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Characterizations of weakly K-analytic and Vašák spaces using projectional skeletons and using separable PRI

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CHARACTERIZATIONS OF WEAKLY $\mathcal{K}$-ANALYTIC AND VAŠÁK SPACES USING PROJECTIONAL SKELETONS AND USING SEPARABLE PRI

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Abstract. We find characterizations of Vašák spaces and weakly $\mathcal{K}$-analytic spaces using the notions of separable projectional resolution of the identity (SPRI) and of projectional skeleton. This in particular solves a problem suggested by M. Fabian and V. Montesinos. Our method of proof also gives similar characterizations of WCG spaces and their subspaces (some aspects of which were known, some are new). Moreover we show that for countably many projectional skeletons on a Banach space there exists a common subskeleton $(P_s)_{s\in\Gamma}$, which is in addition indexed by the ranges of the projections $\{P_s : s \in \Gamma\}$.

1. Introduction

Banach spaces with a projectional skeleton form quite a rich class of Banach spaces (most importantly nonseparable ones), which shares some structural properties with the separable ones. Examples of Banach spaces admitting a projectional skeleton come from several areas of functional analysis such as preduals of Von Neumann algebras [3], preduals of JBW* -triples [5], duals of Asplund spaces [22] and several examples of $\mathcal{C}(K)$ spaces [25].

One of the key aspects behind the definition of a projectional skeleton is the possibility to build a *projectional resolution of the identity* (PRI) on a Banach space, which is a family indexed by the density of the space of bounded projections onto subspaces of smaller densities that enables us to use effectively transfinite induction, see e.g. [16, Sections 6-7] for the definition of PRI and applications. In 1968, D. Amir and J. Lindenstrauss [1] proved that a PRI exists in every *weakly compactly generated space* (WCG). Later there were found several superclasses of WCG spaces, for which a PRI exists as well. Those include the following classes of Banach spaces (where

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inclusions are always strict, SWCG denotes the class of subspaces of WCG spaces and WLD the class of weakly Lindelöf determined spaces)

$$WCG \subset SWCG \subset \text{weakly } \mathcal{K}\text{-analytic} \subset \text{Vašák} \subset WLD \subset \text{Plichko}.$$

A space for which the existence of a PRI is known, but is not Plichko is $\mathcal{C}([0,\kappa])$ for regular cardinals $\kappa \geq \omega_2$, see [19]. We refer the reader e.g. to monographs [12, 16, 17], where more may be found about the classes of Banach spaces mentioned above. In 2009, W. Kubiś [22] introduced the notion of projectional skeletons. The class of Banach spaces that admit a projectional skeleton is contained in the class of Banach spaces admitting a PRI and it contains all Plichko spaces as well as the spaces $\mathcal{C}([0,\kappa])$, for any ordinal κ. Nowadays, it seems that the class of Banach spaces with a projectional skeleton contains all the important classes of Banach spaces admitting PRI, which enables us to provide a uniform treatment for several known results proved previously for each subclass separately. Quite recently, this feature was somehow precised in [21, Theorem 1.1], where it was proved that spaces admitting a 1-projectional skeleton form a "$\mathcal{P}$-class" (that is, they admit PRI with certain additional property). Consequently, they admit LUR renorming or a strong Markushevich basis, see [27] and [17, Theorem 5.1].

Recently, several authors found characterizations of some classes of Banach spaces in terms of projectional skeletons. Those include characterizations of Plichko spaces, WLD spaces, Asplund spaces, WLD Asplund spaces, WCG spaces and SWCG spaces, see [10, 15, 22]. From the classes mentioned previously, characterizations of weakly $\mathcal{K}$-analytic spaces and of Vašák spaces in terms of projectional skeleton were not known and actually those were proposed as a "challenge" in [15]. We accepted this challenge and solved the problem, which might be considered as the main outcome of our paper. Our characterization of weakly $\mathcal{K}$-analytic Banach spaces is the following (see Sections 2 and 4 for the relevant definitions).

Theorem A. *Let X be a Banach space with $\kappa := \operatorname{dens}(X)$. Then the following conditions are equivalent.*

(i) X is weakly $\mathcal{K}$-analytic.
(ii) There exist a projectional skeleton $\mathfrak{s} = (P_s)_{s\in\Gamma}$ on X and a family of non-empty sets $\{A_t \subset B_X : t \in \omega^{<\omega}\}$ satisfying the following conditions
  (a) the set $\mathcal{A} := \bigcup_{\sigma\in\omega^\omega} \bigcap_{i\geq 1} A_{\sigma|i}$ is linearly dense in X and countably supports X^*. Moreover $A_\emptyset = \mathcal{A}$ and $A_t \subset \mathcal{A}$, for every $t \in \omega^{<\omega}$;
  (b) for every $\varepsilon > 0$, $x^* \in X^*$ and $\sigma \in \omega^\omega$, there exists $i \in \omega$ such that $\mathfrak{s}$ is $(A_{\sigma|i}, \varepsilon)$-shrinking in x^*.
(iii) There exist a SPRI $(Q_\alpha)_{\alpha\leq\kappa}$ in X and a family of non-empty sets $\{A_t \subset B_X : t \in \omega^{<\omega}\}$ satisfying the following conditions

(3a) *the set* $\mathcal{A} := \bigcup_{\sigma\in\omega^\omega}\bigcap_{i\geq 1} A_{\sigma|i}$ *is linearly dense in* X *and countably supports* X^**. Moreover* $A_\emptyset = \mathcal{A}$ *and* $A_t \subset \mathcal{A}$*, for every* $t \in \omega^{<\omega}$*;*

(3b) *for every* $x \in \mathcal{A}$ *it holds that* $\{Q_\alpha(x)\colon \alpha \leq \kappa\} \subset \{0, x\}$ *and* $\min\{\alpha \leq \kappa\colon Q_\alpha x = x\}$ *is not a limit ordinal;*

(3c) *for every* $\varepsilon > 0$*,* $x^* \in X^*$*, and* $\sigma \in \omega^\omega$ *there exists* $i \in \omega$ *such that* $(Q_\alpha)_{\alpha\leq\kappa}$ *is* $(A_{\sigma|i}, \varepsilon)$*-shrinking in* x^**.*

It is worth mentioning that there is a rich theory concerning the class of weakly $\mathcal{K}$-analytic spaces, see e.g. [16, Section 4.4] and references therein. From the more recent contributions we should mention the paper [2] where the authors solved a long standing open problem by M. Talagrand by finding an example of a weakly $\mathcal{K}$-analytic space X which is not $F_{\sigma\delta}$ in (X^{**}, w^*). This particular result motivated even some very recent works, see e.g. [18].

The characterization of Vašák spaces is presented in Theorem 27. Note that Vašák spaces are known also under the name *weakly countably determined spaces*. We refer the interested reader to [20] for more information and recent contributions to the study of this class of spaces.

Our characterizations are inspired by the papers [13] and [15]. In [13], characterizations of those classes were found but the notion of a skeleton is not used and "shrinkingness" is replaced with a certain combinatorial property. This probably inspired M. Fabian and V. Montesinos [15] to characterize WCG and SWCG spaces using the notion of a projectional skeleton, where the combinatorial condition from [13] was replaced by the notion of "shrinkingness". The hard part in the proof of the characterizations presented in [15] was to show that a certain property of a projectional skeleton is inherited by the PRI that comes from this skeleton and then a transfinite induction argument was used. However, when dealing with Vašák spaces and weakly $\mathcal{K}$-analytic spaces, this approach fails and we were forced to go further. Namely, we proved that a certain property of a skeleton is actually inherited also by the *separable projectional resolution of the identity* (SPRI) that comes from this skeleton, which is essentially what is contained in the proof of the implication (ii)⇒(iii) in Theorem A. As a consequence, we do not only have a characterization using the notion of a projectional skeleton, but also using SPRI, which seems to be of independent interest. Using the developed techniques we were also able to provide similar characterizations of WCG and SWCG spaces using the notion of SPRI, which is new as well.

We would like to stress out that condition (ii) from Theorem A is not that much dependent on a particular projectional skeleton. This is witnessed by the following consequence of Theorem 18, which seems to be of an independent interest and it might be considered as one of the main outcomes of this paper as well.

Theorem B. *Let* X *be a Banach space and* $\{\mathfrak{s}_n : n \in \omega\}$ *be a countable family of projectional skeletons on* X *inducing the same set* $D \subset X^*$*. Then* *there exists a simple projectional skeleton on X which is isomorphic to a subskeleton of $\mathfrak{s}_n$, for every $n \in \omega$.*

Note that Theorem B implies that some properties of projectional skeletons are not dependent on the particular skeleton but rather on the set it induces. A result of similar purpose was proved already in [21, Lemma 3.5], but our Theorem B is stronger, easier for applications and it generalizes [11, Theorem 4.1], as it shows that up to passing to a subskeleton, we may assume that any projectional skeleton is *simple*, that is, indexed by the ranges of the corresponding projections. We refer the reader to Remark 19 for some more comments concerning the novelty of this result.

Let us now briefly describe the content of each section.

Section 2 contains basic notations and some preliminary results.

In Section 3 we deal with the construction of canonical projections induced by projectional skeletons. To this aim, we use the method of suitable models. The main outcome here is the proof of Theorem B. Some technical constructions used in Section 4 are proved as well, those are concentrated in Subsection 3.3.

In Section 4 we prove our main results, that is, our characterizations of WCG, SWCG, weakly $\mathcal{K}$-analytic and Vašák spaces using the notions of projectional skeletons and SPRI. In Section 5 we suggest further directions for potential research.

2. Notation and Preliminaries

We use standard notation in Banach space theory as can be found in [14]. Let us recall the notion of a projectional skeleton on a Banach space. Let $(\Gamma, \leq)$ be an up-directed partially ordered set. We say that a sequence $(s_n)_{n\in\omega}$ of elements of Γ is increasing if $s_n \leq s_{n+1}$, for every $n \in \omega$. We say that Γ is *σ-complete* if for every increasing sequence $(s_n)_{n\in\omega}$ in Γ there exists $\sup_n s_n$ in Γ. If Γ is σ-complete, then we say that a subset Γ' of Γ is *σ-closed* in Γ if for every increasing sequence $(s_n)_{n\in\omega}$ in Γ', it holds that $\sup_n s_n \in \Gamma'$; finally, given $A \subset \Gamma$ we denote by A_σ the smallest σ-closed subset of Γ containing A.

Definition 1. Let X be a Banach space. A *projectional skeleton* on X is a family of bounded linear projections $\mathfrak{s} = (P_s)_{s\in\Gamma}$ indexed on a up-directed and σ-complete set Γ satisfying:

(1) $P_s[X]$ is separable, for every $s \in \Gamma$;
(2) if $s \leq t$, then $P_s = P_s \circ P_t = P_t \circ P_s$;
(3) if $(s_n)_{n\in\omega}$ is an increasing sequence in Γ and $s = \sup_{n\in\omega} s_n$, then $P_s[X] = \overline{\bigcup_{n\in\omega} P_{s_n}[X]}$;
(4) $X = \bigcup_{s\in\Gamma} P_s[X]$.

We say that $\bigcup_{s\in\Gamma} P_s^*[X^*]$ is the *set induced by* $\mathfrak{s}$ and denote it by $D(\mathfrak{s})$.

We introduce also the following definition, which we use further as well.

Definition 2. Let X be a Banach space and let $\mathfrak{s} = (P_s)_{s\in\Gamma}$ be a projectional skeleton on X.

