{\mathrm{in}}|_\Gamma = \lim_{\delta \to 0+} g(\cdot - \delta \mathbf{n}),\ [[g]]_\Gamma = g^{\mathrm{out}} - g^{\mathrm{in}},\ \{g\}_\Gamma = \frac{g^{\mathrm{out}} + g^{\mathrm{in}}}{2}. \tag{5.17}$$

For $\Gamma_E \subset \partial E$ we simply write g for g^{in}. We also omit the subscript Γ if no confusion arises. Finally, we distinguish two families of faces,

$$\Gamma_{h,\mathrm{ext}} = \left\{ \Gamma \in \Gamma_h \ \middle|\ \Gamma \subset \partial \Omega_h \right\},\ \Gamma_{h,\mathrm{int}} = \Gamma_h \setminus \Gamma_{h,\mathrm{ext}}.$$

Piecewise linear finite elements

We start by introducing the space of piecewise constant functions

$$Q_h(\Omega_h) = \left\{ v \in L^2(\Omega_h) \ \middle|\ v|_E = a_E \in R \text{ for any } E \in E_h \right\},$$

along with the associated projection

$$\Pi_h^Q : L^1(\Omega_h) \to Q_h(\Omega_h),\ \Pi_h^Q[v] \equiv \hat{v},\ \Pi_h^Q[v]|_E = \frac{1}{|E|} \int_E v \ \mathrm{d}x \text{ for any } E \in E_h.$$

From standard Poincaré's inequality we get

$$\left\| v - \Pi_h^Q[v] \right\|_{L^q(\Omega_h)} \stackrel{<}{\sim} h \|\nabla_x v\|_{L^q(\Omega_h;R^3)}, \text{ for any } v \in W^{1,q}(\Omega_h),\ 1 \le q \le \infty, \tag{5.18}$$

$$\left\| v - \frac{1}{|\Gamma_E|} \int_{\Gamma_E} v \ \mathrm{dS}_x \right\|_{L^q(E)} + h^{1/q} \left\| v - \frac{1}{|\Gamma_E|} \int_{\Gamma_E} v \ \mathrm{dS}_x \right\|_{L^q(\Gamma_E)} \stackrel{<}{\sim} h \|\nabla_x v\| |_{L^q(E)}, \tag{5.19}$$

for any $\Gamma_E \subset \partial E$ and $1 \le q < \infty$. The same estimate holds also for $q = \infty$ with obvious modifications.

In order to establish the consistency of the numerical approximation of the heat flux term in (5.3), we shall need another projection

$$\Pi_h^B : C(\overline{\Omega}_h) \to Q_h(\Omega_h),\ \Pi_v^B[v]|_E = v(x_E),\ E \in E_h.$$

Obviously,

$$\|v - \Pi_h^B[v]\|_{L^\infty(\Omega_h)} \stackrel{<}{\sim} h \|\nabla_x v\|_{L^\infty(\Omega_h;R^3)} \text{ for any Lipschitz } v. \tag{5.20}$$

Next, we introduce the *Crouzeix-Raviart finite element spaces*

$$V_h(\Omega_h) = \left\{ v \in L^2(\Omega_h) \middle| \, v|_E = \text{affine}, E \in E_h, \int_\Gamma [[v]] \ \mathrm{dS}_x = 0 \text{ for any } \Gamma \in \Gamma_{h,\text{int}} \right\}, \tag{5.21}$$

$$V_{h,0}(\Omega_h) = \left\{ v \in V_h \ \middle| \ \int_\Gamma v \ \mathrm{dS}_x = 0 \text{ for any } \Gamma \in \Gamma_{h,\text{ext}} \right\}, \tag{5.22}$$

and the projection

$$\Pi_h^V : W^{1,q}(\Omega_h) \to V_h(\Omega_h),\ \int_\Gamma \Pi_h^V[v] \, \mathrm{dS}_x = \int_\Gamma v \, \mathrm{dS}_x \text{ for any } \Gamma \in \Gamma_h.$$

For a differential operator D, we denote

$$D_h v|_E = D(v|_E) \text{ for any } v \text{ differentiable on each element } E \in E_h.$$

It is easy to check that

$$\int_{\Omega_h} \mathrm{div}_h \Pi_h^V[\mathbf{u}] \ w \ \mathrm{d}x = \int_{\Omega_h} \mathrm{div}_h \mathbf{u} \ w \ \mathrm{d}x \text{ for any } w \in Q_h(\Omega_h), \tag{5.23}$$

$$\int_{\Omega_h} \nabla_h v \cdot \nabla_h \Pi_h^V[\varphi] \ \mathrm{d}x = \int_{\Omega_h} \nabla_h v \cdot \nabla_x \varphi \ \mathrm{d}x \text{ if } v \in V_{h,0}(\Omega_h),\ \varphi \in W_0^{1,2}(\Omega_h), \tag{5.24}$$

see Karper [55, Lemma 2.11]. Moreover, as a direct consequence of the shape regularity of the mesh, we record the error estimates

$$\left\| v - \Pi_h^V[v] \right\|_{L^q(\Omega_h)} + h \left\| \nabla_h \left(v - \Pi_h^V[v] \right) \right\|_{L^q(\Omega_h;R^3)} \stackrel{<}{\sim} h^m \left\| \nabla^m v \right\|_{L^q(\Omega_h;R^{3^m})}, \tag{5.25}$$

$m = 1, 2,\ 1 < q < \infty$, for any $v \in W^{m,q}(\Omega_h)$, see Karper [55, Lemma 2.7].

Upwind

We introduce the standard *upwind* operator $\mathrm{Up}[r, \mathbf{u}]$ defined on a face Γ as

$$\mathrm{Up}[r, \mathbf{u}] = r^{\mathrm{in}}[\tilde{\mathbf{u}} \cdot \mathbf{n}]^{+} + r^{\mathrm{out}}[\tilde{\mathbf{u}} \cdot \mathbf{n}]^{-}, \tag{5.26}$$

where we have denoted $[c]^{+} = \max\{c, 0\}$, $[c]^{-} = \min\{c, 0\}$, $\tilde{v} = \frac{1}{|\Gamma|} \int_{\Gamma} v \,\mathrm{dS}_x$. Such a definition makes sense as soon as $r \in Q_h(\Omega_h)$, $\mathbf{u} \in V_h(\Omega_h; R^3)$ and $\Gamma \in \Gamma_{h,\mathrm{int}}$.

After a bit tedious but straightforward manipulation carried over in full detail in [37, Section 2.4, formula (2.17)], we deduce the formula

$$\int_{\Omega_h} r\mathbf{u} \cdot \nabla_x \phi \,\mathrm{d}x = \sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_{\Gamma} \mathrm{Up}[r, \mathbf{u}][[F]] \,\mathrm{dS}_x \tag{5.27}$$

$$+ \sum_{E \in E_h} \sum_{\Gamma_E \subset \partial E} \int_{\Gamma_E} (F - \phi) \; [[r]] \; [\tilde{\mathbf{u}} \cdot \mathbf{n}]^{-} \,\mathrm{dS}_x + \sum_{E \in E_h} \sum_{\Gamma_E \subset \partial E} \int_{\Gamma_E} \phi r (\mathbf{u} - \tilde{\mathbf{u}}) \cdot \mathbf{n} \,\mathrm{dS}_x$$

$$+ \int_{\Omega_h} (F - \phi) r \mathrm{div}_h \mathbf{u} \,\mathrm{d}x$$

for any $r, F \in Q_h(\Omega_h)$, $\mathbf{u} \in V_{h,0}(\Omega_h; R^3)$, $\phi \in C^1(\overline{\Omega}_h)$.

Finally, we recall Jensen's inequality in the form

$$\sum_{\Gamma_E \subset \partial E} \int_{\Gamma_E} |\tilde{v}|^q \,\mathrm{dS}_x \lesssim \sum_{\Gamma_E \subset \partial E} \int_{\Gamma_E} |v|^q \,\mathrm{dS}_x, \; 1 \leq q < \infty \tag{5.28}$$

for any $v \in C(\overline{E})$, $E \in E_h$, together with

$$\sum_{\Gamma \in \Gamma_h} \int_{\Gamma} |v - \tilde{v}|^2 \,\mathrm{dS}_x \lesssim h \int_{\Omega_h} |\nabla_h v|^2 \,\mathrm{d}x \text{ for any } v \in V_{h,0}(\Omega_h; R^3) \tag{5.29}$$

that follows directly from Poincaré's inequality (5.19).

$L^p - L^q$ estimates and traces

Since the mesh is shape regular, we can derive the following estimates by a scaling argument. First, we have

$$\|v\|^q_{L^q(\partial E)} \lesssim \frac{1}{h} \left(\|v\|^q_{L^q(E)} + h^q \|\nabla_x v\|^q_{L^q(E;R^3)} \right), \tag{5.30}$$

$1 \leq q < \infty$, for any $v \in C^1(E)$; whence

$$\|w\|^q_{L^q(\partial E)} \lesssim \frac{1}{h} \|w\|^q_{L^q(E)} \text{ for any } 1 \leq q < \infty, \; w \in P_m, \tag{5.31}$$

where P_m denotes the space of polynomials of order m.

Similarly, we obtain

$$\|w\|_{L^p(E)} \lesssim h^{3\left(\frac{1}{p} - \frac{1}{q}\right)} \|w\|_{L^q(E)} \; 1 \leq q < p \leq \infty, \; w \in P_m, \tag{5.32}$$

and therefore

$$\|w\|_{L^p(\Omega_h)} \leq c h^{3\left(\frac{1}{p}-\frac{1}{q}\right)} \|w\|_{L^q(\Omega_h)}, \tag{5.33}$$

$1 \leq q < p \leq \infty$, for any $w|_E \in P_m(E),\ E \in E_h$. There is an analogue of (5.32) and (5.33) for piecewise smooth functions of the time variable $t \in (0,T)$ for the discretization of order Δt. Specifically, we derive

$$\|w\|_{L^p(0,T)} \stackrel{<}{\sim} (\Delta t)^{\left(\frac{1}{p}-\frac{1}{q}\right)} \|w\|_{L^q(0,T)}\ 1 \leq q < p \leq \infty \tag{5.34}$$

for any w that is constant on any time segment $[j\Delta t, (j+1)\Delta t]$ contained in $[0,T]$.

Discrete Sobolev spaces

For $v \in Q_h(\Omega_h)$, let

$$\|v\|^2_{H^1_{Q_h}(\Omega_h)} = \sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma \frac{[[v]]^2}{h}\ \mathrm{dS}_x$$

be a discrete analogue of the Sobolev gradient semi-norm. Similarly, we introduce

$$\|v\|^2_{H^1_{V_h}(\Omega_h)} = \int_\Omega \left(|\nabla_h v|^2\right)\ \mathrm{d}x \text{ for } v \in V_h(\Omega_h).$$

Recall that

$$\sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma [[v]]^2\, \mathrm{dS}_x \stackrel{<}{\sim} h \|v\|^2_{H^1_{V_h}(\Omega_h)} \text{ for any } v \in V_h(\Omega_h), \tag{5.35}$$

see Gallouet et al. [45, Lemma 2.2].

We report a discrete analogue of the standard Sobolev embedding relations:

$$\|v\|_{L^6(\Omega_h)} \stackrel{<}{\sim} \left(\|v\|_{H^1_{Q_h}(\Omega_h)} + \|v\|_{L^2(\Omega_h)} \right),\ v \in Q_h(\Omega_h), \tag{5.36}$$

see Chenais-Hillairet, Droniou [12, Lemma 6.1], and

$$\|v\|_{L^6(\Omega_h)} \stackrel{<}{\sim} \|v\|_{H^1_{V_h}(\Omega_h)},\ v \in V_{h,0}(\Omega_h), \tag{5.37}$$

see Gallouet et al. [45, Lemma 3.2].

Finally, let $[v]_\delta = v * \omega_\delta$ denote the spatial regularization by a convolution with a family of smooth kernels, specifically $\omega \in C^\infty_c(\{x \in R^3 \mid |x| < 1\})$ satisfying

$$\omega_\delta(y) = \frac{1}{\delta^3} \omega\left(\frac{y}{\delta}\right),\ \omega \geq 0,\ \omega(y) = \omega(|y|),\ \int_{R^3} \omega(y)\ \mathrm{d}y = 1.$$

We have

$$\int_{\{x \in \Omega_h \ \mid\ \mathrm{dist}[x, \partial\Omega_h] > \delta\}} |\nabla_x [v]_\delta|^2\ \ \mathrm{d}x \stackrel{<}{\sim} \frac{h}{\delta} \|v\|^2_{H^1_{Q_h}(\Omega_h)} \text{ for any } v \in Q_h(\Omega_h), \tag{5.38}$$

and

$$\int_{\Omega_h} |\nabla_x [v]_\delta|^2\ \ \mathrm{d}x \stackrel{<}{\sim} \frac{h}{\delta} \|v\|^2_{H^1_{V_h}(\Omega_h)} \text{ for any } v \in V_{h,0}(\Omega_h) \tag{5.39}$$

provided $0 < \delta \leq h$, see Christiansen et al. [16, Proposition 5.67]. Note that the functions from $V_{h,0}$ can be extended to be zero outside Ω_h so that the regularization is well defined.

5.3 Numerical scheme, main result

The numerical scheme is formally the same as in [37], the only difference is that the numerical domains Ω_h depend on the discretization step h. For this reason, it is convenient the initial data ϱ_0, ϑ_0, $\mathbf{u}_0$ to be defined on the whole space R^3, $\mathbf{u}_0$ vanishing outside Ω.

We set

$$\varrho_h^0 = \Pi_h^Q[\varrho_0] \in Q_h(\Omega_h),\ \vartheta_h^0 = \Pi_h^Q[\vartheta_0] \in Q_h(\Omega_h),\ \mathbf{u}_h^0 = \Pi_h^V[\mathbf{u}_0] \in V_{h,0}(\Omega_h; R^3). \tag{5.40}$$

We fix the time step $\Delta t \approx h$ and introduce the discrete time derivative

$$D_t b_h^k = \frac{b_h^k - b_h^{k-1}}{\Delta t}.$$

The numerical solutions $[\varrho_h^k, \vartheta_h^k, \mathbf{u}_h^k]_{h>0}$, $k = 1, 2, \dots,$

$$\varrho_h^k,\ \vartheta_h^k \in Q_h(\Omega_h),\ \mathbf{u}_h^k \in V_{h,0}(\Omega_h; R^3)$$

are defined successively by means of the numerical method:

$$\int_{\Omega_h} D_t\varrho_h^k\phi\ \mathrm{d}x - \sum_{\Gamma\in\Gamma_{h,\mathrm{int}}} \int_\Gamma \mathrm{Up}[\varrho_h^k, \mathbf{u}_h^k]\ [[\phi]]\ \mathrm{dS}_x + h^\alpha \sum_{\Gamma\in\Gamma_{h,\mathrm{int}}} \int_\Gamma [[\varrho_h^k]][[\phi]]\ \mathrm{dS}_x = 0 \tag{5.41}$$

for all $\phi \in Q_h(\Omega_h)$, with a parameter $0 < \alpha < 1$;

$$\int_{\Omega_h} D_t(\varrho_h^k\widehat{\mathbf{u}}_h^k)\cdot\phi\ \mathrm{d}x - \sum_{\Gamma\in\Gamma_{h,\mathrm{int}}} \int_\Gamma \mathrm{Up}[\varrho_h^k\widehat{\mathbf{u}}_h^k, \mathbf{u}_h^k]\cdot[[\widehat{\phi}]]\ \mathrm{dS}_x \tag{5.42}$$

$$+\int_{\Omega_h} \left[\mu\nabla_h\mathbf{u}_h^k : \nabla_h\phi + \lambda\mathrm{div}_h\mathbf{u}_h^k\mathrm{div}_h\phi\right]\ \mathrm{d}x - \int_{\Omega_h} p(\varrho_h^k, \vartheta_h^k)\mathrm{div}_h\phi\ \mathrm{d}x$$

$$+h^\alpha \sum_{\Gamma\in\Gamma_{h,\mathrm{int}}} \int_\Gamma [[\varrho_h^k]]\left\{\widehat{u_h^k}\right\}\cdot[[\widehat{\phi}]]\ \mathrm{dS}_x = 0$$

for any $\phi \in V_{h,0}(\Omega_h; R^3)$;

$$c_v\int_{\Omega_h} D_t(\varrho_h^k\vartheta_h^k)\phi\ \mathrm{d}x - c_v\sum_{\Gamma\in\Gamma_{h,\mathrm{int}}}\int_\Gamma \mathrm{Up}[\varrho_h^k\vartheta_h^k, \mathbf{u}_h^k]\ [[\phi]]\ \mathrm{dS}_x + \sum_{\Gamma\in\Gamma_{h,\mathrm{int}}}\int_\Gamma \frac{1}{d_\Gamma}[[K(\vartheta_h^k)]]\ [[\phi]]\ \mathrm{dS}_x \tag{5.43}$$

$$= \int_{\Omega_h} \left[\mu|\nabla_h\mathbf{u}_h^k|^2 + \lambda|\mathrm{div}_h\mathbf{u}_h^k|^2\right]\phi\ \mathrm{d}x - \int_{\Omega_h} \varrho_h^k\vartheta_h^k\mathrm{div}_h\mathbf{u}_h^k\phi\ \mathrm{d}x$$

for any $\phi \in Q_h(\Omega_h)$.

