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THE TWO-WEIGHT HARDY INEQUALITY: A NEW ELEMENTARY AND UNIVERSAL PROOF

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Abstract. We give a new proof of the known criteria for the inequality

$$\left(\int_0^\infty \left(\int_0^t f\right)^q w(t)\,dt\right)^{\frac{1}{q}} \leq C\left(\int_0^\infty f^p v\right)^{\frac{1}{p}}.$$

The innovation is in the elementary nature of the proof and its versatility.

1. Introduction

Consider the two-weight Hardy inequality

$$\left(\int_0^\infty \left(\int_0^t f\right)^q w(t)\,dt\right)^{\frac{1}{q}} \leq C\left(\int_0^\infty f^p v\right)^{\frac{1}{p}}, \tag{1.1}$$

in which C is a positive constant independent of a nonnegative measurable function f on $(0,\infty)$, v and w are fixed nonnegative measurable functions on $(0,\infty)$ (weights), $p\in[1,\infty)$, and $q\in(0,\infty)$. The requirement $p\in[1,\infty)$ is reasonable since for $p\in(0,1)$ there are functions in weighted L^p which are not locally integrable.

The problem of characterizing pairs of weights for which (1.1) is true has a long and rich history and it would be impossible to mention here every contribution. For $p=q>1$, $v=1$, $w(t)=t^{-q}$ and $C=p'$, it is just the boundedness of the integral averaging operator on $L^p(0,\infty)$, a result almost one century old, which appears in classical Hardy's papers in 1920's, see [5]. The beginning of investigation of a general weighted case goes back to 1950's, and it starts with the paper by Kac and Krein [6] in which a characterization for $p=q=2$ and $v=1$ can be found. In 1950's and 1960's, plenty of partial results were obtained by Beesack, see e.g. [1]. In late 1960's and in 1970's, a boom in the so-called *convex case* ($p\leq q$, named after the convexity of $t\mapsto t^{\frac{q}{p}}$) was seen. For $p=q$, a characterization was obtained by Tomaselli [15], Talenti [14] and Muckenhoupt [9]. It was extended to $p\leq q$ by Bradley [4], the same result is also stated without proof in [7]. Many authors referred further to an untitled and unpublished manuscript by Artola, and in [10], a paper by D.W. Boyd and J.A. Erdős was quoted, which most likely was never published. In any case, (1.1) holds if and only if

$$\sup_{t\in(0,\infty)}\left(\int_t^\infty w\right)^{\frac{1}{q}}\left(\int_0^t v^{1-p'}\right)^{\frac{1}{p'}}<\infty \quad \text{for } 1<p\leq q$$

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and

$$\sup_{t\in(0,\infty)} \left(\int_t^\infty w\right)^{\frac{1}{q}} \operatorname*{ess\,sup}_{s\in(0,t)} \frac{1}{v(s)} < \infty \quad \text{for } 1 = p \le q.$$

Here and throughout, if $p \in (0,\infty]$, then p' denotes the conjugate exponent defined by $\frac{1}{p}+\frac{1}{p'} = 1$. Observe that 1 and ∞ are conjugate exponents and that p' is negative when $p \in (0,1)$.

The non-convex case ($p > q$) turned out to be more difficult to handle, and it had to wait till 1980's and 1990's for appropriate treatment. The first characterization, for $1 \le q < p < \infty$, was obtained by Maz'ya and Rozin, see [8], who proved that a necessary and sufficient condition is

$$\int_0^\infty \left(\int_t^\infty w\right)^{\frac{r}{q}} \left(\int_0^t v^{1-p'}\right)^{\frac{r}{q'}} v(t)^{1-p'}\,dt < \infty,$$

