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SINGULAR EQUIVALENCES TO LOCALLY COHERENT HEARTS OF COMMUTATIVE NOETHERIAN RINGS

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Abstract. We show that Krause's recollement exists for any locally coherent Grothendieck category such that its derived category is compactly generated. As a source of such categories, we consider the hearts of intermediate and restrictable t-structures in the derived category of a commutative noetherian ring. We show that the induced tilting objects in these hearts give rise to an equivalence between the two Krause's recollements, and in particular to a singular equivalence.

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Introduction

It has been known since the dawn of derived categories that the resolutions of unbounded complexes are more delicate then their bounded counterparts. Namely, to be able to lift computations to the homotopy category, it does not in general suffice to replace an unbounded complex by a quasi-isomorphic complex with injective components. The correct concept is that of a dg-injective resolution, as explained by Spaltenstein [42]. Nevertheless, a decade and a half later Krause showed that the homotopy category $\mathsf{K}(\mathrm{Inj})$ of complexes with injective components deserves an attention of its own. In [19], Krause showed that if $\mathcal{G}$ is a locally noetherian Grothendieck category such that its derived category is compactly generated (e.g. the category of quasi-coherent sheaves over a noetherian scheme) then $\mathsf{K}(\mathsf{Inj}(\mathcal{G}))$ is a compactly generated category, and the Verdier localization functor $Q : \mathsf{K}(\mathsf{Inj}(\mathcal{G})) \to \mathsf{D}(\mathcal{G})$ gives rise to a *recollement* $\mathsf{K}_{\mathsf{ac}}(\mathsf{Inj}(\mathcal{G})) \leftrightarrows \mathsf{K}(\mathsf{Inj}(\mathcal{G})) \leftrightarrows \mathsf{D}(\mathcal{G})$. Here, the full subcategory $\mathsf{K}_{\mathsf{ac}}(\mathsf{Inj}(\mathcal{G}))$ of $\mathsf{K}(\mathsf{Inj}(\mathcal{G}))$ consisting of acyclic complexes is called the *stable derived category* of $\mathcal{G}$ in [19]. An important point is that the recollement renders $\mathsf{K}_{\mathsf{ac}}(\mathsf{Inj}(\mathcal{G}))$ compactly generated as well, and in fact its compact objects form a category equivalent (up to retracts) to the *singularity category* $\mathsf{D}^{\mathsf{sg}}(\mathcal{G}) = \mathsf{D}^b(\mathsf{fp}(\mathcal{G}))/\mathsf{D}^c(\mathcal{G})$, an important concept introduced by Buchweitz in order to measure the failure of a scheme to be regular.

Another decade later, Šťovíček in [43] showed that we obtain a similar picture if we replace the locally noetherian condition by a much more general one of being locally coherent. This generalization required employment of very different approach, including model category techniques, viewing $\mathsf{K}(\mathsf{Inj}(\mathcal{G}))$ as the *coderived category* of $\mathcal{G}$ in the sense of Becker [4]. To obtain the Krause's recollement in the locally coherent setting however, an additional hypothesis was used in [43] — $\mathcal{G}$

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is assumed to have a set of finitely presented generators of finite Yoneda projective dimension. The first aim of the present paper is to show that this assumption is unnecessary and the Krause's recollement exists for any locally coherent Grothendieck category with compactly generated derived category (Theorem 2.14). In particular, we define the singularity category of these Grothendieck categories. We also show that the vanishing of the singularity category of a separated coherent scheme admits the expected homological interpretation (Corollary 2.23).

We then proceed to consider a specific source of such Grothendieck categories — the hearts of restrictable t-structures in the unbounded derived category $\mathsf{D}(\mathsf{Mod}\text{-}R)$ of a commutative noetherian ring R. These categories might fail to satisfy the hypothesis used in [43] (Example 3.8) and also, such hearts are rarely locally noetherian [23, Proposition 5.6]. Any such heart $\mathcal{H}$ is induced by a cotilting object of $\mathsf{D}(\mathsf{Mod}\text{-}R)$, and its derived category is known to be triangle equivalent to $\mathsf{D}(\mathsf{Mod}\text{-}R)$. We show that there is also an equivalence between the coderived categories of $\mathcal{H}$ and $\mathsf{Mod}\text{-}R$. In fact, we construct an equivalence between Krause's recollements of the two locally coherent Grothendieck categories (Theorem 4.10). In particular, we obtain an equivalence between their singularity and stable derived categories (Corollary 3.4, Theorem 4.10).

We remark that there is no shortage of the t-structures satisfying our assumptions (see Remark 1.7); for example, whenever R admits a dualizing complex then the *Cohen-Macaulay t-structure* in $\mathsf{D}^b(\mathsf{mod}\text{-}R)$ in the sense of [1] extends to such a t-structure in $\mathsf{D}(\mathsf{Mod}\text{-}R)$. In the literature (see e.g. [17, Lemma 4.1]), singular equivalences are found to be induced by derived equivalences between the bounded derived categories of coherent objects over schemes or rings. In our situation however, the derived equivalences come from the "large" tilting theory as developed in [34], [29], and [44]. This forces us to use different techniques to obtain the singular equivalence, including working with an enhancement in the form of stable derivators.

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1. Preliminaries

1.1. **Compact objects in triangulated categories.** A major role in our discussion will be played by compact objects.

Definition 1.1. Let $\mathcal{T}$ be a triangulated category. An object $C \in \mathcal{T}$ is said to be **compact** if, for every family $(X_i \mid i \in I)$ of objects whose coproduct exists in $\mathcal{T}$, the canonical morphism

$$\coprod_{i\in I} \mathsf{Hom}_{\mathcal{T}}(C, X_i) \to \mathsf{Hom}_{\mathcal{T}}(C, \coprod_{i\in I} X_i)$$

is an isomorphism. The full subcategory of compact objects of $\mathcal{T}$ (which is a thick subcategory) will be denoted by $\mathcal{T}^c$. If $\mathcal{T}$ has all coproducts, it is said to be **compactly generated** if it coincides with its smallest triangulated subcategory closed under coproducts and containing $\mathcal{T}^c$.

We will often employ a *dévissage* argument, which is a standard tool. For the convenience of the reader, we spell out once the application we will use the most.

Lemma 1.2 (Double *dévissage*)**.** *Let $\mathcal{T}, \mathcal{S}$ be compactly generated triangulated categories, and $F\colon \mathcal{T} \to \mathcal{S}$ a triangle functor. Assume that F preserves coproducts, and that it restricts to an equivalence $\mathcal{T}^c \to \mathcal{S}^c$. Then F is an equivalence.*

Proof. We first prove that F is fully faithful. For every X, Y in $\mathcal{T}$, consider the natural map

$$\eta_{X,Y}\colon\ \mathsf{Hom}_{\mathcal{T}}(X, Y) \to \mathsf{Hom}_{\mathcal{S}}(FX, FY)$$

induced by F. Let $\mathcal{Y} \subseteq \mathcal{T}$ be the full subcategory of the objects Y such that $\eta_{C,Y}$ is an isomorphism for every $C \in \mathcal{T}^c$. It is easily seen to be triangulated; moreover, since F preserves coproducts and the objects C and FC are compact in $\mathcal{T}$ and $\mathcal{S}$ respectively, $\mathcal{Y}$ is also closed under coproducts. By hypothesis, $\mathcal{T}^c \subseteq \mathcal{Y}$, and therefore $\mathcal{Y} = \mathcal{T}$. Now, let $\mathcal{X} \subseteq \mathcal{T}$ be the full subcategory of the objects X such that $\eta_{X,Y}$ is an isomorphism for every $Y \in \mathcal{T}$. Again, it is triangulated; this time it is also automatically closed under coproducts, and by the previous discussion $\mathcal{T}^c \subseteq \mathcal{X}$. We conclude that $\mathcal{X} = \mathcal{T}$, i.e. that F is fully faithful. Now, consider the essential image of F in $\mathcal{S}$. Since F is a full triangle functor, its image is a triangulated subcategory (fullness is needed for closure under extensions). Moreover, it is also closed under coproducts, because F preserves them, and contains $\mathcal{S}^c$, by hypothesis. We deduce that F is also essentially surjective, i.e. an equivalence. □

1.2. ***t*-structures.** [6] Let $\mathcal{T}$ be a triangulated category. A pair $\mathbb{T} = (\mathcal{U}, \mathcal{V})$ of full subcategories of $\mathcal{T}$ is a ***t*-structure** provided that the following axioms hold:

(t-1) $\mathsf{Hom}_{\mathcal{T}}(\mathcal{U}, \mathcal{V}) = 0$,
(t-2) $\mathcal{U}$ is closed under the suspension functor, and
(t-3) for any $X \in \mathcal{T}$ there is a triangle $U \to X \to V \to U[1]$ with $U \in \mathcal{U}$ and $V \in \mathcal{V}$.

We call the subcategory $\mathcal{U}$ (resp. $\mathcal{V}$) the **aisle** (resp. the **coaisle**) of the t-structure $\mathbb{T}$. It follows from the axioms that $\mathcal{U} = {}^{\perp_0}\mathcal{V} = \{X \in \mathcal{T} \mid \mathsf{Hom}_{\mathcal{T}}(X, \mathcal{V}) = 0\}$ and $\mathcal{V} = \mathcal{U}^{\perp_0}$, and so any t-structure is uniquely determined by its aisle or by its coaisle. The triangle from the axiom (t-3) is unique and functorial. In fact, the triangle is isomorphic to a triangle of the form $\tau_{\mathcal{U}}(X) \to X \to \tau_{\mathcal{V}}(X) \to \tau_{\mathcal{U}}(X)$, where $\tau_{\mathcal{U}} : \mathcal{T} \to \mathcal{U}$ (resp. $\tau_{\mathcal{U}} : \mathcal{T} \to \mathcal{U}$) is the right (resp. left) adjoint to the inclusion of the aisle (resp. coaisle) into $\mathcal{T}$. The **heart** of the t-structure $\mathbb{T}$ is defined as $\mathcal{H} = \mathcal{U} \cap \mathcal{V}[1]$ and it is an abelian category with the exact structure induced by the triangles of $\mathcal{T}$ whose terms belong to $\mathcal{H}$.

Assuming that $\mathcal{T}$ has a suitable enhancement, see [34, §3, Theorem 3.11] or [44, §4], there is a triangulated functor $\mathsf{real}^b_{\mathbb{T}} : \mathsf{D}^b(\mathcal{H}) \to \mathcal{T}$ which extends the inclusion $\mathcal{H} \subseteq \mathcal{T}$. Any functor satisfying these two properties is called the **(bounded) realization functor** associated to the t-structure $\mathbb{T}$. Realization functors are not uniquely determined in general, but as shown in [34, Proposition 3.17], bounded derived equivalences of abelian categories are always of the form $\mathsf{real}^b_{\mathbb{T}}$ for a suitable t-structure $\mathbb{T}$ up to an exact equivalence of abelian categories.

A t-structure $\mathbb{T} = (\mathcal{U}, \mathcal{V})$ is called **stable** if its aisle $\mathcal{U}$ (equivalently, its coaisle $\mathcal{V}$) is a triangulated subcategory of $\mathcal{T}$. The aisles of stable t-structures are precisely the coreflective thick subcategories of $\mathcal{T}$ ([21, Proposition 4.9.1]). Such subcategories are automatically **localising**, i.e. triangulated and closed under existing coproducts.

1.3. **Recollements and their equivalences.** [6] Let $\mathcal{U}, \mathcal{V}, \mathcal{T}$ be triangulated categories. A **recollement** (of $\mathcal{T}$) is a diagram of triangle functors

(1)
$$\mathcal{V} \; \overset{i^*}{\longleftarrow} \!\!\! \xrightarrow{\;i_*\;} \!\!\! \underset{i^!}{\longleftarrow} \; \mathcal{T} \; \overset{j_!}{\longleftarrow} \!\!\! \xrightarrow{\;j^*\;} \!\!\! \underset{j_*}{\longleftarrow} \; \mathcal{U}$$

such that:

(i) $(i^*, i_*, i^!)$ and $(j_!, j^*, j_*)$ are adjoint triples,
(ii) $i_*, j_!, j_*$ are fully faithful,
(iii) $\mathrm{Im}(i_*) = \mathrm{Ker}(j^*)$.

We say that two recollements $\mathcal{V} \leftrightarrows \mathcal{T} \leftrightarrows \mathcal{U}$ and $\mathcal{V}' \leftrightarrows \mathcal{T}' \leftrightarrows \mathcal{U}'$ are **equivalent** if there are triangle equivalences $F : \mathcal{T} \to \mathcal{T}'$ and $G : \mathcal{U} \to \mathcal{U}'$ such that the diagram

$$\begin{array}{ccc} \mathcal{T} & \xrightarrow{\;j^*\;} & \mathcal{U} \\ {\scriptstyle F}\downarrow{\scriptstyle \cong} & & {\scriptstyle G}\downarrow{\scriptstyle \cong} \\ \mathcal{T}' & \xrightarrow{\;j^{*\prime}\;} & \mathcal{U}' \end{array}$$

is commutative (up to a natural equivalence). It follows from [30] that this situation is enough to induce a triangle equivalence $H : \mathcal{V} \to \mathcal{V}'$ and the commutativity of all of the six possible squares corresponding to the six different functors from the definition of recollement Eq. (1).

It is well-known that any recollement as in Eq. (1) induces a **(stable) TTF triple** $(j_!\mathcal{U}, i_*\mathcal{V}, j_*\mathcal{U})$, that is, a pair of two adjacent (automatically stable) t-structures $(j_!\mathcal{U}, i_*\mathcal{V})$ and $(i_*\mathcal{V}, j_*\mathcal{U})$. In fact, this assignment yields a bijective correspondence between equivalence classes of recollements of $\mathcal{T}$ and TTF triples in $\mathcal{T}$.

1.4. **Categories of complexes and the coderived category.** Let $\mathcal{G}$ be an abelian category. We will deal with many categories whose objects are complexes with terms in $\mathcal{G}$, so we proceed to fix the notation, to recall some less known definitions and to point out the relations among them.

As usual, $\mathsf{C}(\mathcal{G})$ denotes the category of complexes and cochain maps. Inside $\mathsf{C}(\mathcal{G})$, one can consider the acyclic complexes, and among them contractible ones. By forming the quotient over the contractible complexes, one obtains the homotopy category $\mathsf{K}(\mathcal{G})$ of $\mathcal{G}$. Inside $\mathsf{K}(\mathcal{G})$ there are again the acyclic complexes, whose subcategory is denoted by $\mathsf{K}_{\mathsf{ac}}(\mathcal{G})$. The derived category $\mathsf{D}(\mathcal{G})$ of $\mathcal{G}$ is defined as the Verdier localisation $\mathsf{K}(\mathcal{G})/\mathsf{K}_{\mathsf{ac}}(\mathcal{G})$, and in all the occurrences in this paper this construction will result in an honest (triangulated) category. The localisation functor will be denoted by $Q\colon \mathsf{K}(\mathcal{G}) \to \mathsf{D}(\mathcal{G})$. Notice that when $\mathcal{G}$ has exact coproducts, Q commutes with coproducts. We denote by $\mathsf{D}^b(\mathcal{G})$ the bounded derived category of $\mathcal{G}$ — the full triangulated subcategory of $\mathsf{D}(\mathcal{G})$ consisting of objects whose cohomology vanishes in all but finitely many degrees.