Remark 5.3.1. *The terms proportional to h^α are needed for technical reasons explained in detail in [37, Section 7.3]. They represent numerical counterparts of the artificial viscosity regularization used in [32, Chapter 7] and were introduced by Eymard et al. [30] to prove convergence of the momentum scheme (5.42).*

Before stating our main result, it is convenient to extend the numerical solution to be defined for *any* $t \in R$. To this end, we set

$$\varrho_h(t,\cdot) = \varrho_h^0,\ \vartheta_h(t,\cdot) = \vartheta_h^0,\ \mathbf{u}_h(t,\cdot) = \mathbf{u}_h^0 \text{ for } t \le 0,$$

$$\varrho_h(t,\cdot) = \varrho_h^k,\ \vartheta_h(t,\cdot) = \vartheta_h^k,\ \mathbf{u}_h(t,\cdot) = \mathbf{u}_h^k \text{ for } t \in [k\Delta t, (k+1)\Delta t),\ k = 1, 2, \dots,$$

and, accordingly, the discrete time derivative of a quantity v_h is

$$D_t v_h(t,\cdot) = \frac{v_h(t) - v_h(t-\Delta t)}{\Delta t},\ t > 0.$$

The main result of the present paper reads as follows:

Theorem 5.3.1. *Let $\Omega \subset R^3$ be a bounded domain of class C^1 approximated by a family of polyhedral domains $\{\Omega_h\}_{h>0}$ in the sense specified in (5.8), where each Ω_h admits a tetrahedral mesh satisfying the hypotheses introduced in Section 5.2.2. Suppose that $\mu > 0$, $\lambda > 0$, and that the pressure $p = p(\varrho, \vartheta)$ and the heat conductivity coefficient $\kappa = \kappa(\vartheta)$ comply with (5.4), (5.5). Let $[\varrho_h, \vartheta_h, \mathbf{u}_h]_{h>0}$ be a family of numerical solutions resulting from the scheme (5.40 - 5.43), with*

$$\Delta t \approx h$$

such that $\varrho_h > 0$, $\vartheta_h > 0$ for all $h > 0$.

Then, at least for a suitable subsequence,

$$\varrho_h \to \varrho \text{ weakly-(*) in } L^\infty(0,T;L^\gamma(\Omega)) \text{ and strongly in } L^1((0,T)\times\Omega),$$

$$\vartheta_h \to \vartheta \text{ weakly in } L^2(0,T;L^6(\Omega)),$$

$$\mathbf{u}_h \to \mathbf{u} \text{ weakly in } L^2(0,T;L^6(\Omega;R^3)), \nabla_h \mathbf{u}_h \to \nabla_x \mathbf{u} \text{ weakly in } L^2((0,T)\times\Omega;R^{3\times 3}),$$

where $[\varrho, \vartheta, \mathbf{u}]$ is a weak solution of the Navier-Stokes-Fourier system (5.1 – 5.7) in $(0,T)\times\Omega$ in the sense of Definition 5.2.1.

The existence of the numerical solutions $[\varrho_h, \vartheta_h, \mathbf{u}_h]$ was shown in [37, Section 8.1]. The rest of the paper is basically devoted to the proof of Theorem 5.3.1. As some steps are essentially the same as in [37] we omit technicalities and focus only on the necessary modifications to accommodate the variable numerical domains.

5.4 Renormalization

The proof of convergence of the numerical method (5.40 – 5.43) mimics the principal steps of the existence theory developed in [32] based, among other things, on suitable *renormalization* of both the equation of continuity (5.1) and the heat equation (5.3). At the level of numerical solutions, we can deduce the following (see [37, Sections 4.1, 4.2]):

1. **Renormalized continuity scheme.**

$$\int_{\Omega_h} D_t b(\varrho_h^k)\phi \ \mathrm{d}x - \sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma \mathrm{Up}[b(\varrho_h^k), \mathbf{u}_h^k]\ [[\phi]]\ \mathrm{dS}_x \qquad (5.44)$$

$$+\int_{\Omega_h}\phi\ \left(b'(\varrho_h^k)\varrho_h^k - b(\varrho_h^k)\right)\mathrm{div}_h\mathbf{u}_h^k\ \mathrm{d}x = -\int_{\Omega_h}\frac{\Delta t}{2}b''(\xi_{\varrho,h}^k)\left(\frac{\varrho_h^k-\varrho_h^{k-1}}{\Delta t}\right)^2\phi\ \mathrm{d}x$$

$$-h^\alpha\sum_{\Gamma\in\Gamma_{h,\mathrm{int}}}\int_\Gamma\phi\ b''(\overline{\eta}_{\varrho,h}^k)[[\varrho_h^k]]^2\,\mathrm{dS}_x-\frac{1}{2}\sum_{\Gamma\in\Gamma_{h,\mathrm{int}}}\int_\Gamma\phi\ b''(\eta_{\varrho,h}^k)[[\varrho_h^k]]^2|\tilde{\mathbf{u}}_h^k\cdot\mathbf{n}|\,\mathrm{dS}_x$$

for any $\phi\in Q_h(\Omega_h)$, $b\in C^2(0,\infty)$, where $\xi_{\varrho,h}^k\in\mathrm{co}\{\varrho_h^{k-1},\varrho_h^k\}$ on each element $E\in E_h$ and $\eta_{\varrho,h}^k,\overline{\eta}_{\varrho,h}^k\in\mathrm{co}\{\varrho_h^k,(\varrho_h^k)^{\mathrm{out}}\}$ on each face $\Gamma\in\Gamma_{h,\mathrm{int}}$, where $\mathrm{co}\{A,B\}=[\inf\{A,B\},\sup\{A,B\}]$.

2. **Renormalized thermal energy scheme.**

$$c_v\int_{\Omega_h}D_t\left(\varrho_h^k\chi(\vartheta_h^k)\right)\phi\ \mathrm{d}x - c_v\sum_{\Gamma\in\Gamma_{h,\mathrm{int}}}\int_\Gamma\mathrm{Up}(\varrho_h^k\chi(\vartheta_h^k),\mathbf{u}_h^k)\ [[\phi]]\ \mathrm{dS}_x \quad (5.45)$$

$$+\sum_{\Gamma\in\Gamma_{h,\mathrm{int}}}\int_\Gamma\frac{1}{d_\Gamma}[[K(\vartheta_h^k)]]\ [[\chi'(\vartheta_h^k)\phi]]\ \mathrm{dS}_x$$

$$=\int_{\Omega_h}\left(\mu|\nabla_h\mathbf{u}_h^k|^2+\lambda|\mathrm{div}_h\mathbf{u}_h^k|^2\right)\chi'(\vartheta_h^k)\phi\ \mathrm{d}x-\int_{\Omega_h}\chi'(\vartheta_h^k)\varrho_h^k\vartheta_h^k\mathrm{div}_h\mathbf{u}_h^k\phi\ \mathrm{d}x$$

$$-c_v\frac{\Delta t}{2}\int_{\Omega_h}\chi''(\xi_{\vartheta,h}^k)\varrho_h^{k-1}\left(\frac{\vartheta_h^k-\vartheta_h^{k-1}}{\Delta t}\right)^2\phi\ \mathrm{d}x$$

$$+\frac{c_v}{2}\sum_{E\in E_h}\sum_{\Gamma_E\subset\partial E}\int_{\Gamma_E}\phi\chi''(\eta_{\vartheta,h}^k)[[\vartheta_h^k]]^2(\varrho_h^k)^{\mathrm{out}}[\tilde{\mathbf{u}}_h^k\cdot\mathbf{n}]^-\,\mathrm{dS}_x$$

$$-h^\alpha c_v\sum_{\Gamma\in\Gamma_{h,\mathrm{int}}}\int_\Gamma[[\varrho_h^k]]\ [[\left(\chi(\vartheta_h^k)-\chi'(\vartheta_h^k)\vartheta_k^k\right)\phi]]\,\mathrm{dS}_x$$

for any $\phi\in Q_h(\Omega_h)$, $\chi\in C^2(0,\infty)$, with $\xi_{\vartheta,h}^k\in\mathrm{co}\{\vartheta_h^{k-1},\vartheta_h^k\}$ and $\eta_{\vartheta,h}^k\in\mathrm{co}\{\vartheta_h^k,(\vartheta_h^k)^{\mathrm{out}}\}$.

Finally, exactly as in [37, Section 4.3] we may use (5.44), (5.45) and the momentum scheme (5.42) to deduce:

- **Total energy balance.**

$$D_t\int_{\Omega_h}\left[\frac{1}{2}\varrho_h^k|\widehat{\mathbf{u}}_h^k|^2+c_v\varrho_h^k\vartheta_h^k+\frac{a}{\gamma-1}\left(\varrho_h^k\right)^\gamma+b\varrho_h^k\log(\varrho_h^k)\right]\ \mathrm{d}x \quad (5.46)$$

$$+\frac{\Delta t}{2}\int_{\Omega_h}\left(A\left|\frac{\varrho_h^k-\varrho_h^{k-1}}{\Delta t}\right|^2+\varrho_h^{k-1}\left|\frac{\widehat{\mathbf{u}}_h^k-\widehat{\mathbf{u}}_h^{k-1}}{\Delta t}\right|^2\right)\ \mathrm{d}x$$

$$-\sum_{E\in E_h}\sum_{\Gamma_E\subset\partial E}\int_{\Gamma_E}(\varrho_h^k)^{\mathrm{out}}[\tilde{\mathbf{u}}_h^k\cdot\mathbf{n}]^-\frac{\left|\widehat{\mathbf{u}}_h^k-\left(\widehat{\mathbf{u}}_h^k\right)^{\mathrm{out}}\right|^2}{2}\,\mathrm{dS}_x$$

$$+\frac{A}{2}\sum_{\Gamma\in\Gamma_{h,\mathrm{int}}}\int_\Gamma\left(h^\alpha+|\tilde{\mathbf{u}}_h^k\cdot\mathbf{n}|\right)[[\varrho_h^k]]^2\ \mathrm{dS}_x\le 0$$

$$\text{with } A=\min_{\varrho>0}\left\{a\gamma\varrho^{\gamma-2}+\frac{b}{\varrho}\right\}>0.$$

5.5 Stability

Similarly to [37, Section 5] we derive uniform bounds on the family of numerical solutions independent of the step h.

5.5.1 Mass conservation and energy bounds

Taking $\phi \equiv 1$ in the continuity scheme (5.41) we obtain

$$\int_{\Omega_h} \varrho_h(t,\cdot)\ \mathrm{d}x = \int_{\Omega_h} \varrho_h^0\ \mathrm{d}x \approx \int_{\Omega} \varrho_0\ \mathrm{d}x \text{ for any } h > 0, \tag{5.47}$$

meaning the total mass is conserved by the scheme.

The total energy balance (5.46) gives rise to

$$\int_{\Omega_h} \left[\frac{1}{2}\varrho_h|\widehat{\mathbf{u}}_h|^2 + c_v\varrho_h\vartheta_h + \frac{a}{\gamma-1}\left(\varrho_h\right)^\gamma + b\varrho_h\log(\varrho_h)\right](\tau,\cdot)\ \mathrm{d}x \tag{5.48}$$

$$\leq \int_{\Omega_h} \left[\frac{1}{2}\varrho_h|\widehat{\mathbf{u}}_h|^2 + c_v\varrho_h\vartheta_h + \frac{a}{\gamma-1}\left(\varrho_h\right)^\gamma + b\varrho_h\log(\varrho_h)\right](\tau,\cdot)\ \mathrm{d}x$$

$$\leq \int_{\Omega_h} \left[\frac{1}{2}\varrho_h^0|\widehat{\mathbf{u}}_h^0|^2 + c_v\varrho_h^0\vartheta_h^0 + \frac{a}{\gamma-1}\left(\varrho_h^0\right)^\gamma + b\varrho_h^0\log(\varrho_h^0)\right]\ \mathrm{d}x \equiv E_{0,h},\ E_{0,h} \stackrel{<}{\sim} 1.$$

In particular, we deduce the uniform bounds, independent of $h \to 0$:

$$\sup_{\tau\in(0,T)}\|\sqrt{\varrho_h}\widehat{\mathbf{u}}_h(\tau,\cdot)\|_{L^2(\Omega_h)} \stackrel{<}{\sim} 1, \tag{5.49}$$

$$\sup_{\tau\in(0,T)}\|\varrho_h\vartheta_h(\tau,\cdot)\|_{L^1(\Omega_h)} \stackrel{<}{\sim} 1, \tag{5.50}$$

$$\sup_{\tau\in(0,T)}\|\varrho_h[\log\vartheta_h]^+(\tau,\cdot)\|_{L^1(\Omega_h)} \stackrel{<}{\sim} 1, \tag{5.51}$$

$$\sup_{\tau\in(0,T)}\|\varrho_h(\tau,\cdot)\|_{L^\gamma(\Omega_h)} \stackrel{<}{\sim} 1. \tag{5.52}$$

We also record the bounds on the numerical dissipation:

$$\sum_{k\geq 0}\int_{\Omega_h}\left[\left|\varrho_h^k - \varrho_h^{k-1}\right|^2 + \varrho_h^{k-1}\left|\widehat{\mathbf{u}}_h^k - \widehat{\mathbf{u}}_h^{k-1}\right|^2\right]\ \mathrm{d}x \stackrel{<}{\sim} 1, \tag{5.53}$$

$$-\sum_{E\in E_h}\sum_{\Gamma_E\subset\partial E}\int_0^T\int_{\Gamma_E}(\varrho_h)^{\mathrm{out}}[\tilde{\mathbf{u}}_h\cdot\mathbf{n}]^-\left|\widehat{\mathbf{u}}_h - (\widehat{\mathbf{u}}_h)^{\mathrm{out}}\right|^2\mathrm{dS}_x\ \mathrm{d}t \stackrel{<}{\sim} 1 \tag{5.54}$$

and

$$\sum_{\Gamma\in\Gamma_{h,\mathrm{int}}}\int_0^T\int_\Gamma\left(|\tilde{\mathbf{u}}_h\cdot\mathbf{n}| + h^\alpha\right)[[\varrho_h]]^2\ \mathrm{dS}_x\ \mathrm{d}t \stackrel{<}{\sim} 1. \tag{5.55}$$

5.5.2 Entropy bounds

The bounds resulting from the dissipation mechanism encoded in (5.42), (5.43) are obtained by taking $\chi = \log$, $\phi = 1$ in the renormalized thermal energy scheme (5.45). Using the fact that

$$\int_{\Omega_h} \varrho_h^k \mathrm{div}_h \mathbf{u}_h^k \ \mathrm{d}x \le - \int_{\Omega_h} D_t \Big(\varrho_h^k \log(\varrho_h^k) \Big) \ \mathrm{d}x \tag{5.56}$$

(cf. (5.44)), we arrive at

$$c_v \int_{\Omega_h} D_t \left(\varrho_h^k \log(\vartheta_h^k) \right) \ \mathrm{d}x - \int_{\Omega_h} D_t \left(\varrho_h^k \log(\varrho_h^k) \right) \ \mathrm{d}x \ge \tag{5.57}$$

$$- \sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma \frac{1}{d_\Gamma} [[K(\vartheta_h^k)]] \ [[(\vartheta_h^k)^{-1}]] \ \mathrm{dS}_x + \int_{\Omega_h} \left(\mu |\nabla_h \mathbf{u}_h^k|^2 + \lambda |\mathrm{div}_h \mathbf{u}_h^k|^2 \right) \frac{1}{\vartheta_h^k} \ \mathrm{d}x$$

$$+ \frac{\Delta t}{2} c_v \int_{\Omega_h} (\xi_{\vartheta,h}^k)^{-2} \varrho_h^{k-1} \left(\frac{\vartheta_h^k - \vartheta_h^{k-1}}{\Delta t} \right)^2 \ \mathrm{d}x$$

$$- \frac{1}{2} c_v \sum_{E \in E_h} \sum_{\Gamma_E \subset \partial E} \int_{\Gamma_E} (\eta_{\vartheta,h}^k)^{-2} \left(\vartheta_h^k - (\vartheta_h^k)^{\mathrm{out}} \right)^2 \left(\varrho_h^k \right)^{\mathrm{out}} [\tilde{\mathbf{u}}_h^k \cdot \mathbf{n}]^- \ \mathrm{dS}_x$$

$$- h^\alpha c_v \sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma [[\varrho_h^k]] \ [[\log(\vartheta_h^k)]] \ \mathrm{dS}_x,$$

where the parameters appearing in the numerical dissipation are the same as in (5.44), (5.45).