where $r = \frac{pq}{p-q}$. A universal characterization, sheltering both the convex and the non-convex cases and involving more general norms was obtained by Sawyer [11], but the condition in the non-convex case is expressed in terms of a discretized condition. While discretization techniques proved later to be of colossal theoretical importance, conditions expressed in terms of discretizing sequences are difficult to verify. Later, Sinnamon [12] characterized the inequality for $0 < q < 1 < p < \infty$. The criterion turns out to be the same as that of Maz'ya and Rozin but the proof, based on Halperin's level function, is very different. The case $0 < q < p = 1$ was treated by Sinnamon and Stepanov [13], who moreover observed that, unless $p = 1$, Sinnamon's and Mazya-Rozin's results can be proved in a unified manner. The case $p = 1$, however, still required separate treatment. In [3], restriction of (1.1) to the cone of non-increasing functions is studied, together with its discrete version. Some ideas developed there are useful also for the unrestricted case.

In this note we present a short, uniform and elementary proof, in which

- all cases are covered,
- $p > 1$ is not separated from $p = 1$,
- only Fubini's theorem, Hölder's inequality, Minkowski's integral inequality and Hardy's lemma are used.

2. The theorem and its proof

Theorem 2.1. *Let v, w be weights on $(0,\infty)$, $p \in [1,\infty)$ and $q \in (0,\infty)$. For $t \in (0,\infty)$, denote*

$$V(t) = \begin{cases} \left(\int_0^t v^{1-p'}\right)^{\frac{1}{p'}} & \text{if } p \in (1,\infty), \\ \operatorname{ess\,sup}_{s\in(0,t)} \frac{1}{v(s)} & \text{if } p = 1, \end{cases}$$

and

$$W(t) = \int_t^\infty w.$$

Then there exists a positive constant C such that (1.1) *holds for every nonnegative measurable function f on $(0,\infty)$ if and only if $A < \infty$, where*

$$A = \begin{cases} \sup\limits_{t\in(0,\infty)} V(t)W(t)^{\frac{1}{q}} & \text{if } p \le q, \\ \int_0^\infty W^{\frac{p}{p-q}}\,dV^{\frac{pq}{p-q}} & \text{if } p > q, \end{cases}$$

in which the latter integral should be understood in the Lebesgue–Stieltjes sense with respect to the (monotone) function $V^{\frac{pq}{p-q}}$.

Proof. *Sufficiency.* Fix $\varepsilon \in (0,1)$. We claim that, for every nonnegative measurable function f on $(0,\infty)$, one has

$$\int_0^t f \lesssim \left(\int_0^t f^p V^{\varepsilon p} v\right)^{\frac{1}{p}} V(t)^{1-\varepsilon} \quad \text{for } t>0. \tag{2.1}$$

(We write $\lesssim$ when the expression to the left of it is majorized by a constant times that on the right.) To show (2.1), fix $t \in (0,\infty)$. If $p \in (1,\infty)$, then, by Hölder's inequality,

$$\int_0^t f = \int_0^t f V^{\varepsilon} v^{\frac{1}{p}} V^{-\varepsilon} v^{-\frac{1}{p}} \le \left(\int_0^t f^p V^{\varepsilon p} v\right)^{\frac{1}{p}} \left(\int_0^t V^{-\varepsilon p'} v^{1-p'}\right)^{\frac{1}{p'}}.$$

By a change of variables, we obtain

$$\int_0^t V^{-\varepsilon p'} v^{1-p'} = \int_0^t \left(\int_0^s v^{1-p'}\right)^{-\varepsilon} v^{1-p'}\, ds = \frac{1}{1-\varepsilon}\left(\int_0^s v^{1-p'}\right)^{1-\varepsilon} = \frac{1}{1-\varepsilon} V(t)^{(1-\varepsilon)p'},$$

hence

$$\int_0^t f \lesssim \left(\int_0^t f^p V^{\varepsilon p} v\right)^{\frac{1}{p}} V(t)^{1-\varepsilon},$$

and (2.1) follows. If $p=1$, then we get (2.1) from

$$\int_0^t f = \int_0^t f v^{-\varepsilon} v v^{-1+\varepsilon} \le \left(\int_0^t f V^{\varepsilon} v\right) V(t)^{1-\varepsilon}.$$