Now let $\mathcal{G}$ be a Grothendieck abelian category. Inside $\mathsf{K}(\mathcal{G})$ there is the subcategory $\mathsf{K}(\mathsf{Inj}(\mathcal{G}))$ of complexes with injective terms. Its left Hom-orthogonal is the subcategory $\mathsf{K}_{\mathsf{coac}}(\mathcal{G})$ of **coacyclic objects**. These are equivalently defined in $\mathsf{C}(\mathcal{G})$ as those complexes X such that $\mathsf{Ext}^1_{\mathsf{C}(\mathcal{G})}(X, Y) = 0$ for every complex Y with injective terms (see [43, Definition 6.7], and [4, 32] for the original definitions). The pair of subcategories $(\mathsf{K}_{\mathsf{coac}}(\mathcal{G}), \mathsf{K}(\mathsf{Inj}(\mathcal{G})))$ is a stable t-structure in $\mathsf{K}(\mathcal{G})$; the corresponding right truncation will be denoted by $I_\lambda\colon \mathsf{K}(\mathcal{G}) \to \mathsf{K}(\mathsf{Inj}(\mathcal{G}))$ (see [22, Corollary 7 and Example 5]). The **coderived category** (in Becker's sense) of $\mathcal{G}$ is defined as the Verdier localisation $\mathsf{D}^{\mathsf{co}}(\mathcal{G}) := \mathsf{K}(\mathcal{G})/\mathsf{K}_{\mathsf{coac}}(\mathcal{G})$, and it is equivalent to $\mathsf{K}(\mathsf{Inj}(\mathcal{G}))$ via the functor induced by I_λ. Coacyclic complexes are in particular acyclic (otherwise they would have a non-zero morphism to the injective envelopes of their non-zero cohomologies), so there is a localisation $\mathsf{D}^{\mathsf{co}}(\mathcal{G}) \to \mathsf{D}(\mathcal{G})$, which corresponds to Q after identifying $\mathsf{D}^{\mathsf{co}}(\mathcal{G}) \cong \mathsf{K}(\mathsf{Inj}(\mathcal{G}))$.

Remark 1.3. There is a different definition of a coderived category in the literature, which is due to Positselski [32]. The two definitions are known to coincide in many situations, for example if the underlying Grothendieck category is locally noetherian [32, §3.7], but it seems to be an open question even for module categories whether they coincide in general (see e.g. [33, Example 2.5(3)]). However, as we will see in Corollary 3.7, for the locally Grothendieck categories we are most interest in, that is the hearts of intermediate restrictable t-structure over commutative noetherian rings, the two definitions of a coderived category are indeed equivalent, and so there is no need to distinguish them.

1.5. **Derivators.** For some of our arguments to work correctly, we will need to consider $\mathsf{D}(\mathcal{G})$ enhanced with the structure of a stable derivator. For basics about the standard derivator of a Grothendieck category which covers most of what is needed in our application see e.g. [45] or [16, Appendix] and the references therein. Here we recall only some particular aspects and terminology of the theory.

Let CAT denote the large 2-category of all categories, Cat denote the 2-category of all small categories and $\mathcal{G}$ be a Grothendieck category. For any $I \in \mathsf{Cat}$ we consider the Grothendieck category $\mathcal{G}^I$ of all I-shaped diagrams in $\mathcal{G}$, that is, of all functors $I \to \mathcal{G}$. Since $\mathsf{C}(\mathcal{G}^I)$ is naturally isomorphic to $\mathsf{C}(\mathcal{G})^I$, we can view objects of $\mathsf{D}(\mathcal{G}^I)$ as I-shaped diagrams in the category $\mathsf{C}(\mathcal{G})$ of cochain complexes. The **standard derivator** of $\mathcal{G}$ is a 2-functor $\mathfrak{D}_\mathcal{G} : \mathsf{Cat}^{\mathrm{op}} \to \mathsf{CAT}$ satisfying several properties. First, for any small category I, the image $\mathfrak{D}_\mathcal{G}(I)$ is the triangulated category $\mathsf{D}(\mathcal{G}^I)$. In particular, if $\star$ denotes the category with a single object and a single morphism, we see that $\mathfrak{D}_\mathcal{G}(\star)$ recovers the derived category $\mathsf{D}(\mathcal{G})$. Another property is that given any morphism $u : I \to J$ in Cat, the induced functor $\mathfrak{D}_\mathcal{G}(u) : \mathfrak{D}_\mathcal{G}(J) \to \mathfrak{D}_\mathcal{G}(I)$ is triangulated and it admits both the left and the right adjoint which are called the **left and right Kan homotopy extensions** along u. For the full definition of an abstract stable derivator, we refer the reader e.g. to [45, Definition 5.9, Definition 5.11].

For any small category I and any object $i \in I$ let $\iota_i : \star \to I$ denote the unique functor which maps the only object of $\star$ to i. The collection of functors $\mathfrak{D}_{\mathcal{G}}(\iota_i) : \mathfrak{D}_{\mathcal{G}}(I) \to \mathfrak{D}_{\mathcal{G}}(\star)$ induce a functor $\mathrm{diag}_I : \mathfrak{D}_{\mathcal{G}}(I) \to \mathfrak{D}_{\mathcal{G}}(\star)^I = \mathsf{D}(\mathcal{G})^I$ called the **diagram functor**. It is essentially the reason why theory of derivators exists that the diagram functor is usually *not* an equivalence. We call objects of $\mathfrak{D}_{\mathcal{G}}(I)$ the **coherent diagrams** of shape I in $\mathsf{D}(\mathcal{G})$, in contrast with objects of the diagram category $\mathfrak{D}_{\mathcal{G}}(\star)^I$ which are sometimes called **incoherent diagrams**. It is convenient to denote for any coherent diagram $\mathcal{X} \in \mathfrak{D}_{\mathcal{G}}(I)$ by $\mathcal{X}_i$ the i-th coordinate of the incoherent diagram $\mathrm{diag}_I(\mathcal{X})$.

For any small category I, denote the unique functor $I \to \star$ by π_I. The left Kan extension along π_I has a special name — it is the **homotopy colimit** functor $\mathsf{hocolim}_I : \mathfrak{D}_{\mathcal{G}}(I) \to \mathfrak{D}_{\mathcal{G}}(\star) = \mathsf{D}(\mathcal{G})$, and it is equivalent to the left derived functor of the colimit functor $\mathsf{C}(\mathcal{G}^I) = \mathsf{C}(\mathcal{G})^I \to \mathsf{C}(\mathcal{G})$. In particular, if I is a directed category, the associated homotopy colimit functor $\mathsf{hocolim}_I$ is computed on a diagram $\mathcal{X} \in \mathsf{D}(\mathcal{G}^I)$ simply by computing the direct limit $\varinjlim_I(\mathcal{X})$ of the diagram $\mathcal{X} \in \mathsf{C}(\mathcal{G})^I$ inside the Grothendieck category $\mathsf{C}(\mathcal{G})$.

There is also a notion of a morphism and equivalence between derivators, we refer the reader to [45, §5] and [11]. For our purposes, it will be enough to say that if $\mathcal{G}, \mathcal{E}$ are two Grothendieck categories, then a **morphism** of derivators $\eta : \mathfrak{D}_{\mathcal{G}} \to \mathfrak{D}_{\mathcal{E}}$ induces functors $\eta^I : \mathfrak{D}_{\mathcal{G}}(I) \to \mathfrak{D}_{\mathcal{E}}(I)$ such that for each morphism $u : I \to J$ the following square commutes (up to natural equivalence):

$$\begin{array}{ccc} \mathfrak{D}_{\mathcal{G}}(J) & \xrightarrow{\mathfrak{D}_{\mathcal{G}}(u)} & \mathfrak{D}_{\mathcal{G}}(I) \\ {\scriptstyle \eta^J}\downarrow & & {\scriptstyle \eta^I}\downarrow \\ \mathfrak{D}_{\mathcal{E}}(J) & \xrightarrow{\mathfrak{D}_{\mathcal{E}}(u)} & \mathfrak{D}_{\mathcal{E}}(I) \end{array} \tag{2}$$

The morphism of derivators η is an **equivalence** if all the functors η^I are equivalences. If this is the case, then η is an honest equivalence in a suitable category of derivators [11, Proposition 2.11], and all the equivalences η^I are triangle equivalences [45, Proposition 5.12]. Furthermore, if η is an equivalence then one can check by passing to adjoint functors that η is also compatible with left and right Kan extensions along any morphism u in Cat. In particular, we get the commutative square for any $I \in \mathsf{Cat}$:

$$\begin{array}{ccc} \mathfrak{D}_{\mathcal{G}}(I) & \xrightarrow{\mathsf{hocolim}_I} & \mathfrak{D}_{\mathcal{G}}(\star) \\ {\scriptstyle \eta^I}\downarrow{\scriptstyle \cong} & & {\scriptstyle \eta^\star}\downarrow{\scriptstyle \cong} \\ \mathfrak{D}_{\mathcal{E}}(I) & \xrightarrow{\mathsf{hocolim}_I} & \mathfrak{D}_{\mathcal{E}}(\star) \end{array} \tag{3}$$

Note that since cohomology is computed coordinate-wise, an object $\mathcal{X}$ of the bounded derived category $\mathsf{D}^b(\mathcal{G}^I)$ is an I-shaped diagram in $\mathsf{C}(\mathcal{G})$ such that the cohomologies of the coordinates $\mathcal{X}_i$ are uniformly bounded, that is, there are integers $l < m$ such that $H^j(\mathcal{X}_i) = 0$ for all $j < l$ or $j > m$ and all $i \in I$. By the exactness of direct limits in $\mathsf{C}(\mathcal{G})$, we see that for any small directed category I the homotopy colimit functor restricts to a functor $\mathsf{hocolim}_I : \mathsf{D}^b(\mathcal{G}^I) \to \mathsf{D}^b(\mathcal{G})$. We say that an equivalence of standard derivators $\eta : \mathfrak{D}_{\mathcal{G}} \to \mathfrak{D}_{\mathcal{E}}$ is **bounded** if for any small category I the triangle equivalence η^I restricts to a triangle equivalence $\eta^I : \mathsf{D}^b(\mathcal{G}^I) \to \mathsf{D}^b(\mathcal{E}^I)$. If I is directed, the above commutative square restricts to another one:

$$\begin{array}{ccc} \mathsf{D}^b(\mathcal{G}^I) & \xrightarrow{\mathsf{hocolim}_I} & \mathsf{D}^b(\mathcal{G}) \\ {\scriptstyle \eta^I}\downarrow{\scriptstyle \cong} & & {\scriptstyle \eta^\star}\downarrow{\scriptstyle \cong} \\ \mathsf{D}^b(\mathcal{E}^I) & \xrightarrow{\mathsf{hocolim}_I} & \mathsf{D}^b(\mathcal{E}) \end{array}$$

We remark that the bounded property can be naturally reformulated in terms of restriction to bounded standard derivators $\mathfrak{D}^b_{\mathcal{G}}$, as it is done in [44]. These derivators are defined similarly to the standard derivators $\mathfrak{D}_{\mathcal{G}}$, but one needs to replace Cat by the full subcategory of all suitably finite categories to reflect the fact that $\mathsf{D}^b(\mathcal{G})$ is not (co)complete.

In our context, equivalences of standard derivators will appear in the form of enhancements of (unbounded) realization functors. If $\mathbb{T}$ is a t-structure in $\mathsf{D}(\mathcal{G})$ with heart $\mathcal{H}$ satisfying certain assumptions, Virili constructs in [44, Theorem B, §6] a morphism $\mathfrak{real}_{\mathbb{T}} : \mathfrak{D}_{\mathcal{H}} \to \mathfrak{D}_{\mathcal{G}}$ between standard derivators such that the functor $\mathfrak{real}^{\star}_{\mathbb{T}} : \mathsf{D}(\mathcal{H}) \to \mathsf{D}(\mathcal{G})$ is triangulated and restricts to a realization functor $\mathsf{D}^b(\mathcal{H}) \to \mathsf{D}(\mathcal{G})$. We will discuss the cases when this occurs in (co)tilting theory in Section 1.8.

1.6. **Intermediate and standard t-structures in $\mathsf{D}(\mathcal{G})$.** Let $\mathcal{G}$ be an abelian category. For any integer $n \in \mathbb{Z}$, there is a t-structure $(\mathsf{D}^{\leq n}, \mathsf{D}^{>n})$, where $\mathsf{D}^{\leq n} = \{X \in \mathsf{D}(\mathcal{G}) \mid H^i(X) = 0 \ \forall i > n\}$ and $\mathsf{D}^{>n} = \{X \in \mathsf{D}(\mathcal{G}) \mid H^i(X) = 0 \ \forall i \leq n\}$, called the ($n$-th shift of the) **standard t-structure**. The left truncation functor $\tau_{\mathsf{D}^{\leq n}}$ is induced by the **soft truncation** of complexes and denoted simply by $\tau^{\leq n}$, similarly the right truncation is the soft truncation functor $\tau^{>n}$.

A t-structure $\mathbb{T} = (\mathcal{U}, \mathcal{V})$ in $\mathsf{D}(\mathcal{G})$ is **intermediate** if there are integers $l < m$ such that $\mathsf{D}^{\leq l} \subseteq \mathcal{U} \subseteq \mathsf{D}^{\leq m}$, or equivalently, $\mathsf{D}^{>m} \subseteq \mathcal{V} \subseteq \mathsf{D}^{>l}$. It is easy to see that the intermediacy of the t-structure $\mathbb{T}$ yields that the realization functor $\mathsf{real}^b_{\mathbb{T}} : \mathsf{D}^b(\mathcal{H}_{\mathbb{T}}) \to \mathsf{D}(\mathcal{G})$ corestricts to a functor $\mathsf{real}^b_{\mathbb{T}} : \mathsf{D}^b(\mathcal{H}_{\mathbb{T}}) \to \mathsf{D}^b(\mathcal{G})$ between the bounded derived categories.

1.7. **Compactly generated and restrictable t-structures in $\mathsf{D}(\mathsf{Mod}\text{-}R)$.** A t-structure $\mathbb{T} = (\mathcal{U}, \mathcal{V})$ is **compactly generated** if there is a set $\mathcal{S} \subseteq \mathcal{T}^c$ such that $\mathcal{V} = \mathcal{S}^{\perp_0}$, or equivalently, if $\mathcal{V} = (\mathcal{S} \cap \mathcal{T}^c)^{\perp_0}$.

Alonso Tarrío, Jeremías López, and Saorín [1] showed that compactly generated t-structures admit a full classification in geometric terms in the case $\mathcal{T} = \mathsf{D}(\mathsf{Mod}\text{-}R)$, the unbounded derived category of a commutative noetherian ring R. Let $\mathsf{Spec}(R)$ denote the Zariski spectrum of R. A subset V of $\mathsf{Spec}(R)$ is called **specialization closed** if V is a union of Zariski-closed sets (equivalently, V is an upper subset of the poset $(\mathsf{Spec}(R), \subseteq)$). An **sp-filtration** of $\mathsf{Spec}(R)$ is an order-preserving function $\Phi : \mathbb{Z} \to 2^{\mathsf{Spec}(R)}$ such that $\Phi(n)$ is a specialization closed subset for each $n \in \mathbb{Z}$.

Theorem 1.4. ([1, Theorem 3.10]) *Let R be a commutative noetherian ring. There is a bijective correspondence between sp-filtrations Φ of $\mathsf{Spec}(R)$ and the set of compactly generated t-structures in $\mathsf{D}(\mathsf{Mod}\text{-}R)$. The bijection assigns to Φ a t-structure with the aisle $\mathcal{U}_\Phi$ defined as follows:*

$$\mathcal{U}_\Phi = \{X \in \mathsf{D}(\mathsf{Mod}\text{-}R) \mid \mathsf{Supp} H^n(X) \subseteq \Phi(n) \ \forall n \in \mathbb{Z}\}.$$

Definition 1.5. Let $\mathcal{G}$ be a locally coherent Grothendieck category, that is, a locally finitely presented Grothendieck category such that the full subcategory $\mathsf{fp}(\mathcal{G})$ of finitely presented objects forms an abelian subcategory of $\mathcal{G}$. A t-structure $\mathbb{T} = (\mathcal{U}, \mathcal{V})$ in $\mathsf{D}(\mathcal{G})$ is **restrictable** if the pair $(\mathcal{U} \cap \mathsf{D}^b(\mathsf{fp}(\mathcal{G})), \mathcal{V} \cap \mathsf{D}^b(\mathsf{fp}(\mathcal{G})))$ is a t-structure in $\mathsf{D}^b(\mathsf{fp}(\mathcal{G}))$.