- We say that $\mathfrak{s}' = (P_s)_{s\in\Gamma'}$ is a *subskeleton* of $\mathfrak{s}$ if Γ' is a σ-closed subset of Γ and $\mathfrak{s}'$ is a projectional skeleton on X.
- We say that a projectional skeleton $(Q_\lambda)_{\lambda\in\Lambda}$ on X is *isomorphic* to $\mathfrak{s}$ if there exists an order-isomorphism $\phi : \Lambda \to \Gamma$ such that $Q_\lambda = P_{\phi(\lambda)}$, for every $\lambda \in \Lambda$.

Note that if $\mathfrak{s}'$ is a subskeleton of $\mathfrak{s}$, then $D(\mathfrak{s}') = D(\mathfrak{s})$. Indeed, it is clear that $D(\mathfrak{s}') \subset D(\mathfrak{s})$ and thus it follows from [8, Lemma 3.2] that $D(\mathfrak{s}') = D(\mathfrak{s})$.

It follows from [22, Proposition 9 and Lemma 10] that given a projectional skeleton $\mathfrak{s} = (P_s)_{s\in\Gamma}$ on a Banach space X, up to passing to a subskeleton, we can assume that there exists $C \geq 1$ such that $\|P_s\| \leq C$, for every $s \in \Gamma$ and the following stronger version of condition (3) holds:

(3') If $(s_n)_{n\in\omega}$ is an increasing sequence in Γ and $s = \sup_{n\in\omega} s_n$, then $\lim_{n\to\infty} P_{s_n}(x) = P_s(x)$, for every $x \in X$.

In next lemma we recall one aspect of the strong relationship between projectional skeletons and norming subspaces. Given a Banach space X and a real number $C \geq 1$, a subset D of X^* is said to be *C-norming* if for every $x \in X$ it holds that

$$\|x\| \leq C \sup\{|d(x)|/\|d\| : d \in D \setminus \{0\}\}.$$

Lemma 3. *If $\mathfrak{s}$ is a projectional skeleton on a Banach space X, then there exists $C \geq 1$ such that $D(\mathfrak{s})$ is a C-norming subspace of X^*.*

Proof. See [21, Lemma 1.3]. □

In the statement of Theorem B, following [11], we use the notion of a simple projectional skeleton. Let us recall it.

Definition 4. Let X be a Banach space and $\mathcal{F}$ be a family of closed separable subspaces of X. We say $\mathcal{F}$ is *rich* if

(1) each closed separable subspace of X is contained in an element of $\mathcal{F}$ and
(2) for every increasing sequence $(F_n)_{n\in\mathbb{N}}$ in $\mathcal{F}$, we have $\overline{\bigcup_{n\in\mathbb{N}} F_n} \in \mathcal{F}$.

A *simple projectional skeleton* is a family of projections $(P_F)_{F\in\mathcal{F}}$ on a Banach space X indexed by a rich family $\mathcal{F}$ ordered by inclusion such that

(1) for every $F \in \mathcal{F}$ we have $P_F(X) = F$ and
(2) if $E \subset F$ in $\mathcal{F}$, then $P_E = P_E \circ P_F = P_F \circ P_E$.

In our main results we use the notion of SPRI and of shrinkingness. The notion of SPRI is nowadays classical, see e.g. [16, Definition 6.2.6]. The notion of shrinkingness as we introduce below is new and it is inspired by related concepts of shrinkingness introduced in [15].

Definition 5. Let X be a nonseparable Banach space. We say that a family of bounded linear projections $(Q_\alpha)_{\alpha\in[0,\operatorname{dens} X]}$ is a *separable projectional* *resolution of the identity (SPRI)* in X, if $Q_0 = 0$, $Q_{\operatorname{dens} X} = \mathrm{Id}$ and for every $\alpha \in [0, \operatorname{dens} X)$ the following holds

(1) $(Q_{\alpha+1} - Q_\alpha)[X]$ is separable,
(2) $Q_\alpha Q_\beta = Q_\beta Q_\alpha = Q_\beta$ for $\beta \in [0, \alpha]$, and
(3) $x \in \overline{\operatorname{span}}\{(Q_{\alpha+1} - Q_\alpha)x \colon 0 \le \alpha < \operatorname{dens} X\}$ for every $x \in X$.

Let X be a Banach space and $A \subset X$ be a non-empty bounded set. We denote by $\rho_A : X^* \times X^* \to \mathbb{R}$ the pseudo-metric on X^* given by

$$\rho_A(x^*, y^*) := \sup_{x \in A} |x^*(x) - y^*(x)|, \ (x^*, y^*) \in X^* \times X^*.$$

Definition 6. Let X be a Banach space, Γ be a partially ordered set, $\mathfrak{s} = \{P_s : s \in \Gamma\}$ be a family of bounded projections on X and A be a bounded subset of X. Given $\varepsilon \ge 0$ and $x^* \in X^*$, we say that $\mathfrak{s}$ is (A, ε)-*shrinking in* x^* if for every increasing sequence $(s_n)_{n\in\omega}$ of elements of Γ such that $s = \sup_{n\in\omega} s_n$ exists in Γ, it holds that $\limsup_{n\in\omega} \rho_A(P^*_{s_n}(x^*), P^*_s(x^*)) \le \varepsilon\|x^*\|$.

Given a bounded subset $\mathcal{A}$ of X, we define:

$$\mathcal{T}(\mathfrak{s}, \mathcal{A}) := \{(\varepsilon, A, x^*) : \varepsilon \ge 0, A \subset \mathcal{A}, x^* \in X^* \text{ and } \mathfrak{s} \text{ is } (A, \varepsilon)\text{-shrinking in } x^*\}.$$

Finally, we collect some notions related to the class of Plichko spaces. Given a subset $\mathcal{A}$ of a Banach space X and $x^* \in X^*$, we say that $\mathcal{A}$ *countably supports* x^* if $\operatorname{suppt}_{\mathcal{A}}(x^*) := \{x \in \mathcal{A} : x^*(x) \neq 0\}$ is countable and we say that $\mathcal{A}$ *countably supports* a subset D of X^* if $\mathcal{A}$ countably supports every element of D. We recall that a Banach space is *Plichko* if there exists a C-norming set $D \subset X^*$ and a linearly dense set $\mathcal{A} \subset X$ such that $\mathcal{A}$ countably supports D. It is nowadays well-known that a Banach spaces admits a commutative projectional skeleton if and only if it is a Plichko space, see [22, Theorem 27] for a proof using suitable models and [21, Theorem 3.1] for a proof not using suitable models.

3. Projections associated to suitable models

In this Section we try to convince the reader that the natural approach when studying Banach spaces with a projectional skeleton is to use the method of suitable models. The results which are needed in further Sections are concentrated in the last subsection.

3.1. **Preliminaries concerning suitable models.**

Here we settle the notation and give some basic observations concerning suitable models. We refer the interested reader to [7], where more details about this method may be found (warning: in [7] only **countable** models were considered, while here we consider suitable models which are not necessarily countable).

Let M be a fixed set and ϕ be a formula in the language of ZFC. Then the *relativization of* ϕ *to* M is the formula ϕ^M which is obtained from ϕ by replacing each quantifier of the form "$\exists x$" by "$\exists x \in M$" and each quantifier of the form "$\forall x$" by "$\forall x \in M$". If $\phi(x_1, \ldots, x_n)$ is a formula with all free variables shown, then ϕ *is absolute for* M if

$$\forall a_1, \ldots, a_n \in M \quad (\phi^M(a_1, \ldots, a_n) \leftrightarrow \phi(a_1, \ldots, a_n)).$$

Definition 7. Let Φ be a finite list of formulas and X be any set. If M is a set such that $X \subset M$ and every formula of Φ is absolute for M, then we say that M is a *suitable model for* Φ *containing* X and we write $M \prec (\Phi; X)$.

Note that the Reflexion Theorem [23, Theorem IV.7.4] ensures that given a set X and a finite list of formulas Φ, there exists a set M such that $M \prec (\Phi; X)$. Actually, we have the following stronger result that follows from [23, Theorem IV.7.8].

Theorem 8. *Let Φ be a finite list of formulas and X be any set. Then there exists a set R such that $R \prec (\Phi; X)$, $|R| \leq \max(\omega, |X|))$ and moreover, for every countable set $Z \subset R$ there exists $M \subset R$ such that $M \prec (\Phi;\ Z)$ and M is countable.*

The fact that a certain formula is absolute for M will always be used exclusively in order to satisfy the assumption of the following lemma. Using this lemma we can force the model M to contain all the needed objects created (uniquely) from elements of M.

Lemma 9. *Let $\phi(y, x_1, \ldots, x_n)$ be a formula with all free variables shown and let M be a set such that ϕ and $\exists y \phi(y, x_1, \ldots, x_n)$ are absolute for M. If $a_1, \ldots, a_n \in M$ are such that there exists a set u satisfying $\phi(u, a_1, \ldots, a_n)$, then there exists a set $v \in M$ satisfying $\phi(v, a_1, \ldots, a_n)$. Moreover, if there exists a unique set u such that $\phi(u, a_1, \ldots, a_n)$, then $u \in M$.*

Proof. See [6, Lemma 5]. □

Convention 10. Whenever we say "*for any suitable model M (the following holds ...)*" we mean that "*there exists a finite list of formulas Φ and a countable set S such that for every $M \prec (\Phi; S)$ (the following holds ...)*".

Given a topological space X and a set M, we denote by X_M the subset $\overline{X \cap M}$ of X. If M is a set and $\langle X, \tau \rangle$ is a topological space or $\langle X, d \rangle$ is a metric space or $\langle X, +, \cdot, \| \cdot \| \rangle$ is a normed linear space, then we say that M *contains* X if $\langle X, \tau \rangle \in M$, $\langle X, d \rangle \in M$ or $\langle X, +, \cdot, \| \cdot \| \rangle \in M$, respectively.

Given an up-directed and σ-complete partially ordered set $(\Gamma, \leq)$ and a projectional skeleton $\mathfrak{s} = (P_s)_{s \in \Gamma}$ on a Banach space X, we say that a set M *contains* $\mathfrak{s}$ if $\{(\Gamma, \leq), P\} \subset M$, where $P : \Gamma \to \mathcal{L}(X, X)$ is the mapping given by $P(s) = P_s$, for every $s \in \Gamma$.

Basic properties of suitable models were presented in [6, Lemma 7 and Lemma 8]. In the next lemma we summarize further properties that will be used in this work.

Lemma 11. *For every suitable model M the following holds:*

(1) If $f : T \to W$ is a homeomorphism between topological spaces and $f \in M$, then $f[\overline{T \cap M}] = \overline{W \cap M}$.

(2) Let X be a Banach space such that M contains X.
(i) The operator span *which assigns to every subset of X its linear hull belongs to M.*
(ii) If $\mathcal{A} \subset X$ is linearly dense and $\mathcal{A} \in M$, then $\overline{\operatorname{span}}(\mathcal{A} \cap M) = X_M$.
(iii) If D is a linear subspace of X^* and M contains D, then $\overline{D \cap M}^{w*} = ((D \cap M)_\perp)^\perp$.

Proof. Let S be the union of the countable sets from the statements of [6, Lemma 7 and Lemma 8] and let Φ be the union of the finite lists of formulas from the statements of [6, Lemma 7 and Lemma 8] enriched by the formulas (and their subformulas) marked by $(*)$ in the proof below. Let $M \prec (\Phi; S)$.