Let $p \le q$. Then $A<\infty$ implies $V \lesssim W^{-\frac{1}{q}}$. Using this and (2.1), we get

$$\int_0^t f \lesssim \left(\int_0^t f^p W^{-\frac{\varepsilon p}{q}} v\right)^{\frac{1}{p}} W(t)^{\frac{\varepsilon-1}{q}} \quad \text{for } t>0.$$

Raising to q and integrating with respect to $w(t)\,dt$, we obtain

$$\int_0^\infty \left(\int_0^t f\right)^q w(t)\,dt \lesssim \int_0^\infty \left(\int_0^t f(s)^p W(s)^{-\frac{\varepsilon p}{q}} v(s)\,ds\right)^{\frac{q}{p}} W(t)^{\varepsilon-1} w(t)\,dt.$$

Next we apply Minkowski's integral inequality (note that $\frac{q}{p} \ge 1$ and all the expressions in the play are nonnegative) in the form

$$\int_0^\infty \left(\int_0^\infty F(s,t)\,d\mu_1(s)\right)^{\frac{q}{p}} d\mu_2(t) \le \left(\int_0^\infty \left(\int_0^\infty F(s,t)^{\frac{q}{p}}\,d\mu_2(t)\right)^{\frac{p}{q}} d\mu_1(s)\right)^{\frac{q}{p}},$$

in which $F(s,t)=\chi_{(0,t)}(s)f(s)^p$, χ denotes the characteristic function, $d\mu_1(s)=W(s)^{-\frac{\varepsilon p}{q}}v(s)ds$ and $d\mu_2(t)=W(t)^{\varepsilon-1}w(t)\,dt$. We thus obtain

$$\begin{aligned}&\int_0^\infty\left(\int_0^t f(s)^pW(s)^{-\frac{\varepsilon p}{q}}v(s)\,ds\right)^{\frac{q}{p}}W(t)^{\varepsilon-1}w(t)\,dt\\ &\le\left(\int_0^\infty f(s)^pW(s)^{-\frac{\varepsilon p}{q}}v(s)\left(\int_s^\infty W(t)^{\varepsilon-1}w(t)\,dt\right)^{\frac{p}{q}}ds\right)^{\frac{q}{p}}\\ &\approx\left(\int_0^\infty f^pv\right)^{\frac{q}{p}}.\end{aligned}$$

(We write $\approx$ when both $\lesssim$ and $\gtrsim$ apply.) Altogether, we arrive at

$$\int_0^\infty\left(\int_0^t f\right)^q w(t)\,dt\lesssim\left(\int_0^\infty f^pv\right)^{\frac{q}{p}},$$

and (1.1) follows.

Let $p>q$. Fix $\alpha\in(0,\infty)$. We shall use the symbol $V(\infty)$ for $\lim_{t\to\infty}V(t)$ (this limit always exists, either finite or infinite, owing to the monotonicity of V). By (2.1),

$$\begin{aligned}\int_0^\infty\left(\int_0^t f\right)^q w(t)\,dt&\lesssim\int_0^\infty\left(\int_0^t f^pV^{\varepsilon p}v\right)^{\frac{q}{p}}V(t)^{-\alpha q}V(t)^{(1-\varepsilon+\alpha)q}w(t)\,dt\\ &\lesssim\int_0^\infty\left(\int_0^t f^pV^{\varepsilon p}v\right)^{\frac{q}{p}}\left(V(t)^{-\alpha p}-V(\infty)^{-\alpha p}\right)^{\frac{q}{p}}V(t)^{(1-\varepsilon+\alpha)q}w(t)\,dt\\ &+\int_0^\infty\left(\int_0^t f^pV^{\varepsilon p}v\right)^{\frac{q}{p}}V(t)^{(1-\varepsilon+\alpha)q}w(t)\,dt\cdot V(\infty)^{-\alpha q}=I+II.\end{aligned}$$