Under mild assumption on a commutative noetherian ring R, the restrictability of a compactly generated t-structure in $\mathsf{D}(\mathsf{Mod}\text{-}R)$ can be read rather directly from the associated sp-filtration. For the definition of a pointwise dualizing complex we refer the reader e.g. to [1, § 6], in particular, any (classical) dualizing complex is a pointwise dualizing complex.

Theorem 1.6. ([1, Corollary 4.5, Theorem 6.9]) *Let R be a commutative noetherian ring and let $\mathbb{T}$ be the compactly generated t-structure corresponding to an sp-filtration Φ. Consider the following two conditions:*

(i) $\mathbb{T}$ *is restrictable (to* $\mathsf{D}^b(\mathsf{mod}\text{-}R)$*),*

(ii) Φ *satisfies the **weak Cousin condition**, that is, whenever $\mathfrak{p} \subsetneq \mathfrak{q}$ are prime ideals such that $\mathfrak{q}$ is minimal over $\mathfrak{p}$, then for any $n \in \mathbb{Z}$ the implication $\mathfrak{q} \in \Phi(n) \implies \mathfrak{p} \in \Phi(n-1)$ holds.*

Then (i) $\implies$ (ii) holds. Furthermore, if R admits a pointwise dualizing complex then also (ii) $\implies$ (i) holds.

Remark 1.7. Let R be a commutative noetherian ring. Restrictable t-structures in $\mathsf{D}(\mathsf{Mod}\text{-}R)$ are ubiquitous:

- [31, Theorem 2.16, Remark 2.7] A Happel-Reiten-Smalø (HRS) t-structure obtained from a hereditary torsion pair in $\mathsf{Mod}\text{-}R$ is compactly generated and restrictable. In view of Theorem 1.4, these t-structures correspond to sp-filtrations Φ such that $\Phi(n) = \mathsf{Spec}(R)$ for all $n < 0$ and $\Phi(n) = \emptyset$ for all $n > 0$.
- Assume that d is a codimension function on $\mathsf{Spec}(R)$, that is, a function $\mathsf{d} : \mathsf{Spec}(R) \to \mathbb{Z}$ such that $\mathsf{d}(\mathfrak{q}) = \mathsf{d}(\mathfrak{p}) + 1$ whenever $\mathfrak{p} \subsetneq \mathfrak{q}$ are primes with $\mathfrak{q}$ minimal over $\mathfrak{p}$. Then the assignment $\Phi_{\mathsf{d}}(n) = \{\mathfrak{p} \in \mathsf{Spec}(R) \mid \mathsf{d}(\mathfrak{p}) > n\}$ defines an sp-filtration which satisfies the weak Cousin condition.

  Furthermore, any pointwise dualizing complex D induces a codimension function d_D [13, p. 287], and therefore a restrictable t-structure, see [1, §6].
- If R admits a dualizing complex D, the restrictable t-structure induced by the codimension function d_D has a particularly nice description, we follow [1, §6.4]. The functor $\mathbb{R}\mathsf{Hom}_R(-, D)$ induces a duality functor on the category $\mathsf{D}^b(\mathsf{mod}\text{-}R)$, and therefore it sends the standard t-structure to another t-structure on $\mathsf{D}^b(\mathsf{mod}\text{-}R)$, called the **Cohen-Macaulay t-structure**. This t-structure then naturally lifts to a restrictable t-structure in $\mathsf{D}(\mathsf{Mod}\text{-}R)$, see [25, §3], and coincides with the compactly generated t-structure corresponding to the sp-filtration Φ_{d_D}.

1.8. **Silting and cosilting t-structures and realization functors.** Let $\mathcal{T}$ be a triangulated category and $M \in \mathcal{T}$. We define the full subcategories $M^{\perp_{>0}} = \{X \in \mathcal{T} \mid \mathsf{Hom}_{\mathcal{T}}(M, X[i]) \ \forall i > 0\}$ and ${}^{\perp_{>0}}M = \{X \in \mathcal{T} \mid \mathsf{Hom}_{\mathcal{T}}(X, M[i]) \ \forall i > 0\}$, the subcategories $M^{\perp_{\leq 0}}, M^{\perp_{<0}}$ and ${}^{\perp_{\leq 0}}M, {}^{\perp_{<0}}M$ are defined analogously.

Following Psaroudakis-Vitória [34] and Nicolás-Saorín-Zvonareva [29], an object T in $\mathcal{T}$ is **silting** if the pair $(T^{\perp_{>0}}, T^{\perp_{\leq 0}})$ is a t-structure in $\mathcal{T}$, which we call a **silting t-structure**. A silting object C (as well as the induced t-structure) is called **tilting** if $\mathsf{Add}(T) \subseteq T^{\perp_{<0}}$, where $\mathsf{Add}(T)$ is the smallest subcategory of $\mathcal{T}$ containing T and closed under all coproducts and retracts. Dually, an object $C \in \mathcal{T}$ is **cosilting** if the pair $({}^{\perp_{\leq 0}}C, {}^{\perp_{>0}}C)$ is a t-structure in $\mathcal{T}$, which we call a **cosilting t-structure**. A cosilting object C (as well as the induced t-structure) is called **cotilting** if $\mathsf{Prod}(C) \subseteq {}^{\perp_{<0}}C$, where $\mathsf{Prod}(C)$ is the smallest subcategory of $\mathcal{T}$ containing C and closed under all products and retracts.

(Co)silting and (co)tilting objects serve to study triangle equivalences, often induced by the realization functors associated to the induced (co)silting t-structures. Let us specialize now to the case $\mathcal{T} = \mathsf{D}(\mathcal{G})$, where $\mathcal{G}$ is a Grothendieck category. Given a (co)silting object $M \in \mathsf{D}(\mathcal{G})$ denote the heart of the silting t-structure $\mathbb{T}_M$ by $\mathcal{H}_M$ and the induced realization functor as $\mathsf{real}_M : \mathsf{D}^b(\mathcal{H}_M) \to \mathsf{D}(\mathcal{G})$. We call a (co)silting object in $\mathsf{D}(\mathcal{G})$ **bounded** if the induced (co)silting t-structure in is intermediate. Recall that the intermediacy implies that the realization functor factors through $\mathsf{D}^b(\mathcal{G})$. Specializing the result of Psaroudakis and Vitória to Grothendieck categories, we have the following tilting theorem.

Theorem 1.8. [34, Corollary 5.2] *Let $\mathcal{G}$ be a Grothendieck category and $M \in \mathsf{D}(\mathcal{G})$ a bounded (co)silting object. Then $\mathsf{real}_M : \mathsf{D}^b(\mathcal{H}_M) \to \mathsf{D}^b(\mathcal{G})$ is a triangle equivalence if and only if the object M is (co)tilting.*

We remark that if T is a silting object then $\mathcal{H}_T$ is an abelian category with a projective generator [2]. If T is (additively equivalent to) a compact object of $\mathsf{D}(\mathcal{G})$ then it follows that $\mathcal{H}_T$ is equivalent to a module category, and if in addition T is tilting then we have $\mathcal{H}_T \cong \mathsf{Mod}\text{-}\mathsf{End}_{\mathsf{D}(\mathcal{G})}(T)$ [34, Corollary 4.7]. On the other hand, consider a module category $\mathsf{Mod}\text{-}R$ and a bounded cosilting object $C \in \mathsf{D}(\mathsf{Mod}\text{-}R)$. Then it is known that the heart $\mathcal{H}_C$ is a Grothendieck category [24, Proposition 3.10].

In [44], Virili extended the (co)tilting realization functors to the unbounded level by constructing realization equivalences of standard derivators. See also the formulation [44, Theorem E] characterizing restrictable derived equivalences.

Theorem 1.9. [44, Theorem C, D] *Let $\mathcal{G}$ be a Grothendieck category and $M \in \mathsf{D}(\mathcal{G})$ a bounded tilting (resp. cotilting) object. Then there is an equivalence $\mathfrak{real}_M : \mathfrak{D}_{\mathcal{H}_M} \to \mathfrak{D}_{\mathcal{G}}$ of derivators which is bounded.*

In the situation of Theorem 1.9, we denote the triangle equivalence on the base as $\mathsf{real}_M := \mathfrak{real}_M^\star$. Then the triangle equivalence $\mathsf{real}_M : \mathsf{D}(\mathcal{H}_M) \to \mathsf{D}(\mathcal{G})$ is an unbounded realization functor [44, Theorem 7.7, Theorem 7.9] which restricts to a bounded realization functor $\mathsf{D}^b(\mathcal{H}_M) \to \mathsf{D}^b(\mathcal{G})$ which is an equivalence.

A compilation of known results gives a nice characterization of t-structures in $\mathsf{D}(\mathsf{Mod}\text{-}R)$ induced by bounded cotilting objects amongst all intermediate t-structures when R is commutative noetherian.

Theorem 1.10. ([34],[16]) *Let R be a commutative noetherian ring and $\mathbb{T}$ an intermediate t-structure in $\mathsf{D}(\mathsf{Mod}\text{-}R)$. The following are equivalent:*

(i) there is a triangle equivalence $\mathsf{D}(\mathcal{H}_\mathbb{T}) \to \mathsf{D}(\mathsf{Mod}\text{-}R)$ which restricts to the bounded level and $\mathcal{H}_\mathbb{T}$ is a locally finitely presented Grothendieck category,

(ii) the realization functor $\mathsf{real}^b_\mathbb{T} : \mathsf{D}^b(\mathcal{H}_\mathbb{T}) \to \mathsf{D}^b(\mathsf{Mod}\text{-}R)$ is an equivalence and $\mathcal{H}_\mathbb{T}$ is a Grothendieck category,

(iii) $\mathbb{T}$ is a cotilting t-structure.

Proof. $(i) \Rightarrow (ii)$: Clear.

$(ii) \Rightarrow (iii)$: This is Theorem 1.8.

$(iii) \Rightarrow (i)$: The first part follows by Theorem 1.9. Since R is commutative noetherian, it is is known that $\mathbb{T}_\mathbb{T}$ is a compactly generated t-structure [15],[16] and the heart $\mathcal{H}_\mathbb{T}$ is a locally finitely presentable Grothendieck category [40]. □

Finally, we record a recently established strong connection between the cotilting property and restrictability of the associated t-structure.

Theorem 1.11 ([31, Corollary 6.18])**.** *Let R be a commutative noetherian ring and $\mathbb{T}$ be an intermediate, compactly generated, and restrictable t-structure in $\mathsf{D}(\mathsf{Mod}\text{-}R)$. Then $\mathbb{T}$ is a cotilting t-structure.*

2. Krause's recollement for locally coherent Grothendieck categories

Let $\mathcal{G}$ be a locally coherent Grothendieck category such that $\mathsf{D}(\mathcal{G})$ is compactly generated. Our goal is to extend [19, Corollary 4.3] from the locally noetherian to the locally coherent case, that is, to show that Krause's recollement exists for $\mathcal{G}$.

This was shown by Šťovíček [43, Theorem 7.7] under the additional assumption that $\mathcal{G}$ admits a set of finitely presented generators of finite Yoneda projective dimension (see [43, Hypothesis 7.1]). This assumption implies that $\mathsf{D}(\mathcal{G})$ is compactly generated, but it is strictly stronger: we demonstrate an example, which is a Happel-Reiten-Smalø tilt in the derived category of a commutative noetherian ring, in Example 3.8. Our approach here is closer to the original one of Krause, but relies on some of the results of Šťovíček [43, Section 6] (these do not depend on the aforementioned [43, Hypothesis 7.1]).

Recall that $\mathsf{fp}(\mathcal{G})$ denotes the subcategory of all finitely presented objects of $\mathcal{G}$, an exact abelian subcategory in case $\mathcal{G}$ is locally coherent. Our starting point is the following result of Šťovíček.

Theorem 2.1. [43, Corollary 6.13] *Let $\mathcal{G}$ be a locally coherent Grothendieck category. Then $\mathsf{K}(\mathsf{Inj}(\mathcal{G}))$ is compactly generated and the functor assigning to an object of $\mathsf{D}^b(\mathsf{fp}(\mathcal{G}))$ its injective resolution induces an equivalence $\mathsf{D}^b(\mathsf{fp}(\mathcal{G})) \cong \mathsf{K}(\mathsf{Inj}(\mathcal{G}))^c$.*

Corollary 2.2. *Let $\mathcal{G}$ be a locally coherent Grothendieck category. Then the functor $Q : \mathsf{K}(\mathsf{Inj}(\mathcal{G})) \to \mathsf{D}(\mathcal{G})$ admits a right adjoint Q_r.*

Furthermore, the equivalence $\mathsf{D}^b(\mathsf{fp}(\mathcal{G})) \cong \mathsf{K}(\mathsf{Inj}(\mathcal{G}))^c$ of Theorem 2.1 is induced by the restrictions of the adjoint functors Q_r and Q.

Proof. By Theorem 2.1, $\mathsf{K}(\mathsf{Inj}(\mathcal{G}))$ is compactly generated. Since $\mathcal{G}$ has exact coproducts, the functor Q preserves coproducts, and so [28, Theorem 4.1] applies and produces the desired right adjoint.

It follows directly from the adjunction that for any $X \in \mathsf{D}(\mathcal{G})$, $Q_r(X)$ is homotopy equivalent to a dg-injective resolution of X (which exists by [41]). By Theorem 2.1 we have that $Q_r(X)$ restricts to the equivalence $\mathsf{D}^b(\mathsf{fp}(\mathcal{G})) \cong \mathsf{K}(\mathsf{Inj}(\mathcal{G}))^c$ with the inverse equivalence being the restriction of Q to $\mathsf{K}(\mathsf{Inj}(\mathcal{G}))^c$. □

2.1. **Compact objects of $\mathsf{D}(\mathcal{G})$ and the (small) singularity category.** The main obstacle in extending Krause's proof to the locally coherent case is showing that any compact object of $\mathsf{D}(\mathcal{G})$ belongs to $\mathsf{D}^b(\mathsf{fp}(\mathcal{G}))$, and therefore representes a compact object also in $\mathsf{K}(\mathsf{Inj}(\mathcal{G}))$ via Q_r; the proof in the locally noetherian case [19, Lemma 4.1] does not generalize directly.

Following Gillespie [8], an object M of a Grothendieck category $\mathcal{G}$ is said to be *of type* FP_∞ if the functor $\mathsf{Ext}^i_{\mathcal{G}}(M, -)$ naturally preserves direct limits for all $i \geq 0$. It will be convenient for our purposes to extend this notion to any object of the bounded derived category.

Definition 2.3. Let $\mathcal{G}$ be a Grothendieck category. An object $X \in \mathsf{D}^b(\mathcal{G})$ is **of type** FP_∞ if for any direct system $(M_i \mid i \in I)$ in $\mathcal{G}$ and any $n \in \mathbb{Z}$ the natural map

$$\varinjlim_{i \in I} \mathsf{Hom}_{\mathsf{D}^b(\mathcal{G})}(X, M_i[n]) \to \mathsf{Hom}_{\mathsf{D}^b(\mathcal{G})}(X, \varinjlim_{i \in I} M_i[n])$$

is an isomorphism.