Let $f : T \to W$ be as in (1). Fixed $w \in W \cap M$, using [6, Lemma 7(2)], Lemma 9 and the absoluteness of the following formula (and its subformulas)

$$\exists t \in T : \quad w = f(t), \tag{$*$}$$

we conclude that there exists $t \in T \cap M$ such that $w = f(t)$. This implies that $W \cap M \subset f[\overline{T \cap M}]$ and thus $\overline{W \cap M} \subset f[\overline{T \cap M}]$. Note that [6, Lemma 8(3)] ensures that $f^{-1} \in M$. Thus it follows from the proved above that $\overline{T \cap M} \subset f^{-1}[\overline{W \cap M}]$, which implies that $f[\overline{T \cap M}] \subset \overline{W \cap M}$.

Now let X be a Banach space such that M contains X. Item (i) follows from Lemma 9 and the absoluteness of the following formula (and its subformulas)

$$\exists\, \operatorname{span} : \mathcal{P}(X) \to \mathcal{P}(X) : \quad (\forall A \subset X : \operatorname{span} A \text{ is the minimal linear subspace containing the set } A). \tag{$*$}$$

In order to prove (ii), note that [6, Lemma 7(7)] ensures that X_M is a closed linear subspace of X and thus $\overline{\operatorname{span}}(\mathcal{A} \cap M) \subset X_M$. On the other hand, for every $x \in X \cap M$, using the fact that $\mathcal{A}$ is linearly dense in X, item (i), Lemma 9 and the absoluteness of the following formula (and its subformulas)

$$\exists S \subset \mathcal{A} : \qquad (S \text{ is countable and } x \in \overline{\operatorname{span}} S), \tag{$*$}$$

we conclude that there exists a countable set $S \subset \mathcal{A}$ such that $S \in M$ and $x \in \overline{\operatorname{span}} S$. It follows from [6, Lemma 7(4)] that $S \subset M$ and thus we obtain that $x \in \overline{\operatorname{span}}(\mathcal{A} \cap M)$. This shows that $X \cap M \subset \overline{\operatorname{span}}(\mathcal{A} \cap M)$ and therefore $X_M \subset \overline{\operatorname{span}}(\mathcal{A} \cap M)$. Now let us prove (iii). Note that since the linear operations of D belong to M, using the fact that $\mathbb{Q} \subset M$ and [6, Lemma 7(2)], we conclude that $D \cap M$ is a $\mathbb{Q}$-linear subspace of X^* and therefore $\overline{D \cap M}^{w^*}$ is a linear subspace of X^*. Therefore, we have that

$$((D \cap M)_\perp)^\perp = \overline{\operatorname{span}}^{w^*}(D \cap M) = \overline{D \cap M}^{w^*}.$$ □

In the rest of the paper we will use several times arguments similar to those presented in the proof of Lemma 11. To simplify the presentation, we adopt the following terminology. Whenever we say that "it follows from absoluteness that some object Ω is in M" or that "it follows from absoluteness that there exists an object Ω in M with a certain property", we mean that the existence of such Ω in M can be established using Lemma 9 for an appropriate formula ϕ (which either uniquely defines Ω or states the property that Ω should satisfy) and that the formulas ϕ and $\exists\Omega\phi$ should be added to the finite list of formulas that are absolute for M.

Next Lemma describes a very concrete construction of suitable models. The reason for including it here is twofold. First, it shows that there is nothing too abstract when working with suitable models and this terminology just enables us to replace complicated inductive constructions by an abstract notion. Second, it is a one of the key tools in the proof of Theorem 18. The construction is more-or-less standard and described also in [11, Lemma 2.4] (where it is formulated for countable models only), for the convenience of the reader we provide a full proof below.

Lemma 12. *Let Φ be a finite subformula closed list of formulas and R be a set such that $R \prec (\Phi;\emptyset)$. Then there is a mapping $\psi : \mathcal{P}(R) \to \mathcal{P}(R)$ (called the* Skolem function*) satisfying the following*

(1)
$$\forall A \subset R : \quad \psi(A) \prec (\Phi; A) \text{ and } |\psi(A)| \leq \max(\omega, |A|),$$
(2) The mapping ψ is monotone, i.e., $\psi(A) \subset \psi(B)$ whenever $A, B \in \mathcal{P}(R)$ are such that $A \subset B$.
(3) The mapping ψ is idempotent, i.e., $\psi \circ \psi = \psi$.
(4) For every $\mathcal{F} \subset \mathcal{P}(R)$ such that $\{\psi(F) \colon F \in \mathcal{F}\}$ is up-directed, we have that $\psi(\bigcup \mathcal{F}) = \bigcup_{F\in\mathcal{F}} \psi(F)$.
(5) Let $J \subset R$ be an arbitrary set. Then, for every $A \subset J$ and $B \subset R$, we have that
$$\psi(A) \cap J \subset \psi(B) \iff \psi(A) \subset \psi(B).$$
(6) For every $\mathcal{F} \subset \mathcal{P}(R)$, we have that
$$\psi\Big(\bigcap_{F\in\mathcal{F}} \psi(F)\Big) = \bigcap_{F\in\mathcal{F}} \psi(F).$$

Proof. Fix a well-ordering $\triangleleft$ on the set R and let $\varphi_1, \ldots, \varphi_n$ be the formulas from the list Φ. For every $i \in \{1, \ldots, n\}$, denote by l_i the number of all the free variables of the formula φ_i and consider the mapping $H_i : R^{l_i} \to R$ defined as follows:

- if $l_i = 0$, then $R^{l_i} = \{\emptyset\}$ and $H_i(\emptyset)$ is the $\triangleleft$-least element of R.
- if $l_i > 0$ and $(r_1, \ldots, r_{l_i}) \in R^{l_i}$ is fixed, then:
  - if there exists $j \in \{1, \ldots, n\}$ such that $\varphi_i = \exists x \varphi_j(x, y_1, \ldots, y_{l_i})$ and there exists $r \in R$ such that $\varphi_j(r, r_1, \ldots, r_{l_i})$ holds, then $H_i(r_1, \ldots, r_{l_i})$ is the $\triangleleft$-least of such elements.
  - in all the other cases, $H_i(r_1, \ldots, r_{l_i})$ is the $\triangleleft$-least element of R.

Fixed $A \in \mathcal{P}(R)$, we define $\psi(A) := \bigcup_{k\in\omega} A_k$, where $(A_k)_{k\in\omega}$ is the increasing sequence of subsets of R built by recursion such that $A_0 := A$ and
$$A_{k+1} := A_k \cup \bigcup\{H_i(a_1, \ldots, a_{l_i}) : \ i = 1, \ldots, n \ , (a_1, \ldots, a_{l_i}) \in (A_k)^{l_i}\},$$

for every $k \in \omega$. Note that it follows immediately from the definition of the Skolem function ψ that $|\psi(A)| \leq \max(\omega, |A|)$ and that (2) and (3) hold. Moreover using [7, Lemma 2.1] and the fact that every formula of Φ is absolute for R, we conclude that $\psi(A) \prec (\Phi; A)$, since clearly $A \subset \psi(A)$.

Let $\mathcal{F}$ be as in (4). Then it follows from (2) that $\psi(\bigcup \mathcal{F}) \supset \bigcup_{F\in\mathcal{F}} \psi(F)$. In order to prove the other inclusion, firstly we claim that $\psi(\bigcup_{F\in\mathcal{F}} \psi(F)) = \bigcup_{F\in\mathcal{F}} \psi(F)$. Indeed, let $(A_k)_{k\in\omega}$ be the sequence used in the definition of $\psi(\bigcup_{F\in\mathcal{F}} \psi(F))$ and let us show by induction that $A_k = \bigcup_{F\in\mathcal{F}} \psi(F)$, for every $k \in \omega$. It is clear that $A_0 = \bigcup_{F\in\mathcal{F}} \psi(F)$. Now fix $k \in \omega$ and assume that $A_k = \bigcup_{F\in\mathcal{F}} \psi(F)$. Note that fixed $i \in \{1, \ldots, n\}$ and $a_1, \ldots, a_{l_i} \in A_k = \bigcup_{F\in\mathcal{F}} \psi(F)$, there exists $F \in \mathcal{F}$ such that $a_1, \ldots, a_{l_i} \in \psi(F)$, since $\{\psi(F) \colon F \in \mathcal{F}\}$ is up-directed. This implies that $H_i(a_1, \ldots, a_{l_i}) \in \psi(F)$ and thus $A_{k+1} = \bigcup_{F\in\mathcal{F}} \psi(F)$. Since $\psi(\bigcup_{F\in\mathcal{F}} \psi(F)) = \bigcup_{k\in\omega} A_k$, we conclude the claim. Finally, since $\bigcup \mathcal{F} \subset \bigcup_{F\in\mathcal{F}} \psi(F)$, using (2) we obtain that $\psi(\bigcup \mathcal{F}) \subset \psi(\bigcup_{F\in\mathcal{F}} \psi(F)) = \bigcup_{F\in\mathcal{F}} \psi(F)$.

Let J, A, B be as in (5). It is obvious that if $\psi(A) \subset \psi(B)$, then $\psi(A) \cap J \subset \psi(B)$. Now suppose that $\psi(A) \cap J \subset \psi(B)$ and note that this implies that $A \subset \psi(B)$. Indeed, using the fact that $A \subset \psi(A)$, we obtain that:

$$A = A \cap J \subset \psi(A) \cap J \subset \psi(B).$$

Therefore, using (2) and (3), we conclude that $\psi(A) \subset \psi(B)$.

Finally, fix $\mathcal{F} \subset \mathcal{P}(R)$ and let $(A_k)_{k\in\omega}$ be the sequence used in the definition of $\psi\Big(\bigcap_{F\in\mathcal{F}} \psi(F)\Big)$. By an inductive argument similar to the presented in the proof of (4), we conclude that $A_k = \bigcap_{F\in\mathcal{F}} \psi(F)$, for every $k \in \omega$. This establishes (6), since $\psi\Big(\bigcap_{F\in\mathcal{F}} \psi(F)\Big) = \bigcup_{k\in\omega} A_k$. $\square$

The following proposition goes essentially back to the proof of [9, Theorem 15]. We record it here for further use in Subsection 3.3. As usual we denote the density of a topological space X by $\operatorname{dens} X$.

Proposition 13. *There exist a countable set S and a finite list of formulas Φ such that the following holds: Let X be a topological space which is homeomorphic to a Banach space with $\kappa := \operatorname{dens} X$. Then there exists a countable set $S' \supset S$ such that*

$$\forall M, N \prec (\Phi; S') : \quad M \cap \kappa \subset N \cap \kappa \Leftrightarrow X_M \subset X_N.$$

Proof. Let S be the union of the countable sets from the statements of Lemma 11 and [6, Lemma 7 and Lemma 8] and let Φ be the union of the finite lists of formulas from the statements of Lemma 11 and [6, Lemma 7 and Lemma 8]. By the result of H. Toruńczyk [26], all infinite-dimensional Banach spaces with the same density are topologically homeomorphic. Thus, there exists a (not necessarily linear) homeomorphism $f : \ell_2(\kappa) \to X$. Let $(e_i)_{i\in\kappa}$ be the canonical orthonormal basis of $\ell_2(\kappa)$, $e : \kappa \to \ell_2(\kappa)$ be the map given by $e(i) := e_i$, for every $i < \kappa$ and let S' be a countable set that contains X, $\ell_2(\kappa)$ and such that $S \cup \{f, e\} \subset S'$.