Now, exactly as in [37, Section 5], inequality (5.57), together with the bounds already established, gives rise to the following estimates:

$$\sup_{\tau \in (0,T)} \| \varrho_h \log(\vartheta_h)(\tau, \cdot) \|_{L^1(\Omega_h)} \lesssim 1, \tag{5.58}$$

$$\int_0^T \int_{\Omega_h} \frac{1}{\vartheta_h} |\nabla_h \mathbf{u}_h|^2 \ \mathrm{d}x \ \ \mathrm{d}t \lesssim 1, \tag{5.59}$$

$$\sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_0^T \int_\Gamma \frac{[[\vartheta_h^\beta]]^2}{d_\Gamma} \mathrm{dS}_x \ \ \mathrm{d}t \lesssim 1, \qquad \sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_0^T \int_\Gamma \frac{[[\log(\vartheta_h)]]^2}{d_\Gamma} \mathrm{dS}_x \ \ \mathrm{d}t \lesssim 1, \tag{5.60}$$

where $0 \le \beta \le 1$ and

$$\| \vartheta_h \|_{L^2(0,T;L^6(\Omega_h))} + \| \log(\vartheta_h) \|_{L^2(0,T;L^6(\Omega_h))} \lesssim 1. \tag{5.61}$$

We have also bounds on the numerical dissipation:

$$\sum_{k \ge 0} \int_{\Omega_h} (\xi_{\vartheta,h}^k)^{-2} \varrho_h^{k-1} \left(\vartheta_h^k - \vartheta_h^{k-1} \right)^2 \ \mathrm{d}x \lesssim 1, \ \xi_{\vartheta,h}^k \in \mathrm{co}\{ \vartheta_h^{k-1}, \vartheta_h^k \}, \tag{5.62}$$

$$- \sum_{E \in E_h} \sum_{\Gamma_E \subset \partial E} \int_0^T \int_{\Gamma_E} (\eta_{\vartheta,h})^{-2} [[\vartheta_h]]^2 \left(\varrho_h \right)^{\mathrm{out}} [\tilde{\mathbf{u}}_h \cdot \mathbf{n}]^- \ \mathrm{dS}_x \ \mathrm{d}t \lesssim 1, \ \eta_{\vartheta,h} \in \mathrm{co}\{ \vartheta_h, \vartheta_h^{\mathrm{out}} \}. \tag{5.63}$$

5.5.3 Temperature estimates

Revisiting the thermal energy balance (5.45) for $\chi(\vartheta_h^k) = (\vartheta_h^k)^\beta$, $0 < \beta < 1$, and with the test function $\phi = 1$, we obtain

$$-\beta \sum_{\Gamma \in \Gamma_{h,\rm int}} \int_\Gamma \frac{1}{d_\Gamma} [[K(\vartheta_h)]] \ [[(\vartheta_h)^{\beta-1}]] \ {\rm dS}_x + \beta\mu \int_{\Omega_h} \vartheta_h^{\beta-1} |\nabla_h \mathbf{u}_h|^2 \ {\rm d}x \ {\rm d}t \tag{5.64}$$

$$+c_v \beta(1-\beta) \frac{\Delta t}{2} \sum_{k=1} \int_{\Omega_h} (\xi_{\vartheta,h}^k)^{\beta-2} \varrho_h^{k-1} \left(\frac{\vartheta_h^k - \vartheta_h^{k-1}}{\Delta t} \right)^2 \ {\rm d}x$$

$$+\frac{c_v}{2} \beta(1-\beta) \sum_{E \in E_h} \sum_{\Gamma_E \subset \partial E} \int_\Gamma (\eta_{\vartheta,h}^k)^{\beta-2} [[\vartheta_h^k]]^2 \left(\varrho_h^k \right)^{\rm out} [\tilde{\mathbf{u}}_h^k \cdot \mathbf{n}]^- \ {\rm dS}_x$$

$$\lesssim c_v \int_{\Omega_h} D_t(\varrho_h^k (\vartheta_h^k)^\beta) \ {\rm d}x + \beta \int_{\Omega_h} \varrho_h^k (\vartheta_h^k)^\beta {\rm div}_h \mathbf{u}_h^k \ {\rm d}x$$
$$+ h^\alpha c_v (1-\beta) \sum_{\Gamma \in \Gamma_{h,\rm int}} \int_\Gamma [[\varrho_h^k]] \ [[(\vartheta_h^k)^\beta]] \, {\rm dS}_x.$$

Arguing as in [37, Section 5.3] we deduce from (5.64) the following estimates:

$$- \sum_{\Gamma \in \Gamma_{h,\rm int}} \int_\Gamma \int_0^T \frac{1}{d_\Gamma} [[K(\vartheta_h)]] \ [[(\vartheta_h)^{\beta-1}]] \ {\rm dS}_x \lesssim 1 \text{ for all } 0 < \beta < 1, \tag{5.65}$$

$$\sum_{\Gamma \in \Gamma_h} \int_0^T \int_\Gamma \frac{[[\vartheta_h^{1+\frac{\beta}{2}}]]^2}{h} \ {\rm dS}_x \lesssim 1 \text{ for all } 0 \le \beta < 1; \tag{5.66}$$

whence, in accordance with (5.36),

$$\|\vartheta_h\|_{L^p(0,T;L^q(\Omega_h))} \lesssim 1 \text{ for any } 1 \le p < 3, \ 1 \le q < 9. \tag{5.67}$$

Finally, returning to the thermal energy scheme (5.43) with $\phi = 1$, we may use the previous estimates to conclude

$$\int_0^T \int_{\Omega_h} |\nabla_h \mathbf{u}_h|^2 \ {\rm d}x \ {\rm d}t \lesssim 1, \tag{5.68}$$

and, in accordance with (5.37),

$$\|\mathbf{u}_h\|^2_{L^2(0,T;L^6(\Omega_h;R^3))} \lesssim 1. \tag{5.69}$$

5.6 Consistency and convergence

Our goal is to check that **(i)** the numerical method is *consistent* with the original weak formulation, **(ii)** the numerical solutions converge, modulo a suitable subsequence, to a weak solution of the problem as stated in Theorem 5.3.1.

5.6.1 Consistency

To begin, we claim that the proof of consistency for the continuity scheme (5.41) and the momentum scheme (5.42) is exactly the same as in [37, Sections 6.1, 6.2], where the upwind terms may be handled by means of formula (5.27).

Continuity and momentum scheme

Taking $\Pi_h^Q[\phi]$, $\phi \in C_c^\infty(R^3)$ as a test function in the continuity scheme (5.41) gives rise to

$$\int_{R^3} \left[D_t \varrho_h - \varrho_h \mathbf{u}_h \cdot \nabla_x \phi \right] \ \mathrm{d}x = \int_{R^3} R_h^1(t,\cdot) \cdot \nabla_x \phi \ \mathrm{d}x \tag{5.70}$$

for any $\phi \in C_c^\infty(R^3)$ provided ϱ_h, $\mathbf{u}_h$ were extended to be zero outside Ω_h. The remainder satisfies (see [36, Section 6.1])

$$\left\| R_h^1 \right\|_{L^2(0,T;L^{\frac{6\gamma}{5\gamma-6}(R^3;R^3)}(R^3;R^3))} \lesssim h^\beta \text{ for some } \beta > 0. \tag{5.71}$$

The choice $\Pi_h^V[\phi]$, $\phi \in C_c^\infty(\Omega; R^3)$ as a test function in the momentum balance (5.42) gives rise to

$$\int_\Omega D_t(\varrho_h \widehat{\mathbf{u}}_h) \cdot \phi \ \mathrm{d}x - \int_\Omega (\varrho_h \widehat{\mathbf{u}}_h \otimes \mathbf{u}_h) : \nabla_x \phi \ \mathrm{d}x \tag{5.72}$$

$$+ \int_\Omega \left[\mu \nabla_h \mathbf{u}_h : \nabla_x \phi + \lambda \mathrm{div}_h \mathbf{u}_h \mathrm{div}_x \phi \right] \ \mathrm{d}x - \int_\Omega p(\varrho_h, \vartheta_h) \mathrm{div}_x \phi \ \mathrm{d}x = \int_\Omega \mathbb{R}_h^2 : \nabla_x \phi \ \mathrm{d}x$$

for any $\phi \in C_c^\infty(\Omega; R^3)$, where the remainder satisfies (see [36, Section 6.2])

$$\left\| \mathbb{R}_h^2 \right\|_{L^1(0,T;L^{\frac{\gamma}{\gamma-1}}(\Omega;R^{3\times 3}))} \lesssim h^\beta \text{ for some } \beta > 0. \tag{5.73}$$

Since, $\Omega \subset \Omega_h$ for any h and ϕ has compact support in Ω, all terms in (5.72) are well defined.

Consistency for the thermal energy balance

Instead of working directly with the thermal energy scheme (5.43), we consider its renormalized variant (5.45). Motivated by [37, Section 6.3], we take the nonlinearities χ belonging to the class

$$\chi \in W^{2,\infty}[0,\infty), \ \chi'(\vartheta) \geq 0, \ \chi''(\vartheta) \leq 0, \quad \chi(\vartheta) = \text{const for all } \vartheta > \vartheta_\chi. \tag{5.74}$$

We start by rewriting

$$\sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma \frac{1}{d_\Gamma} [[K(\vartheta_h^k)]] \ [[\chi'(\vartheta_h^k)\phi]] \ \mathrm{dS}_x \tag{5.75}$$

$$= \sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma \frac{1}{d_\Gamma} \{\phi\} \, [[K(\vartheta_h^k)]] \ [[\chi'(\vartheta_h^k)]] \mathrm{dS}_x + \sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma \frac{1}{d_\Gamma} \left\{ \chi'\left(\vartheta_h^k\right) \right\} [[K(\vartheta_h^k)]] \ [[\phi]] \mathrm{dS}_x$$

for any $\phi \in Q_h(\Omega_h)$.

Next, take $\phi \in C^2(R^3)$ such that $\nabla_x \phi \cdot \mathbf{n} = 0$ on $\partial\Omega$, and use $\Pi_h^B[\phi]$ as a test function in the renormalized thermal energy scheme (5.45). In view of (5.75), we obtain

$$c_v \int_{\Omega_h} D_t \left(\varrho_h^k \chi(\vartheta_h^k)\right) \Pi_h^B[\phi] \, \mathrm{d}x - c_v \sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma \mathrm{Up}(\varrho_h^k \chi(\vartheta_h^k), \mathbf{u}_h^k) \, [[\Pi_h^B[\phi] \,]] \, \mathrm{dS}_x \tag{5.76}$$
$$+ \sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma \frac{1}{d_\Gamma} \left\{\chi'\left(\vartheta_h^k\right)\right\} [[K(\vartheta_h^k)]] \, [[\Pi_h^B[\phi] \,]] \mathrm{dS}_x$$
$$= \int_{\Omega_h} \left(\mu |\nabla_h \mathbf{u}_h^k|^2 + \lambda |\mathrm{div}_h \mathbf{u}_h^k|^2\right) \chi'(\vartheta_h^k) \Pi_h^B[\phi] \, \mathrm{d}x - \int_{\Omega_h} \chi'(\vartheta_h^k) \vartheta_h^k \varrho_h^k \mathrm{div}_h \mathbf{u}_h^k \Pi_h^B[\phi] \, \mathrm{d}x$$
$$- h^\alpha c_v \sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma [[\varrho_h^k]] \, [[\left(\chi(\vartheta_h^k) - \chi'(\vartheta_h^k)\vartheta_k^k\right) \Pi_h^B[\phi] \,]] \, \mathrm{dS}_x + \langle D_h, \phi \rangle \,,$$

where

$$\langle D_h(t), \phi \rangle = - \sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma \frac{1}{d_\Gamma} \left\{\Pi_h^B[\phi]\right\} [[K(\vartheta_h^k)]] \, [[\chi'(\vartheta_h^k)]] \mathrm{dS}_x$$
$$- c_v \frac{\Delta t}{2} \int_{\Omega_h} \chi''(\xi_{\vartheta,h}^k) \varrho_h^{k-1} \left(\frac{\vartheta_h^k - \vartheta_h^{k-1}}{\Delta t}\right)^2 \Pi_h^B[\phi] \, \mathrm{d}x$$
$$+ \frac{c_v}{2} \sum_{E \in E_h} \sum_{\Gamma_E \subset \partial E} \int_{\Gamma_E} \Pi_h^B[\phi] \chi''(\eta_{\vartheta,h}^k) [[\vartheta_h^k]]^2 \left(\varrho_h^k\right)^{\mathrm{out}} [\tilde{\mathbf{u}}_h^k \cdot \mathbf{n}]^- \, \mathrm{dS}_x.$$

As χ satisfies (5.74), it is easy to check that $\langle D_h(t), \phi \rangle \geq 0$ whenever $\phi \geq 0$. Moreover, applying (5.76) with $\phi = 1$ we get

$$0 \leq \langle D_h(t), 1 \rangle \leq h^\alpha c_v \sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma [[\varrho_h^k]] \, [[\left(\chi(\vartheta_h^k) - \chi'(\vartheta_h^k)\vartheta_k^k\right)]] \, \mathrm{dS}_x$$
$$+ \int_{\Omega_h} \chi'(\vartheta_h^k) \vartheta_h^k \varrho_h^k \mathrm{div}_h \mathbf{u}_h^k \, \mathrm{d}x + c_v \int_{\Omega_h} D_t \left(\varrho_h^k \chi(\vartheta_h^k)\right) \, \mathrm{d}x$$

where the three integrals on the right-hand side are controlled by the estimates (5.50), (5.52), (5.55), (5.60), and (5.68). We may therefore conclude that

$$0 \leq \langle D_h(t), \phi \rangle \stackrel{<}{\sim} R_h^3(t) \|\phi\|_{L^\infty(\Omega_h)}, \ \|R_h^3\|_{L^1(0,T)} \stackrel{<}{\sim} 1 \text{ whenever } \phi \geq 0. \tag{5.77}$$

Note that (5.77) as well as other estimates derived in this section depend on the structural properties of the function χ postulated in (5.74).

Now, the discrete time derivative can be written as

$$\int_{\Omega_h} D_t \left(\varrho_h^k \chi(\vartheta_h^k)\right) \Pi_h^B[\phi] \, \mathrm{d}x = \int_{\Omega_h} D_t \left(\varrho_h^k \chi(\vartheta_h^k)\right) \phi \, \mathrm{d}x$$
$$+ \int_{\Omega_h} \frac{\varrho_h^k - \varrho_h^{k-1}}{\Delta t} \chi(\vartheta_h^k) \left(\Pi_h^B[\phi] - \phi\right) \, \mathrm{d}x + \int_{\Omega_h} \varrho_h^{k-1} \frac{\chi(\vartheta_h^k) - \chi(\vartheta_h^{k-1})}{\Delta t} \left(\Pi_h^B[\phi] - \phi\right) \, \mathrm{d}x.$$

As χ is bounded and $\Delta t \approx h$, we may use (5.20) to deduce

$$\left| \int_{\Omega_h} \frac{\varrho_h^k - \varrho_h^{k-1}}{\Delta t} \chi(\vartheta_h^k) \left(\Pi_h^B[\phi] - \phi \right) \mathrm{d}x \right|$$
$$\lesssim \left(\int_{\Omega_h} \frac{\left(\varrho_h^k - \varrho_h^{k-1} \right)^2}{\Delta t} \ \mathrm{d}x \right)^{1/2} \sqrt{h} \|\nabla_x \phi\|_{L^\infty(\Omega_h;R^3)},$$

where the right-hand side is controlled by (5.53).

Similarly,

$$\left| \int_{\Omega_h} \sqrt{\varrho_h^{k-1}} \sqrt{\varrho_h^{k-1}} \frac{\chi(\vartheta_h^k) - \chi(\vartheta_h^{k-1})}{\Delta t} \left(\Pi_h^B[\phi] - \phi \right) \ \mathrm{d}x \right|$$

$$\lesssim \left(\Delta t \int_{\Omega_h} \varrho_h^{k-1} \left(\frac{\chi(\vartheta_h^k) - \chi(\vartheta_h^{k-1})}{\Delta t} \right)^2 \ \mathrm{d}x \right)^{1/2} \sqrt{h} \|\nabla_x \phi\|_{L^\infty(\Omega_h;R^3)} \|\varrho_h^{k-1}\|_{L^\gamma(\Omega_h)}^{1/2},$$

which can be bounded by means of (5.62). Indeed it is enough to check that

$$\chi(A) - \chi(B) \lesssim \frac{A-B}{A} \text{ whenever } A>B \geq 0$$

as long as χ belongs to the class (5.74).