If $V(\infty)=\infty$, one has $II=0$. Since

$$V(t)^{(1-\varepsilon+\alpha)q}\approx\int_0^t V^{(1-\varepsilon+\alpha)q-\frac{pq}{p-q}}d(V^{\frac{pq}{p-q}})\quad\text{for }t>0$$

and

$$V(t)^{-\alpha p}-V(\infty)^{-\alpha p}=\int_t^\infty d(-V^{-\alpha p})\quad\text{for }t>0,$$

monotonicity and Fubini's theorem yield

$$\begin{aligned}I&\lesssim\int_0^\infty\left(\int_t^\infty\left(\int_0^s f^pV^{\varepsilon p}v\right)d(-V^{-\alpha p})(s)\right)^{\frac{q}{p}}\left(\int_0^t V^{(1-\varepsilon+\alpha)q-\frac{pq}{p-q}}\,dV^{\frac{pq}{p-q}}\right)w(t)\,dt\\ &\lesssim\int_0^\infty\left(\int_0^t\left(\int_s^\infty\left(\int_0^\tau f^pV^{\varepsilon p}v\right)d(-V^{-\alpha p})(\tau)\right)^{\frac{q}{p}}V(s)^{(1-\varepsilon+\alpha)q-\frac{pq}{p-q}}dV^{\frac{pq}{p-q}}(s)\right)w(t)\,dt\\ &=\int_0^\infty\left(\int_s^\infty\left(\int_0^\tau f^pV^{\varepsilon p}v\right)d(-V^{-\alpha p})(\tau)\right)^{\frac{q}{p}}V(s)^{(1-\varepsilon+\alpha)q-\frac{pq}{p-q}}W(s)dV^{\frac{pq}{p-q}}(s).\end{aligned}$$

Thus, owing to $A<\infty$, Hölder's inequality, and Fubini's theorem,

$$I \lesssim \left(\int_0^\infty W^{\frac{p}{p-q}} dV^{\frac{pq}{p-q}}\right)^{\frac{p-q}{p}} \left(\int_0^\infty \left(\int_s^\infty \left(\int_0^\tau f^p V^{\varepsilon p} v\right) d(-V^{-\alpha p})(\tau)\right) V(s)^{(1-\varepsilon+\alpha)p-\frac{p^2}{p-q}} dV^{\frac{pq}{p-q}}(s)\right)^{\frac{q}{p}}$$

$$\lesssim \left(\int_0^\infty \left(\int_0^\tau f^p V^{\varepsilon p} v\right) \left(\int_0^\tau V^{(1-\varepsilon+\alpha)p-\frac{p^2}{p-q}} dV^{\frac{pq}{p-q}}\right) d(-V^{-\alpha p})(\tau)\right)^{\frac{q}{p}}$$

$$\approx \left(\int_0^\infty \left(\int_0^\tau f^p V^{\varepsilon p} v\right) V(\tau)^{(\alpha-\varepsilon)p} d(-V^{-\alpha p})(\tau)\right)^{\frac{q}{p}}$$

$$= \left(\int_0^\infty f(y)^p V(y)^{\varepsilon p} v(y) \int_y^\infty V^{(\alpha-\varepsilon)p} d(-V^{-\alpha p}) dy\right)^{\frac{q}{p}} \approx \left(\int_0^\infty f^p v\right)^{\frac{q}{p}}.$$

If $V(\infty)<\infty$, we have

$$II \le \int_0^\infty \left(\int_0^t f^p v\right)^{\frac{q}{p}} V(t)^{(1+\alpha)q} w(t) dt \cdot V(\infty)^{-\alpha q} \le \left(\int_0^\infty f^p v\right)^{\frac{q}{p}} \left(\int_0^\infty V^{(1+\alpha)q} w\right) V(\infty)^{-\alpha q}.$$