Note that $X_M = f\big[\overline{\mathrm{span}}\{e_i\colon i\in\kappa\cap M\}\big]$, for every $M\prec(\Phi;S')$. Indeed, it follows from Lemma 11(2)(ii) and [6, Lemma 7(2)] that $\overline{\ell_2(\kappa)\cap M} = \overline{\mathrm{span}}\big(\{e_i : i\in\kappa\}\cap M\big)$. Thus using Lemma 11(1) and [6, Lemma 8(4)], we conclude that $X_M = f\big[\overline{\mathrm{span}}\{e_i\colon i\in\kappa\cap M\}\big]$. Now fix $M,N\prec(\Phi;S')$. By the above, it is easy to observe that $M\cap\kappa\subset N\cap\kappa$ implies $X_M\subset X_N$. On the other hand, if $X_M\subset X_N$ and $i\in M\cap\kappa$ is given, then $f(e_i)\in X_M\subset X_N = f\big[\overline{\mathrm{span}}\{e_j\colon j\in N\cap\kappa\}\big]$ and so we have that $i\in N\cap\kappa$, since f is a homeomorphism and $e_i\notin\overline{\mathrm{span}}\{e_j\colon j\neq i, j<\kappa\}$. $\square$

3.2. **Canonical projections in spaces with a projectional skeleton.** Next lemma is a key tool for the construction of projections in spaces with a projectional skeleton. This is more-or-less known to the experts, see [22, Lemma 4] and [4, Lemma 5.1], where proofs are essentially given for suitable models which are countable. Since this is a crucial tool for us and we need it for uncountable models as well, for the convenience of the reader we include the full proof here.

Lemma 14. *For every suitable model M the following holds: Let X be a Banach space, $C\geq 1$ and $D\subset X^*$ be a C-norming subspace such that M contains X and $D\in M$. Then*

- *(i) $X_M\cap(D\cap M)_\perp=\{0\}$,*
- *(ii) $X_M+(D\cap M)_\perp$ is a closed subspace of X and the canonical linear projection from $X_M\oplus(D\cap M)_\perp$ onto X_M has norm less or equal to C,*

and the following conditions are equivalent

- *(1) $X_M+(D\cap M)_\perp = X$,*
- *(2) $X\cap M$ separates the points of $\overline{D\cap M}^{w^*}$,*
- *(3) there exists a unique projection $P_M : X\to X$ with $P_M[X]=X_M$ and $\ker P_M=(D\cap M)_\perp$,*
- *(4) there exists a unique projection $P_M : X\to X$ with $P_M[X]=X_M$ and $d=d\circ P_M$ for every $d\in D\cap M$.*

Moreover, for any projection $P: X\to X$ with $P[X]=X_M$ we have $\ker P = (D\cap M)_\perp$ if and only if $d=d\circ P$ for every $d\in D\cap M$.

Proof. Let S be the union of the countable sets from the statements of Lemma 11 and [6, Lemma 7] and let Φ be the union of the finite lists of formulas from the statements of Lemma 11 and [6, Lemma 7], enriched by the finitely many formulas used in the proof below. Let $M\prec(\Phi;S)$ be such that $D\in M$ and M contains X.

We claim that for every $x\in X_M$ and $y\in(D\cap M)_\perp$, it holds that $\|x\|\leq C\|x+y\|$. Indeed, fix $x\in X\cap M$, $n\in\mathbb{N}$ and a rational number $q>C$. By Lemma 9 and the absoluteness of the following formula (and its subformulas)

$$\exists d\in D:\qquad \|x\|<q\tfrac{|d(x)|}{\|d\|}+\tfrac{1}{n},$$

there is $d \in D \cap M$ such that $\|x\| < q\frac{|d(x)|}{\|d\|} + \frac{1}{n}$. Thus for every $y \in (D \cap M)_\perp$ we have that

$$\|x\| < q\frac{|d(x)|}{\|d\|} + \frac{1}{n} = q\frac{|d(x+y)|}{\|d\|} + \frac{1}{n} \leq q\|x+y\| + \frac{1}{n}.$$

Since $n \in \mathbb{N}$ and $q \in (C, \infty) \cap \mathbb{Q}$ were arbitrary, we obtain that $\|x\| \leq C\|x+y\|$, for every $x \in X \cap M$ and $y \in (D \cap M)_\perp$, which implies the claim. It is easy to see that (i) and (ii) follow from this claim.

(1)⇒(2): If (1) holds then for every $x^* \in \overline{D \cap M}^{w*}$ with $x^* \neq 0$, there exists $x \in X_M$ and $y \in (D \cap M)_\perp$ such that $x^*(x+y) \neq 0$. Note that Lemma 11(2)(iii) ensures that $\overline{D \cap M}^{w*} = ((D \cap M)_\perp)^\perp$. Therefore, we have that $x^*(x) = x^*(x+y) \neq 0$, which implies that X_M separates the points of $\overline{D \cap M}^{w*}$ and thus (2) follows from the density of $X \cap M$ in X_M.

(2)⇒(1): If (2) holds, then we have

$$(X_M + (D \cap M)_\perp)^\perp \subset (X_M)^\perp \cap ((D \cap M)_\perp)^\perp = (X_M)^\perp \cap \overline{D \cap M}^{w*} = \{0\},$$

so we obtain that $X_M + (D \cap M)_\perp \supset \{0\}_\perp = X$ and (1) holds.

The equivalence between (1) and (3) is obvious.

Finally, in order to conlude the proof, let us show that if $P : X \to X$ is a projection with $P[X] = X_M$, then $\ker P = (D \cap M)_\perp$, if and only if $d = d \circ P$, for every $d \in D \cap M$. It is easy to see that if $\ker P = (D \cap M)_\perp$, then $d = d \circ P$, for every $d \in D \cap M$ and the converse follows from (i). □

It was shown in [22, Theorem 15] that a Banach space X admits a projectional skeleton if and only if there exist $C \geq 1$ and a C-norming subspace D of X^* such that $X = X_M + (D \cap M)_\perp$, for every **countable** suitable model M. It follows from Proposition 15 that we do not need to assume countability of the model and moreover Proposition 15 provides us a formula for the canonical projection from Lemma 14, when D is a set induced by a projectional skeleton.

Proposition 15. *For every suitable model M, the following holds: Let X be a Banach space and $\mathfrak{s} = (P_s)_{s \in \Gamma}$ be a projectional skeleton on X. If M contains X and $\mathfrak{s}$, then $\Gamma \cap M$ is up-directed and the mapping $P_M : X \to X$ given by*

$$P_M(x) = \lim_{s \in \Gamma \cap M} P_s(x), \quad \forall x \in X$$

is a well-defined bounded projection such that $P_M[X] = X_M$ and $\ker P_M = (D(\mathfrak{s}) \cap M)_\perp$.

Proof. Let S be the union of the countable sets from the statements of [6, Lemma 7 and Lemma 8] and Lemma 14 and let Φ be the union of the finite lists of formulas from the statements of [6, Lemma 7 and Lemma 8] and Lemma 14 enriched by the finitely many formulas needed in the proof below. Let $M \prec (\Phi; S)$ containing X and $\mathfrak{s}$. Note that it follows from [6, Lemma 8(1)] that $\Gamma \cap M$ is up-directed and so [22, Lemma 11] ensures that P_M is a well-defined bounded projection on X with $P_M[X] = \overline{\bigcup_{s \in \Gamma \cap M} P_s[X]}$.

First, we *claim* that $P_M[X] = X_M$. Indeed, given $s \in \Gamma \cap M$, since $P_s[X]$ is separable, by the absoluteness of the following formula (and its subformulas)

$$\exists S \subset X : \quad (S \text{ is countable and } \overline{P_s[S]} = P_s[X]),$$

we conclude that there exists a countable set $S \in M$ such that $\overline{P_s[S]} = P_s[X]$ and using [6, Lemma 7 and 8] we have that $S \subset M$ and $P_s[S] \subset X \cap M$, which implies that $P_s[X] \subset X_M$ and since $s \in \Gamma \cap M$ was arbitrary, we obtain $P_M[X] \subset X_M$. On the other hand, fixed $x \in X \cap M$, it follows from absoluteness that there exists $s \in \Gamma \cap M$ such that $x \in P_s[X] \subset P_M[X]$; hence, we also have that $X_M \subset P_M[X]$. This proves the claim.

Note that since M contains $\mathfrak{s}$, it follows from absoluteness that $D(\mathfrak{s}) \in M$ and thus by Lemmas 3 and 14, in order to conclude that $\ker P_M = (D(\mathfrak{s}) \cap M)_\perp$, it suffices to show that $d = d \circ P_M$, for every $d \in D(\mathfrak{s}) \cap M$. Fixed $d \in D(\mathfrak{s}) \cap M$, it follows from absoluteness that there exists $s \in \Gamma \cap M$ such that $d = P_s^*(d) = d \circ P_s$, which implies that $d = d \circ P_M$. $\square$

We believe that the projections from Lemma 14 that were constructed in Proposition 15 are the key ingredients to handle inductive arguments in spaces with a projectional skeleton. Let us give it a name.

Definition 16. Let X be a Banach space, $C \geq 1$ and D be a C-norming subspace of X^*. Given a set M, we say that P_M is the *canonical projection associated to M, X and D* if it is the unique projection on X satisfying $P_M[X] = X_M$ and $\ker P_M = (D \cap M)_\perp$. We say that a set M *admits canonical projection associated to X and D* if there exists the canonical projection associated to M, X and D.

In Plichko spaces we may apply also the following lemma which we record here for further use.

Lemma 17. *Let X be a Banach space, $\mathfrak{s}$ be a projectional skeleton on X and $\mathcal{A}$ be a subset of X that countably supports $D(\mathfrak{s})$. Then there exist a countable set S and a finite list of formulas Φ such that if $M \prec (\Phi; S \cup \{\mathcal{A}\})$ and M contains X and $\mathfrak{s}$, then M admits canonical projection P_M associated to X and $D(\mathfrak{s})$ and it holds that $P_M(x) \in \{0, x\}$, for every $x \in \mathcal{A}$.*

Proof. Let S be the union of the countable sets from the statements of Proposition 15 and [6, Lemma 7] and let Φ be the union of the finite lists of formulas from the statements of Proposition 15 and [6, Lemma 7] enriched by the finitely many formulas used in the proof below. Fix $M \prec (\Phi; S \cup \{\mathcal{A}\})$ such that M contains X and $\mathfrak{s}$ and note that Proposition 15 ensures that M admits canonical projection P_M associated to X and $D(\mathfrak{s})$. It is clear that $\mathcal{A} \cap M \subset P_M[X]$. Fixed $x^* \in D(\mathfrak{s}) \cap M$, by absoluteness we have that $\operatorname{suppt}_{\mathcal{A}}(x^*) \in M$ and thus it follows from the countability of $\operatorname{suppt}_{\mathcal{A}}(x^*)$ and [6, Lemma 7(4)] that $\operatorname{suppt}_{\mathcal{A}}(x^*) \subset M$. This implies that $\mathcal{A} \setminus M \subset \ker P_M = (D(\mathfrak{s}) \cap M)_\perp$ and concludes the proof. $\square$

3.3. **Proof of Theorem B and other results used in further sections.** In this subsection we concentrate on the outcomes of Section 3 that will be applied in Section 4. The first outcome is Theorem B. We formulate a slightly more general result from which Theorem B immediately follows.