Not very surprisingly, Definition 2.3 admits a somewhat more internal characterization using homotopy colimits of bounded directed coherent diagrams, which in turn provides a "bounded" version of the following notion from the theory of stable derivators.

Definition 2.4 ([39, Definition 5.1])**.** Given a directed small category I, $X \in \mathsf{D}(\mathcal{G})$, and $\mathcal{Y} \in \mathsf{D}(\mathcal{G}^I)$, there is a natural map (see [45, Definition 6.5])

$$\varinjlim_{i \in I} \mathsf{Hom}_{\mathsf{D}(\mathcal{G})}(X, \mathcal{Y}_i) \to \mathsf{Hom}_{\mathsf{D}(\mathcal{G})}(X, \mathsf{hocolim}_I \mathcal{Y}).$$

An object $X \in \mathsf{D}(\mathcal{G})$ is called **homotopically finitely presented** if the map above is an isomorphism for any choice of I and $\mathcal{Y}$.

Lemma 2.5. *Let $\mathcal{G}$ be a Grothendieck category. An object $X \in \mathsf{D}^b(\mathcal{G})$ is of type FP_∞ if and only if for any directed small category I and any coherent diagram $\mathcal{Y} \in \mathsf{D}^b(\mathcal{G}^I)$ the natural map*

$$\varinjlim_{i \in I} \mathsf{Hom}_{\mathsf{D}^b(\mathcal{G})}(X, \mathcal{Y}_i) \to \mathsf{Hom}_{\mathsf{D}^b(\mathcal{G})}(X, \mathsf{hocolim}_I \mathcal{Y})$$

is an isomorphism.

Proof. Since $\mathcal{Y} \in \mathsf{D}^b(\mathcal{G}^I)$, the coherent diagram $\mathcal{Y}$ is represented by a direct system $(Y_i \mid i \in I)$ in $\mathsf{C}(\mathcal{G})$ such that the cohomology of the complexes Y_i is uniformly bounded. Therefore, there is $n \in \mathbb{Z}$ and $k \geq 0$ such that for all $i \in I$, the cohomology of Y_i vanishes outside of degrees $n, \dots, n+k$. If $k = 0$, by applying the soft truncation we may assume that $\mathcal{Y}$ is such that $(Y_i \mid i \in I)$ is a direct system of stalk complexes in degree n, and therefore the required isomorphism is provided by the definition of an object of type FP_∞. The general case follows by induction on $k > 0$. Indeed, applying $\mathsf{hocolim}_I$ to the soft truncation triangle of $\mathcal{Y}$ in $\mathsf{D}^b(\mathcal{G}^I)$ we obtain the triangle

$$\mathsf{hocolim}_I \tau^{\leq n} \mathcal{Y} \to \mathsf{hocolim}_I \mathcal{Y} \to \mathsf{hocolim}_I \tau^{> n} \mathcal{Y} \xrightarrow{+}$$

in $\mathsf{D}^b(\mathcal{G})$. Notice that soft truncations commute naturally with the component functors $(-)_i$, and we have triangles in $\mathsf{D}^b(\mathcal{G})$

$$\tau^{\leq n} \mathcal{Y}_i \to \mathcal{Y}_i \to \tau^{> n} \mathcal{Y}_i \xrightarrow{+} .$$

Then there is following commutative diagram, in which the horizontal maps are induced by the two triangles above and the vertical ones are the natural maps (we write $(A, B) := \mathsf{Hom}_{\mathsf{D}(\mathcal{G})}(A, B)$, to lighten the notation):

$$\begin{array}{ccccccccc}
\varinjlim_{i\in I}(X,\tau^{>n}\mathcal{Y}_i[-1]) & \longrightarrow & \varinjlim_{i\in I}(X,\tau^{\leq n}\mathcal{Y}_i) & \longrightarrow & \varinjlim_{i\in I}(X,\mathcal{Y}_i) & \longrightarrow & \varinjlim_{i\in I}(X,\tau^{>n}\mathcal{Y}_i) & \longrightarrow & \varinjlim_{i\in I}(X,\tau^{\leq n}\mathcal{Y}_i[1]) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
(X,\mathsf{hocolim}_I\tau^{>n}\mathcal{Y}[-1]) & \to & (X,\mathsf{hocolim}_I\tau^{\leq n}\mathcal{Y}) & \to & (X,\mathsf{hocolim}_I\mathcal{Y}) & \to & (X,\mathsf{hocolim}_I\tau^{>n}\mathcal{Y}) & \to & (X,\mathsf{hocolim}_I\tau^{\leq n}\mathcal{Y}[1])
\end{array}$$

Then the induction step follows directly by Five lemma, as both the coherent diagrams $\tau^{>n}\mathcal{Y}$ and $\tau^{\leq n}\mathcal{Y}$ are subject to the induction hypothesis for $k-1$. □

Lemma 2.6. *Let $\mathcal{G}$ be a Grothendieck category. The objects of type FP_∞ of $X \in \mathsf{D}^b(\mathcal{G})$ form a thick subcategory of $\mathsf{D}^b(\mathcal{G})^c$.*

Proof. By exactness of coproducts in $\mathcal{G}$, the coproducts in $\mathsf{D}^b(\mathcal{G})$ are precisely the coproducts of collections of objects with uniformly bounded cohomology computed in $\mathsf{D}(\mathcal{G})$. Therefore, any coproduct in $\mathsf{D}^b(\mathcal{G})$ can be realized as a directed homotopy colimit of a suitable diagram of $\mathsf{D}^b(\mathcal{G}^I)$ whose components are finite subcoproducts. In this way Lemma 2.5 shows that any object of type FP_∞ in $\mathsf{D}^b(\mathcal{G})$ is compact in $\mathsf{D}^b(\mathcal{G})$. The fact that objects of type FP_∞ form a triangulated subcategory follows from the Five lemma similarly as in the proof of Lemma 2.5, the closure under retracts is clear. □

Lemma 2.7. *Let $\mathcal{G}$ be a locally coherent Grothendieck category. An object $X \in \mathsf{D}^b(\mathcal{G})$ is of type FP_∞ if and only if $X \in \mathsf{D}^b(\mathsf{fp}(\mathcal{G}))$.*

Proof. An object $F \in \mathsf{fp}(\mathcal{G})$ is of type FP_∞ as an object in $\mathsf{D}^b(\mathcal{G})$, see [8, Theorem 3.21]. By Lemma 2.6, any object in the thick closure of $\mathsf{fp}(\mathcal{G})$ in $\mathsf{D}^b(\mathcal{G})$ is of type FP_∞, which shows that $X \in \mathsf{D}^b(\mathsf{fp}(\mathcal{G}))$ implies that X is of type FP_∞.

For the converse implication, let $X \in \mathsf{D}^b(\mathcal{G})$ be of type FP_∞ and let n be a maximal integer such that $H^n(X) \neq 0$. For any $M \in \mathcal{G}$ the soft truncation yields a natural isomorphism $\mathsf{Hom}_{\mathsf{D}^b(\mathcal{G})}(X, M[-n]) \cong \mathsf{Hom}_\mathcal{G}(H^n(X), M)$. Since X is of type FP_∞, it follows that the functor $\mathsf{Hom}_\mathcal{G}(H^n(X), -) : \mathcal{G} \to \mathsf{Mod}\text{-}\mathbb{Z}$ preserves direct limits, and so $H^n(X)$ belongs to $\mathsf{fp}(\mathcal{G})$. Using the previous paragraph and Lemma 2.6 we infer that the soft truncation $\tau^{<n}X$ is of type FP_∞. Continuing by finite induction we conclude that all cohomologies of X belong to $\mathsf{fp}(\mathcal{G})$, and so $X \in \mathsf{D}^b(\mathsf{fp}(\mathcal{G}))$, see e.g. [18, Theorem 15.3.1]. □

Remark 2.8. Combining Lemma 2.7 and Lemma 2.6 we obtain the inclusion $\mathsf{D}^b(\mathsf{fp}(\mathcal{G})) \subseteq \mathsf{D}^b(\mathcal{G})^c$. We do not know whether the converse inclusion holds true in general for a locally coherent Grothendieck category such that $\mathsf{D}(\mathcal{G})$ is compactly generated. However, in Section 3, we will show that this these two categories coincide in case $\mathcal{G}$ is the heart of an intermediate cotilting t-structure over a commutative noetherian ring.

Proposition 2.9. *Let $\mathcal{G}$ be a locally coherent Grothendieck category. There is an inclusion $\mathsf{D}(\mathcal{G})^c \subseteq \mathsf{D}^b(\mathsf{fp}(\mathcal{G}))$.*

Proof. Let C be a compact object of $\mathsf{D}(\mathcal{G})$. For each $n \in \mathbb{Z}$ there is a natural map $C \to E(H^n(C))[-n]$ in $\mathsf{D}(\mathcal{G})$ to a shift of the injective envelope of $H^n(C)$. This induces a morphism $C \to \prod_{n\in\mathbb{Z}} E(H^n(C))[-n]$. Products in $\mathsf{D}(\mathcal{G})$ are computed as component-wise products of dg-injective resolutions; so in this case, the compontent-wise product of the $E(H^n(C))[-n]$. In this particular case, it coincides with the component-wise coproduct. This is also the coproduct in $\mathsf{D}(\mathcal{G})$, since $\mathcal{G}$ has exact coproducts. Therefore we obtain a morphism $C \to \coprod_{n\in\mathbb{Z}} E(H^n(C))[-n]$ in $\mathsf{D}(\mathcal{G})$. By compactness of C, this map factors through a finite subcoproduct. It follows that C has finitely many non-zero cohomologies.

By [39, Proposition 5.4], C is homotopically finitely presented in $\mathsf{D}(\mathcal{G})$. In particular, C is of type FP_∞ in $\mathsf{D}^b(\mathcal{G})$. Therefore, $C \in \mathsf{D}^b(\mathsf{fp}(\mathcal{G}))$ by Lemma 2.7. □

Remark 2.10. Let $\mathcal{G}$ be a locally coherent Grothendieck category. Proposition 2.9 shows that $\mathsf{D}(\mathcal{G})^c$ is a thick subcategory of $\mathsf{D}^b(\mathsf{fp}(\mathcal{G}))$, and therefore we can form the Verdier quotient $\mathsf{D}^{\mathsf{sg}}(\mathcal{G}) = \mathsf{D}^b(\mathsf{fp}(\mathcal{G}))/\mathsf{D}(\mathcal{G})^c$. Following the locally noetherian case [19], we call $\mathsf{D}^{\mathsf{sg}}(\mathcal{G})$ the **(small) singularity category** of $\mathcal{G}$.

2.2. The left adjoint.

Lemma 2.11. *Let $\mathcal{G}$ be a locally coherent Grothendieck category. For any $C \in \mathsf{D}(\mathcal{G})^c$ and any $Y \in \mathsf{K}(\mathsf{Inj}(\mathcal{G}))$, there is a natural isomorphism*

$$\mathsf{Hom}_{\mathsf{D}(\mathcal{G})}(C, QY) \cong \mathsf{Hom}_{\mathsf{K}(\mathsf{Inj}(\mathcal{G}))}(Q_r C, Y).$$

Proof. Consider the natural transformation

$$\eta_{C,Y} : \mathsf{Hom}_{\mathsf{K}(\mathsf{Inj}(\mathcal{G}))}(Q_r C, Y) \to \mathsf{Hom}_{\mathsf{D}(\mathcal{G})}(QQ_r C, QY)$$

induced by Q. By Corollary 2.2, the functors Q_r and Q induce an equivalence $\mathsf{D}^b(\mathsf{fp}(\mathcal{G})) \cong \mathsf{K}(\mathsf{Inj}(\mathcal{G}))^c$. We see that $QQ_r C$ is naturally isomorphic to C and also, in view of Proposition 2.9, that $\eta_{C,Y}$ is an isomorphism whenever $Y \in \mathsf{K}(\mathsf{Inj}(\mathcal{G}))^c$. Consider the subcategory $\mathcal{K}$ of $\mathsf{K}(\mathsf{Inj}(\mathcal{G}))$ consisting of all objects Y such that $\eta_{C,Y}$ is an isomorphism for all $C \in \mathsf{D}(\mathcal{G})^c$. A standard argument shows that $\mathcal{K}$ is a triangulated subcategory. Since C is compact in $\mathsf{D}(\mathcal{G})$ and $Q_r C$ is compact in $\mathsf{K}(\mathsf{Inj}(\mathcal{G}))$, the subcategory $\mathcal{K}$ is closed under coproducts. Then $\mathcal{K}$ is a localizing subcategory of $\mathsf{K}(\mathsf{Inj}(\mathcal{G}))$ containing all compact objects, and therefore $\mathcal{K} = \mathsf{K}(\mathsf{Inj}(\mathcal{G}))$ by Theorem 2.1. □

Lemma 2.12. *Let $\mathcal{G}$ be a locally coherent Grothendieck category such that $\mathsf{D}(\mathcal{G})$ is compactly generated. Then the functor $Q : \mathsf{K}(\mathsf{Inj}(\mathcal{G})) \to \mathsf{D}(\mathcal{G})$ admits a left adjoint Q_l.*

Proof. Let $\mathcal{L}$ be the localizing subcategory of $\mathsf{K}(\mathsf{Inj}(\mathcal{G}))$ generated by $Q_r(\mathsf{D}(\mathcal{G})^c)$. Then $\mathcal{L}$ is a compactly generated triangulated category, and the restriction $Q_{\restriction \mathcal{L}} : \mathcal{L} \to \mathsf{D}(\mathcal{G})$ is a functor between compactly generated triangulated categories that preserves coproducts and by Corollary 2.2 restricts further to an equivalence $\mathcal{L}^c \cong \mathsf{D}(\mathcal{G})^c$. Then $Q_{\restriction \mathcal{L}}$ is an equivalence by Lemma 1.2, and so there is an inverse equivalence $P : \mathsf{D}(\mathcal{G}) \xrightarrow{\sim} \mathcal{L}$. We define Q_l as the composition of P and the inclusion $\iota : \mathcal{L} \hookrightarrow \mathsf{K}(\mathsf{Inj}(\mathcal{G}))$.

The inclusion ι of $\mathcal{L}$ into $\mathsf{K}(\mathsf{Inj}(\mathcal{G}))$ has a right adjoint $\tau : \mathsf{K}(\mathsf{Inj}(\mathcal{G})) \to \mathcal{L}$, see e.g. [28, Theorem 4.1]. It follows that $Q_l = \iota \circ P$ has a right adjoint $Q \circ \tau$. It remains to show that $Q \circ \tau$ is naturally equivalent to Q. Applying Q to the counit transformation $\iota \circ \tau \to \mathrm{id}_{\mathsf{K}(\mathsf{Inj}(\mathcal{G}))}$ we see that it is enough to show that any object of $\mathcal{L}^{\perp_0}$ is sent to zero by Q, i.e. $\mathcal{L}^{\perp_0} \subseteq \mathsf{K}_{\mathsf{ac}}(\mathsf{Inj}(\mathcal{G}))$. If $Y \in \mathcal{L}^{\perp_0}$ then $\mathsf{Hom}_{\mathsf{K}(\mathsf{Inj}(\mathcal{G}))}(Q_r C, Y) = 0$ for all $C \in \mathsf{D}(\mathcal{G})^c$. By Lemma 2.11, this implies $\mathsf{Hom}_{\mathsf{D}(\mathcal{G})}(C, QY) = 0$ for all $C \in \mathsf{D}(\mathcal{G})^c$, and since $\mathsf{D}(\mathcal{G})$ is compactly generated, we have $QY = 0$, as desired. □

We record the following auxiliary property of the adjoints of Q for later use.