Owing to $A<\infty$, Fubini's theorem, and Hölder's inequality, we get

$$\int_0^\infty V^{(1+\alpha)q} w \approx \int_0^\infty \left(\int_0^t V^{\alpha q+q-p'} v^{1-p'}\right) w(t)\, dt = \int_0^\infty V^{\alpha q+q-p'} v^{1-p'} W$$

$$\lesssim \left(\int_0^\infty V^{\alpha p-p'} v^{1-p'}\right)^{\frac{q}{p}} \left(\int_0^\infty W(t)^{\frac{p}{p-q}}\, dV^{\frac{pq}{p-q}}\right)^{\frac{p-q}{p}} \lesssim V(\infty)^{\alpha q},$$

establishing $II \lesssim \left(\int_0^\infty f^p v\right)^{\frac{q}{p}}$. This shows sufficiency.

Necessity. Let $p\le q$ and assume that (1.1) holds. Fix $t\in(0,\infty)$. Then

$$\int_0^\infty \left(\int_0^s f\right)^q w(s)\, ds \ge \int_t^\infty \left(\int_0^s f\right)^q w(s)\, ds \ge W(t) \left(\int_0^t f\right)^q.$$

Therefore, (1.1) yields

$$C \ge W(t)^{\frac{1}{q}} \sup_{f\ge 0} \frac{\int_0^t f}{\left(\int_0^\infty f^p v\right)^{\frac{1}{p}}}. \tag{2.2}$$

We claim that

$$\sup_{f\ge 0} \frac{\int_0^t f}{\left(\int_0^\infty f^p v\right)^{\frac{1}{p}}} = V(t). \tag{2.3}$$

Indeed, if $p>1$, then we have, by Hölder's inequality,

$$\int_0^t f = \int_0^t f v^{\frac{1}{p}} v^{-\frac{1}{p}} \le \left(\int_0^t f^p v\right)^{\frac{1}{p}} \left(\int_0^t v^{-\frac{p'}{p}}\right)^{\frac{1}{p'}} \le \left(\int_0^\infty f^p v\right)^{\frac{1}{p}} V(t)$$

for every measurable $f\ge 0$. On the other hand, this inequality is saturated by the choice $f=v^{1-p'}\chi_{(0,t)}$, since $f^p v = f$, and, consequently,

$$\int_0^t f = \left(\int_0^t f^p v\right)^{\frac{1}{p}} \left(\int_0^t f^p v\right)^{\frac{1}{p'}} = \left(\int_0^\infty f^p v\right)^{\frac{1}{p}} V(t).$$

If $p=1$, then we, once again, obtain

$$\int_0^t f=\int_0^t fvv^{-1}\le V(t)\int_0^t fv\le V(t)\int_0^\infty fv$$

for every measurable $f\ge 0$. In order to saturate this inequality, fix any $\lambda<V(t)$. Then there exists a set $E\subset(0,t)$ of positive measure such that $\frac{1}{v}\ge\lambda$ on E. Set $f=\frac{\chi_E}{v}$. Then

$$\int_0^t f=\int_E\frac{1}{v}\ge\lambda|E|=\lambda\int_0^\infty fv.$$

On letting $\lambda\to V(t)_-$, we get

$$\int_0^t f\ge V(t)\int_0^\infty fv.$$

In any case, (2.3) follows. Since t was arbitrary, plugging (2.3) into (2.2) yields

$$C\ge\sup_{t\in(0,\infty)}W(t)^{\frac{1}{q}}\sup_{f\ge 0}\frac{\int_0^t f}{\left(\int_0^\infty f^pv\right)^{\frac{1}{p}}}=\sup_{t\in(0,\infty)}W(t)^{\frac{1}{q}}V(t),$$

establishing $A<\infty$.