Theorem 18. *Let X be a Banach space and $\{\mathfrak{s}_n : n \in \omega\}$ be a countable family of projectional skeletons on X inducing the same set $D \subset X^*$. Then there exists a simple projectional skeleton on X which is isomorphic to a subskeleton of $\mathfrak{s}_n$, for every $n \in \omega$.*

Moreover, for every countable set S and every finite list of formulas Φ, there exists a family $\mathcal{M}$ consisting of countable suitable models for Φ containing S such that every $M \in \mathcal{M}$ admits canonical projection P_M associated to X and D and if $\mathcal{M}$ is ordered by inclusion, then the following holds:

- *$\mathcal{F} = \{X_M : M \in \mathcal{M}\}$ is a rich family and if we set $P_F := P_M$, whenever $F = X_M$ with $M \in \mathcal{M}$, then $(P_F)_{F \in \mathcal{F}}$ is simple projectional skeleton on X isomorphic to $(P_M)_{M\in\mathcal{M}}$, via the order isomorphism $\mathcal{M} \ni M \mapsto X_M \in \mathcal{F}$.*
- *$(P_M)_{M\in\mathcal{M}}$ is isomorphic to a subskeleton of $\mathfrak{s}_n$, for every $n \in \omega$.*

Proof. Let $\kappa = \operatorname{dens} X$ and $d : \kappa \to X$ be such that $\{d(i) : i < \kappa\}$ is dense in X. Fix a countable set S and a finite list of formulas Φ. Let Φ' be a finite subformula closed list of formulas that contains Φ and the union of the lists of formulas from the statements of Proposition 15, Proposition 13 and [6, Lemma 7]. Let S' be a countable set such that $S \subset S'$, S' contains X, d, $\mathfrak{s}_n$, for every $n \in \omega$ and S' contains the union of the countable sets from the statements of Proposition 15, Proposition 13 and [6, Lemma 7]. By [23, Theorem IV.7.4], there exists a set R such that $R \prec (\Phi'; S' \cup \kappa)$. Let $\psi : \mathcal{P}(R) \to \mathcal{P}(R)$ be the Skolem function given by Lemma 12. Consider the following family of countable subsets of R

$$\mathcal{M} := \{\psi(A \cup S')\colon A \in [\kappa]^{\leq\omega}\}.$$

Note that for every $M \in \mathcal{M}$ it holds that $M \prec (\Phi'; S')$ and therefore it follows from Proposition 15 that M admits canonical projection P_M associated to X and D. In order to see that the mapping $\mathcal{M} \ni M \mapsto X_M \in \mathcal{F}$ is an order-isomorphism, if $\mathcal{M}$ and $\mathcal{F}$ are ordered by inclusion, note that it follows from Lemma 12(5) applied to $J = \kappa \cup S'$ that for every $M, N \in \mathcal{M}$, it holds that $M \subset N$ if and only if $M \cap \kappa \subset N \cap \kappa$, which is by Proposition 13 equivalent to $X_M \subset X_N$.

Now let us show that $\mathcal{F}$ is a rich family. If $Y \subset X$ is a separable space, then there exists $A \in [\kappa]^{\leq\omega}$ such that $Y \subset \overline{\{d(i)\colon i \in A\}}$ and by [6, Lemma 7(2)] we have that $\overline{\{d(i)\colon i \in A\}} \subset X_{\psi(A\cup S')}$. Now let $(M_n)_{n\in\mathbb{N}}$ be an increasing sequence in $\mathcal{M}$ and set $M_\infty := \bigcup_{n\in\mathbb{N}} M_n$. It follows from Lemma 12(4) that M_∞ belongs to $\mathcal{M}$ and it is easy to check that $X_{M_\infty} = \overline{\bigcup_{n\in\mathbb{N}} X_{M_n}}$. Thus, $\mathcal{F}$ is a rich family. Moreover, if $M, N \in \mathcal{M}$ satisfy $X_M \subset X_N$, then $P_M(X) = X_M \subset X_N = P_N(X)$ and $\ker P_M = (D \cap M)_\perp \supset$

$(D \cap N)_\perp = \ker P_N$, which implies that $P_M = P_N \circ P_M = P_M \circ P_N$. Therefore, we conclude that $(P_F)_{F\in\mathcal{F}}$ is a simple projectional skeleton on X that is isomorphic to $(P_M)_{M\in\mathcal{M}}$.

Finally, fix $n \in \mathbb{N}$ and let us show that $(P_M)_{M\in\mathcal{M}}$ is isomorphic to a subskeleton of $\mathfrak{s}_n$. Consider the mapping $\phi : \mathcal{M} \to \Gamma_n$ given by $\phi(M) = \sup(\Gamma_n \cap M)$, where $\mathfrak{s}_n = (P^n_s)_{s\in\Gamma_n}$. Note that $P_M = P^n_{\phi(M)}$, for every $M \in \mathcal{M}$. Indeed, since $\Gamma_n \cap M$ is countable and up-directed we find an increasing sequence $(s_k)_{k\in\omega}$ from $\Gamma_n \cap M$ with $\sup_k s_k = s = \phi(M)$. Then, using that the sequence $\{s_k \colon k \in \omega\}$ is cofinal in $\Gamma_n \cap M$, we obtain that $P_M = P_{\{s_k\colon k\in\omega\}} = P_{\phi(M)}$ as claimed above. To see that ϕ is an order-isomorphism onto its image, fix $M, N \in \mathcal{M}$. If $M \subset N$, then it is clear that $\phi(M) \le \phi(N)$. On the other hand, if $\phi(M) \le \phi(N)$, then $X_M = P_{\phi_n(M)}[X] \subset P_{\phi_n(N)}[X] = X_N$ and thus $M \subset N$. It remains to show that $\phi[\mathcal{M}]$ is a σ-closed subset of Γ_n. Let $(M_k)_{k\in\mathbb{N}}$ be an increasing sequence in $\mathcal{M}$. As discussed previously we have that $M_\infty := \bigcup_{n\in\mathbb{N}} M_n \in \mathcal{M}$ and it is easy to see that $\phi(M_\infty) = \sup_{n\in\mathbb{N}} \phi(M_n)$. □

Remark 19. As mentioned above, Theorem 18 is a generalization of [11, Theorem 4.1]. Note however that in the proof of [11, Theorem 4.1] there is a gap which was fixed only later. Namely, the authors used the fact that any Banach space with a projectional skeleton has a Markushevich basis, which was not known at the time and it was proved later in [21]. Our proof of Theorem 18 is different from the proof of [11, Theorem 4.1] - the difference is that, inspired by the proof of [9, Theorem 4], instead of the existence of Markushevich basis we use the deep result by H. Toruńczyk that all infinite-dimensional Banach spaces of the same density are topologically homeomorphic.

The following seems to be the crucial property of projectional skeletons we need further in our inductive arguments.

Corollary 20. *Let X be a Banach space, $\mathfrak{s}$ be a projectional skeleton on X and $\mathcal{A}$ be a subset of X that countably supports $D(\mathfrak{s})$. Then there exists a projectional skeleton $(P_s)_{s\in\Gamma}$ on X that is isomorphic to a subskeleton of $\mathfrak{s}$ and such that $P_s(x) \in \{0, x\}$, for every $x \in \mathcal{A}$ and every $s \in \Gamma$.*

Proof. This is a consequence of Lemma 17 and Theorem 18. □

Remark 21. Note that if there exist a projectional skeleton $(P_s)_{s\in\Gamma}$ on a Banach space X and a linearly dense set $\mathcal{A} \subset X$ such that $\{P_s x \colon s \in \Gamma\} \subset \{0, x\}$ for every $x \in \mathcal{A}$, then we easily obtain that $P_s P_t x = P_t P_s x$, for every $x \in \mathcal{A}$, which implies that the skeleton is commutative. Therefore, it seems that our methods apply to subclasses of Plichko spaces (equivalently, as mentioned in Section 2, spaces admitting a commutative projectional skeleton).

In the following technical result, we construct a transfinite sequence of projections using the method of suitable model. An alternative way of constructing this sequence would be to use the approach from [21, Section 2].

One could deduce many other properties of the projections constructed in the proof of Proposition 22, but we try to formulate only the ones which will be used further.

Proposition 22. *Let X be a Banach space with $\kappa := \operatorname{dens} X$, $\mathcal{A} \subset X$ be a bounded set and let $\mathfrak{s} = (P_s)_{s\in\Gamma}$ be a projectional skeleton on X such that $\mathcal{A}$ countably supports $D(\mathfrak{s})$ and $P_s x \in \{0, x\}$, for every $x \in \mathcal{A}$ and every $s \in \Gamma$. Then there exists a sequence of bounded projections $\mathfrak{p} = (P_\alpha)_{\alpha\leq\kappa}$ satisfying the following properties*

- *(P1)* $P_0 = 0$ *and* $P_\kappa = Id$.
- *(P2) For every $\alpha < \beta$, we have that $P_\alpha \circ P_\beta = P_\beta \circ P_\alpha = P_\alpha$.*
- *(P3) Let $\alpha \leq \kappa$, let $\eta : [0, \alpha) \to \kappa$ be an increasing function and let $\xi \leq \kappa$ be a limit ordinal with $\sup_{\beta<\alpha} \eta(\beta) = \xi$. Then $\lim_{\beta<\alpha} P_{\eta(\beta)}(x) = P_\xi(x)$, for every $x \in X$.*
- *(P4) For every $\alpha \in [0, \kappa)$, we have that $\operatorname{dens}(P_\alpha[X]) < \kappa$.*
- *(P5) $P_\alpha x \in \{0, x\}$, for every $\alpha \leq \kappa$ and $x \in \mathcal{A}$;*
- *(P6) For every $\alpha < \kappa$ there exists a cofinal σ-closed subset $\Gamma_\alpha \subset (\Gamma \cap M_{\alpha+1})_\sigma$ such that the family $\mathfrak{s}_\alpha = (P_s|_{(P_{\alpha+1}-P_\alpha)[X]})_{s\in\Gamma_\alpha}$ is a projectional skeleton on $(P_{\alpha+1} - P_\alpha)[X]$.*

  Moreover, $(P_{\alpha+1} - P_\alpha)[\mathcal{A}]$ countably supports $D(\mathfrak{s}_\alpha)$.
- *(P7) $\mathcal{T}(\mathfrak{s}, \mathcal{A}) \subset \mathcal{T}(\mathfrak{p}, \mathcal{A})$.*

Proof. Let S be union of the countable sets from the statements of Proposition 15 and Lemma 17 and let Φ be the union of the finite lists of formulas from the statements of Proposition 15 and Lemma 17 enriched by the finitely many formulas used in the proof below. Let $\{d_\alpha : \alpha < \kappa\}$ be a dense set in X. Using Theorem 8, we construct a transfinite sequence of suitable models $(M_\alpha)_{\alpha\leq\kappa}$ such that for every $\alpha \in [0, \kappa]$ we have

- $|M_\alpha| \leq \max(\omega, |\alpha|)$, $M_\alpha \prec (\Phi; S \cup \{\mathcal{A}\})$ and M_α contains X and $\mathfrak{s}$,
- $M_{\alpha+1} \supset M_\alpha \cup \{d_\alpha\}$ whenever $\alpha \neq \kappa$,
- $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$, if α is a limit ordinal (in order to see that an increasing union of suitable models is a suitable model we use e.g. [7, Lemma 2.1]).