Summing up the previous estimates, we may infer that

$$\left| \int_{\Omega_h} D_t \left(\varrho_h \chi(\vartheta_h) \right) \left(\Pi_h^B[\phi] - \phi \right) \ \mathrm{d}x \right| \lesssim \sqrt{h} \ R_h^4(t) \|\nabla_x \phi\|_{L^\infty(\Omega_h;R^3)}, \ \|R_h^4\|_{L^2(0,T)} \lesssim 1. \tag{5.78}$$

To handle the upwind term, we use formula (5.27) yielding

$$\sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma \mathrm{Up}[\varrho_h^k \chi(\vartheta_h^k), \mathbf{u}_h^k] \ [[\Pi_h^B[\phi] \ ]] \ \mathrm{dS}_x \tag{5.79}$$

$$= \int_{\Omega_h} \varrho_h^k \chi(\vartheta_h^k) \mathbf{u}_h^k \cdot \nabla_x \phi \ \mathrm{d}x - \sum_{E \in E_h} \sum_{\Gamma_E \subset \partial E} \int_{\Gamma_E} \left(\Pi_h^B[\phi] - \phi \right) [[\varrho_h^k \chi(\vartheta_h^k)]] [\tilde{\mathbf{u}}_h^k \cdot \mathbf{n}]^- \, \mathrm{dS}_x$$

$$+ \sum_{E \in E_h} \sum_{\Gamma_E \subset \partial E} \int_{\Gamma_E} \varrho_h^k \chi(\vartheta_h^k) \phi (\tilde{\mathbf{u}} - \mathbf{u}) \cdot \mathbf{n} \, \mathrm{dS}_x + \sum_{E \in E_h} \int_{E_h} \varrho_h^k \chi(\vartheta_h^k) \mathrm{div}_h \mathbf{u}_h^k \left(\phi - \Pi_h^B \phi \right) \ \mathrm{d}x.$$

We write

$$\sum_{E \in E_h} \sum_{\Gamma_E \subset \partial E} \int_{\Gamma_E} \left(\Pi_h^B[\phi] - \phi \right) [[\varrho_h^k \chi(\vartheta_h^k)]] [\tilde{\mathbf{u}}_h^k \cdot \mathbf{n}]^- \, \mathrm{dS}_x =$$

$$\sum_{E \in E_h} \sum_{\Gamma_E \subset \partial E} \int_{\Gamma_E} \left(\Pi_h^B[\phi] - \phi \right) \varrho_h^k [[\chi(\vartheta_h^k)]] [\tilde{\mathbf{u}}_h^k \cdot \mathbf{n}]^- \, \mathrm{dS}_x$$

$$+ \sum_{E \in E_h} \sum_{\Gamma_E \subset \partial E} \int_{\Gamma_E} \left(\Pi_h^B[\phi] - \phi \right) [[\varrho_h^k]] \chi((\vartheta_h^k)^{\mathrm{out}}) [\tilde{\mathbf{u}}_h^k \cdot \mathbf{n}]^- \, \mathrm{dS}_x,$$

where, by means of Hölder's and Jensen's inequalities, the error estimates (5.20), and the trace estimates (5.31),

$$\left| \sum_{E \in E_h} \sum_{\Gamma_E \subset \partial E} \int_{\Gamma_E} \left(\Pi_h^B[\phi] - \phi \right) (\varrho_h^k) [[\chi(\vartheta_h^k)]] [\tilde{\mathbf{u}}_h^k \cdot \mathbf{n}]^- \, \mathrm{dS}_x \right|$$

$$\lesssim h^{3/2} \|\nabla_x \phi\|_{L^\infty(\Omega_h;R^3)} \left(\sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma \frac{[[\chi(\vartheta_h^k)]]^2}{h} \, \mathrm{dS}_x \right)^{1/2} \left(\sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma |\varrho_h^k|^2 |\tilde{\mathbf{u}}_h^k \cdot \mathbf{n}|^2 \, \mathrm{dS}_x \right)^{1/2}$$

$$\lesssim h^{3/2} \|\nabla_x \phi\|_{L^\infty(\Omega_h;R^3)} \left(\sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma \frac{[[\chi(\vartheta_h^k)]]^2}{h} \, \mathrm{dS}_x \right)^{1/2} \left(\sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma |\varrho_h^k|^2 |\mathbf{u}_h^k|^2 \, \mathrm{dS}_x \right)^{1/2}$$

$$\lesssim h \|\nabla_x \phi\|_{L^\infty(\Omega_h;R^3)} \left(\sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma \frac{[[\chi(\vartheta_h^k)]]^2}{h} \, \mathrm{dS}_x \right)^{1/2} \left(\sum_{E \in E_h} \int_E |\varrho_h^k|^2 |\mathbf{u}_h^k|^2 \mathrm{d}x \right)^{1/2}.$$

Now, the relations (5.52), (5.60), and (5.69) may be used to control both integrals on the right-hand side in $L^2(0,T)$.

Furthermore, as χ is bounded, the integral

$$\sum_{E \in E_h} \sum_{\Gamma_E \subset \partial E} \int_{\Gamma_E} \left(\Pi_h^B[\phi] - \phi \right) [[\varrho_h^k]] \chi((\vartheta_h^k)^{\mathrm{out}}) [\tilde{\mathbf{u}}_h^k \cdot \mathbf{n}]^- \, \mathrm{dS}_x$$

can be handled with the help of the energy estimate (5.55), (5.69), and (5.20).

Finally, we observe that the remaining two integrals on the right-hand side of (5.79) can be estimated by means of (5.19) and the available energy bounds (5.52), (5.68). Thus we conclude that

$$\left| \sum_{\Gamma \in \Gamma_h} \int_\Gamma \mathrm{Up}[\varrho_h^k \chi(\vartheta_h^k), \mathbf{u}_h^k] \, [[\Pi_h^B[\phi] \,]] \, \mathrm{dS}_x - \int_{\Omega_h} \varrho_h^k \chi(\vartheta_h^k) \mathbf{u}_h^k \cdot \nabla_x \phi \, \mathrm{d}x \right| \tag{5.80}$$

$$\lesssim h^{\frac{\gamma-2}{\gamma}} R_h^5(t) \|\nabla_x \phi\|_{L^\infty(\Omega_h;R^3)}, \ \|R_h^5\|_{L^1(0,T)} \lesssim 1.$$

The most delicate part of the proof of consistency of the thermal energy scheme ((5.43)) is the heat-flux term. We need the following auxiliary result.

Lemma 5.6.1. *Let $\phi \in C^2(R^3)$ such that $\nabla_x \phi \cdot \mathbf{n}|_{\partial\Omega} = 0$.*
Then

$$\left| \sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma \frac{1}{d_\Gamma} [[v]] \, [[\ \Pi_h^B[\phi] \ ]] \mathrm{dS}_x + \int_\Omega v \Delta\phi \ \mathrm{d}x \right| \lesssim \sqrt{h} \left(\|v\|_{H^1_{Q_h}(\Omega_h)} + \|v\|_{L^\infty(\Omega_h)} \right) \|\phi\|_{C^2}$$

for any $v \in Q_h(\Omega_h)$.

Proof:
First, by Gauss-Green theorem,

$$\int_{\Omega_h} v \Delta\phi \ \mathrm{d}x = \sum_{E \in E_h} \int_E v \Delta\phi \ \mathrm{d}x = \sum_{E \in E_h} \int_{\partial E} v \nabla_x \phi \cdot \mathbf{n} \ \mathrm{dS}_x$$

$$= - \sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma [[v]] \nabla_x \phi \cdot \mathbf{n} \, \mathrm{dS}_x + \int_{\partial \Omega_h} v \nabla_x \phi \cdot \mathbf{n} \ \mathrm{dS}_x,$$

where, furthermore,

$$\left| \int_{\Omega_h} v \Delta \phi \ \mathrm{d}x - \int_\Omega v \Delta \phi \ \mathrm{d}x \right| \leq \left| \int_{\Omega_h \setminus \Omega} |v| |\Delta \phi| \ \mathrm{d}x \right| \stackrel{<}{\sim} h \|v\|_{L^\infty(\Omega_h)} \|\phi\|_{C^2(R^3)}. \tag{5.81}$$

Next, going back to the definition of the projection Π_h^B, we get

$$\left| \nabla_x \phi \cdot \mathbf{n} - \frac{[[\Pi_h^B \phi]]}{d_\Gamma} \right| \stackrel{<}{\sim} h \|\phi\|_{C^2(\overline{\Omega})} \text{ on any face } \Gamma,$$

and, by Hölder's inequality,

$$\sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma |\ [[v]]\ | \, \mathrm{dS}_x \leq \left(\sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma \frac{[[v]]^2}{d_\Gamma} \ \mathrm{dS}_x \right)^{1/2} \left(\sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma d_\Gamma \mathrm{dS}_x \right)^{1/2} \stackrel{<}{\sim} \|v\|_{H^1_{Q_h}(\Omega)} |\Omega|^{1/2}. \tag{5.82}$$

Thus it remains to control the integral $\int_{\partial \Omega_h} v \nabla_x \phi \cdot \mathbf{n} \ \mathrm{dS}_x$. To this end, write

$$\int_{\Omega_h \setminus \Omega} v \Delta \phi \ \mathrm{d}x = \sum_{E \in E_h, E \not\subset \overline{\Omega}} \int_{E \setminus \Omega} v \Delta \phi \ \mathrm{d}x,$$

where the left-hand side is small in view of (5.81). Moreover, by Gauss-Green theorem,

$$\sum_{E \in E_h, E \not\subset \overline{\Omega}} \int_{E \setminus \Omega} v \Delta \phi \ \mathrm{d}x = \int_{\partial \Omega_h} v \nabla_x \phi \cdot \mathbf{n} \ \mathrm{dS}_x + \sum_{E \in E_h, E \not\subset \overline{\Omega}} \int_{\partial (E \setminus \Omega) \setminus \partial \Omega_h} v \nabla_x \phi \cdot \mathbf{n} \ \mathrm{dS}_x.$$

Seeing that $\nabla_x \phi \cdot \mathbf{n}|_{\partial \Omega} = 0$ we may infer that

$$\sum_{E \in E_h, E \not\subset \overline{\Omega}} \int_{\partial (E \setminus \Omega) \setminus \partial \Omega_h} v \nabla_x \phi \cdot \mathbf{n} \ \mathrm{dS}_x = - \sum_{\Gamma \in \Gamma_{h,\mathrm{int}},\ \Gamma \subset \partial E,\ E \not\subset \overline{\Omega}} \int_{\Gamma \setminus \Omega} [[v]] \nabla_x \phi \cdot \mathbf{n} \ \mathrm{dS}_x,$$

where, similarly to (5.82),

$$\left| \sum_{\Gamma \in \Gamma_{h,\mathrm{int}},\ \Gamma \subset \partial E,\ E \not\subset \overline{\Omega}} \int_{\Gamma \setminus \Omega} [[v]] \nabla_x \phi \cdot \mathbf{n} \ \mathrm{dS}_x \right|$$

$$\stackrel{<}{\sim} \|\phi\|_{C^1(R^3)} \left(\sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma \frac{[[v]]^2}{d_\Gamma} \mathrm{dS}_x \right)^{1/2} \left(\sum_{\Gamma \in \Gamma_{h,\mathrm{int}},\ \Gamma \subset \partial E,\ E \not\subset \overline{\Omega}} \int_\Gamma d_\Gamma \, \mathrm{dS}_x \right)^{1/2}$$

$$\stackrel{<}{\sim} \|\phi\|_{C^1(R^3)} \|v\|_{H^1_{Q_h}(\Omega_h)} \left| \left\{ x \in R^3 \ \middle| \ \mathrm{dist}[x, \partial \Omega_h] < 2h \right\} \right|^{1/2} \approx h^{1/2} \|\phi\|_{C^1} \|v\|_{H^1_{Q_h}(\Omega_h)}.$$

□

Now, we are ready to deal with the diffusion term

$$\sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma \frac{1}{d_\Gamma} \left\{ \chi' \left(\vartheta_h^k \right) \right\} [[K(\vartheta_h^k)]] \ [[\Pi_h^B [\phi]]] \mathrm{dS}_x.$$

Introducing a new function K_χ, $K'_\chi(\vartheta) = \chi'(\vartheta)K'(\vartheta)$, we rewrite the diffusive term with the help of the mean-value theorem as

$$\left\{\chi'\left(\vartheta_h^k\right)\right\}[[K(\vartheta_h^k)]] = [[K_\chi(\vartheta_h^k)]] + c_h^k(x)[[\vartheta_n^k]]^2,$$

where c_h^k is uniformly bounded. Consequently, we get

$$\sum_{\Gamma\in\Gamma_{h,\mathrm{int}}} \int_\Gamma \frac{1}{d_\Gamma}\left\{\chi'\left(\vartheta_h^k\right)\right\}[[K(\vartheta_h^k)]]\ [[\Pi_h^B[\phi]]]\mathrm{dS}_x$$

$$= \sum_{\Gamma\in\Gamma_{h,\mathrm{int}}} \int_\Gamma \frac{1}{d_\Gamma}[[K_\chi(\vartheta_h^k)]]\ [[\Pi_h^B[\phi]]]\mathrm{dS}_x + \sum_{\Gamma\in\Gamma_{h,\mathrm{int}}} \int_\Gamma c_h^k\frac{[[\vartheta_h^k]]^2}{d_\Gamma}[[\Pi_h^B[\phi]]]\mathrm{dS}_x.$$

Seeing that $|[[\Pi_h^B\phi]]| \leq h\|\nabla_x\phi\|_{L^\infty(\Omega_h;R^3)}$, we can estimate the last integral using the entropy bounds (5.60), while the first integral can be "replaced" by $\int_\Omega K_\chi(\vartheta_h^k)\Delta\phi\ \mathrm{d}x$ in view of Lemma 5.6.1.

Finally, observing that the remaining terms in (5.76) can be treated in a similar way, we sum up the previous estimates to obtain

$$\int_{\Omega_h} D_t\left(\varrho_h^k\chi(\vartheta_h^k)\right)\phi\ \mathrm{d}x - \int_{\Omega_h}\varrho_h^k\chi(\vartheta_h^k)\mathbf{u}_h^k\cdot\nabla_x\phi\ \mathrm{d}x - \int_\Omega K_\chi(\vartheta_h^k)\Delta\phi\ \mathrm{d}x \tag{5.83}$$

$$= \int_{\Omega_h}\left(\mu|\nabla_h\mathbf{u}_h^k|^2 + \lambda|\mathrm{div}_h\mathbf{u}_h^k|^2\right)\chi'(\vartheta_h^k)\phi\ \mathrm{d}x - \int_{\Omega_h}\chi'(\vartheta_h^k)\vartheta_h^k\varrho_h^k\mathrm{div}_h\mathbf{u}_h^k\phi\ \mathrm{d}x$$

$$+ \langle D_h,\phi\rangle + h^\beta\left\langle R_h^6,\phi\right\rangle,$$

for a certain $\beta > 0$, where

$$\left|\left\langle R_h^6(t),\phi\right\rangle\right| \lesssim R_h^7(t)\|\phi\|_{C^2(R^3)},\ \|R_h^7\|_{L^1(0,T)} \lesssim 1. \tag{5.84}$$

Relation (5.83) holds for any test function $\phi\in C^2(R^2)$ such that $\nabla_x\phi\cdot\mathbf{n}|_{\partial\Omega} = 0$, and any χ enjoying the properties stated in (5.74). The quantity D_h is a bounded measure satisfying (5.77).

We conclude by a simple observation that (5.83) gives rise to

$$\int_{\Omega_h} D_t\left(\varrho_h^k\chi(\vartheta_h^k)\right)\phi\ \mathrm{d}x - \int_\Omega\varrho_h^k\chi(\vartheta_h^k)\mathbf{u}_h^k\cdot\nabla_x\phi\ \mathrm{d}x - \int_\Omega K_\chi(\vartheta_h^k)\Delta\phi\ \mathrm{d}x \tag{5.85}$$

$$=\int_\Omega\left(\mu|\nabla_h\mathbf{u}_h^k|^2 + \lambda|\mathrm{div}_h\mathbf{u}_h^k|^2\right)\chi'(\vartheta_h^k)\phi\ \mathrm{d}x - \int_\Omega\chi'(\vartheta_h^k)\vartheta_h^k\varrho_h^k\mathrm{div}_h\mathbf{u}_h^k\phi\ \mathrm{d}x$$

$$+ \langle D_h,\phi\rangle + h^\beta\left\langle R_h^6,\phi\right\rangle,$$

where the integrals over the complements $\Omega_h\setminus\Omega$ were incorporated in D_h and R_h^6. As for the discrete time derivative, we claim that

$$\int_0^T\psi(t)\int_{\Omega_h} D_t\left(\varrho_h\chi(\vartheta_h)\right)\phi\ \mathrm{d}x\ \mathrm{d}t \tag{5.86}$$

$$= \psi(0)\int_{\Omega_h}\varrho_h^0\chi(\vartheta_h^0)\phi\ \mathrm{d}x - \int_0^T\int_{\Omega_h}\left(\frac{\psi(t+\Delta t)-\psi(t)}{\Delta t}\right)\varrho_h\chi(\vartheta_h)\phi\ \mathrm{d}x$$

for any $\psi\in C_c^\infty[0,T)$, where, by the mean-value theorem,

$$\left|\left(\frac{\psi(t+\Delta t)-\psi(t)}{\Delta t}\right) - \partial_t\psi\right| \lesssim \Delta t\sup_{s\in[0,T]}|\psi''(s)|.$$

Thus, with (5.86) in mind, we observe that (5.85) coincides with its analogue proved in [37, Section 6.3, formula (6.25)].

5.6.2 Convergence

As observed above, the consistency formulation (5.72), (5.73), (5.85), and (5.86) is the same as in [37]; whence the proof of convergence can be carried over by means of the arguments specified in [37, Section 7]. We have proved Theorem 5.3.1.