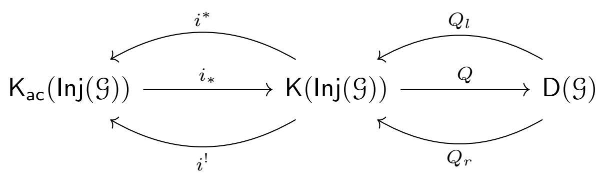
Lemma 2.13. *In the setting of Lemma 2.12 we have an isomorphism $Q_r C \cong Q_l C$ for all $C \in \mathsf{D}(\mathcal{G})^c$.*

Proof. By Lemma 2.12 and Lemma 2.11, there are natural isomorphisms for all $Y \in \mathsf{K}(\mathsf{Inj}(\mathcal{G}))$

$$\mathsf{Hom}_{\mathsf{K}(\mathsf{Inj}(\mathcal{G}))}(Q_l C, Y) \cong \mathsf{Hom}_{\mathsf{D}(\mathcal{G})}(C, QY) \cong \mathsf{Hom}_{\mathsf{K}(\mathsf{Inj}(\mathcal{G}))}(Q_r C, Y).$$

The isomorphism $Q_r C \cong Q_l C$ thus follows from the Yoneda lemma. □

Theorem 2.14. *Let $\mathcal{G}$ be a locally coherent Grothendieck category such that $\mathsf{D}(\mathcal{G})$ is compactly generated. Then there is a recollement:*



Proof. Recall that the functor Q is a Verdier localization functor whose kernel is the full subcategory $\mathsf{K}_{\mathsf{ac}}(\mathsf{Inj}(\mathcal{G}))$. By a standard argument (see [19, Lemma 3.2]), it is enough to establish that Q admits both left and right adjoint functors, which we showed in Corollary 2.2 and Lemma 2.12. □

Corollary 2.15. *In the setting of Theorem 2.14, the category $\mathsf{K}_{\mathsf{ac}}(\mathsf{Inj}(\mathcal{G}))$ is compactly generated and the subcategory of compact objects $\mathsf{K}_{\mathsf{ac}}(\mathsf{Inj}(\mathcal{G}))^c$ is equivalent up to retracts to the singularity category $\mathsf{D}^{\mathsf{sg}}(\mathcal{G})$ of $\mathcal{G}$.*

Proof. This follows directly from [27, Theorem 2.1] applied to the situation of Theorem 2.14. □

Corollary 2.16 (cf. [19, Corollary 4.4])**.** *Let $\mathcal{G}$ be a locally coherent Grothendieck category such that $\mathsf{D}(\mathcal{G})$ is compactly generated. Then any product of acyclic complexes of injective objects is acyclic.*

Remark 2.17. In the locally noetherian situation [19], the category $\mathsf{K}_{\mathsf{ac}}(\mathsf{Inj}(\mathcal{G}))$ is called the **stable derived category** of $\mathcal{G}$ and denoted by $\mathsf{S}(\mathcal{G})$, while other sources [4], [43] call it the **(large) singularity category** of $\mathcal{G}$. In the latter two citations, it is shown that $\mathsf{S}(\mathcal{G})$ is a homotopy category of $\mathsf{C}(\mathcal{G})$ endowed with a suitable abelian model structure. It is also explained in [43, §7] that $\mathsf{S}(\mathcal{G})$ naturally identifies with the subcategory of all acyclic complexes of the coderived category $\mathsf{D}^{\mathsf{co}}(\mathcal{G})$ via the equivalence $\mathsf{K}(\mathsf{Inj}(\mathcal{G})) \cong \mathsf{D}^{\mathsf{co}}(\mathcal{G})$, and the same equivalence identifies the recollement of Theorem 2.14 with the recollement of the form

$$\mathsf{S}(\mathcal{G}) \xrightarrow{\ \subseteq\ } \mathsf{D}^{\mathsf{co}}(\mathcal{G}) \xrightarrow{\ Q\ } \mathsf{D}(\mathcal{G})$$

2.3. **The singularity category of a locally coherent Grothendieck category.** The next goal is to interpret the vanishing of the singularity category $\mathsf{D}^{\mathsf{sg}}(\mathcal{G})$ in terms of homological dimension of objects of $\mathcal{G}$. For this, we need to impose a relatively mild condition on $\mathcal{G}$. Following Roos [36], a Grothendieck category $\mathcal{G}$ is $\mathsf{Ab4}^*{-}d$ for a non-negative integer d if for any set I, the product functor $\prod_I : \mathcal{G}^I \to \mathcal{G}$ has cohomological dimension at most d, we refer the reader to [14] for further details. In particular, $\mathcal{G}$ satisfies $\mathsf{Ab4}^*{-}0$ if and only if the products are exact in $\mathcal{G}$. Recall that the product $\prod_{i\in I} M_i$ in $\mathsf{D}(\mathcal{G})$ is computed as the product $\prod_{i\in I} E_i$ in $\mathsf{C}(\mathcal{G})$ for any choice of injective resolutions E_i of M_i. Therefore, $\mathcal{G}$ satisfies $\mathsf{Ab4}^*{-}d$ if and only if the (component-wise) product $\prod_{i\in I} E_i$ belongs to $\mathsf{D}^{\leq d}$ whenever E_i are complexes of injective objects of $\mathcal{G}$ concentrated in non-negative degrees with the only non-vanishing cohomology in degree zero.

Lemma 2.18. *Let $\mathcal{G}$ be Grothendieck category which is $\mathsf{Ab4}^*{-}d$ for some $d \geq 0$. Then $\prod_{i\in I} X_i \in \mathsf{D}^{\leq d}$ for any collection of objects $X_i \in \mathsf{D}^{\leq 0}$ (in other words, the standard t-structure $(\mathsf{D}^{\leq 0}, \mathsf{D}^{>0})$ in $\mathsf{D}(\mathcal{G})$ is d-**cosmashing**, see* [44, Definition 5.4]*).*

Proof. Let E_i be a dg-injective replacement of X_i for any $i \in I$ [41]. Now we need to show that the (component-wise) product $\prod_{i\in I} E_i$ is in $\mathsf{D}^{\leq d}$. First, let us assume that there is $k \leq 0$ such that all components of E_i in degrees below k are zero. If $k = 0$ then E_i are injective resolutions of objects of $\mathcal{G}$ and so $\prod_{i\in I} E_i \in \mathsf{D}^{\leq d}$ by the definition of the $\mathsf{Ab4}^*{-}d$ property. The case of $k < 0$ is proved using induction and brutal truncations, using the fact that brutal truncations commute with component-wise products. Finally, if there is no such k we argue using the following isomorphism: $\prod_{i\in I} E_i \cong \varinjlim_{n<0} \sigma^{>n} \prod_{i\in I} E_i \cong \varinjlim_{n<0} \prod_{i\in I} \sigma^{>n} E_i$, where $\sigma^{>n}$ is the brutal truncation to degrees above n. Then $\prod_{i\in I} \sigma^{>n} E_i \in \mathsf{D}^{\leq d}$ by the previous case, and the directed colimit also stays in $\mathsf{D}^{\leq d}$ by exactness. □

Lemma 2.19. (cf. [20, §1.6], [14, Theorem 1.3]) *Let $\mathcal{G}$ be a Grothendieck category satisfying $\mathsf{Ab4}^*{-}d$ for some $d \geq 0$. Then for any collection $M_n \in \mathcal{G}$ of objects indexed by $n \in \mathbb{Z}$ we have an isomorphism $\coprod_{n\in\mathbb{Z}} M_n[n] \cong \prod_{n\in\mathbb{Z}} M_n[n]$ in $\mathsf{D}(\mathcal{G})$ (the product is computed in $\mathsf{D}(\mathcal{G})$).*

Proof. Consider the canonical morphism $\eta : \coprod_{n\in\mathbb{Z}} M_n[n] \to \prod_{n\in\mathbb{Z}} M_n[n]$ in $\mathsf{D}(\mathcal{G})$ and let us show that η is a an isomorphism. Let $l \in \mathbb{Z}$ and let us compute $H^l(\eta)$. Since coproducts are exact in $\mathcal{G}$, the coproduct is equivalently computed in $\mathsf{C}(\mathcal{G})$ and $H^l(\coprod_{n\in\mathbb{Z}} M_n[n])$ is clearly just M_n. On the other hand, the product $\prod_{n\in\mathbb{Z}} M_n[n]$ isomorphic in $\mathsf{D}(\mathcal{G})$ to the product $\prod_{n\in\mathbb{Z}} E_n[n]$

computed in $\mathsf{C}(\mathcal{G})$, where E_n is an injective resolution of M_n for each n. Consider the decomposition $\prod_{n\in\mathbb{Z}} E_n[n] = \prod_{n>-l+d} E_n[n] \times \prod_{n=-l,\dots,-l+d} E_n[n] \times \prod_{n<-l} E_n[n]$. Clearly, $\prod_{n<-l} E_n[n] \in \mathsf{D}^{>l}$. For any $n > -l + d$, we have $E_n[n] \cong M_n[n] \in \mathsf{D}^{<l-d}$, and using Lemma 2.18 we conclude that $\prod_{n>-l+d} E_n[n] \in \mathsf{D}^{<l}$. It follows that $H^l(\eta)$ factors as a map $H^l(\eta) : M_l \to H^l(\prod_{n=-l,\dots,-l+d} E_n[n]) = \prod_{n=-l,\dots,-l+d} H^l(E_n[n]) = H^l(E_l[l])$, and this is clearly an isomorphism. □

The following definition is not necessarily standard.

Definition 2.20. An object X of $\mathsf{D}(\mathcal{G})$ is of **finite projective dimension** if there is $n \in \mathbb{Z}$ such that $\mathsf{Hom}_{\mathsf{D}(\mathcal{G})}(X, \mathsf{D}^{\leq n}) = 0$.

Remark 2.21. If $X \in \mathsf{D}(\mathcal{G})$ is of finite projective dimension in the sense of Definition 2.20, then in particular we have that $\mathsf{Hom}_{\mathsf{D}(\mathcal{G})}(X, \mathcal{G}[i]) = 0$ for $i \gg 0$. If X is an object of $\mathcal{G}$, this means precisely that $\mathsf{Ext}^i_{\mathcal{G}}(X, -) = 0$ for $i \gg 0$, i.e. that X has finite Yoneda projective dimension; see e.g. [43, Hypothesis 7.1]. The converse implication is however not so clear, at least without further assumptions on $\mathcal{G}$. If $\mathsf{D}(\mathcal{G})$ is **left-complete** [34, Definition 6.2], that is if any $X \in \mathsf{D}(\mathcal{G})$ is isomorphic to the homotopy limit of its soft truncations from below, then a standard argument similar to the one in the following paragraph shows that these two definitions are equivalent. It is shown in [14, Theorem 1.3] that $\mathsf{D}(\mathcal{G})$ is left-complete whenever $\mathcal{G}$ satisfies $\mathsf{Ab4}^*{-}d$ for some $d \geq 0$.

Even without additional assumptions on $\mathcal{G}$, the two definitions coincide whenever X is a compact object of $\mathsf{D}(\mathcal{G})$. Indeed, then the condition $\mathsf{Hom}_{\mathsf{D}(\mathcal{G})}(X, \mathsf{D}^{\leq n}) = 0$ can be checked just on bounded complexes from $\mathsf{D}^{\leq n}$ by a standard argument using homotopy colimits of hard truncations and the fact that X is compact. Arguing by dimension shifting and finite extensions, $\mathsf{Hom}_{\mathsf{D}(\mathcal{G})}(X, \mathsf{D}^{\leq n}) = 0$ is then equivalent to $\mathsf{Hom}_{\mathsf{D}(\mathcal{G})}(X, \mathcal{G}[-n]) = 0$.

Proposition 2.22. *Let $\mathcal{G}$ be a locally coherent Grothendieck category satisfying $\mathsf{Ab4}^*{-}d$ for some $d \geq 0$ such that $\mathsf{D}(\mathcal{G})$ is compactly generated. Then the following conditions are equivalent:*

(i) $\mathsf{D}^{\mathsf{sg}}(\mathcal{G}) = 0$,
(ii) $\mathsf{S}(\mathcal{G}) = 0$,
(iii) any object $F \in \mathsf{fp}(\mathcal{G})$ has finite projective dimension.

Proof. The equivalence of (i) and (ii) is clear from Corollary 2.15.

$(i) \Rightarrow (iii)$: Suppose that there is $F \in \mathsf{fp}(\mathcal{G})$ which is not of finite projective dimension, and assume towards contradiction that F is compact as an object of $\mathsf{D}(\mathcal{G})$. In view of Remark 2.21, there are objects $M_n \in \mathcal{G}$ for all $n \geq 0$ such that $\mathsf{Ext}^n_{\mathcal{H}}(F, M_n) \neq 0$. By Lemma 2.19 there is an isomorphism $\coprod_{n\geq 0} M_n[n] \cong \prod_{n\geq 0} M_n[n]$ in $\mathsf{D}(\mathcal{G})$, and so there is a morphism $F \Rightarrow \coprod_{n\geq 0} M_n[n]$ which does not factor through any finite subcoproduct of $\coprod_{n\geq 0} M_n[n]$ in $\mathsf{D}(\mathcal{G})$ (cf. [45, Remark 1.11]). It follows that F is not compact in $\mathsf{D}(\mathcal{G})$, and therefore F is a non-zero object of $\mathsf{D}^{\mathsf{sg}}(\mathcal{G}) = \mathsf{D}^b(\mathsf{fp}(\mathcal{G}))/\mathsf{D}(\mathcal{G})^c$.

$(iii) \Rightarrow (i)$: It is enough to show that any $F \in \mathsf{fp}(\mathcal{G})$ is compact as an object of $\mathsf{D}(\mathcal{G})$. Because F is of finite projective dimension, there is an $n \leq 0$ such that we can use soft truncations to obtain for any collection of objects $X_i \in \mathsf{D}(\mathcal{G}), i \in I$ a chain of natural isomorphisms $\mathsf{Hom}_{\mathsf{D}(\mathcal{G})}(F, \coprod_{i\in I} X_i) \cong \mathsf{Hom}_{\mathsf{D}(\mathcal{G})}(F, \tau^{\geq n}\tau^{\leq 0} \coprod_{i\in I} X_i) \cong \mathsf{Hom}_{\mathsf{D}(\mathcal{G})}(F, \coprod_{i\in I} \tau^{\geq n}\tau^{\leq 0} X_i)$. It follows that F is compact in $\mathsf{D}(\mathcal{G})$ if and only if it is compact in $\mathsf{D}^b(\mathcal{G})$. But $F \in (\mathsf{D}^b(\mathcal{G}))^c$ by Remark 2.8. □

We conclude this section by showing that Proposition 2.22 specializes neatly to the case of the category of quasicoherent sheaves over a scheme. Following [10], a quasicompact and quasiseparated scheme X is **coherent** if it admits a cover $X = \bigcup_{i\in I} \mathsf{Spec}(R_i)$ by open affine sets such that R_i is a coherent commutative ring for all $i \in I$. By a standard argument [12, Corollary 2.1], this is equivalent to any open affine subset $\mathsf{Spec}(R)$ of X being such that the ring R is coherent. By [10, Proposition 9.2], X is coherent if and only if the Grothendieck category $\mathsf{Qcoh}\text{-}X$ of quasicoherent sheaves is locally coherent. It follows that $\mathsf{fp}(\mathsf{Qcoh}\text{-}X) = \mathsf{coh}\text{-}X$, the category of coherent sheaves, and $\mathsf{D}^b(\mathsf{fp}(\mathsf{Qcoh}\text{-}X)) = \mathsf{D}^b(\mathsf{coh}\text{-}X)$.