Set $P_0 := 0$ and for $\alpha \in (0, \kappa]$ we let P_α be the canonical projection associated to M_α, X and $D(\mathfrak{s})$ given by Proposition 15, that is, it holds that $P_\alpha[X] = X_{M_\alpha}$, $\ker P_\alpha = (D(\mathfrak{s}) \cap M_\alpha)_\perp$ and

$$P_\alpha x = \lim_{s\in\Gamma\cap M_\alpha} P_s x, \qquad x \in X, \tag{1}$$

where $\Gamma \cap M_\alpha$ is an up-directed set. Since $M_\kappa \supset \{d_\alpha : \alpha < \kappa\}$, we have that $P_\kappa[X] = X$ and thus (P1) holds. Note that (P2) follows immediately from [21, Lemma 2.2], since the sequence $(\Gamma \cap M_\alpha)_{\alpha\leq\kappa}$ is increasing. In order to prove (P3), pick α, η as in (P3) and fix $x \in X$ and $\varepsilon > 0$. Then there exists $s_0 \in \Gamma \cap M_\xi$ such that for every $s \geq s_0$, $s \in M_\xi \cap \Gamma$ we have $\|P_s x - P_\xi x\| \leq \varepsilon$. Since $M_\xi = \bigcup_{\beta<\alpha} M_{\eta(\beta)}$, there exists $\beta_0 < \alpha$ with $s_0 \in M_{\eta(\beta_0)}$. Then for every $\beta \in [\beta_0, \alpha)$ we have

$$\|P_{\eta(\beta)}(x) - P_\xi x\| = \lim_{s\in\Gamma\cap M_{\eta(\beta)}, s\geq s_0} \|P_s(x) - P_\xi x\| \leq \varepsilon,$$

which implies that $\lim_{\beta<\alpha} P_{\eta(\beta)}(x) = P_{M_\xi}(x)$ and so (P3) holds. Clearly (P4) follows from the fact that $P_\alpha[X] = X_{M_\alpha}$ and $|M_\alpha| < \kappa$ for $\alpha < \kappa$ and (P5) is a consequence of Lemma 17.

In order to prove (P6), fix $\alpha < \kappa$. By [21, Proposition 2.3 (iii)], the family $(P_s|_{P_{\alpha+1}[X]})_{s\in(\Gamma\cap M_{\alpha+1})_\sigma}$ is a projectional skeleton on $P_{\alpha+1}[X]$ with induced set given by

$$D_{\alpha+1} := \bigcup_{s\in(\Gamma\cap M_{\alpha+1})_\sigma} \{P_s^* x^*|_{P_{\alpha+1}[X]} \colon x^* \in X^*\}.$$

Observe that it follows from absoluteness that given $d \in D(\mathfrak{s}) \cap M_\alpha$ there exists $s \in \Gamma \cap M_{\alpha+1}$ such that $d \in P_s^* X^*$ and thus $d|_{P_{\alpha+1}[X]} \in D_{\alpha+1}$. This implies that the set

$$(P_{\alpha+1} - P_\alpha)[X] = P_{\alpha+1}[X] \cap \ker P_\alpha = \bigcap_{d\in D(\mathfrak{s})\cap M_\alpha} \ker d|_{P_{\alpha+1}[X]}$$

is $\sigma(P_{\alpha+1}[X], D_{\alpha+1})$-closed and then the existence of Γ_α as in (P6) follows from [21, Lemma 2.1]. The moreover part of (P6) follows from (P5), since $(P_{\alpha+1} - P_\alpha)[\mathcal{A}] \subset \mathcal{A} \cup \{0\}$.

Finally let us prove (P7). Let $\varepsilon > 0$, $A \subset \mathcal{A}$ and $x^* \in X^*$ be such that $\mathfrak{s}$ is (A, ε)-shrinking for x^*, pick an increasing sequence $(\alpha_k)_{k\in\omega}$ in $[0, \kappa)$ and put $\alpha = \sup_k \alpha_k$. In order to get a contradiction assume that $\limsup_k \rho_A(P^*_{\alpha_k}(x^*), P^*_\alpha(x^*)) > \varepsilon\|x^*\|$ in which case, up to passing to a subsequence, we may assume that there is $\delta > 0$ such that

$$\rho_A(P^*_{\alpha_k}(x^*), P^*_\alpha(x^*)) > \varepsilon\|x^*\| + \delta, \quad k \in \omega.$$

Fixed $k \in \omega$, by (P5) we have that

$$\begin{aligned}\rho_A(P^*_{\alpha_k}(x^*), P^*_\alpha(x^*)) &= \sup_{x\in A} |x^*(P_{\alpha_k}(x)) - x^*(P_\alpha(x))| \\ &= \sup_{x\in A} \begin{cases} 0 & \text{if } P_{\alpha_k}x = P_\alpha x \\ |x^*(x)| & \text{if } P_{\alpha_k}x \neq P_\alpha x, \end{cases}\end{aligned}$$

which implies that there exists $x_k \in A$ such that $|x^*(x_k)| \geq \varepsilon\|x^*\| + \delta$. Moreover, for the same $x_k \in A$, it follows from (P2) that $P_\alpha(x_k) = x_k$ and $P_{\alpha_k}(x_k) = 0$. Now, using that $P_s x \in \{0, x\}$, for every $s \in \Gamma$ and $x \in A$, we may recursively construct two increasing sequences $\{s_j\}_{j\in\omega} \subset M_\alpha \cap \Gamma$, $\{k_j\}_{j\in\omega} \subset \mathbb{N}$ satisfying the following properties

- $s_{2j+1} \in M_{\alpha_{k_j}}$ and $P_{s_{2j+1}} x_{k_j} = 0$, for every $j \in \omega$;
- $s_{2j+2} \in M_{\alpha_{k_{j+1}}}$ and $P_{s_i} x_{k_j} = x_{k_j}$, for every $j \in \omega$ and $i \geq 2j + 2$.

Indeed, in the initial step of the induction we put $k_0 = 1$. If $k_0, \ldots, k_j$ are defined, using (1) we find $s_{2j+1} \in M_{\alpha_{k_j}}$ such that $P_{s_{2j+1}} x_{k_j} = P_{\alpha_{k_j}} x_{k_j} = 0$ and we find $s_{2j+2} \in M_\alpha \cap \Gamma = \bigcup_{k\in\omega} M_{\alpha_k} \cap \Gamma$ such that $s_{2j+2} \geq s_{2j+1}$ and

$P_{s_i}x_{k_j} = P_\alpha x_{k_j} = x_{k_j}$, for every $i \geq 2j+2$. Finally, we pick $k_{j+1} > k_j$ such that $s_{2j+2} \in M_{\alpha_{k_{j+1}}}$, which finishes the inductive step.

If $s := \sup_j s_j$, then we have that $P_{s_{2j+1}}x_{k_j} = 0$ and $P_s x_{k_j} = x_{k_j}$ for every $j \in \omega$. Therefore, for every $j \in \omega$, we have that

$$\rho_A(P^*_{s_{2j+1}}(x^*), P^*_s(x^*)) \geq |x^*(x_{k_j})| \geq \varepsilon\|x^*\| + \delta.$$

But this contradicts the fact that $\mathfrak{s}$ is (A,ε)-shrinking in x^*. Thus, (P7) holds. $\square$

4. Main results

This section contains the proofs of our main results. The main technical parts are concentrated in Subsection 4.1 and the proofs of our characterizations are then presented in the remainder of the section.

4.1. **Shrinkingness passes from projectional skeletons to SPRI.** In this first subsection we aim at proving Theorem 24, which enables us to pass shrinking-like properties from projectional skeletons to SPRI.

Lemma 23. *Let X be a Banach space with a projectional skeleton $\mathfrak{s}$ and $\kappa := \operatorname{dens}(X)$. Let $\mathcal{A} \subset X$ be a bounded set and let $\mathfrak{p} = (P_\alpha)_{\alpha\leq\kappa}$ be a sequence of bounded projections satisfying (P1), (P2), (P3), (P5) and (P7). Assume that for every $\alpha < \kappa$ the space $(P_{\alpha+1} - P_\alpha)[X]$ is either separable or it admits a SPRI $\mathfrak{S}_\alpha = (Q^\alpha_\beta)_{\beta\leq\mu_\alpha}$, which satisfies:*

1. *$Q^\alpha_\beta x \in \{0, x\}$, for every $x \in (P_{\alpha+1} - P_\alpha)[\mathcal{A}]$ and $\beta \in [0, \mu_\alpha]$;*
2. *for every $x \in (P_{\alpha+1} - P_\alpha)[\mathcal{A}]$, $\min\{\beta \leq \mu_\alpha : Q^\alpha_\beta x = x\}$ is not a limit ordinal;*
3. *for every $(\varepsilon, A, x^*) \in \mathcal{T}(\mathfrak{s}, \mathcal{A})$, we have that*
$$(\varepsilon_\alpha, A_\alpha, x^*_\alpha) \in \mathcal{T}(\mathfrak{S}_\alpha, \mathcal{A}_\alpha),$$
where $\mathcal{A}_\alpha := (P_{\alpha+1} - P_\alpha)[\mathcal{A}]$, $A_\alpha := (P_{\alpha+1} - P_\alpha)[A]$, $x^*_\alpha := x^*|_{(P_{\alpha+1}-P_\alpha)[X]}$, $\varepsilon_\alpha := \varepsilon\frac{\|x^*\|}{\|x^*_\alpha\|}$, if $x^*_\alpha \neq 0$ and $\varepsilon_\alpha := 0$, if $x^*_\alpha = 0$.

Then the whole space X admits a SPRI $\mathfrak{S} = (Q_\alpha)_{\alpha\leq\kappa}$ such that

4. *$Q_\alpha x \in \{0, x\}$, for every $x \in \mathcal{A}$ and $\alpha \in [0, \kappa]$;*
5. *for every $x \in \mathcal{A}$, $\min\{\alpha \leq \kappa : Q_\alpha x = x\}$ is not a limit ordinal;*
6. *$\mathcal{T}(\mathfrak{s}, \mathcal{A}) \subset \mathcal{T}(\mathfrak{S}, \mathcal{A})$.*

Proof. We argue as in [16, Proposition 6.2.7]. Let $\alpha < \kappa$. If $(P_{\alpha+1} - P_\alpha)[X]$ is separable, then we put $\mu_\alpha = \omega$, $Q^\alpha_0 \equiv 0$, and $Q^\alpha_\beta = P_{\alpha+1} - P_\alpha$, for all $0 < \beta < \omega$. If $(P_{\alpha+1} - P_\alpha)[X]$ is nonseparable, let $(Q^\alpha_\beta)_{\beta\leq\mu_\alpha}$ be the SPRI given by the hypothesis. We start by defining $Q_{(\kappa,0)} = Id$,

$$Q_{(\alpha,\beta)} = Q^\alpha_\beta(P_{\alpha+1} - P_\alpha) + P_\alpha,\ 0 \leq \beta < \mu_\alpha,\ 0 \leq \alpha < \mu,$$
$$\Lambda = \{(\alpha,\beta) : \beta < \mu_\alpha, \alpha < \mu\} \cup \{(\mu, 0)\}.$$

We endow Λ with the lexicographical order, i.e $(\alpha,\beta) \leq (\alpha',\beta')$ if and only if $\alpha < \alpha'$ or $\alpha = \alpha'$ and $\beta \leq \beta'$. With the same proof as the one presented in [16, Proposition 6.2.7], we conclude that $\mathfrak{S} := (Q_{(\alpha,\beta)})_{(\alpha,\beta)\in\Lambda}$ is a SPRI on X. Let us show $\mathfrak{S}$ satisfies the additional properties (4), (5) and (6).