5.7 Unconditional convergence

If the initial data $[\varrho_0, \vartheta_0, \mathbf{u}_0]$ are regular and the physical domain has sufficiently smooth boundary, the Navier-Stokes-Fourier system is known to admit strong solutions, at least on a possibly short time interval. If

$$\varrho_0,\ \vartheta_0 \in W^{3,2}(\Omega),\ \varrho_0 > 0,\ \vartheta_0 > 0,\ \mathbf{u}_0 \in W^{3,2}(\Omega; R^3) \tag{5.87}$$

are the initial data satisfying the relevant *compatibility conditions*, and if Ω is of class $C^{2+\nu}$, then the problem (5.1 – 5.7) admits a (classical) solution

$$\varrho, \vartheta \in C([0, T_{\max}); W^{3,2}(\Omega)),\ \mathbf{u}_0 \in C([0, T_{\max}); W^{3,2}(\Omega; R^3)) \tag{5.88}$$

on a maximal time interval $[0, T_{\max})$, see Valli [78], [77] , Valli and Zajaczkowski [76].

On the other hand, as shown in [32, Chapter 7], the problem (5.1 – 5.7) endowed with the regular initial data (5.87) possesses a global in time weak solution in the sense of Definition 5.2.1. Weak and strong solutions emanating from the same initial data *should* coincide on their common existence time interval. As a matter of fact, the answer is not completely straightforward, however, the following result holds, see [43, Lemma 3.2].

Proposition 5.7.1. *In addition to the hypotheses of Theorem 5.3.1, suppose that $\Omega \subset R^3$ is a bounded domain, $\partial\Omega \in C^{2,\nu}$, and that the initial data satisfy (5.87). Let $[\varrho, \vartheta, \mathbf{u}]$ be a weak solution of the Navier-Stokes-Fourier system (5.1 – 5.7) enjoying extra regularity*

$$\varrho,\ \vartheta,\ \mathrm{div}_x \mathbf{u} \in L^\infty((0,T) \times \Omega),\ \mathbf{u} \in L^\infty((0,T) \times \Omega; R^3).$$

Then $[\varrho, \vartheta, \mathbf{u}]$ coincides with the strong solution of the same problem as long as the latter exists.

It turns out that the weak solutions possessing the regularity claimed in Proposition 5.7.1 are in fact strong. More specifically, we report the following assertion, see [43, Theorem 2.2]:

Proposition 5.7.2. *Under the hypotheses of Proposition 5.7.1, let $[\varrho, \vartheta, \mathbf{u}]$ be a weak solution of the Navier-Stokes-Fourier system, emanating from regular initial data satisfying (5.87), and enjoying the extra regularity*

$$\varrho,\ \vartheta,\ \mathrm{div}_x \mathbf{u} \in L^\infty((0,T) \times \Omega),\ \mathbf{u} \in L^\infty((0,T) \times \Omega; R^3).$$

Then $[\varrho, \vartheta, \mathbf{u}]$ is a strong (classical) solution of the problem in $(0,T) \times \Omega$.

Combining the previous results with Theorem 5.3.1, we obtain the following statement concerning *unconditional convergence* of the numerical scheme (5.40 – 5.43).

Theorem 5.7.1. *In addition to the hypotheses of Theorem 5.3.1, suppose that* $\Omega \subset R^3$ *is a bounded domain,* $\partial\Omega \in C^{2,\nu}$*, and the initial data satisfy (5.87). Let* $[\varrho_h, \vartheta_h, \mathbf{u}_h]_{h>0}$ *be a family of numerical solutions constructed by means of the scheme (5.40 – 5.43) such that*

$$\varrho_h > 0,\ \vartheta_h > 0,\ \text{and } \varrho_h,\ \vartheta_h,\ |\mathbf{u}_h|,\ |\mathrm{div}_h \mathbf{u}_h| \leq M$$

a.a. in $(0,T) \times \Omega$ *for a certain constant* M *independent of* h.

Then

$$\varrho_h \to \varrho \text{ weakly-(*) in } L^\infty(0,T;L^\gamma(\Omega)) \text{ and strongly in } L^1((0,T)\times\Omega),$$

$$\vartheta_h \to \vartheta \text{ weakly in } L^2(0,T;L^6(\Omega)),$$

$$\mathbf{u}_h \to \mathbf{u} \text{ weakly in } L^2(0,T;L^6(\Omega;R^3)),\ \nabla_h \mathbf{u}_h \to \nabla_x \mathbf{u} \text{ weakly in } L^2((0,T)\times\Omega;R^{3\times 3}),$$

where $[\varrho, \vartheta, \mathbf{u}]$ *is the (unique) strong solution of the Navier-Stokes-Fourier system (5.1 – 5.7) in* $(0,T)\times\Omega$ *emanating from the initial data (5.87).*

6. Existence and non–uniqueness of global weak solutions to inviscid primitive and Boussinesq equations

Corresponds to the article

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Abstract

We consider the initial value problem for the inviscid Primitive and Boussinesq equations in three spatial dimensions. We recast both systems as an abstract Euler-type system and apply the methods of convex integration of De Lellis and Székelyhidi to show the existence of infinitely many global weak solutions of the studied equations for general initial data. We also introduce an appropriate notion of dissipative solutions and show the existence of suitable initial data which generate infinitely many dissipative solutions.

6.1 Introduction

The Boussinesq equations are used to model the behaviour of oceans. Recall that the Boussinesq approximation consists in neglecting changes of density except in the buoyancy terms and results in a system coupling the incompressible Navier-Stokes equations (for an unknown velocity field $\mathbf{u} = \mathbf{u}(t, \mathbf{x}) = (u, v, w)$ and pressure $p = p(t, \mathbf{x})$) with the convection-diffusion equation (for an unknown temperature $\vartheta = \vartheta(t, \mathbf{x})$). Physically relevant references can be found in [61]. We will also consider the effect of the Coriolis force in the form $\Omega \times \mathbf{u}$ for a vector function $\Omega = (\Omega_x, \Omega_y, \Omega_z)$ and neglect the effect of viscosity. The inviscid Boussinesq equations then read as

$$\partial_t u + \mathbf{u} \cdot \nabla_{\mathbf{x}} u + \Omega_y w - \Omega_z v + \partial_x p = 0, \tag{6.1a}$$

$$\partial_t v + \mathbf{u} \cdot \nabla_{\mathbf{x}} v - \Omega_x w + \Omega_z u + \partial_y p = 0, \tag{6.1b}$$

$$\partial_t w + \mathbf{u} \cdot \nabla_{\mathbf{x}} w + \Omega_x v - \Omega_y u + \partial_z p = -\vartheta, \tag{6.1c}$$

$$\operatorname{div}_{\mathbf{x}} \mathbf{u} = 0, \tag{6.1d}$$

$$\partial_t \vartheta + \mathbf{u} \cdot \nabla_{\mathbf{x}} \vartheta - \lambda_1 (\partial^2_{xx} + \partial^2_{yy}) \vartheta - \lambda_2 \partial^2_{zz} \vartheta = 0 \tag{6.1e}$$

for unknown functions u, v, w, p and $\vartheta\colon [0, T) \times U \to \mathbb{R}$. We consider $U \subseteq \mathbb{R}^3$ an open bounded set. The parameters T, λ_1, and λ_2 are given positive real constants without additional restrictions. We will also denote $Q = (0, T) \times U$.

When modeling the large scale behaviour of oceans or the atmosphere, one spatial scale (vertical) is essentially smaller than the other (horizontal) ones. The primitive equations, which are also considered, can be obtained as a formal singular limit of the Boussinesq equations in the way that the convective derivative of the vertical velocity coordinate is neglected. The momentum equation for the vertical velocity component is then replaced by the hydrostatic approximation. Under the already given notation, the inviscid primitive equations consist of the following system of partial differential equations:

$$\partial_t u + \mathbf{u} \cdot \nabla_{\mathbf{x}} u - \Omega_z v + \Omega_y w + \partial_x p = 0, \tag{6.2a}$$

$$\partial_t v + \mathbf{u} \cdot \nabla_{\mathbf{x}} v + \Omega_z u - \Omega_x w + \partial_y p = 0, \tag{6.2b}$$

$$\partial_z p = -\vartheta, \tag{6.2c}$$

$$\operatorname{div} \mathbf{u} = 0, \tag{6.2d}$$

$$\partial_t \vartheta + \mathbf{u} \cdot \nabla_{\mathbf{x}} \vartheta - \lambda_1(\partial^2_{xx} + \partial^2_{yy})\vartheta - \lambda_2 \partial^2_{zz}\vartheta = 0. \tag{6.2e}$$

To complete both systems, we assume the "no-flow" boundary condition for (u, v, w) and homogeneous Dirichlet condition for ϑ:

$$\mathbf{u}(t, \mathbf{x}) \cdot \eta(t, \mathbf{x}) = 0 \quad \text{on } (0, T) \times \partial U, \tag{6.3a}$$

$$\vartheta(t, \mathbf{x}) = 0 \quad \text{on } (0, T) \times \partial U, \tag{6.3b}$$

where η denotes the exterior normal to the boundary ∂U. Both systems describe the time evolution of u, v and ϑ and therefore it makes sense to prescribe initial conditions for these quantities. We assume that

$$u(0, \cdot) = u_0, v(0, \cdot) = v_0 \text{ and } \vartheta(0, \cdot) = \vartheta_0 \quad \text{in } U. \tag{6.4}$$

For the Boussinesq equations, we also prescribe the initial vertical velocity

$$w(0, \cdot) = w_0 \quad \text{in } U. \tag{6.5}$$

Remark. The main results of the article hold also for other boundary conditions for ϑ for which solutions of (6.1e) or (6.2e) with $\mathbf{u} \in L^\infty(Q; \mathbb{R}^3)$ belong to $C(\overline{Q})$.

Let us recall known results about the mentioned systems. The system (6.1) shares many similarities with the Euler system (i. e. when $\vartheta = 0$). In two dimensions, the global well–posedness for regular initial data was established in [11] (see also [19] for the recent development).

To the best of our knowledge, the question of the existence of global solutions of (6.1) in 3D remains open. We give a positive answer to this question in the case of weak solutions. A similar system, namely (6.1) in dimension 2 with $\lambda_1 = \lambda_2 = 0$ and without the temperature in (6.1c), was treated in [7] using a slightly different approach.

Considering three spatial dimensions, there are a few mathematical results connected to the *viscid* (Navier-Stokes-like) analogue of system (6.2). The local in time existence of regular solutions was presented in [61], where a proof of the global in time existence of weak solutions can also be found. The existence of global regular solutions under the assumption that initial data are slowly varying in the z–variable was proved in [52]. Cao and Titi demonstrated in [9] that the solutions emanating from regular initial data stay regular for all times $t > 0$. The regularity was also shown in the case of homogeneous Dirichlet boundary conditions in [57]. Very recently, global strong well-posedness in L^p was given in [49]. These results should be put into comparison with the similar Navier-Stokes system for which the question on the global regularity is still open. Very recently, it was shown in [10] that if one drops the viscous term and considers the system (6.2) with suitable boundary conditions then a finite time blow-up occurs for some specific regular initial data. Local existence of regular solutions for inviscid primitive equations in 2D was given in [4].

A question remaining open is whether there exist global weak solutions for any (suitably regular) initial data for (6.2) with (6.3). At first glance, there is almost no hope in any kind of positive answer. The inviscid primitive equations differ notably from the incompressible Euler system. In comparison to the Euler system, primitive equations are degenerate with respect to w. It is known that the system is not hyperbolic and that the boundary value problem is ill-posed for pointwise boundary conditions, see [70] and also [72]. On the other hand, we recall that De Lellis and Székelyhidi (see e. g. [22], [23]) extended the possibility to use techniques of convex integration on the Euler system. They constructed infinitely many "oscillatory" weak solutions satisfying even different admissibility criteria, yet exceptionally non-unique. The aim of this paper is to demonstrate that the inviscid primitive equations also admit such oscillatory solutions. We will employ the recent refinements of the De Lellis and Székelyhidi approach, carried out in [27] and in [14] for the Euler-Fourier system or in [34] for Savage-Hutter model.

We give the reader the outline of the rest of the article: in Section 6.2 we define the notion of weak solution and formulate the main results. In Section 6.3 we give a reformulation of the given systems into an abstract Euler-type problem. In Sections 6.4 and 6.5 we present the proof of existence of infinitely weak solutions with general initial conditions for the abstract problem combining approaches from [22], [23], [13], [14] and [33]. The result is extended in Section 6.6, where the existence of some suitable initial data allowing for infinitely many dissipative weak solutions is proven. For the reader's convenience, Section 6.7 contains some auxiliary results which are employed in the article.

6.2 The main results

We will denote by $C([0,T];X_w)$ the set of continuous functions from $[0,T]$ with values in a Banach space X equipped with the weak topology. For $B \subseteq \mathbb{R}^d$ open we denote $\mathcal{D}(B)$ the topological vector space of smooth functions with compact support in B and $\mathcal{D}'(B)$ its topological dual.

6.2.1 The Boussinesq equations

We start with introducing the definition of weak solutions to the problem (6.1) supplemented by (6.3) and (6.4), (6.5).

Definition 1. We call the quintet of functions (u, v, w, p, ϑ) a *weak solution of the inviscid Boussinesq equations* with (6.3), (6.4) if

- $u,\, v,\, w \in C([0,T]; L^2_w(U))$, $p \in L^1(Q)$ and equations

$$\int_0^T \int_U u\partial_t \phi_1 \,\mathrm{d}\mathbf{x}\,\mathrm{d}t + \int_0^T \int_U u\mathbf{u}\cdot\nabla_{\mathbf{x}}\phi_1 \,\mathrm{d}\mathbf{x}\,\mathrm{d}t + \int_U u_0(\cdot)\phi_1(0,\cdot)\,\mathrm{d}\mathbf{x} + \int_0^T \int_U (-\Omega_y w + \Omega_z v)\phi_1 \,\mathrm{d}\mathbf{x}\,\mathrm{d}t + \int_0^T \int_U p\partial_x\phi_1 \,\mathrm{d}\mathbf{x}\,\mathrm{d}t = 0, \tag{6.6}$$

$$\int_0^T \int_U v\partial_t \phi_2 \,\mathrm{d}\mathbf{x}\,\mathrm{d}t + \int_0^T \int_U v\mathbf{u}\cdot\nabla_{\mathbf{x}}\phi_2 \,\mathrm{d}\mathbf{x}\,\mathrm{d}t + \int_U v_0(\cdot)\phi_2(0,\cdot)\,\mathrm{d}\mathbf{x} + \int_0^T \int_U (\Omega_x w - \Omega_z u)\phi_2 \,\mathrm{d}\mathbf{x}\,\mathrm{d}t + \int_0^T \int_U p\partial_y\phi_2 \,\mathrm{d}\mathbf{x}\,\mathrm{d}t = 0, \tag{6.7}$$

$$\int_0^T \int_U w\partial_t \phi_3 \,\mathrm{d}\mathbf{x}\,\mathrm{d}t + \int_0^T \int_U w\mathbf{u}\cdot\nabla_{\mathbf{x}}\phi_3 \,\mathrm{d}\mathbf{x}\,\mathrm{d}t + \int_U w_0(\cdot)\phi_3(0,\cdot)\,\mathrm{d}\mathbf{x} + \int_0^T \int_U (-\Omega_x v + \Omega_y u)\phi_3 \,\mathrm{d}\mathbf{x}\,\mathrm{d}t + \int_0^T \int_U p\partial_z\phi_3 \,\mathrm{d}\mathbf{x}\,\mathrm{d}t = \int_0^T \int_U \vartheta\phi_3 \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \tag{6.8}$$

  are satisfied for any ϕ_1, ϕ_2, $\phi_3 \in \mathcal{D}([0,T)\times U)$,

- $\mathbf{u}\chi_Q$ solves (6.1d) in $\mathcal{D}'((0,T)\times\mathbb{R}^3)$, i. e.

$$\int_0^T \int_U \mathbf{u}\cdot\nabla_{\mathbf{x}}\phi \,\mathrm{d}\mathbf{x}\,\mathrm{d}t = 0 \text{ for every } \phi \in \mathcal{D}((0,T)\times\mathbb{R}^3), \tag{6.9}$$

- $\vartheta \in W^{1,p}((0,T); L^p(U)) \cap L^p\left((0,T); W^{2,p}(U)\cap W^{1,p}_0(U)\right)$ for a $p \in (1,\infty)$ and (6.1e) holds almost everywhere in Q and $\vartheta(0,\cdot) = \vartheta_0(\cdot)$.

Theorem 6.2.1. *Let $T > 0$, U be a bounded open set with $\partial U \in C^2$, $\mathbf{u}_0 \in L^\infty(U;\mathbb{R}^3)\cap C(U;\mathbb{R}^3)$ satisfy (6.9), $\vartheta_0 \in L^\infty(U)\cap C^2(U)$ satisfy (6.3b) and $\Omega = \Omega(\mathbf{x}) \in L^\infty(U;\mathbb{R}^3)$. Then there exist infinitely many weak solutions to the Boussinesq equations in the sense of Definition 1.*

For the Boussinesq equations, the total energy is defined as the sum of the kinetic and potential energy:

$$E_{Bous}(t) = \int_U \frac{1}{2}|\mathbf{u}(t,\mathbf{x})|^2 + z\vartheta(t,\mathbf{x})\,\mathrm{d}\mathbf{x},$$

see also [80]. Referring to [24], we recall that in the case of homogeneous Dirichlet boundary conditions for ϑ, the Boussinesq equations violate the principle of conservation of total energy. On the other hand, the choice of homogeneous Neumann boundary conditions for ϑ would imply the conservation of E_{Bous}.