The classical notion of a regular noetherian ring admits the following generalization to coherent rings, here we follow [5] and [9]. A coherent commutative ring R is **regular** if any finitely generated ideal has finite projective dimension. By [9], this is equivalent to any finitely presented R-module being of finite projective dimension.

It is then natural to call a coherent scheme X **regular** if it admits a cover $X = \bigcup_{i\in I} \mathsf{Spec}(R_i)$ where R_i are regular coherent rings. Since regular coherent rings descent along faithfully flat morphisms [9, Theorem 6.2.5], this is equivalent to any open affine subset $\mathsf{Spec}(R)$ of X being such that R is regular coherent.

Corollary 2.23. *Let X be a separated coherent scheme. Then the followng are equivalent:*

(i) $\mathsf{D}^{\mathsf{sg}}(\mathsf{Qcoh}\text{-}X) = 0$,
(ii) *X is regular,*
(iii) *any coherent sheaf has finite projective dimension in* $\mathsf{Qcoh}\text{-}X$.

Proof. Since X is separated, the derived category $\mathsf{D}(\mathsf{Qcoh}\text{-}X)$ is compactly generated by [7, §3], and the compact objects are up to isomorphism precisely the perfect complexes, that is, complexes which are locally quasi-isomorphic to bounded complexes of vector bundles.

$(i) \Leftrightarrow (ii)$: If $\mathcal{F} \in \mathsf{D}^b(\mathsf{coh}\text{-}X)$, we can check whether $\mathcal{F} \in \mathsf{D}(\mathsf{Qcoh}\text{-}X)^c$ locally on an open affine cover, and any such restriction becomes a bounded complex of finitely presented modules. Therefore, since regularity is also a local property, the task reduces to the case of X being an affine scheme. But this case follows directly from Proposition 2.22, because for an affine scheme X the category $\mathsf{Qcoh}\text{-}X$ is equivalent to a module category, and thus has exact products.

$(i) \Leftrightarrow (iii)$: Since $\mathsf{coh}\text{-}X = \mathsf{fp}(\mathsf{Qcoh}\text{-}X)$, this follows from Proposition 2.22 because $\mathsf{Qcoh}\text{-}X$ satisfies $\mathsf{Ab4}^*{-}d$ for some $d \geq 0$ by [14, Remark 3.3]. □

3. Restrictable t-structures

Recall from Theorem 1.10 that if R is a commutative noetherian ring and $\mathbb{T}$ is an intermediate cotilting t-structure, then $\mathbb{T}$ is compactly generated and $\mathcal{H}$ is a locally finitely presentable Grothendieck category by [40, Theorem 1.6]. In view of the previous section, we are mostly interested in the case when $\mathcal{H}$ is in addition locally coherent. Therefore, in this section we consider the following setting.

Setting 3.1. *Let R be a commutative noetherian ring. Let $\mathbb{T}_C$ be a t-structure, with heart $\mathcal{H}_C$, such that:*

(C1) *$\mathbb{T}_C$ is the cotilting t-structure associated to a cotilting object C.*
(C2) *$\mathbb{T}_C$ is intermediate.*
(C3) *$\mathcal{H}_C$ is a locally coherent Grothendieck category.*

Condition (C2) is equivalent to the requirement that $C \in \mathsf{K}^b(\mathsf{Inj}(R))$, which is sometimes included in the definition of a cotilting object. The fact that C is cotilting provides us with a triangle equivalence

$$\mathsf{real}_C \colon \mathsf{D}(\mathcal{H}_C) \to \mathsf{D}(\mathsf{Mod}\text{-}R)$$

which restricts to the level of bounded derived categories and which lifts to an equivalence between the standard derivators, see Section 1.8.

The main goal of this section is to characterize Setting 3.1 using the restrictability of the t-structure $\mathbb{T}_C$. To do that, we first need to better understand the compact objects in the bounded derived category of $\mathcal{H}_C$. Recall from Remark 2.8 that we have an inclusion $\mathsf{D}^b(\mathsf{fp}(\mathcal{H}_C)) \subseteq \mathsf{D}^b(\mathcal{H}_C)^c$. We will use the derived equivalence to $\mathsf{Mod}\text{-}R$ to show that this inclusion is an equality.

Lemma 3.2. *Let $\mathcal{G}$ and $\mathcal{E}$ be Grothendieck categories and $\eta : \mathfrak{D}_\mathcal{G} \to \mathfrak{D}_\mathcal{E}$ a bounded equivalence of derivators. Then for an object $X \in \mathsf{D}^b(\mathcal{G})$ is of type FP_∞ if and only if $\eta^\star(X)$ is of type FP_∞ in $\mathsf{D}^b(\mathcal{E})$.*

Proof. Let I be a directed small category and $\mathcal{Y} \in \mathsf{D}^b(\mathcal{G}_C^I)$. Then there is the following commutative square induced by application of the equivalence η between derivators, where all of the maps are the naturally induced ones:

$$\begin{array}{ccc} \varinjlim_{i\in I} \mathsf{Hom}_{\mathsf{D}(\mathcal{G})}(X, \mathcal{Y}_i) & \xrightarrow{\cong} & \varinjlim_{i\in I} \mathsf{Hom}_{\mathsf{D}(\mathcal{E})}(\eta^\star X, (\eta^I \mathcal{Y})_i) \\ \downarrow & & \downarrow \\ \mathsf{Hom}_{\mathsf{D}(\mathcal{G})}(X, \mathsf{hocolim}_I \mathcal{Y}) & \xrightarrow{\cong} & \mathsf{Hom}_{\mathsf{D}(\mathcal{E})}(\eta^\star X, \mathsf{hocolim}_I(\eta^I \mathcal{Y})) \end{array}$$

Note that both the horizontal isomorphisms are induced by the triangle equivalence $\eta^\star$. Indeed, this follows from the two canonical isomorphisms induced by the derivator equivalence η:

$$\mathsf{hocolim}_I(\eta^I \mathcal{Y}) \cong \eta^\star(\mathsf{hocolim}_I \mathcal{Y}) \text{ and } (\eta^I \mathcal{Y})_i \cong \eta^\star(\mathcal{Y}_i),$$

see Eq. (3) and Eq. (2). Since the equivalence η is bounded, $\eta^I \mathcal{Y} \in \mathsf{D}^b(\mathcal{E}^I)$. Therefore, if $\eta^\star$ is of type FP_∞ then the right vertical map is an isomorphism by Lemma 2.5. Then the square implies that the left vertical map is an isomorphism for any choice of $\mathcal{Y} \in \mathsf{D}^b(\mathcal{G}^I)$, and so X is of type FP_∞. The converse implication follows similarly using the fact that $\eta^\star$ and η^I are equivalences between the bounded derived categories. □

Lemma 3.3. *In Setting 3.1, we have* $(\mathsf{D}^b(\mathcal{H}_C))^c = \mathsf{D}^b(\mathsf{fp}(\mathcal{H}_C))$. *In particular, the derived equivalence* real_C *restricts to an equivalence* $\mathsf{D}^b(\mathsf{fp}(\mathcal{H}_C)) \to \mathsf{D}^b(\mathsf{mod}\text{-}R)$.

Proof. Recall from Theorem 1.9 that the cotilting t-structure $\mathbb{T}$ induces a bounded equivalence $\mathfrak{real}_C : \mathsf{D}_{\mathcal{H}_C} \to \mathsf{D}_{\mathsf{Mod}\text{-}R}$ of derivators. In particular, we have a triangle equivalence $\mathsf{D}^b(\mathcal{H}_C) \xrightarrow{\sim} \mathsf{D}^b(\mathsf{Mod}\text{-}R)$ obtained by restriction of $\mathfrak{real}_C^\star : \mathsf{D}(\mathcal{H}_C) \xrightarrow{\sim} \mathsf{D}(\mathsf{Mod}\text{-}R)$. Then $\mathfrak{real}_C^\star$ further restricts to an equivalence $\mathsf{D}^b(\mathcal{H}_C)^c \xrightarrow{\sim} \mathsf{D}^b(\mathsf{Mod}\text{-}R)^c$ between the categories of compact objects. Since R is noetherian, $\mathsf{D}^b(\mathsf{Mod}\text{-}R)^c = \mathsf{D}^b(\mathsf{mod}\text{-}R)$ by [37, Corollary 6.17], and $\mathsf{D}^b(\mathsf{mod}\text{-}R)$ is also precisely the subcategory of $\mathsf{D}^b(\mathsf{Mod}\text{-}R)$ consisting of objects of type FP_∞, see Lemma 2.7. Then Lemma 3.2 applies and shows that $\mathsf{D}^b(\mathcal{H}_C)^c$ coincides with the subcategory of all objects of type FP_∞ of $\mathsf{D}^b(\mathcal{H}_C)$. But by Lemma 2.7 this is precisely the subcategory $\mathsf{D}^b(\mathsf{fp}(\mathcal{H}_C))$.

Finally, note that we proved the second statement along the way, since $\mathsf{real}_C = \mathfrak{real}_C^\star$. □

Corollary 3.4. *In Setting 3.1, the functor* real_C *induces a triangle equivalence* $\mathsf{D}^{\mathsf{sg}}(\mathcal{H}_C) \to \mathsf{D}^{\mathsf{sg}}(\mathsf{Mod}\text{-}R)$ *between singularity categories.*

Proof. By Lemma 3.3, the derived equivalnce $\mathsf{real}_C : \mathsf{D}(\mathcal{H}_C) \to \mathsf{D}(\mathsf{Mod}\text{-}R)$ restricts to an equivalence $\mathsf{D}^b(\mathsf{fp}(\mathcal{H}_C)) \to \mathsf{D}^b(\mathsf{mod}\text{-}R)$. Since real_C also restricts to an equivalence $\mathsf{D}(\mathcal{H}_C)^c \to \mathsf{D}(\mathsf{Mod}\text{-}R)^c$ between the subcategories of compact objects, the result follows formally by passing to Verdier quotients. □

Now we are ready to formulate the main result of this section, that is, to characterize the case in which the heart $\mathcal{H}_C$ is a locally coherent category. Our results can be seen as a refinement of the characterization of the locally coherent property of hearts due to Marks and Zvonareva [25, Corollary 4.2], but only in the special case of intermediate compactly generated t-structures in $\mathsf{D}(\mathsf{Mod}\text{-}R)$.

Theorem 3.5. *Let R be a commutative noetherian ring and $\mathbb{T}$ be an intermediate compactly generated t-structure in* $\mathsf{D}(\mathsf{Mod}\text{-}R)$ *with heart $\mathcal{H}$. Then the following are equivalent:*

(i) we are in Setting 3.1, that is, $\mathsf{real}_{\mathbb{T}}^b$ *is an equivalence and $\mathcal{H}$ is locally coherent;*
(ii) the t-structure $\mathbb{T}$ restricts to $\mathsf{D}^b(\mathsf{mod}\text{-}R)$.

Proof. Recall that $\mathsf{real}_{\mathbb{T}}^b$ being an equivalence amounts to $\mathbb{T}$ being induced by a cotilting object C by Theorem 1.10, and therefore the description in (i) indeed corresponds to Setting 3.1.

The two claims of the implication $(ii) \Rightarrow (i)$ are proven in [31, Corollary 6.17] and [38, Theorem 6.3], respectively.

It remains to show $(i) \Rightarrow (ii)$. Assume now that $\mathcal{H}$ is locally coherent. To establish that $\mathbb{T}$ is restrictable, we just need to recall from Lemma 3.3 that the derived equivalence real_C :

$\mathsf{D}(\mathcal{H}) \xrightarrow{\sim} \mathsf{D}(\mathsf{Mod}\text{-}R)$ restricts to an equivalence $\mathsf{D}^b(\mathsf{fp}(\mathcal{H})) \xrightarrow{\sim} \mathsf{D}^b(\mathsf{mod}\text{-}R)$. The t-structure $\mathbb{T}$ corresponds under real to the standard t-structure on $\mathsf{D}(\mathcal{H})$, which clearly restricts to a t-structure in $\mathsf{D}^b(\mathsf{fp}(\mathcal{H}))$. □

As another application of Lemma 3.3, we can show that the two versions of coderived categories of $\mathcal{H}_C$ due to Becker and Positselski coincide. Recall that an object $M \in \mathcal{H}_C$ is **fp-injective** if $\mathsf{Ext}^1_{\mathcal{H}_C}(F, M) = 0$ for all $F \in \mathsf{fp}(\mathcal{H}_C)$. Furthermore, $M \in \mathcal{H}_C$ is of **finite fp-injective dimension** if M is isomorphic in $\mathsf{D}(\mathcal{H}_C)$ to a bounded complex of fp-injective objects concentrated in non-negative degrees.

Lemma 3.6. *In Setting 3.1, any object in $\mathcal{H}_C$ of finite fp-injective dimension is of finite injective dimension.*

Proof. Let $M \in \mathcal{H}_C$, put $X = \mathsf{real}_C(M) \in \mathsf{D}^b(\mathsf{Mod}\text{-}R)$, and let us denote the converse equivalence to real_C as $\mathsf{real}_C^{-1} : \mathsf{D}(\mathsf{Mod}\text{-}R) \xrightarrow{\sim} \mathsf{D}(\mathcal{H}_C)$. Since the t-structure $\mathbb{T}$ is intermediate, and using Lemma 3.3, there is an integer $n \in \mathbb{Z}$ such that $\mathsf{real}_C^{-1}(\mathsf{mod}\text{-}R) \subseteq \mathsf{D}(\mathsf{fp}(\mathcal{H}_C)) \cap \mathsf{D}^{\geq n}$. If M is of finite fp-injective dimension then $\mathsf{Hom}_{\mathsf{D}(\mathcal{H}_C)}(\mathsf{D}(\mathsf{fp}(\mathcal{H}_C))^{\geq n}, M[i]) = 0$ for all $i \gg 0$. Applying real_C we therefore obtain $\mathsf{Hom}_{\mathsf{D}(\mathsf{Mod}\text{-}R)}(\mathsf{mod}\text{-}R, X[i]) = 0$ for all $i \gg 0$, which amounts to $X \in \mathsf{D}^b(\mathsf{Mod}\text{-}R)$ being of finite injective dimension in $\mathsf{D}(\mathsf{Mod}\text{-}R)$, since R is noetherian. Equivalently, we have $\mathsf{Hom}_{\mathsf{D}(\mathsf{Mod}\text{-}R)}(\mathsf{D}(\mathsf{Mod}\text{-}R)^{\geq 0}, X[i]) = 0$ for $i \gg 0$. But using the intermediacy of $\mathbb{T}$ again, we know that $\mathsf{real}_C\mathcal{H}_C[j] \subseteq \mathsf{D}(\mathsf{Mod}\text{-}R)^{\geq 0}$ for $j \ll 0$, and so it follows by applying real_C^{-1} that $\mathsf{Hom}_{\mathsf{D}(\mathcal{H}_C)}(\mathcal{H}_C, M[i+j]) = 0$ for $i + j \gg 0$, which in turn implies that M is of finite injective dimension in $\mathcal{H}_C$. □

Corollary 3.7. *In Setting 3.1, the coderived category $\mathsf{K}(\mathsf{Inj}(\mathcal{H}_C))$ (in Becker's sense) is equivalent to the coderived category in Positselski's sense.*

Proof. This follows directly from [32, §3.7, Theorem] in view of Lemma 3.6 . □

We finish this section with an example of a locally coherent Grothendieck category which does not satisfy [43, Hypothesis 7.1] even though its derived category is compactly generated. In fact, we obtain it as a heart $\mathcal{H}_\mathbb{T}$ in $\mathsf{D}(\mathsf{Mod}\text{-}R)$ induced by a compactly generated, intermediate and restrictable t-structure.