(4): Fix $x \in \mathcal{A}$ and $\alpha \in [0, \kappa)$. Note that by (P2) and (P5) we have that $(P_{\alpha+1} - P_\alpha)x \in \{0, x\}$. Thus, if $\mu_\alpha = \omega$, then (4) follows. Otherwise, if $P_\alpha x = 0$, then since $(P_{\alpha+1} - P_\alpha)x \in (P_{\alpha+1} - P_\alpha)[\mathcal{A}]$, applying (1) we obtain that $Q^\alpha_\beta(P_{\alpha+1} - P_\alpha)(x) \in \{0, (P_{\alpha+1} - P_\alpha)(x)\} = \{0, x\}$, therefore $Q_{(\alpha,\beta)}x \in \{0, x\}$. On the other hand if $P_\alpha x = x$, then $P_{\alpha+1}x = x$ as well and so $Q_{(\alpha,\beta)}x = P_\alpha x = x$. This proves (4).

(5): Fix $x \in \mathcal{A}$. Note that if $x = 0$, then $\min\{(\alpha, \beta) \in \Lambda : Q_{(\alpha,\beta)}x = x\} = (0, 0)$. Now assume that $x \neq 0$ and let us consider the following cases:

- $\min\{(\alpha, \beta) \in \Lambda : Q_{(\alpha,\beta)}x = x\} = (\gamma + 1, 0)$, for some $\gamma < \kappa$. Let us first notice that $x = Q_{(\gamma+1,0)}x = P_{\gamma+1}x$ and $0 = Q_{(\gamma,0)}x = P_\gamma x$. Moreover for $\beta < \mu_\gamma$ we have that
$$0 = Q_{(\gamma,\beta)}x = Q^\gamma_\beta((P_{\gamma+1} - P_\gamma)(x)) + P_\gamma(x) = Q^\gamma_\beta x.$$
If $\mu_\gamma = \omega$, then $Q^\gamma_1 x = (P_{\gamma+1} - P_\gamma)(x) = x$, which is a contradiction. Otherwise, since $(P_{\gamma+1} - P_\gamma)(x) = x$ and $(Q^\gamma_\beta)_{\beta\leq\mu_\gamma}$ is a SPRI on $(P_{\gamma+1} - P_\gamma)[X]$ satisfying (1), there exists $\beta < \mu_\gamma$ such that $Q^\gamma_\beta x = x$, which is a contradiction.
- $\min\{(\alpha, \beta) \in \Lambda : Q_{(\alpha,\beta)}x = x\} = (\gamma, 0)$ for some $\gamma \leq \kappa$ limit ordinal. Observing that $x = Q_{(\gamma,0)}x = P_\gamma x$, by (P3) and (P5) there exists $\delta < \gamma$ such that $P_{\delta+1}x = x$. Thus, $x = P_{\delta+1}x = Q_{(\delta+1,0)}x$, which is in contradiction with the minimality of $(\gamma, 0)$.

Since the cases above were leading to a contradiction, it holds that $\min\{(\alpha, \beta) \in \Lambda : Q_{(\alpha,\beta)}x = x\} = (\gamma, \delta)$ for some $\gamma < \kappa$ and $0 < \delta < \mu_\gamma$. If $\mu_\gamma = \omega$, then δ is successor, so (γ, δ) is successor. Assume that $\mu_\gamma > \omega$. If $P_\gamma x = x$, then $Q_{(\gamma,0)}x = x$. But this contradicts the minimality of (γ, δ). So, by (P5) we have that $P_\gamma x = 0$ and since $x = Q_{(\gamma,\delta)}x$ we cannot have $P_{\gamma+1}x = 0$ and therefore, by (P5) we have $P_{\gamma+1}x = x$. Then $x = Q_{(\gamma,\delta)}x = Q^\gamma_\delta P_{\gamma+1}x = Q^\gamma_\delta x$, therefore (2) ensures that δ is a successor ordinal and so (γ, δ) is a successor ordinal. This establishes (5).

(6): Fix $(\varepsilon, A, x^*) \in \mathcal{T}(\mathfrak{s}, \mathcal{A})$ and let $((\alpha_k, \beta_k))_{k\in\omega}$ be an increasing sequence in Λ such that $\sup_k(\alpha_k, \beta_k) = (\alpha, \beta)$. Only three cases are possible.

Case 1: $\beta \neq 0$.

In this case α_k is eventually constant and $\sup_k \beta_k = \beta < \mu_\alpha$. Thus we may assume that $\alpha_k = \alpha$, for every $k \in \omega$. If $\mu_\alpha = \omega$, then the result follows easily. Now assume that $\mu_\alpha > \omega$ and note that fixed $x \in A$, we have that

$$\begin{aligned}
&|Q^*_{(\alpha,\beta_k)}(x^*)(x) - Q^*_{(\alpha,\beta)}(x^*)(x)| \\
&\qquad = |(Q^\alpha_{\beta_k})^*(x^*)((P_{\alpha+1} - P_\alpha)(x)) - (Q^\alpha_\beta)^*(x^*)((P_{\alpha+1} - P_\alpha)(x))| \\
&\qquad \leq \rho_{A_\alpha}((Q^\alpha_{\beta_k})^*(x^*_\alpha), (Q^\alpha_\beta)^*(x^*_\alpha)).
\end{aligned}$$

Thus,

$$\rho_A(Q^*_{(\alpha,\beta_k)}(x^*), Q^*_{(\alpha,\beta)}(x^*)) \leq \rho_{A_\alpha}((Q^\alpha_{\beta_k})^*(x^*_\alpha), (Q^\alpha_\beta)^*(x^*_\alpha))$$

and by (3) we get

$$\limsup_k \rho_A(Q^*_{(\alpha,\beta_k)}(x^*), Q^*_{(\alpha,\beta)}(x^*)) \leq \varepsilon_\alpha \|x^*_\alpha\| \leq \varepsilon \|x^*\|.$$

Therefore we conclude that $(\varepsilon, A, x^*) \in \mathcal{T}(\mathfrak{S}, \mathcal{A})$.

Case 2: $\beta = 0$ and α is limit.

In this case $\alpha = \sup_k \alpha_k$. Fix $x \in A$ and recall that $Q_{(\alpha,0)} = P_\alpha$. It follows from (P3) and (P5) that $(P_{\alpha_k+1} - P_{\alpha_k})(x)$ is eventually equal to zero. Therefore we get

$$\begin{aligned}
|Q^*_{(\alpha_k,\beta_k)}&(x^*)(x) - Q^*_{(\alpha,0)}(x^*)(x)| \\
&= |(Q^{\alpha_k}_{\beta_k})^*(x^*)((P_{\alpha_k+1} - P_{\alpha_k})(x)) + x^*(P_{\alpha_k} x) - x^*(P_\alpha x)| \\
&= |x^*(P_{\alpha_k} x) - x^*(P_\alpha x)| \leq \rho_A(P^*_{\alpha_k}(x^*), P^*_\alpha(x^*)).
\end{aligned}$$

From which we get

$$\rho_A(Q^*_{(\alpha_k,\beta_k)}(x^*), Q^*_{(\alpha,\beta)}(x^*)) \leq \rho_A((P^*_{\alpha_k}(x^*), P^*_\alpha(x^*)).$$

Note that (P7) ensures that $(\varepsilon, A, x^*) \in \mathcal{T}(\mathfrak{p}, \mathcal{A})$ and thus we obtain that

$$\limsup_k \rho_A(Q^*_{(\alpha_k,\beta_k)}(x^*), Q^*_{(\alpha,\beta)}(x^*)) \leq \varepsilon \|x^*\|.$$

This shows that $(\varepsilon, A, x^*) \in \mathcal{T}(\mathfrak{S}, \mathcal{A})$.

Case 3: $\beta = 0$ and $\alpha = \gamma + 1$, for some $\gamma < \kappa$.

In this case α_k is eventually equal to γ and $\sup_k \beta_k = \mu_\gamma$. So we may assume that $\alpha_k = \gamma$, for every $k \in \omega$. Fix $x \in A$ and recall that $Q_{(\gamma+1,0)} = P_{\gamma+1}$. If $\mu_\gamma = \omega$, then we observe that $Q_{(\gamma,\beta_k)} = P_{\gamma+1}$ whenever $\beta_k \neq 0$, so eventually we have $Q_{(\alpha_k,\beta_k)} = Q_{(\alpha,\beta)}$. Now assume that $\mu_\gamma > \omega$. Using that $Q^\gamma_{\mu_\gamma}$ is the identity map on $(P_{\gamma+1} - P_\gamma)[X]$, we obtain

$$\begin{aligned}
|Q^*_{(\gamma,\beta_k)}&(x^*)(x) - Q^*_{(\gamma+1,0)}(x^*)(x)| \\
&= |(Q^\gamma_{\beta_k})^*(x^*)((P_{\gamma+1} - P_\gamma)(x)) - x^*((P_{\gamma+1} - P_\gamma)(x)))| \\
&= |(Q^\gamma_{\beta_k})^*(x^*)((P_{\gamma+1} - P_\gamma)(x)) - (Q^\gamma_{\mu_\gamma})^*(x^*)((P_{\gamma+1} - P_\gamma)(x))| \\
&\leq \rho_{A_\gamma}((Q^\gamma_{\beta_k})^*(x^*_\gamma), (Q^\gamma_{\mu_\gamma})^*(x^*_\gamma)).
\end{aligned}$$

Therefore

$$\rho_A(Q^*_{(\alpha_k,\beta_k)}(x^*), Q^*_{(\alpha,0)}(x^*)) \leq \rho_{A_\gamma}((Q^\gamma_{\beta_k})^*(x^*_\gamma), (Q^\gamma_{\mu_\gamma})^*(x^*_\gamma))$$

and using (3), we obtain that $\limsup_k \rho_A(Q^*_{(\alpha_k,\beta_k)}(x^*), Q^*_{(\alpha,0)}(x^*)) \leq \varepsilon \|x^*\|$. This shows that $(\varepsilon, A, x^*) \in \mathcal{T}(\mathfrak{S}, \mathcal{A})$. $\square$

Theorem 24. *Let X be a nonseparable Banach space with a projectional skeleton $\mathfrak{s} = (P_s)_{s\in\Gamma}$ and $\kappa := \operatorname{dens} X$. If $\mathcal{A}$ is a bounded subset of X countably supporting $D(\mathfrak{s})$, then X admits a SPRI $\mathfrak{S} = (Q_\alpha)_{\alpha\leq\kappa}$ satisfying the following conditions*

(a) For every $x \in \mathcal{A}$ and $\alpha \in [0, \kappa]$, $Q_\alpha x \in \{0, x\}$;
(b) For every $x \in \mathcal{A}$, $\min\{\alpha \leq \kappa \colon Q_\alpha x = x\}$ is not a limit ordinal;
(c) $\mathcal{T}(\mathfrak{s}, \mathcal{A}) \subset \mathcal{T}(\mathfrak{S}, \mathcal{A})$.