Definition 2. We say that a weak solution of the Boussinesq equations satisfies the *strong energy inequality* if $E_{Bous}(t)$ is non-increasing on $[0,T)$. We also call such solutions *dissipative*.

Let us mention that the weak solutions given by Theorem 6.2.1 are violating the strong energy inequality, particularly

$$\liminf_{t\to 0^+} E_{Bous}(t) > E_{Bous}(0).$$

We remark that this property is a consequence of the method of convex integration used to construct the weak solutions of Theorem 6.2.1.

Theorem 6.2.2. *Let $\Omega \in L^\infty(U;\mathbb{R}^3)$ and $\vartheta_0 \in L^\infty(U)\cap C^2(U)$. Then there exists $\mathbf{u}_0 \in L^\infty(U;\mathbb{R}^3)$ for which we can find infinitely many weak dissipative solutions of the Boussinesq equations emanating from $\mathbf{u}_0$.*

6.2.2 The primitive equations

Analogously, we present the definition of the weak solutions to (6.2):

Definition 3. We call the quintet of functions (u, v, w, p, ϑ) a *weak solution of the inviscid primitive equations* with (6.3), (6.4) if

- $\mathbf{u} = (u, v, w) \in L^2(Q;\mathbb{R}^3)$, u, $v \in C([0, T]; L^2_w(U))$, $p \in L^1(Q)$, $\partial_z p \in L^1(Q)$ and equations (6.6) and (6.7) are satisfied for any ϕ_1, $\phi_2 \in \mathcal{D}([0, T) \times U)$,
- $\mathbf{u}$ satisfies (6.9),
- $\vartheta \in W^{1,p}((0, T); L^p(U)) \cap L^p\left((0, T); W^{2,p}(U) \cap W^{1,p}_0(U)\right)$ for a $p \in (1, \infty)$ and (6.2e) holds almost everywhere in Q and $\vartheta(0, \cdot) = \vartheta_0(\cdot)$.
- equation (6.2c) holds for the weak derivative of p almost everywhere in Q.

As we will see, the problem of finding weak solutions of the primitive equations is highly underdetermined. Let us fix a function p and supplement the system by an equation describing the evolution of w:

$$\partial_t w + \mathbf{u} \cdot \nabla_{\mathbf{x}} w + \Omega_x v - \Omega_y u + \partial_z p = 0. \tag{6.10}$$

The equations (6.2a), (6.2b) and (6.10) can be recast in the usual vector form

$$\partial_t \mathbf{u} + \operatorname{div}_{\mathbf{x}}(\mathbf{u} \otimes \mathbf{u}) + \Omega \times \mathbf{u} + \nabla_{\mathbf{x}} p = 0,$$

where $\mathbf{u} = (u, v, w)$. We use the notion *extended primitive equations* for the system (6.2) coupled with (6.10) together with an additional initial condition for w. For the sake of completeness, we present the definition of the corresponding weak solution.

Definition 4. We call $(\mathbf{u}, p, \vartheta)$ a *weak solution of the extended primitive equations* with (6.3), (6.4), (6.5) if

- $(\mathbf{u}, p, \vartheta)$ is a weak solution of the inviscid primitive equations with (6.3), (6.4)
- $w \in C([0, T]; L^2_w(U))$ and

$$\begin{aligned}\int_0^T \int_U w\partial_t\phi_3 \,\mathrm{d}\mathbf{x}\,\mathrm{d}t + \int_0^T \int_U w\mathbf{u} \cdot \nabla_{\mathbf{x}}\phi_3 \,\mathrm{d}\mathbf{x}\,\mathrm{d}t + \int_U w_0(\cdot)\phi_3(0, \cdot)\,\mathrm{d}\mathbf{x} \\ + \int_0^T \int_U (-\Omega_x v + \Omega_y u)\phi_3 \,\mathrm{d}\mathbf{x}\,\mathrm{d}t + \int_0^T \int_U p\partial_z\phi_3 \,\mathrm{d}\mathbf{x}\,\mathrm{d}t = 0\end{aligned} \tag{6.11}$$

  is satisfied for any $\phi_3 \in \mathcal{D}([0, T) \times U)$.

Remark. Because of the well-posedness result, it seems to be unreasonable to work with the extended primitive equations. However, all the results following the approach of De Lellis and Székelyhidi (see e. g. [22], [23], [27] or [14]) are foreshadowing that weak formulations of inviscid problems in fluid dynamics are surprisingly highly underdetermined. The main results of this paper, namely Theorem 6.2.3, 6.2.5 and Corollary 6.2.4, are in agreement with this observation.

Theorem 6.2.3. *Let $T > 0$, U be a bounded open set with $\partial U \in C^2$, $\mathbf{u}_0 \in L^\infty(U;\mathbb{R}^3) \cap C(U;\mathbb{R}^3)$ satisfy* (6.9), *$\vartheta_0 \in L^\infty(U) \cap C^2(U)$ satisfy* (6.3b) *and $\Omega \in L^\infty(U;\mathbb{R}^3)$. Then there exist infinitely many weak solutions to the extended primitive equations with* (6.3), (6.4) *and* (6.5).

Corollary 6.2.4. *Let $\vartheta_0 \in L^\infty(U) \cap C^2(U)$ satisfy* (6.3b), *$\Omega = \Omega(\mathbf{x}) \in L^\infty(U;\mathbb{R}^3)$ and let u_0, $v_0 \in L^\infty(U) \cap C(U)$ be such that exists $w_0 \in L^\infty(U) \cap C(U)$ satisfying*

$$\int_U (u_0, v_0, w_0) \cdot \nabla_{\mathbf{x}} \phi \, \mathrm{d}\mathbf{x} = 0 \text{ for every } \phi \in \mathcal{D}(\mathbb{R}^3). \tag{6.12}$$

Then there exist infinitely many weak solutions to the primitive equations with (6.3) *and* (6.4).

Remark. The technical assumption on u_0 and v_0 is needed only because we are considering boundary conditions (6.3a). If we took $U = \mathbb{T}^3$ then the additional condition leading to (6.12) would be $\partial_x u_0 + \partial_y v_0 \in L^\infty(U) \cap C(U)$.

To the best of our knowledge, there are no a priori estimates on (u, v, w) in the case of inviscid primitive equations. Still, it is possible to find initial data for which there exist infinitely many weak solutions of the primitive equations satisfying the conservation of the kinetic energy or which are dissipating the mechanical energy. Let us define

$$E_{Prim}(t) = \int_U \frac{1}{2} \left(|u(t,\mathbf{x})|^2 + |v(t,\mathbf{x})|^2 + |w(t,\mathbf{x})|^2 \right) \mathrm{d}\mathbf{x}.$$

Definition 5. We say that a weak solution of the extended primitive equations or primitive equations satisfies the *strong energy inequality* if $E_{Prim}(t)$ is non-increasing on $[0, T)$. We also call such solutions *dissipative*.

Similarly to the previous section, the particular weak solutions constructed in Theorem 6.2.3 are violating the strong energy inequality, specifically

$$\liminf_{t \to 0^+} E_{Prim}(t) > E_{Prim}(0).$$

Theorem 6.2.5. *There exists $\mathbf{u}_0 \in L^\infty(U;\mathbb{R}^3)$ for which we can find infinitely many weak dissipative solutions of the extended primitive equations emanating from $\mathbf{u}_0$.*

Remark. To obtain the largest possible space for initial temperatures in which the given method holds, we can apply the theory of maximal regularity for parabolic equations (see e. g. [1]). Particularly, ϑ_0 can be taken arbitrarily from the interpolation space $[L^p, W^{2,p}]_\alpha$ for a suitable $p \in (1, \infty)$ and $\alpha \in (0, 1)$.

6.3 Abstract Euler-type system

To use the techniques from [22], [23], we will follow [34], [33] and reformulate the Boussinesq and the extended primitive equations as an Euler-type equation. Let us denote by $\mathbf{u}\odot\mathbf{u} = \mathbf{u}\otimes\mathbf{u}-\frac{1}{3}|\mathbf{u}|^2\mathbb{I}$ the traceless part of the symmetric matrix $\mathbf{u}\otimes\mathbf{u}$. We will introduce operators $\mathbb{H}\colon L^\infty(Q;\mathbb{R}^3) \to L^1(Q;\mathbb{R}^{3\times 3}_{0,sym})$, $\Pi\colon L^\infty(Q;\mathbb{R}^3) \to L^1(Q)$ and consider the abstract Euler-type system

$$\partial_t \mathbf{u} + \operatorname{div}_{\mathbf{x}} (\mathbf{u}\odot\mathbf{u} + \mathbb{H}(\mathbf{u})) + \nabla_{\mathbf{x}} \left(\Pi[\mathbf{u}] + \frac{1}{3}|\mathbf{u}|^2\right) = 0 \text{ in } Q, \tag{6.13a}$$

$$\operatorname{div}_{\mathbf{x}}(\mathbf{u}) = 0 \text{ in } Q, \tag{6.13b}$$

$$\mathbf{u}\cdot\eta = 0 \text{ on } (0,T)\times\partial U, \tag{6.13c}$$

$$\mathbf{u}(0) = \mathbf{u}_0 \text{ in } U, \tag{6.13d}$$

For the sake of completeness, we add a definition of weak solutions of (6.13):

Definition 6. We say that $\mathbf{u}\colon Q \to \mathbb{R}^3$ is a weak solution of the abstract Euler system (6.13) if

- $\mathbf{u} \in C([0,T]; L^2_w(U))$,
- $\mathbf{u}$ satisfies (6.13a) in $\mathcal{D}'(Q)$,
- $\mathbf{u}$ satisfies (6.9),
- $\mathbf{u}(0) = \mathbf{u}_0$.

Theorem 6.3.1. *Let $T > 0$, $\mathbf{u}_0 \in L^\infty(U;\mathbb{R}^3) \cap C(U;\mathbb{R}^3)$ satisfy (6.9). Assume that the $\mathbb{H}$ and Π have the following properties:*

- *$\mathbb{H}$ is continuous from $C([0,T]; L^q_w(U))$ to $C(\overline{Q};\mathbb{R}^{3\times3}_{0,sym})$ and mapping bounded sets to bounded sets (with respect to the mentioned topologies).*
- *Π is continuous from $C([0,T]; L^q_w(U))$ to $C(\overline{Q})$ and there exists $\overline{\Pi} \in \mathbb{R}$ such that*
$$\Pi[\mathbf{u}] < \overline{\Pi} \quad \text{for every } \mathbf{u} \in L^\infty(Q;\mathbb{R}^3). \tag{6.14}$$
- *For $\mathbf{u}$, $\mathbf{w} \in L^\infty(Q;\mathbb{R}^3)$ with $\operatorname{supp}\mathbf{w} \subseteq (\tau,T)\times\overline{U}$*
$$\Pi[\mathbf{u}+\mathbf{w}] = \Pi[\mathbf{u}],\ \mathbb{H}[\mathbf{u}+\mathbf{w}] = \mathbb{H}[\mathbf{u}],\ \text{almost everywhere in } (0,\tau)\times U. \tag{6.15}$$

Then there exist infinitely many weak solutions to (6.13) which moreover satisfy

$$\frac{3}{2}\Pi[\mathbf{u}](t,\mathbf{x}) + \frac{1}{2}|\mathbf{u}(t,\mathbf{x})|^2 = \frac{3}{2}Z(t) \quad \text{for every } t\in(0,T), \text{ almost everywhere in } U, \tag{6.16}$$

for any function $Z(t)$ continuous on $[0,T]$ satisfying $\sup_{t\in[0,T]} Z(t) > \overline{\Pi}$.

In the first part of this Section, we will show that Theorem 6.2.1 and 6.2.3 follow directly from Theorem 6.3.1 after suitable choices of $\mathbb{H}$ and Π. In the second part, we will present the notion of subsolution for (6.13) and important properties of the set of subsolutions.

6.3.1 Reformulation of the Boussinesq equations

Assume that ϑ_0 and Ω comply with the assumptions of Theorem 6.2.1. Using the classical theory of parabolic equations (see e. g. Lemma 6.7.3) we can define an operator $\Theta = \Theta[\mathbf{u}]$ from $L^\infty(Q;\mathbb{R}^3)$ to $C([0,T]\times\overline{U})$ such that $\vartheta = \Theta[\mathbf{u}]$ solves (6.1e) in the sense of Definition 1. The operator $\mathbf{u} \mapsto \Theta[\mathbf{u}]$ is continuous from $C([0,T];L^q_w(U))$ to $C([0,T]\times\overline{U})$ (for q large enough).

Using Corollary 6.7.2, we obtain the existence of a linear operator $\mathbb{H}_{Bous} = \mathbb{H}_{Bous}[\mathbf{u}]$ from $L^\infty(Q;\mathbb{R}^3)$ to $C(\overline{Q};\mathbb{R}^{3\times 3}_{0,sym})$ such that

$$\operatorname{div}_{\mathbf{x}}(\mathbb{H}_{Bous}[\mathbf{u}]) = \Omega \times \mathbf{u} + \begin{pmatrix} 0 \\ 0 \\ \Theta[\mathbf{u}] \end{pmatrix} - \nabla\left(\frac{2}{3} z\Theta[\mathbf{u}]\right).$$

Define $\Pi_{Bous}[\mathbf{u}] = \frac{2}{3}z\Theta[\mathbf{u}]$. The operator Π_{Bous} is continuous from $C([0,T];L^q_w(U))$ to $C(\overline{Q})$ and $\mathbb{H}_{Bous}$ is continuous from $C([0,T];L^q_w(U;\mathbb{R}^3))$ to $C(\overline{Q})$ for any $q > 3$. Both operators are mapping bounded sets from $L^\infty(Q)$ on bounded sets in $C(\overline{Q})$. Using the maximum principle for (6.2e), see Lemma 6.7.3, we obtain

$$|\Pi_{Bous}[u](t,\mathbf{x})| \leq \frac{2}{3}\|z\cdot\vartheta_0\|_{L^\infty(U)} < \infty \text{ for every } (t,\mathbf{x}) \in Q.$$

The condition (6.15) holds from the defition of the operators. Indeed, $\Theta\colon L^\infty(Q;\mathbb{R}^3)$ is a unique solution of an evolutionary equation and $\mathbf{v} = 0$ is the only solution of (6.31) with $\mathbf{g} = 0$ with zero boundary conditions. If $\mathbf{u}$ is a weak solution of (6.13) then the triplet $(\mathbf{u}, p, \vartheta) = (\mathbf{u}, 0, \Theta[\mathbf{u}])$ is a weak solution of (6.1), hence Theorem 6.2.1 is a corollary of Theorem 6.3.1.

6.3.2 Reformulation of the extended primitive equations

Assume that ϑ_0, $\mathbf{u}_0$, P and Ω comply with the assumptions of Theorem 6.2.3. For a given $\mathbf{u}$ we can extend $\Theta[\mathbf{u}]$ continuously with respect to space on $[0,T]\times\mathbb{R}^3$. As U is bounded, one can define an extension $\Theta[\mathbf{u}]$ such that it has compact support. Let us take arbitrary function $P \in C([0,T]\times\mathbb{R}^2)$. The function $p\colon Q \to \mathbb{R}$ defined by

$$p(t,x,y,z) = P(t,x,y) - \int_{-\infty}^{z} \Theta[\mathbf{u}](t,x,y,s)\,\mathrm{d}s$$

satisfies (6.2c) and the operator $\Gamma_{Prim} = \Gamma_{Prim}[\mathbf{u}]\colon \mathbf{u} \mapsto p$ maps functions from $L^\infty(Q;\mathbb{R}^3)$ to $C(\overline{Q})$. The operator Γ_{Prim} is continuous from $C([0,T];L^q_w(U)))$ to $C(\overline{Q})$. Using Corollary 6.7.2, we obtain existence of a linear operator $\mathbb{H}_{Prim} = \mathbb{H}_{Prim}[\mathbf{u}]$ from $L^\infty(Q;\mathbb{R}^3)$ to $C(\overline{Q};\mathbb{R}^{3\times 3}_{0,sym})$ such that

$$\operatorname{div}_{\mathbf{x}}(\mathbb{H}_{Prim}[\mathbf{u}]) = \Omega \times \mathbf{u} + \nabla\Gamma_{Prim}[\mathbf{u}].$$

Similarly to the Boussinesq equations, $\mathbb{H}_{Prim}$ is a continuous operator from $C([0,T];L^q_w(U;\mathbb{R}^3))$ to $C(\overline{Q})$ for any $q > 3$. Both operators are mapping bounded sets from $L^\infty(Q)$ on bounded sets in $C(\overline{Q})$. For any weak solution $\mathbf{u}$ of (6.13) the triplet $(\mathbf{u}, p, \vartheta) = (\mathbf{u}, \Gamma_{Prim}, \Theta[\mathbf{u}])$ is a weak solution of the extended primitive equations. Therefore also Theorem 6.2.3 is a corollary of Theorem 6.3.1.