Let $p>q$ and $p>1$, denote $r=\frac{pq}{p-q}$ and $B=\int_0^\infty V^rW^{\frac{r}{p}}w$. Let $\theta\in(\frac{r}{p'},\infty)$ and set

$$f(t)=\left(\int_t^\infty W^{\frac{r}{p}}wV^{r-\theta p'}\right)^{\frac{1}{p}}V(t)^{(\theta-1)(p'-1)}v(t)^{1-p'}\quad\text{for } t>0.$$

By Fubini's theorem,

$$\begin{aligned}\int_0^\infty f^pv&=\int_0^\infty\left(\int_t^\infty W^{\frac{r}{p}}wV^{r-\theta p'}\right)V(t)^{(\theta-1)p'}v(t)^{1-p'}dt\\&=\int_0^\infty W(s)^{\frac{r}{p}}w(s)V(s)^{r-\theta p'}\left(\int_0^s V^{(\theta-1)p'}v^{1-p'}\right)ds\approx B.\end{aligned}$$

On the other hand, by monotonicity,

$$\begin{aligned}&\int_0^\infty\left(\int_0^t f\right)^q w(t)dt\ge\int_0^\infty\left(\int_0^t V^{(\theta-1)(p'-1)}v^{1-p'}\right)^q\left(\int_t^\infty W^{\frac{r}{p}}wV^{r-\theta p'}\right)^{\frac{q}{p}}w(t)dt\\&\ge\int_0^\infty\left(\int_0^t V^{(\theta-1)(p'-1)+\frac{r}{p}-\frac{\theta p'}{p}}v^{1-p'}\right)^q\left(\int_t^\infty W^{\frac{r}{p}}w\right)^{\frac{q}{p}}w(t)dt\approx B.\end{aligned}$$

Altogether, (1.1) implies $B^{\frac{1}{q}}\lesssim B^{\frac{1}{p}}$. Using a standard approximation argument, we obtain $B^{\frac{1}{r}}<\infty$, hence $B<\infty$. Since $A\approx B$ owing to integration by parts, we get $A<\infty$.

Finally, let $p=1$ and $p>q$. Fix some $\sigma>1$ and define

$$E_k=\{t\in(0,\infty):\sigma^k<V(t)\le\sigma^{k+1}\}\quad\text{for } k\in\mathbb{Z}.$$

Set $\mathbb{A}=\{k\in\mathbb{Z}:E_k\ne\emptyset\}$. Then $(0,\infty)=\bigcup_{k\in\mathbb{A}}E_k$, in which the union is disjoint and each E_k is a nondegenerate interval (which could be either open or closed at each end) with endpoints a_k and b_k, $a_k<b_k$. For every $k\in\mathbb{A}$, we find $\delta_k>0$ so that $a_k+\delta_k<b_k$ and

$$\int_{a_k}^{b_k}W^{\frac{q}{1-q}}w\le\sigma\int_{a_k+\delta_k}^{b_k}W^{\frac{q}{1-q}}w,\tag{2.4}$$

which is clearly possible, and then we define the set

$$G_k = \left\{ t \in (a_k, a_k + \delta_k) : \frac{1}{v(t)} > \sigma^k \right\}.$$

Since V is non-decreasing and left-continuous, $|G_k| > 0$ for every $k \in \mathbb{A}$. Set $h = \sum_{k \in \mathbb{A}} \frac{\chi_{G_k}}{|G_k|}$. Then, for every $k \in \mathbb{A}$, one has

$$\int_0^{a_k+\delta_k} hv^{-1}V^{\frac{q}{1-q}} \geq \int_{a_k}^{a_k+\delta_k} hv^{-1}V^{\frac{q}{1-q}} = \frac{1}{|G_k|}\int_{G_k} V^{\frac{q}{1-q}}v^{-1} \geq \sigma^{\frac{k}{1-q}}. \tag{2.5}$$