Example 3.8. Let $(R, \mathfrak{m})$ be a commutative and noetherian local ring, of dimension 1, which is not Cohen-Macaulay; for example, take R to be the localisation of $k[x, y]/(x^2, xy)$, for an algebraically closed field k, at the maximal ideal $\mathfrak{m} = (x, y)$. In particular, we have $1 = \mathsf{dim}(R) > \mathsf{depth}(R) = 0$; and then, by the Auslander-Buchsbaum formula, every non-zero finitely generated module is projective or has infinite projective dimension (in other words, the small finitistic global dimension of R is 0). Moreover, since R is not Cohen-Macaulay, $\mathfrak{m}$ is an associated prime of R; and the other primes are minimal, so they are associated as well, i.e. $\mathsf{Ass}(R) = \mathsf{Spec}(R)$. Therefore, every cyclic module $R/\mathfrak{p}R$ for a prime $\mathfrak{p}$ is a subobject of a projective module (R itself). It follows from Matlis' Theorem and [3, Theorem 7.1] that the finitistic injective global dimension of R and, by duality, also the finitistic weak global dimension of R are 0. We recall that this means that any R-module of finite flat dimension is automatically flat.

Let $V = \{\mathfrak{m}\}$, consider the associated hereditary torsion pair $\mathbf{t} = (\mathcal{T}, \mathcal{F})$ in $\mathsf{Mod}\text{-}R$, and let $\mathcal{H}$ be the HRS-tilt of $\mathsf{Mod}\text{-}R$ with respect to $\mathbf{t}$; namely, $\mathcal{H} = \mathcal{F}[1] * \mathcal{T}$, we refer to [31] for terminology and details. Notice that since $\mathsf{D}(\mathcal{H}) \cong \mathsf{D}(\mathsf{Mod}\text{-}R)$ (by [31, Corollary 5.11]) the former is compactly generated. Also, $\mathcal{H}$ is the heart of the Happel-Reiten-Smalø t-structure corresponding to the torsion pair $(\mathcal{T}, \mathcal{F})$, and this is an intermediate t-structure which is compactly generated and restrictable ([31, Remark 4.8 and Theorem 2.16(3)]).

Nonetheless, we shall show that there are no non-zero finitely presented objects of finite (Yoneda) projective dimension in $\mathcal{H}$, and therefore [43, Hypothesis 7.1] is not satisfied.

Since R has dimension 1, every subset of $\mathsf{Spec}(R)$ is coherent, and therefore V corresponds to a flat ring epimorphism $R \to S$; given our choice of V, S will be a regular ring of dimension 0. In $\mathcal{H}$, there is a hereditary torsion pair $\mathsf{s} = (\mathcal{T}, \mathsf{Mod}\text{-}S[1])$ (see [31, §4.2]).

Let X be a finitely presented object of $\mathcal{H}$, i.e. $X \in \mathcal{H} \cap \mathsf{D}^b(\mathsf{mod}\text{-}R)$, and assume it has finite projective dimension. Note that this implies that X is of finite projective dimension also as an object of $\mathsf{D}(\mathsf{Mod}\text{-}R)$. Consider its approximation sequence with respect to $\mathbf{s}$ in $\mathcal{H}$, i.e. the triangle

$$T \to X \to L[1] \to T[1]$$

with $T \in \mathcal{T}$ and L an S-module. In particular, since $\mathrm{gl.dim}(S) = 0$, L is a projective S module; since S is a flat R-module, it has finite projective dimension over R [35, Seconde partie, Corollaire 3.2.7], and then so does L. From the triangle above, we deduce that T has finite projective dimension as well. Then, its flat dimension in $\mathsf{Mod}\text{-}R$ is also finite, and since the finitistic weak global dimension of R is 0, T is a flat R-module. Now, we claim that this implies $T = 0$. Indeed, consider a presentation

$$0 \to K \to F \to T \to 0$$

with $F = R^{(\alpha)}$ a free R-module. Since T is flat, this sequence is pure exact, and therefore the torsion radical t of $\mathbf{t}$ gives a short exact sequence

$$0 \to tK \to tF \to T \to 0.$$

By construction, tR is supported on $V = \{\mathfrak{m}\}$, and since it is finitely generated, this means that $V(\mathsf{ann}(tR)) = \{\mathfrak{m}\}$. Hence $\mathfrak{m} = \sqrt{\mathsf{ann}(tR)}$, and since R is noetherian it follows that there exists n such that $\mathfrak{m}^n tR = 0$. Therefore, $tR, tF = (tR)^{(\alpha)}$ and also T are $R/\mathfrak{m}^n$-modules. T is also flat over $R/\mathfrak{m}^n$, and since this is an artinian local ring, T is free, i.e. $T \cong (R/\mathfrak{m}^n)^{(\beta)}$. But then, if $T \neq 0$, its direct summand $R/\mathfrak{m}^n$ should be a finitely presented flat R-module, and therefore projective, which is a contradiction because it would force R to be artinian (and therefore 0-dimensional).

It follows that our finitely presented object X of $\mathcal{H}$ is isomorphic to $L[1]$. But then, L is a finitely presented R-module of finite projective dimension, hence it is projective, hence free. Now, since L is also an S-module, if $L \neq 0$ this would imply that $R \in \mathsf{Mod}\text{-}S$. In particular, R would be torsion-free with respect to $\mathbf{t}$, which is not the case since $\mathfrak{m} \in \mathsf{Ass}(R)$. We conclude that $X \cong L[1] = 0$.

4. The equivalence of recollements

Let R be a commutative noetherian ring. By Theorem 3.5, Setting 3.1 characterises the case in which we have an intermediate compactly generated restrictable t-structure $\mathbb{T}$.

Consider now the following seemingly new situation.

Setting 4.1. *Let $\mathcal{H}$ be a locally coherent Grothendieck category, and assume that there exists an object T in $\mathsf{D}(\mathcal{H})$ such that:*

(T1) T is compact tilting.

(T2) T has finite projective dimension, i.e. $\mathsf{Hom}_{\mathsf{D}(\mathcal{H})}(T, \mathcal{H}[i]) = 0$ for $i \gg 0$ (cf. Remark 2.21).

(T3) $\mathsf{End}_{\mathsf{D}(\mathcal{H})}(T)$ is isomorphic to a commutative noetherian ring R.

Condition (T1) assures that $\mathsf{D}(\mathcal{H})$ is compactly generated. Since T is compact, it belongs to $\mathsf{D}^b(\mathcal{G})$, see Proposition 2.9. Under this assumption, similarly to before, condition (T2) is equivalent to requiring the tilting t-structure $\mathbb{T}_T$ of $\mathsf{D}(\mathcal{H})$ associated to T to be intermediate. Conditions (T1) and (T3) imply that its heart $\mathcal{H}_T$ is isomorphic to $\mathsf{Mod}\text{-}R$, and we have a triangle equivalence

$$\mathsf{real}_T \colon \mathsf{D}(\mathsf{Mod}\text{-}R) = \mathsf{D}(\mathcal{H}_T) \to \mathsf{D}(\mathcal{H}).$$

Using the equivalences real_C and real_T, we see that these two settings are the two sides of the same picture: starting from Setting 3.1, the choices $\mathcal{H} := \mathcal{H}_C$ and $T := \mathsf{real}_C^{-1}(R)$ fit Setting 4.1; conversely, taking $C := \mathsf{real}_T^{-1}(W)$ for an injective cogenerator W of $\mathcal{H}$, one obtains the t-structure $\mathbb{T}_C$ as the pullback along real_T of the standard t-structure of $\mathsf{D}(\mathcal{H})$. In the following we will work with Setting 4.1, with Setting 3.1 serving as motivation.

Since $\mathcal{H}$ is locally coherent and $\mathsf{D}(\mathcal{H})$ is compactly generated (by T), we have the recollement of Theorem 2.14; and there is also the Krause's recollement for $\mathsf{Mod}\text{-}R$:

$$\mathsf{K_{ac}}(\mathsf{Inj}(\mathcal{H})) \leftrightarrows \mathsf{K}(\mathsf{Inj}(\mathcal{H})) \leftrightarrows \mathsf{D}(\mathcal{H}) \qquad \mathsf{K_{ac}}(\mathsf{Inj}(R)) \leftrightarrows \mathsf{K}(\mathsf{Inj}(R)) \leftrightarrows \mathsf{D}(\mathsf{Mod}\text{-}R)$$

Our goal is to construct an equivalence between these two recollements. In order to do that, we replace the derived equivalence real_C by another one which we are able to lift to the coderived level. We start by fixing a convenient resolution of T.

Lemma 4.2. *Up to shift, T admits a resolution $T := (F_{-n} \to F_{-n+1} \to \cdots \to F_0)$ with finitely presented objects $F_i \in \mathsf{fp}(\mathcal{H})$.*

Proof. Since T is compact, by Proposition 2.9 it belongs to $\mathsf{D}^b(\mathsf{fp}(\mathcal{H}))$, so it is quasi-isomorphic to a complex over the abelian category $\mathsf{fp}(\mathcal{H})$. By taking soft truncations this complex can be made strictly bounded. □

Now we consider the functor $\mathcal{H}\mathrm{om}(T,-)\colon \mathsf{C}(\mathcal{H}) \to \mathsf{C}(\mathbb{Z})$, defined as the totalisation of the bicomplex $\mathcal{H}\mathrm{om}^{\bullet,\bullet}(T,-)$. Notice that this bicomplex is always bounded along the direction of T (because we chose a strictly bounded resolution of T).

Since R is commutative, $\mathsf{D}(\mathcal{H}) \cong \mathsf{D}(\mathsf{Mod}\text{-}R)$ is an R-linear category, and then so is $\mathcal{H}$. The bicomplex $\mathcal{H}\mathrm{om}^{\bullet,\bullet}(T,-)$ and its totalisation $\mathcal{H}\mathrm{om}(T,-)$ have therefore terms in $\mathsf{Mod}\text{-}R$ and R-linear differentials; this gives us a functor

$$\Psi := \mathcal{H}\mathrm{om}(T,-)\colon \mathsf{C}(\mathcal{H}) \to \mathsf{C}(R). \tag{4}$$

Moreover, if $X \in \mathsf{C}(\mathcal{H})$ is contractible, then the rows of $\mathcal{H}\mathrm{om}^{\bullet,\bullet}(T,X)$ are also contractible, since $\mathsf{Hom}_{\mathcal{H}}(F_i,-)$ is an additive functor for all $-n \leq i \leq 0$. It follows that $\mathcal{H}\mathrm{om}(T,X) \in \mathsf{C}(R)$ is also contractible, which gives us a functor

$$\Psi := \mathcal{H}\mathrm{om}(T,-)\colon \mathsf{K}(\mathcal{H}) \to \mathsf{K}(R). \tag{5}$$

In particular, by restriction of the domain, Ψ induces functors on the subcategories $\mathsf{K}(\mathsf{Inj}(\mathcal{H})) \subseteq \mathsf{K}(\mathsf{fpInj}\text{−}\mathcal{H}) \subseteq \mathsf{K}(\mathcal{H})$, which we will continue to denote by Ψ.

We record immediately that Ψ induces a derived equivalence $\mathsf{D}(\mathcal{H}) \cong \mathsf{D}(\mathsf{Mod}\text{-}R)$.

Lemma 4.3. *The functor $\mathbb{R}\mathsf{Hom}_{\mathcal{H}}(T,-) := Q\Psi Q_r\colon \mathsf{D}(\mathcal{H}) \to \mathsf{D}(\mathsf{Mod}\text{-}R)$ is an equivalence. Moreover, it restricts to an equivalence $\mathsf{D}^b(\mathcal{H}) \to \mathsf{D}^b(\mathsf{Mod}\text{-}R)$, and also to an equivalence $\mathsf{D}^b(\mathsf{fp}(\mathcal{H})) \to \mathsf{D}^b(\mathsf{mod}\text{-}R)$.*

Proof. By (T1) and (T3) we have $\mathbb{R}\mathsf{Hom}_{\mathcal{H}}(T,T) \cong \mathsf{End}_{\mathsf{D}(\mathcal{H})}(T) \cong R$, so the functor $\mathbb{R}\mathsf{Hom}_{\mathcal{H}}(T,-)$ sends a compact generator of $\mathsf{D}(\mathcal{H})$ to a compact generator of $\mathsf{D}(\mathsf{Mod}\text{-}R)$. Moreover, since $\mathbb{R}\mathsf{Hom}_{\mathcal{H}}(T,-)$ is R-linear on Hom-sets, it must induce the isomorphism $\mathsf{End}_{\mathsf{D}(\mathcal{H})}(T) \cong \mathsf{End}_R(R) = R$ of endomorphism rings. Since T is a compact generator of $\mathsf{D}(\mathcal{H})$ and R is a compact generator of $\mathsf{D}(\mathsf{Mod}\text{-}R)$, a standard arguments shows that $\mathbb{R}\mathsf{Hom}_{\mathcal{H}}(T,-)$ induces an equivalence $\mathsf{D}(\mathcal{H})^c \xrightarrow{\sim} \mathsf{D}(\mathsf{Mod}\text{-}R)^c$ between the categories of compact objects (see e.g. [26, Proposition 6]). Lastly, $\mathbb{R}\mathsf{Hom}_{\mathcal{H}}(T,-)$ preserves coproducts, since T is compact. Then, the derived equivalence is established by double *dévissage* (Lemma 1.2).

For the claim about the bounded equivalence, let $X \in \mathsf{D}(\mathcal{H})$. Then its image $\mathbb{R}\mathsf{Hom}_{\mathcal{H}}(T,X)$ belongs to $\mathsf{D}^b(\mathsf{Mod}\text{-}R)$ if and only if $\mathsf{Hom}_{\mathsf{D}(\mathcal{H})}(T,X[i]) = 0$ for all but finitely many $i \in \mathbb{Z}$; and this means that X has finitely many cohomologies with respect to $\mathbb{T}_T$. Since $\mathbb{T}_T$ is intermediate, this is equivalent to X belonging to $\mathsf{D}^b(\mathcal{H})$. Therefore, $\mathbb{R}\mathsf{Hom}_{\mathcal{H}}(T,-)$ restricts to an equivalence $\mathsf{D}^b(\mathcal{H}) \xrightarrow{\sim} \mathsf{D}^b(\mathsf{Mod}\text{-}R)$, and therefore also to an equivalence $\mathsf{D}^b(\mathcal{H})^c \xrightarrow{\sim} \mathsf{D}^b(\mathsf{Mod}\text{-}R)^c$ between compact objects of the bounded derived categories. By Lemma 3.3 and [37, Corollary 6.17], this last equivalence is the same as the desired equivalence $\mathsf{D}^b(\mathsf{fp}(\mathcal{H})) \xrightarrow{\sim} \mathsf{D}^b(\mathsf{mod}\text{-}R)$. □

Lemma 4.4. *$\Psi\colon \mathsf{C}(\mathcal{H}) \to \mathsf{C}(R)$ preserves direct limits (and in particular coproducts). Therefore, also the induced functor $\Psi\colon \mathsf{K}(\mathcal{H}) \to \mathsf{K}(R)$ and its restriction $\Psi\colon \mathsf{K}(\mathsf{fpInj}\text{−}\mathcal{H}) \to \mathsf{K}(R)$ preserve coproducts.*

Proof. Coproducts in $\mathsf{K}(\mathcal{H})$ are computed termwise, as in $\mathsf{C}(\mathcal{H})$. Moreover, since $\mathsf{fpInj}\text{−}\mathcal{H}$ is closed under coproducts in $\mathcal{H}$, coproducts in $\mathsf{K}(\mathsf{fpInj}\text{−}\mathcal{H})$ are computed as in $\mathsf{K}(\mathcal{H})$. It is then enough to prove the claim for $\Psi\colon \mathsf{C}(\mathcal{H}) \to \mathsf{C}(R)$.