6.3.3 Subsolutions for the abstract Euler equation

Let $\mathbf{u}_0$ comply with the assumptions of Theorem 6.3.1. Observe that the abstract system is invariant with respect to adding a continuous function $Z = Z(t)$ to the pressure term $\Pi[\mathbf{u}]$. Moreover, $\tilde{\Pi}[\mathbf{u}] = \Pi[\mathbf{u}] + Z$ satisfies the same qualitative properties as Π in Theorem 6.3.1.

Let us fix $Z = Z(t)$ continuous on $[0, T]$ such that

$$\Pi[\mathbf{v}(t, \mathbf{x})] < Z(t) \quad \text{for } (t, \mathbf{x}) \in Q$$

for any $\mathbf{v} \in L^\infty(Q; \mathbb{R}^3)$. Such function Z exists due to the boundedness of Π. We restrict our attention only on the so-called pressureless case, i. e. when solutions are satisfying

$$\Pi[\mathbf{u}] + \frac{1}{3}|\mathbf{u}|^2 - Z(t) = 0 \quad \text{in } Q.$$

Mimicking the strategy of De Lellis and Székelyhidi we recast the abstract system into a linear system supplemented by implicit constitutive (possibly non-algebraic) relations:

$$\partial_t \mathbf{u} + \operatorname{div}_{\mathbf{x}} \mathbb{V} = 0, \tag{6.17a}$$

$$\operatorname{div}_{\mathbf{x}} \mathbf{u} = 0, \tag{6.17b}$$

$$\mathbb{V} = \mathbf{u} \odot \mathbf{u} + \mathbb{H}(\mathbf{u}), \tag{6.17c}$$

$$\frac{1}{2}|\mathbf{u}|^2 = \frac{3}{2}\left(Z(t) - \Pi[\mathbf{u}]\right). \tag{6.17d}$$

To introduce a suitable notion of subsolution we put

$$\bar{e}[\mathbf{u}] = \frac{3}{2}\left(Z(t) - \Pi[\mathbf{u}]\right)$$

and

$$e(\mathbf{u}, \mathbb{V}) = \frac{3}{2}\lambda_{\max}[\mathbf{u} \otimes \mathbf{u} + \mathbb{H}[\mathbf{u}] - \mathbb{V}],$$

where $\lambda_{\max}(\mathbb{U})$ denotes the maximal eigenvalue of $\mathbb{U} \in \mathbb{R}^{3\times 3}_{sym}$. One has for any $\mathbf{v} \in \mathbb{R}^3$ and $\mathbb{U} \in \mathbb{R}^{3\times 3}_{0,sym}$ the following inequality

$$\frac{1}{2}|\mathbf{v}|^2 \leq \frac{3}{2}\lambda_{\max}\left(\mathbf{v} \otimes \mathbf{v} + \mathbb{U}\right) \tag{6.18}$$

and the equality holds if and only if

$$\mathbb{U} = \mathbf{v} \otimes \mathbf{v} - \frac{1}{3}|\mathbf{v}|^2 \mathbb{I}. \tag{6.19}$$

It is possible to estimate $\mathbb{U}$ by the means of $|e(\mathbf{v}, \mathbb{U})|$, particularly

$$|\mathbb{U}|_{\ell^\infty} \leq 2|\lambda_{\min}(\mathbb{U})| \leq \frac{4}{3}e(\mathbf{v}, \mathbb{U}). \tag{6.20}$$

Analogously to [33]:

Definition 7. We call a pair $(\mathbf{u}, \mathbb{V})$ a *subsolution of the abstract Euler system* (or briefly a *subsolution*) if

1. $\mathbf{u} \in C([0,T]; L^2_w(U;\mathbb{R}^3)) \cap C(Q;\mathbb{R}^3)$ and $\mathbb{V} \in L^\infty \cap C(Q; \mathbb{R}^{3\times 3}_{0,sym})$,
2. the pair $(\mathbf{u}, \mathbb{V})$ satisfies (6.17a) in the sense of distributions on Q and $\mathbf{u}$ satisfies (6.9),
3. $\mathbf{u}(0) = \mathbf{u}_0$,
4. for every $0 < \tau < T$ $\operatorname{ess\,inf}_{t\in(\tau,T),\,\mathbf{x}\in U} \left(\bar{e}[\mathbf{u}](t,\mathbf{x}) - e(\mathbf{u}(t,\mathbf{x}), \mathbb{V}(t,\mathbf{x}))\right) > 0$.

We denote X_0 the set of all $\mathbf{u}$ for which exist $\mathbb{V}$ such that $(\mathbf{u}, \mathbb{V})$ is a subsolution of the abstract Euler type system. Let us remark that there exists a constant $E > 0$ such that $\bar{e}[\mathbf{u}] \leq E$ for every $\mathbf{u} \in X_0$. Observe that (6.14), (6.18) and (6.20) imply the boundedness of X_0 in $L^\infty(Q;\mathbb{R}^3)$.

We consider for each $\tau \in (0, T/2)$ a negative functional I_τ on X_0 defined by

$$I_\tau(\mathbf{u}) = \inf_{t\in(\tau,T-\tau)} \int_U \frac{1}{2}|\mathbf{u}(t,\mathbf{x})|^2 - \bar{e}[\mathbf{u}(t,\mathbf{x})]\,\mathrm{d}\mathbf{x}.$$

Lemma 6.3.2. *Let $\{(\mathbf{u}_n, \mathbb{V}_n)\}_{n\in\mathbb{N}}$ be subsolutions. Then there exists a pair $(\mathbf{u}, \mathbb{V})$ such that for a suitable subsequence (not relabeled)*

$$\mathbf{u}_n \to \mathbf{u} \text{ strongly in } C([0,T]; L^2_w(U;\mathbb{R}^3)) \text{ and weakly-}* \in L^\infty(Q;\mathbb{R}^3) \tag{6.21}$$
$$\mathbb{V}_n \to \mathbb{V} \text{ weakly-}* \text{ in } L^\infty(Q;\mathbb{R}^{3\times 3}_{0,sym}) \tag{6.22}$$

holds. The limit $(\mathbf{u}, \mathbb{V})$ is satisfying all conditions on subsolutions except condition 4, where only (nonstrict) inequality holds. Moreover, if

$$I_\tau(\mathbf{u}) = 0 \text{ for each } \tau > 0 \tag{6.23}$$

then $\mathbf{u}$ *is a weak solution of the abstract Euler system* (6.13) *satisfying* (6.16) *for suitable functions Z.*

Proof. As X_0 consists of functions bounded in $L^\infty(Q;\mathbb{R}^3)$, $\mathbf{u}_n$ resp. $\mathbb{V}_n$ are also uniformly bounded in $L^\infty(Q;\mathbb{R}^3)$ resp. $L^\infty(Q;\mathbb{R}^{3\times 3}_{0,sym})$. The standard time regularity for weak solutions together with the Arzelà-Ascoli theorem implies the existence of $\mathbf{u}$ and $\mathbb{V}$ such that (6.21) holds for a suitable subsequence. With respect to Lemma 6.7.4, the pair $(\mathbf{u}, \mathbb{V})$ satisfies all conditions on subsolutions except condition 4.

If, moreover, $I_\tau(\mathbf{u}) = 0$ for every $\tau \in (0,T)$ then

$$\frac{1}{2}|\mathbf{u}|^2 = \bar{e}[\mathbf{u}] = \frac{3}{2}\left(Z(t) - \Pi[\mathbf{u}]\right) \tag{6.24}$$

everywhere in $(0,T)$ and a.e. in U and (6.19) hold almost everywhere in $(0,T)\times\Omega$. Hence, thanks to the hypothesis of the lemma

$$\mathbb{V} = \mathbb{H}[\mathbf{u}] + \mathbf{u}\otimes\mathbf{u} - \frac{1}{3}|\mathbf{u}|^2\mathbb{I} \quad \text{a.e. in } (0,T)\times\Omega$$

and $\mathbf{u}$ is a weak solution of (6.17). □

6.4 Existence result for the abstract Euler-type system

An important step on the way to find $\mathbf{u} \in X$ satisfying (6.23) is the following possibility to appropriately perturb any subsolution so that I increases. The proof of the following lemma is postponed until Section 6.5.

Lemma 6.4.1 (Oscillatory lemma). *Let $\mathbf{u} \in X_0$ and $(\mathbf{u}, \mathbb{V})$ be a subsolution and $\tau > 0$. Then there exist sequences $\{\mathbf{w}_n\}_{n\in\mathbb{N}} \subseteq \mathcal{D}((\tau, T) \times U; \mathbb{R}^3)$ and $\{\mathbb{W}_n\}_{n\in\mathbb{N}} \subseteq \mathcal{D}((\tau, T) \times U; \mathbb{R}^{3\times3}_{0,sym})$ such that:*

- *$(\mathbf{u} + \mathbf{w}_n, \mathbb{V} + \mathbb{W}_n)$ are subsolutions,*
- *$\mathbf{w}_n \to 0$ in $C([0, T]; L^2_w(U))$,*
- *there exists $c = c(E) > 0$ such that*

$$\liminf_{n\to\infty} I_\tau(\mathbf{u} + \mathbf{w}_n) \geq I_\tau(\mathbf{u}) + c(E) \left(I_\tau(\mathbf{u})\right)^2 . \tag{6.25}$$

Remark. The constant $c(E)$ does not depend on $\mathbf{u}$ or τ.

The existence of infinitely many weak solutions is then concluded from a Baire category argument similar to e. g. [22].

Lemma 6.4.2. *Let (X, d) be a complete metric space, $I\colon X \to (-\infty; 0]$ a function of Baire class 1. Let X_0 be a nonempty dense subset of X with the following property: for any $\beta < 0$ there exists $\alpha = \alpha(\beta) > 0$ such that for any $x \in X_0$ satisfying $I(x) < \beta < 0$ there exists $x_n \in X_0$ with*

- *$x_n \to x$ in (X, d) and*
- *$\liminf_{n\to\infty} I(x_n) \geq I(x) + \alpha(\beta)$.*

Then there exists a residual set $S \subseteq X$ such that $I(x) = 0$ on S.

Proof. As (X, d) is complete, the set of points of continuity of functions of Baire class 1 on X is residual. To complete the proof, it is sufficient to show that $I = 0$ on the set of points of continuity. We prove that by contradiction. Let x be a point of continuity of I such that $I(x) < \beta < 0$. Then from the density of X_0, there exists a sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq X_0$, converging to x and $I(x_n) \to I(x)$. Without loss of generality, we may assume that $I(x_n) < \beta$. For each $n \in \mathbb{N}$ there exists sequence $\{x_{n,k}\}_{k\in\mathbb{N}}$ satisfying the conditions given by the hypothesis of the lemma. By a diagonal argument we can find a subsequence $\{x_{n,k(n)}\}_{n\in\mathbb{N}} \subseteq X_0$ such that $x_{n,k(n)} \to x$ and

$$\liminf_{n\to\infty} I(x_{n,k(n)}) \leq I(x_n) + \alpha\left(\beta\right).$$

This contradicts the assumption that x is a point of continuity of I. □

Proof of Theorem 6.3.1. Let X_0 be the set of subsolutions to the abstract Euler system. X_0 consists of functions $\mathbf{u} : [0, T] \to L^2(U)$ taking values in a bounded subset Y of $L^2(U)$. Hence Y is metrizable with respect to the weak topology of

L^2. Correspondingly, we consider the metric d naturally defined on $C([0,T];Y)$ which induces a topology equivalent to the topology of $C([0,T];Y)$ as a subset of $C([0,T];L^2_w(U))$. We denote by X the completion of X_0 in $C([0,T];L^2_w(U))$ with respect to the metric d. Obviously, X is bounded in $L^\infty(Q;\mathbb{R}^3)$. The set X_0 is non-empty as it makes no difficulty to check that $\mathbf{u}(t) = \mathbf{u}_0$ with $\mathbb{V} = \mathbf{0}$ defines a subsolution.

For each $\tau \in (0, T/2)$, I_τ can be extended on a lower-semicontinuous functional on X and therefore is of Baire class 1. Indeed, observe that $\bar{\mathbf{e}}$ is continuous from (X,d) to $C(\overline{Q})$, hence, the semicontinuity follows from the case when $\bar{\mathbf{e}}$ is a constant function (see [23, Lemma 5]).

Finally, a combination of Lemma 6.4.1, 6.4.2 and 6.3.2 implies the existence of residual sets

$$S_\tau = \{x \in X \colon I_\tau(x) = 0\}.$$

The set $S = \cap_{n=1}^{\infty} S_{\frac{1}{n}}$ is also residual, especially nonempty and of infinite cardinality. Due to Lemma 6.3.2, all functions in S are weak solutions to the abstract Euler problem with $\mathbf{u}(0,\mathbf{x}) = \mathbf{u}_0$ and such that (6.16) holds. □

6.5 Proof of Lemma 6.4.1

We start with a special case of the oscillatory lemma when the operators $\mathbb{H}$ and $\bar{e}$ are not depending on $\mathbf{u}$. Let us define $\tilde{e} \colon \mathbb{R}^3 \times \mathbb{R}^{3\times 3}_{0,sym} \times \mathbb{R}^{3\times 3}_{0,sym} \to \mathbb{R}$ by

$$\tilde{e}(\mathbf{u}, \mathbb{V}, \mathbb{G}) = \frac{3}{2} \lambda_{\max}\left(\mathbf{u} \otimes \mathbf{u} + \mathbb{G} - \mathbb{V}\right).$$

For $f \in L^\infty \cap C(Q)$ and $\mathbb{G} \in L^\infty \cap C(Q; \mathbb{R}^{3\times 3}_{0,sym})$ we denote $X_{0,\mathbb{G},f}$ the set of all functions $\mathbf{u}$ satisfying

1. $\mathbf{u} \in C([0,T]; L^2_w(U;\mathbb{R}^3)) \cap C((0,T) \times U; \mathbb{R}^3)$,
2. exists $\mathbb{V} \in L^\infty \cap C((0,T) \times U; \mathbb{R}^{3\times 3}_{0,sym})$ such that the pair $(\mathbf{u}, \mathbb{V})$ satisfies (6.17a) in the sense of distributions on $(0,T) \times U$ and $\mathbf{u}$ satisfies (6.9),
3. $\mathbf{u}(0) = \mathbf{u}_0$,
4. for every $\tau > 0$ $\inf_{t\in(\tau,T),\, \mathbf{x}\in U} \left(f(t,\mathbf{x}) - \tilde{e}(\mathbf{u}(t,\mathbf{x}), \mathbb{V}(t,\mathbf{x}), \mathbb{G}(t,\mathbf{x}))\right) > 0$.

The following auxiliary result was proven in [27] for I_τ defined using integrals with respect to time and space. In our case, the proof remains the same and we will omit it (see also [23] where the functional setting is the same as ours).

Lemma 6.5.1. *Let $O = (\tau_1, \tau_2) \times U \subseteq Q$ be an open set, $\mathbb{G}$ and f be as above with $f > 0$ in Q. Assume that $\mathbf{u} \in X_{0,f,\mathbb{G}}$. Then exists $\Lambda > 0$ and sequences $\{\mathbf{w}_n\}_{n\in\mathbb{N}} \subseteq \mathcal{D}(O;\mathbb{R}^3)$ and $\{\mathbb{W}_n\}_{n\in\mathbb{N}} \subseteq \mathcal{D}(O;\mathbb{R}^{3\times 3}_{0,sym})$ such that $(\mathbf{u}+\mathbf{w}_n, \mathbb{U}+\mathbb{W}_n) \in X_{0,f,\mathbb{G}}$,*

$$\mathbf{w}_n \to 0 \textit{ in } C([0,T]; L^2_w(U))$$

and

$$\liminf_{n\to\infty} \inf_{t\in(\tau_1,\tau_2)} \int_U |\mathbf{u}+\mathbf{w}_n|^2 \,\mathrm{d}x \geq \inf_{t\in(\tau_1,\tau_2)} \int_U |\mathbf{u}|^2 \,\mathrm{d}\mathbf{x} + \Lambda \left(\inf_{t\in(\tau_1,\tau_2)} \int_U f - \frac{1}{2}|\mathbf{u}|^2 \,\mathrm{d}\mathbf{x} \right)^2, \tag{6.26}$$

where $\Lambda = \Lambda(\sup_{(t,\mathbf{x})\in Q} |f|)$ (namely, Λ does not depend on O or $\mathbf{u}_n$).

Let us show that Lemma 6.4.1 follows from Lemma 6.5.1 using a perturbation argument.