Fix $t \in (0,\infty)$. Then there is a uniquely defined $k \in \mathbb{A}$ such that $t \in (a_k, b_k]$. Consequently,

$$\int_0^t hV^{\frac{q}{1-q}} \leq \sum_{j\in\mathbb{A},\ j\leq k} \frac{1}{|G_j|}\int_{G_j} V^{\frac{q}{1-q}} \leq \sum_{j=-\infty}^{k} \sigma^{\frac{q(j+1)}{1-q}} = \frac{\sigma^{\frac{q(k+2)}{1-q}}}{\sigma^{\frac{q}{1-q}}-1}.$$

On the other hand,

$$\int_0^t dV^{\frac{q}{1-q}} \geq \int_0^{a_k} dV^{\frac{q}{1-q}} = V(a_k)^{\frac{q}{1-q}} \geq \sigma^{\frac{qk}{1-q}}.$$

The last two estimates yield

$$\int_0^t hV^{\frac{q}{1-q}} \lesssim \int_0^t dV^{\frac{q}{1-q}} \quad \text{for } t > 0. \tag{2.6}$$

Since $W^{\frac{1}{1-q}}$ is non-increasing, we can apply Hardy's lemma (whose version for Lebesgue integrals can be found in [2, Chapter 2, Proposition 3.6] - note that the proof presented there works verbatim for Lebesgue–Stieltjes integrals) to (2.6) and get

$$\int_0^\infty hV^{\frac{q}{1-q}}W^{\frac{1}{1-q}} \lesssim \int_0^\infty W^{\frac{1}{1-q}}dV^{\frac{q}{1-q}}. \tag{2.7}$$

Finally, using subsequently integration by parts, decomposition of $(0,\infty)$ into $\bigcup_{k\in\mathbb{A}} E_k$, the definition of E_k, the fact that each E_k is an interval with endpoints a_k, b_k, (2.4), (2.5), monotonicity of functions given by integrals, (1.1) applied to $p=1$ and $f = hv^{-1}V^{\frac{q}{1-q}}W^{\frac{1}{1-q}}$, and (2.7), we get

$$\begin{aligned}
&\int_0^\infty W^{\frac{1}{1-q}}dV^{\frac{q}{1-q}} \leq 2\int_0^\infty V^{\frac{q}{1-q}}W^{\frac{q}{1-q}}w = 2\sum_{k\in\mathbb{A}}\int_{E_k} V^{\frac{q}{1-q}}W^{\frac{q}{1-q}}w \lesssim \sum_{k\in\mathbb{A}} \sigma^{\frac{(k+1)q}{1-q}} \int_{E_k} W^{\frac{q}{1-q}}w \\
&\lesssim \sum_{k\in\mathbb{A}} \sigma^{\frac{(k+1)q}{1-q}} \int_{a_k+\delta_k}^{b_k} W^{\frac{q}{1-q}}w \lesssim \sum_{k\in\mathbb{A}} \left(\int_0^{a_k+\delta_k} hv^{-1}V^{\frac{q}{1-q}}\right)^q \int_{a_k+\delta_k}^{b_k} W^{\frac{q}{1-q}}w \\
&\lesssim \sum_{k\in\mathbb{A}} \int_{a_k+\delta_k}^{b_k} \left(\int_0^t hv^{-1}V^{\frac{q}{1-q}}\right)^q W(t)^{\frac{q}{1-q}}w(t)\,dt \lesssim \int_0^\infty \left(\int_0^t hv^{-1}V^{\frac{q}{1-q}}W^{\frac{1}{1-q}}\right)^q w(t)\,dt \\
&\lesssim \left(\int_0^\infty hV^{\frac{q}{1-q}}W^{\frac{1}{1-q}}\right)^q \lesssim \left(\int_0^\infty W^{\frac{1}{1-q}}\,dV^{\frac{q}{1-q}}\right)^q,
\end{aligned}$$

in which the multiplicative constants depend only on C and q. This establishes, via a standard approximation argument, that $A^{1-q} < \infty$, which in turn yields $A < \infty$. The proof is complete. □

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