Now, let $X_\alpha := (\cdots \to X_\alpha^i \to X_\alpha^{i+1} \to \cdots) \in \mathsf{C}(\mathcal{H})$ be a direct system of objects, and consider their direct limit $\varinjlim X_\alpha = (\cdots \to \varinjlim X_\alpha^i \to \varinjlim X_\alpha^{i+1} \to \cdots)$. Ψ sends it to the totalisation of the bicomplex

$$\begin{array}{ccccccc}
\cdots \longrightarrow & \mathsf{Hom}_{\mathcal{H}}(F_0, \varinjlim X_\alpha^i) & \longrightarrow & \mathsf{Hom}_{\mathcal{H}}(F_0, \varinjlim X_\alpha^{i+1}) & \longrightarrow \cdots \\
& \downarrow & & \downarrow & \\
\cdots \longrightarrow & \mathsf{Hom}_{\mathcal{H}}(F_{-1}, \varinjlim X_\alpha^i) & \longrightarrow & \mathsf{Hom}_{\mathcal{H}}(F_{-1}, \varinjlim X_\alpha^{i+1}) & \longrightarrow \cdots \\
& \downarrow & & \downarrow & \\
& \vdots & & \vdots & \\
& \downarrow & & \downarrow & \\
\cdots \longrightarrow & \mathsf{Hom}_{\mathcal{H}}(F_{-n}, \varinjlim X_\alpha^i) & \longrightarrow & \mathsf{Hom}_{\mathcal{H}}(F_{-n}, \varinjlim X_\alpha^{i+1}) & \longrightarrow \cdots
\end{array}$$

Since the F_i's are finitely presented in $\mathcal{H}$, the functors $\mathsf{Hom}_{\mathcal{H}}(F_i, -)$ commute naturally with the direct limits, so $\mathcal{H}\mathrm{om}^{\bullet,\bullet}(T, \varinjlim X_\alpha)$ is isomorphic in $\mathsf{C}(\mathsf{C}(\mathcal{H}))$ to the direct limit of the bicomplexes $\mathcal{H}\mathrm{om}^{\bullet,\bullet}(T, X_\alpha)$. Totalisation also commutes with direct limits, and so Ψ preserves them. □

In order to obtain a functor between the coderived categories, we want Ψ to preserve coacyclicity. Recall that a locally finitely presentable Grothendieck category $\mathcal{G}$ admits a natural notion of a pure exact sequence, and that a complex in $\mathsf{C}(\mathcal{G})$ is **pure-acyclic** if it is acyclic and in addition, each exact sequence $0 \to Z^i(X) \to X^i \to Z^{i+1}(X) \to 0$ induced by the cocycles is pure exact in $\mathcal{G}$.

We start by recalling the following fact.

Proposition 4.5 ([43])**.** *Over a locally coherent Grothendieck category, pure-acyclic complexes are coacyclic.*

Proof. This follows mainly from [43, §6.2]; we recollect the argument for the convenience of the reader. Let $\mathcal{G}$ be a locally coherent Grothendieck category, and X a complex in $\mathsf{C}(\mathcal{G})$. Then X corresponds to a coacyclic object of $\mathsf{K}(\mathcal{G})$ if and only if it is $\mathsf{Ext}^1_{\mathsf{C}}$-orthogonal to $\mathsf{C}(\mathsf{Inj}(\mathcal{G}))$, i.e. if it belongs to the left class of the functorially complete cotorsion pair generated by disks of $\mathsf{fp}(\mathcal{G})$. Now, this left class is closed under retracts and transfinite extensions, and pure-acyclic complexes are (retracts of) transfinite extensions of disks of $\mathsf{fp}(\mathcal{G})$ in $\mathsf{C}(\mathcal{G})$ by [43, Lemma 5.6]. □

Lemma 4.6. *The restriction* $\Psi\colon \mathsf{K}(\mathsf{fpInj}{-}\mathcal{H}) \to \mathsf{K}(R)$ *preserves coacyclic complexes.*

Proof. As a partial converse of Proposition 4.5, a complex $X \in \mathsf{C}(\mathcal{H})$ of fp-injectives is coacyclic in $\mathsf{K}(\mathcal{H})$ if and only if it is pure-acyclic [43, Proposition 6.11]. By [43, Lemma 4.14], a complex X in $\mathsf{C}(\mathcal{H})$ is pure-acyclic if and only if it is a direct limit of bounded contractible complexes. Since $\Psi\colon \mathsf{C}(\mathcal{H}) \to \mathsf{C}(R)$ preserves both direct limits (Lemma 4.4) and contractibility, $\Psi(X)$ will also be pure-acyclic by the same characterisation. Then we conclude by Proposition 4.5 that $\Psi(X)$ is also coacyclic. □

In view of the equivalences

$$\mathsf{K}(\mathsf{Inj}(\mathcal{H})) \xrightarrow[\cong]{\subseteq} \mathsf{K}(\mathsf{fpInj}{-}\mathcal{H})/\{\text{pure acyclics}\} \xrightarrow[\cong]{} \mathsf{D}^{\mathsf{co}}(\mathcal{H})$$

by Lemma 4.6 we deduce that Ψ induces a functor

$$\mathbb{R}^{\mathsf{co}}\Psi\colon \mathsf{D}^{\mathsf{co}}(\mathcal{H}) \to \mathsf{D}^{\mathsf{co}}(R). \tag{6}$$

On an object $X \in \mathsf{D}^{\mathsf{co}}(\mathcal{H})$, $\mathbb{R}^{\mathsf{co}}\Psi$ is computed by first resolving X by a complex of fp-injectives (or even injectives), then applying Ψ and considering the resulting complex as an object of $\mathsf{D}^{\mathsf{co}}(R)$. When identifying $\mathsf{D}^{\mathsf{co}}(\mathcal{H}) \cong \mathsf{K}(\mathsf{Inj}(\mathcal{H}))$ and $\mathsf{D}^{\mathsf{co}}(R) \cong \mathsf{K}(\mathsf{Inj}(R))$, $\mathbb{R}^{\mathsf{co}}\Psi$ is then the composition

$$\mathbb{R}^{\mathsf{co}}\Psi\colon \quad \mathsf{K}(\mathsf{Inj}(\mathcal{H})) \xrightarrow{\subseteq} \mathsf{K}(\mathcal{H}) \xrightarrow{\Psi} \mathsf{K}(R) \xrightarrow{I_\lambda} \mathsf{K}(\mathsf{Inj}(R)). \tag{7}$$

Proposition 4.7. $\mathbb{R}^{\mathsf{co}}\Psi\colon \mathsf{D}^{\mathsf{co}}(\mathcal{H}) \to \mathsf{D}^{\mathsf{co}}(R)$ *is an equivalence.*

Proof. We want to argue by double *dévissage*.

First, $\mathbb{R}^{\mathsf{co}}\Psi\colon \mathsf{D}^{\mathsf{co}}(\mathcal{H}) \to \mathsf{D}^{\mathsf{co}}(R)$ preserves coproducts, since Ψ does (Lemma 4.4).

Now we show that it induces an equivalence between the subcategories of compact objects. In view of the identification $\mathsf{D}^{\mathsf{co}}(\mathcal{H}) \cong \mathsf{K}(\mathsf{Inj}(\mathcal{H}))$, a compact object of $\mathsf{D}^{\mathsf{co}}(\mathcal{H})$ is identified with the dg-injective resolution X of an object in $\mathsf{D}^b(\mathsf{fp}(\mathcal{H}))$; in particular, this is a bounded below complex. When we apply Ψ and then I_λ, as in (7), we obtain again a bounded below complex, first in $\mathsf{K}(R)$ and then in $\mathsf{K}(\mathsf{Inj}(R))$. This last object $Y := I_\lambda\Psi(X)$, in particular, is a dg-injective complex. Since we have $X \cong Q_rQX$ and $Y \cong Q_rQY$ in $\mathsf{K}(\mathsf{Inj}(\mathcal{H}))$ and $\mathsf{K}(\mathsf{Inj}(R))$, respectively, we can write

$$\mathbb{R}^{\mathsf{co}}\Psi(X) = Y \cong Q_rQY = Q_rQI_\lambda\Psi X = Q_rQ\Psi X \cong Q_rQ\Psi Q_rQX =: (*)$$

Now, by definition, $\mathbb{R}\mathsf{Hom}_{\mathcal{H}}(T,-) := Q\Psi Q_r$, so we can continue

$$(*) = Q_r\,\mathbb{R}\mathsf{Hom}_{\mathcal{H}}(T, QX)$$

It is therefore sufficient to show that $Q_r\,\mathbb{R}\mathsf{Hom}_{\mathcal{H}}(T, Q-)$ is an equivalence between $\mathsf{K}(\mathsf{Inj}(\mathcal{H}))^c$ and $\mathsf{K}(\mathsf{Inj}(R))^c$. Now, $Q\colon \mathsf{K}(\mathsf{Inj}(\mathcal{H}))^c \to \mathsf{D}^b(\mathsf{fp}(\mathcal{H}))$ and $Q_r\colon \mathsf{D}^b(\mathsf{mod}\text{-}R) \to K(\mathsf{Inj}(R))^c$ are equivalences, and $\mathbb{R}\mathsf{Hom}_{\mathcal{H}}(T,-)\colon \mathsf{D}^b(\mathsf{fp}(\mathcal{H})) \to \mathsf{D}^b(\mathsf{mod}\text{-}R)$ is an equivalence by Lemma 4.3. □

Now that we have the equivalence between the coderived categories, we show that it preserves the recollements. First we need a technical lemma.

Lemma 4.8. *$I_\lambda T \cong Q_lQI_\lambda T$ in $\mathsf{D}^{\mathsf{co}}(\mathcal{H}) \cong \mathsf{K}(\mathsf{Inj}(\mathcal{H}))$.*

Proof. Let E be the dg-injective resolution of T; we have a triangle in $\mathsf{K}(\mathcal{H})$

$$A \to T \to E \to A[1]$$

with A acyclic. Since T is bounded below, E and then A are also bounded below. A is therefore also coacyclic. This means that $E \cong I_\lambda T$ in $\mathsf{K}(\mathsf{Inj}(\mathcal{H}))$. Now, since E is dg-injective we have $E \cong Q_rQE \cong Q_rQT$; but QT is compact in $\mathsf{D}(\mathcal{H})$, and therefore $Q_rQT \cong Q_lQT$ by Lemma 2.13. We conclude as wanted that $I_\lambda T \cong Q_lQT$ in $\mathsf{K}(\mathsf{Inj}(\mathcal{H}))$. □

Lemma 4.9. *$\mathbb{R}^{\mathsf{co}}\Psi\colon \mathsf{D}^{\mathsf{co}}(\mathcal{H}) \to \mathsf{D}^{\mathsf{co}}(R)$ preserves acyclics.*

Proof. Identifying $\mathsf{D}^{\mathsf{co}}(\mathcal{H}) \cong \mathsf{K}(\mathsf{Inj}(\mathcal{H}))$ and in view of (7), let $X \in \mathsf{K}(\mathsf{Inj}(\mathcal{H}))$ be acyclic. For every $n \in \mathbb{Z}$ we have

$$\begin{aligned} H^nI_\lambda\Psi X &\cong H^n\Psi X = H^n\,\mathcal{H}\mathrm{om}(T,X) \cong \\ &\cong \mathsf{Hom}_{\mathsf{K}(\mathcal{H})}(T, X[n]) \cong \mathsf{Hom}_{\mathsf{K}(\mathsf{Inj}(\mathcal{H}))}(I_\lambda T, X[n]) \overset{(1)}{\cong} \\ &\cong \mathsf{Hom}_{\mathsf{K}(\mathsf{Inj}(\mathcal{H}))}(Q_lQT, X[n]) \cong \mathsf{Hom}_{\mathsf{D}(\mathcal{H})}(QT, QX[n]) \overset{(2)}{=} 0 \end{aligned}$$

where (1) is by Lemma 4.8 and (2) because $QX = 0$. □

Theorem 4.10. *$\mathbb{R}^{\mathsf{co}}\Psi\colon \mathsf{D}^{\mathsf{co}}(\mathcal{H}) \to \mathsf{D}^{\mathsf{co}}(R)$ induces an equivalence of recollements, that is, there is a diagram*

$$\begin{array}{ccccc} \mathsf{S}(\mathcal{H}) & \Longleftrightarrow & \mathsf{D}^{\mathsf{co}}(\mathcal{H}) & \Longleftrightarrow & \mathsf{D}(\mathcal{H}) \\ {\scriptstyle \mathbb{S}\Psi}\downarrow{\scriptstyle\cong} & & {\scriptstyle \mathbb{R}^{\mathsf{co}}\Psi}\downarrow{\scriptstyle\cong} & & {\scriptstyle \mathbb{R}\Psi}\downarrow{\scriptstyle\cong} \\ \mathsf{S}(\mathsf{Mod}\text{-}R) & \Longleftrightarrow & \mathsf{D}^{\mathsf{co}}(\mathsf{Mod}\text{-}R) & \Longleftrightarrow & \mathsf{D}(\mathsf{Mod}\text{-}R) \end{array}$$

in which the rows are the recollements from Remark 2.17 of $\mathcal{H}$ and $\mathsf{Mod}\text{-}R$ and such that all the six obvious squares commute.

Proof. Identify $\mathsf{D}^{\mathsf{co}}(\mathcal{H}) \cong \mathsf{K}(\mathsf{Inj}(\mathcal{H}))$ and $\mathsf{D}^{\mathsf{co}}(R) \cong \mathsf{K}(\mathsf{Inj}(R))$. By Proposition 4.7, $\mathbb{R}^{\mathsf{co}}\Psi$ is an equivalence. By Lemma 4.9, it preserves acyclicity. In view of basic results on recollement equivalences (see Section 1.3), it is enough to show that the following square is commutative up to equivalence

$$\begin{array}{ccc} \mathsf{K}(\mathsf{Inj}(\mathcal{H})) & \xrightarrow{Q} & \mathsf{D}(\mathcal{H}) \\ {\scriptstyle \mathbb{R}^{\mathsf{co}}\Psi}\downarrow\cong & & {\scriptstyle \mathbb{R}\Psi}\downarrow\cong \\ \mathsf{K}(\mathsf{Inj}(R)) & \xrightarrow{Q} & \mathsf{D}(\mathsf{Mod}\text{-}R) \end{array}$$

where $\mathbb{R}\Psi = \mathbb{R}\mathsf{Hom}_{\mathcal{H}}(T, -)$. Since $\mathbb{R}^{\mathsf{co}}\Psi$ preserves acyclics, the composition $Q\mathbb{R}^{\mathsf{co}}\Psi$ kills objects from $\mathsf{K}_{\mathsf{ac}}(\mathsf{Inj}(\mathcal{H}))$, and thus the approximation triangle with respect to the stable t-structure $(\mathsf{K}_{\mathsf{ac}}(\mathsf{Inj}(\mathcal{G})), Q_r(\mathsf{D}(\mathcal{G}))$ in $\mathsf{K}(\mathsf{Inj}(\mathcal{G}))$ yields a natural equivalence $Q\mathbb{R}^{\mathsf{co}}\Psi \cong Q\mathbb{R}^{\mathsf{co}}\Psi Q_r Q$. Then we can compute similarly as in Proposition 4.7:

$$Q\mathbb{R}^{\mathsf{co}}\Psi Q_r Q = QI_\lambda \Psi Q_r Q \cong Q\Psi Q_r Q = \mathbb{R}\Psi Q.$$

The rest follows by denoting the induced triangle equivalence $\mathsf{S}(\mathcal{H}) \to \mathsf{S}(\mathsf{Mod}\text{-}R)$ by $\mathbb{S}\Psi$. □

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