Proof of Lemma 6.4.1. If $\mathbf{u} \in X_0$ then there exists an increasing continuous function $\delta\colon (0,T) \to (0,+\infty)$ such that for any $s \in (0,T)$

$$\inf_{t\in(s,T),\, x\in U} (\bar{e}[\mathbf{u}] - e(\mathbf{u}, \mathbb{V})) > \delta(s).$$

Hence, $\mathbf{u} \in X_{0,\bar{e}[\mathbf{u}]-\delta,\mathbb{H}[\mathbf{u}]}$ and we obtain sequences $\{\mathbf{w}_n\}_{n\in\mathbb{N}}$ and $\{\mathbb{W}_n\}_{n\in\mathbb{N}}$ satisfying Lemma 6.5.1 with $f = \bar{e}[\mathbf{u}] - \delta$ and $\mathbb{G} = \mathbb{H}[\mathbf{u}]$. Moreover, due to the boundedness of $\mathbf{w}_n$ and $\mathbb{W}_n$, we have

$$\mathbf{w}_n \to 0 \text{ in } C([0,T]; L^p_w(U)) \quad \text{for any } p \in [1,\infty),$$

see Lemma 6.7.4. Inequality (6.25) follows directly from (6.26) as $\bar{e}[\mathbf{u}+\mathbf{w}_n] \to \bar{e}[\mathbf{u}]$ uniformly in Q. Hence, to finish the proof, it is sufficient to check that $\mathbf{u}+\mathbf{w}_n \in X_0$ at least for indices large enough. As $\mathbf{u}+\mathbf{w}_n \in X_{0,\bar{e}[\mathbf{u}]-\delta,\mathbb{H}[\mathbf{u}]}$, we get

$$e(\mathbf{u}+\mathbf{w}_n, \mathbb{V}+\mathbb{W}_n) = \tilde{e}(\mathbf{u}+\mathbf{w}_n, \mathbb{V}+\mathbb{W}_n, \mathbb{H}[\mathbf{u}]) + r_n < \bar{e}[\mathbf{u}] - \delta + r_n \tag{6.27}$$
$$= \bar{e}[\mathbf{u}+\mathbf{w_n}] - \delta + r_n + t_n, \tag{6.28}$$

where

$$r_n = \tilde{e}(\mathbf{u}+\mathbf{w}_n, \mathbb{V}+\mathbb{W}_n, \mathbb{H}[\mathbf{u}+\mathbf{w}_n]) - \tilde{e}(\mathbf{u}+\mathbf{w}_n, \mathbb{V}+\mathbb{W}_n, \mathbb{H}[\mathbf{u}])$$

and

$$t_n = \bar{e}[\mathbf{u}] - \bar{e}[\mathbf{u}+\mathbf{w_n}].$$

The function $\mathbb{A} \mapsto \lambda_{max}(\mathbb{A})$ restricted on the symmetric positive semidefinite matrices is equal to $\ell^2 \to \ell^2$ operator norm, hence it is 1–Lipschitz. Thus, using the continuity of $\mathbb{H}$ and $\bar{e}$, we obtain

$$r_n + t_n \to 0 \text{ uniformly in } [\tau, T] \times U.$$

Having in mind (6.15), we claim that $r_n + t_n = 0$ for $(t,\mathbf{x}) \in (0,\tau) \times U$, therefore

$$e(\mathbf{u}+\mathbf{w}_n, \mathbb{V}+\mathbb{W}_n) < \bar{e}[\mathbf{u}+\mathbf{w_n}] - \frac{\delta}{2}$$

holds on Q for sufficiently large n. □

6.6 Dissipative solutions

This Section is devoted to the proofs of Theorems 6.2.2 and 6.2.5. Thanks to the reformulation of the Boussinesq and the extended primitive equations in the framework of abstract Euler–type systems carried out in Section 6.3, Theorems 6.2.2 and 6.2.5 can be reduced to prove the following more general theorem on the abstract system.

Theorem 6.6.1. *Under the same hypotheses on $\mathbb{H}$ and Π of Theorem 6.3.1, there exists $\mathbf{u}_0 \in L^\infty(U;\mathbb{R}^3)$ for which we can find infinitely many weak solutions to (6.13) emanating from $\mathbf{u}_0$ and such that the functional*

$$E_{abs}(t) := \int_U \left(\frac{3}{2}\Pi[\mathbf{u}](t,\mathbf{x}) + \frac{1}{2}|\mathbf{u}(t,\mathbf{x})|^2\right) \mathrm{d}\mathbf{x} \quad \textit{is non–increasing on } [0,T). \tag{6.29}$$

Remark. Thanks to Theorem 6.3.1, and in particular to the property (6.16), the conclusion that the functional $E_{abs}(t)$ is non–increasing on $(0, T)$ can be achieved for any $\mathbf{u}_0 \in L^\infty(U; \mathbb{R}^3) \cap C(U; \mathbb{R}^3)$ with u_0 satisfying (6.9) by simply choosing the function $Z(t)$ to be non–increasing on $(0, T)$. But in order to obtain dissipative solutions for the Boussinesq and primitive equations the property (6.29) is required up to time $t = 0$: this forces the construction of suitable initial data $\mathbf{u}_0 \in L^\infty(U; \mathbb{R}^3)$.

We now show how Theorems 6.2.2 and 6.2.5 follow from Theorem 6.6.1.

Proofs of Theorems 6.2.2 and 6.2.5. Due to the reformulations of the Boussinesq and extended primitive equations of Section 6.3, the respective choices for Π are $\Pi_{Bous}[\mathbf{u}] = \frac{2}{3} z\Theta[\mathbf{u}]$ and $\Pi_{Prim} = 0$. These choices allow to obtain from E_{abs} exactly E_{Bous} and E_{Prim} respectively. Hence the conclusion of Theorem 6.6.1 implies the existence of infinitely many dissipative solutions to the Boussinesq and extended primitive equations starting from suitably constructed initial data $\mathbf{u}_0$, as stated in Theorems 6.2.2 and 6.2.5. □

6.6.1 Construction of initial data

The abstract Euler system (6.13) fits the framework introduced by Feireisl in [33]. In particular, we can apply Theorem 6.1 therein to obtain strong continuity in L^2 at time $t = 0$. For the sake of completeness, we report here a version of [33, Theorem 6.1] adapted to our context. For other variants of the following result we refer to [23] and also to [13], [14].

Lemma 6.6.2. *Let $\mathbb{H}$ and Π satisfy the hypotheses of Theorem 6.3.1. Then there exist a set of times $\mathcal{R} \subset (0, T)$ dense in $(0, T)$ such that for any $\tau \in \mathcal{R}$ there is $\mathbf{u} \in X$ with the following properties*

(i) $\mathbf{u} \in C(((0, \tau) \cup (\tau, T)) \times U) \cap C([0, T], L^2_w)$, $\mathbf{u}(0, \cdot) = \mathbf{0}$,

(ii) there exists $\mathbb{V} \in C(((0, \tau) \cup (\tau, T)) \times U; \mathbb{R}^{3\times 3}_{0,sym})$ *the pair* $(\mathbf{u}, \mathbb{V})$ *satisfies* (6.17a) *in the sense of distributions on* Q *and* $\mathbf{u}$ *satisfies* (6.9),

(iii) $(\bar{e}[\mathbf{u}](t, \mathbf{x}) - e(\mathbf{u}(t, \mathbf{x}), \mathbb{V}(t, \mathbf{x}))) > 0$ *for all* $(t, \mathbf{x}) \in ((0, \tau) \cup (\tau, T)) \times U$,

(iv) $\frac{1}{2}|\mathbf{u}(\tau, \mathbf{x})|^2 = \bar{e}[\mathbf{u}](\tau, \mathbf{x})$ *a.e. in* U

where we recall that

$$\bar{e}[\mathbf{u}](t, \mathbf{x}) = \frac{3}{2}\left(Z(t) - \Pi[\mathbf{u}](t, \mathbf{x})\right)$$

for a continuous function $Z(t)$ *satisfying* $\sup_{t\in[0,T]} Z(t) > \overline{\Pi}$.

Remark. Lemma 6.6.2 provides subsolutions which are strongly continuous at the point τ and allow to obtain the desired strong energy conditions.

For the proof we refer the reader to [33].

Proof of Theorem 6.6.1. The proof consists in finding an initial datum

$$\mathbf{u}_0 \in L^\infty(U;\mathbb{R}^3)$$

and a function $Z(t)$ with the following properties

$$\frac{1}{2}|\mathbf{u}_0(\mathbf{x})|^2 = \bar{e}[\mathbf{u}_0](\mathbf{x}) \quad \text{a.e. in } U; \tag{6.30}$$

- $Z(t)$ continuous on $[0,T]$ and $\sup_{t\in[0,T]} Z(t) > \overline{\Pi}$,
- $Z'(t) \leq 0$ for all $t \in [0,T)$

and such that the set X_0 of subsolutions $\mathbf{u}$ associated to this datum is non–empty. First of all, we notice that we can easily choose $Z(t) = C_Z$ for some constant $C_Z > \overline{\Pi}$. Once Z has been chosen, then we apply Lemma 6.6.2 which provides the existence of a time $\tau \in \mathcal{R}$ and a function $\mathbf{u}$ for which (i)–(iv) hold. We define the initial datum $\mathbf{u}_0$ to be $\mathbf{u}_0(\cdot) = \mathbf{u}(\tau,\cdot)$ in U. To such a datum we associate, as in Section 6.3, the set of subsolutions X_0 and we can prove that it is non–empty by choosing as eligible element the following subsolution

$$\bar{\mathbf{u}}(t,\mathbf{x}) = \begin{cases} \mathbf{u}(t+\tau,\mathbf{x}) & \text{for}\, t \in [0, T-\tau] \\ \mathbf{u}(t-(T-\tau),\mathbf{x}) & \text{for}\, t \in [T-\tau, T] \end{cases}$$

with relative matrix field $\bar{\mathbb{V}}$ analogously defined. Indeed from (6.30), redoing the proof of Theorem 6.3.1, we would now obtain infinitely many solutions to (6.13) emanating from $\mathbf{u}_0$ and such that

$$\left(\frac{3}{2}\Pi[\mathbf{u}](t,\mathbf{x}) + \frac{1}{2}|\mathbf{u}(t,\mathbf{x})|^2\right) = \frac{3}{2}C_Z \quad \text{for all } t \in [0,T) \text{ and a.e. in } U$$

(we remark that the equality now holds up to time $t = 0$) which implies Theorem 6.6.1.

□

6.7 Appendix

6.7.1 The Lamé system

Let us denote

$$\mathbb{D}_0(\mathbf{v}) = \left(\nabla \mathbf{v} + \nabla \mathbf{v}^T - \frac{2}{3}\operatorname{div}_{\mathbf{x}} \mathbf{v}\mathbb{I}\right)$$

and let $\mathbf{g}\colon U \to \mathbb{R}^3$. The Dirichlet boundary value problem for the Lamé system is a whether there exists a function $\mathbf{v}\colon U \to \mathbb{R}^3$ with zero trace on $\partial\Omega$ such that

$$\operatorname{div}_{\mathbf{x}} \mathbb{D}_0(\mathbf{v}) = \mathbf{g}. \tag{6.31}$$

Lemma 6.7.1. *Let $U \subseteq \mathbb{R}^3$, $\partial U \in C^2$, $\mathbf{g} \in L^p(U,\mathbb{R}^3)$ and $p \in (1,\infty)$. Then there exists a unique $\mathbf{v} \in W^{2,p}(U,\mathbb{R}^3)$ with zero trace satisfying* (6.31) *almost everywhere in U and the operator $\mathbf{g} \mapsto \mathbf{v}\colon L^p(U;\mathbb{R}^3) \to W^{2,p}(U;\mathbb{R}^3)$ is continuous.*

Proof. We only show that the elliptic operator in (6.31) satisfies the Legendre-Hadamard conditions. As the operator has constant coefficients, the existence, uniqueness and regularity follows directly from the standard theory of elliptic systems (see for example [46]). Let us denote $A^{\alpha,\beta}_{i,j}$, where α, β, i, $j \in \{1,2,3\}$ the coefficients of the elliptic system (6.31) (for the notation check [46]). Then

$$\sum_{\alpha,\beta,i,j=1}^{3} A^{\alpha,\beta}_{i,j}\xi_\alpha\xi_\beta\eta^i\eta^j = \xi\otimes\eta : \left(\xi\otimes\eta + (\xi\otimes\eta)^T - \frac{2}{3}\xi\cdot\eta\mathbb{I}\right)$$
$$= |\xi|^2|\eta|^2 + \frac{1}{3}|\xi\cdot\eta|^2 \geq |\xi|^2|\eta|^2.$$

□

Corollary 6.7.2. *Let $U \subseteq \mathbb{R}^3$, $\partial U \in C^2$ and $p \in (1,\infty)$. Then there exists a continuous operator $\mathbb{G}\colon L^p(U,\mathbb{R}^3) \to W^{1,p}(U;\mathbb{R}^{3\times3}_{sym,0})$ such that*

$$\mathrm{div}_{\mathbf{x}}(\mathbb{G}[\mathbf{g}])) = \mathbf{g}.$$

6.7.2 Parabolic regularity

The standard regularity result for parabolic equations (see e. g. [1]) gives

$$W^{1,q}\big((0,T);L^q(U)\big)\cap L^q\big((0,T);W^{2,q}(U)\cap W^{1,q}_0(U)\big) \tag{6.32}$$
$$\hookrightarrow C([0,T];W^{1,q}_0(U)) \hookrightarrow C([0,T]\times\overline{U}) \tag{6.33}$$

whenever $q > 3$. Therefore we have:

Lemma 6.7.3. *Assume that $q \in (3,\infty)$. Let $\vartheta_0 \in W^{1,q}_0(U)$ and $\mathbf{u} \in L^\infty(Q;\mathbb{R}^3)$. Then exists a unique*

$$\vartheta \in W^{1,q}((0,T);L^q(U))\cap L^q((0,T);W^{2,q}(U)\cap W^{1,q}_0(U))$$

which satisfies (6.2e) *almost everywhere and $\vartheta(0) = \vartheta_0$. Moreover, the operator $u \mapsto \Theta[\mathbf{u}]$ is continuous from $L^\infty(Q)$ to $C([0,T]\times\overline{U})$ and the comparison principle holds, i. e. for two solutions ϑ^1, ϑ^2 emanating from ϑ^1_0, ϑ^2_0 holds that*

$$\text{if } \vartheta^1_0 \leq \vartheta^2_0 \text{ a. e. in } U \text{ then } \vartheta^1(t,\mathbf{x}) \leq \vartheta^2(t,\mathbf{x}) \text{ a. e. in } Q.$$

Moreover,

$$\|\vartheta\|_{W^{1,q}((0,T);L^q(U))\cap L^q((0,T);W^{2,q}(U)\cap W^{1,q}_0(U))} \leq C(\|\vartheta_0\|_{W^{1,q}} + \|\mathbf{u}\|_{L^\infty}),$$

therefore given $\vartheta_0 \in W^{1,q}$, the solving operator

$$\mathbf{u} \mapsto \Theta[\mathbf{u}]$$

is continuous from $C([0,T];L^q_w(\Omega)$ to $C([0,T]\times\overline{U})$.

6.7.3 Convergence in linear conservation laws

For the reader's convenience, we also recall the following standard weak compactness result for linear conservation laws.

Lemma 6.7.4. *Let* $\{\mathbf{u}_{n,0}\}_{n\in\mathbb{N}}$ *converges weakly-∗ in* $L^\infty(U;\mathbb{R}^3)$ *to* $\mathbf{u}_0$*. Let* $\{\mathbf{u}_n\}_{n\in\mathbb{N}}$ *be a bounded sequence in* $(L^\infty(Q;\mathbb{R}^3))$ *and* $\{\mathbb{V}_n\}_{n\in\mathbb{N}}$ *be a bounded sequence in* $(L^\infty(Q;\mathbb{R}^{3\times 3}_{0,sym}))$ *satisfying*

$$\int_0^T \int_U \mathbf{u}_n \cdot \partial_t \psi + \mathbb{V}_n \nabla \psi \, \mathrm{d}\mathbf{x}\, \mathrm{d}t = 0 \quad \text{for every } \psi \in \mathcal{D}(Q;\mathbb{R}^3), \tag{6.34}$$

$$\int_0^T \int_U \mathbf{u}_n \cdot \nabla_{\mathbf{x}} \phi \, \mathrm{d}\mathbf{x}\, \mathrm{d}t = 0 \quad \text{for every } \phi \in \mathcal{D}(\mathbb{R}^3), \tag{6.35}$$

$$\mathbf{u}_n(0) = \mathbf{u}_{n,0}. \tag{6.36}$$

Then $\{\mathbf{u}_n\}_{n\in\mathbb{N}}$ *is precompact in* $C([0,T];L^p_w(U))$ *for every* $p \in [1,\infty)$*. Moreover, if* $(\mathbf{u},\mathbb{V})$ *is a limit of any weakly-∗ convergent subsequence of* $\{(\mathbf{u}_n,\mathbb{V}_n)\}_{n\in\mathbb{N}}$ *then* $(\mathbf{u},\mathbb{V})$ *satisfies* (6.34)*,* (6.35) *and* $\mathbf{u}(0,\cdot) = \mathbf{u}_0(\cdot)$*.*

Conclusion

The presented results of the thesis extend the nowadays knowledge on the compressible models of fluids and contribute to the recent communication in mathematical physics.

There are a few possible directions for the future development of the presented results. Let us mention at least some which are related to the compressible Navier–Stokes with entropy transport. It is expected that at least some of results which are known for the isentropic compressible Navier–Stokes system can be extended to the case with variable entropy. One achievable goal might be to provide the weak–strong uniqueness for the system with entropy - showing that weak solutions emanating from "regular" initial data are equal to strong solutions as long as the latter exist. Another feasible objective is to synthesize the ideas of Chapter 4 and Chapter 5 and construct a numerical scheme converging to a weak solution of (4.1). The rigorous analysis might be then supplemented by a numerical simulation.

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