Proof. It follows from Corollary 20 and the fact that if $\mathfrak{s}'$ is a subskeleton of $\mathfrak{s}$, then $D(\mathfrak{s}') = D(\mathfrak{s})$ and $\mathcal{T}(\mathfrak{s}, \mathcal{A}) \subset \mathcal{T}(\mathfrak{s}', \mathcal{A})$ that we may without loss of generality assume that $P_s x \in \{0, x\}$ for every $x \in \mathcal{A}$ and $s \in \Gamma$. We will prove the result by induction on $\kappa \geq \omega_1$. Let $\mathfrak{p} = (P_\alpha)_{\alpha\leq\kappa}$ be the sequence of bounded projections satisfying (P1)-(P7) given by Proposition 22. If $\kappa = \omega_1$, then using (P1), (P2), (P3) and (P4), we conclude that the family $\mathfrak{S} := \mathfrak{p}$ is a SPRI on X. Moreover, it follows from (P3), (P5) and (P7) that $\mathfrak{S}$ satisfies conditions (a), (b) and (c).

Assume now that $\kappa > \omega_1$ and that the result holds for every nonseparable Banach space with density strictly smaller than κ. Let us verify that $\mathfrak{p}$ satisfies the assumptions of Lemma 23. Fix $\alpha \in \kappa$ and suppose that $(P_{\alpha+1} - P_\alpha)[X]$ is not separable. By (P6), there exists a σ-closed and cofinal subset Γ_α of $(\Gamma \cap M_{\alpha+1})_\sigma$ such that $\mathfrak{s}_\alpha := (P_s|_{(P_{\alpha+1}-P_\alpha)[X]})_{s\in\Gamma_\alpha}$ is a projectional skeleton on $(P_{\alpha+1} - P_\alpha)[X]$ and $\mathcal{A}_\alpha := (P_{\alpha+1} - P_\alpha)[\mathcal{A}]$ countably supports $D(\mathfrak{s}_\alpha)$. Therefore it follows from (P4) and the induction hypothesis that $(P_{\alpha+1} - P_\alpha)[X]$ admits a SPRI $\mathfrak{S}_\alpha$ satisfying the appropriate versions of (a), (b) and (c). It is clear that $\mathfrak{S}_\alpha$ satisfies conditions (1) and (2) from Lemma 23. In order to check that $\mathfrak{S}_\alpha$ satisfies condition (3) from Lemma 23, fix $(\varepsilon, A, x^*) \in \mathcal{T}(\mathfrak{s}, \mathcal{A})$ and let $(s_n)_{n\in\omega}$ be an increasing sequence in Γ_α with $s = \sup_n s_n$. Using that (P5) ensures that $A_\alpha \subset A \cup \{0\}$, we obtain that

$$\begin{aligned}
&\limsup_n \rho_{A_\alpha}\big((P_{s_n}|_{(P_{\alpha+1}-P_\alpha)[X]})^*(x^*_\alpha), (P_s|_{(P_{\alpha+1}-P_\alpha)[X]})^*(x^*_\alpha)\big) \\
&\qquad = \limsup_n \sup_{x\in A} \Big| P^*_{s_n}(x^*((P_{\alpha+1} - P_\alpha)x)) - P^*_s(x^*(P_{\alpha+1} - P_\alpha x)) \Big| \\
&\qquad \leq \limsup_n \rho_A(P^*_{s_n}(x^*), P^*_s(x^*)) \leq \varepsilon\|x^*\|,
\end{aligned}$$

which implies that $(\varepsilon_\alpha, A_\alpha, x^*_\alpha) \in \mathcal{T}(\mathfrak{s}_\alpha, \mathcal{A}_\alpha)$. Since the induction hypothesis ensures that $\mathcal{T}(\mathfrak{s}_\alpha, \mathcal{A}_\alpha) \subset \mathcal{T}(\mathfrak{S}_\alpha, \mathcal{A}_\alpha)$, we conclude that $(\varepsilon_\alpha, A_\alpha, x^*_\alpha) \in \mathcal{T}(\mathfrak{S}_\alpha, \mathcal{A}_\alpha)$. Thus $\mathfrak{S}_\alpha$ satisfies condition (3) from Lemma 23. Now the result follows from Lemma 23. $\square$

4.2. **Weakly $\mathcal{K}$-analytic Banach spaces.** In what follows, we denote by $\omega^{<\omega}$ the collection of functions defined on a natural number and taking values in ω, i.e., $\omega^{<\omega} = \bigcup_{i\in\omega} \omega^i$.

Following [13], we say that a Banach space X is *weakly $\mathcal{K}$-analytic* if there exists a family $\{K_t : t \in \omega^{<\omega}\}$ of weak* compact subsets of X^{**} such that $X = \bigcup_{\sigma\in\omega^\omega} \bigcap_{i\geq 1} K_{\sigma|i}$. The key ingredients to establish the characterization of weakly $\mathcal{K}$-analytic Banach spaces given in Theorem A are Theorem 24 and the characterization of weakly $\mathcal{K}$-analyticity presented in [13, Theorem 4] that we recall below.

Lemma 25. *For a Banach space X, the following conditions are equivalent*

(i) X is weakly $\mathcal{K}$-analytic;

(ii) there exist a set $\mathcal{A} \subset B_X$ that is linearly dense in X and a family $\{A_t : t \in \omega^{<\omega}\}$ of subsets of $\mathcal{A}$ such that $A_\emptyset = \mathcal{A} = \bigcup_{\sigma\in\omega^\omega} \bigcap_{i\geq 1} A_{\sigma|i}$ and for every $\varepsilon > 0$, $x^* \in X^*$, $\sigma \in \omega^\omega$, there exists $i \in \omega$ such that

$$|\{x \in A_{\sigma|i} : |x^*(x)| > \varepsilon\}| < \omega.$$

Moreover, for $\mathcal{A}$ we can take any subset of B_X that is linearly dense in X and countably supports X^*.

Here as usual, given $n, i \in \omega$ and $t \in \omega^i$, we denote by $n^\frown t$ the element of ω^{i+1} given by $n^\frown t(0) = n$ and if $i \geq 1$, $n^\frown t(j) = t(j-1)$, for every $1 \leq j \leq i$.

Proof of Theorem A. (i)$\Rightarrow$ (ii) Since X is WLD, there exists a set $\mathcal{A} \subset B_X$ that is linearly dense in X and countably supports X^*. Thus, it follows from Lemma 25 that there exists a family $\{A_t \subset \mathcal{A} : t \in \omega^{<\omega}\}$ such that $\mathcal{A} = A_\emptyset = \bigcup_{\sigma\in\omega^\omega} \bigcap_{i\geq 1} A_{\sigma|i}$ and for every $\varepsilon > 0$, $x^* \in X^*$ and $\sigma \in \omega^{<\omega}$ there exists $i \in \omega$ such that

$$|\{x \in A_{\sigma|i} \colon |x^*(x)| > \varepsilon\}| < \omega.$$

Since X is Plichko, we have that X admits a projectional skeleton $\mathfrak{s} = (P_s)_{s\in\Gamma}$ and thus it follows from Corollary 20 that we may assume without loss of generality that $P_s(x) \in \{0, x\}$, for every $x \in \mathcal{A}$ and $s \in \Gamma$.
Now fix $\varepsilon > 0$, $x^* \in X^* \setminus \{0\}$ and $\sigma \in \omega^\omega$. Let $i \in \omega$ be such that the set $Y := \{x \in A_{\sigma|i} \colon |x^*(x)| > \varepsilon\|x^*\|\}$ is finite and let us show that $\mathfrak{s}$ is $(A_{\sigma|i}, \varepsilon)$-shrinking in x^*. Fix an increasing sequence $(s_k)_{k\in\omega}$ in Γ with $s = \sup_{k\in\omega} s_k$ and note that

$$|P^*_{s_k}(x^*)(x) - P^*_s(x^*)(x)| = |x^*(P_{s_k}x) - x^*(P_s x)| = \begin{cases} 0 & \text{if } P_{s_k}x = P_s x, \\ |x^*(x)| & \text{if } P_{s_k}x \neq P_s x, \end{cases}$$

for every $k \in \omega$ and $x \in \mathcal{A}$. Thus we have that

$$\forall x \in A_{\sigma|i} \setminus Y : \qquad |P^*_{s_k}(x^*)(x) - P^*_s(x^*)(x)| \leq \varepsilon\|x^*\|. \qquad (I)$$

Moreover, since $(P^*_{s_k}(x^*))_{k\in\omega}$ converges to $P^*_s(x^*)$ in the weak*-topology of X^* and Y is finite, there exists $k_0 \in \omega$ such that

$$\forall k \geq k_0,\ \forall x \in Y : \quad |P^*_{s_k}(x^*)(x) - P^*_s(x^*)(x)| \leq \varepsilon\|x^*\|. \qquad (II)$$

Therefore using (I) and (II), we conclude that $\limsup_k \rho_{A_{\sigma|i}}(P^*_{s_k}x^*, P^*_s x^*) \leq \varepsilon\|x^*\|$.
(ii) $\Rightarrow$ (iii) If X is separable, then the result is trivial and in the case when X is nonseparable, the result follows directly from Theorem 24.
(iii) $\Rightarrow$ (i) For every $\alpha \in \kappa$, set $R_\alpha := Q_{\alpha+1} - Q_\alpha$ and note that since $R_\alpha[X]$ is separable and $R_\alpha[\mathcal{A}]$ is linearly dense in $R_\alpha[X]$, there exists a countable set $\{v^\alpha_n : n \in \omega\} \subset R_\alpha[\mathcal{A}] \setminus \{0\}$ that is linearly dense in $R_\alpha[X]$.

Consider the family $\{B_\sigma : \sigma \in \omega^{<\omega}\}$ defined as follows:

- $B_{n^\frown t} := A_t \cap \{v_n^\alpha : \alpha \in \kappa\} \subset B_X$, for every $n \in \omega$ and $t \in \omega^{<\omega}$;
- $B_\emptyset := \bigcup_{\sigma\in\omega^\omega} \bigcap_{i\geq 1} B_{\sigma|i}$.

In order to conclude that X is weakly $\mathcal{K}$-analytic, let us show that the subset $\mathcal{B} := \bigcup_{\sigma\in\omega^\omega} \bigcap_{i\geq 1} B_{\sigma|i}$ of B_X and the family $\{B_\sigma : \sigma \in \omega^{<\omega}\}$ satisfy condition (ii) of Lemma 25. Using that $A_t \subset \mathcal{A}$, for every $t \in \omega^{<\omega}$ and that $A_\emptyset = \mathcal{A}$, it is easy to see that $B_{n^\frown t} \subset \mathcal{B}$, for every $n \in \omega$ and $t \in \omega^{<\omega}$. In order to prove that $\mathcal{B}$ is linearly dense in X, we claim that $\{v_n^\alpha : \alpha \in \kappa, n \in \omega\} \subset \mathcal{B}$. Indeed, fixed $\alpha \in \kappa$ and $n \in \omega$, it follows from (3b) that $v_n^\alpha \in \mathcal{A}$ and thus there exists $\sigma \in \omega^\omega$ such that $v_n^\alpha \in \bigcap_{i\geq 1} A_{\sigma|i}$, which implies that $v_n^\alpha \in \bigcap_{i\geq 1} B_{\sigma'|i} \subset \mathcal{B}$, where $\sigma' \in \omega^\omega$ is defined as $\sigma'(0) = n$ and $\sigma'(i) = \sigma(i-1)$, for every $i \geq 1$. This proves the claim and ensures that $\mathcal{B}$ is linearly dense in X, since $\{v_n^\alpha : \alpha \in \kappa, n \in \omega\}$ is linearly dense in $\bigcup_{\alpha\in\kappa} R_\alpha[X]$ and $\bigcup_{\alpha\in\kappa} R_\alpha[X]$ is linearly dense in X. Now fix $\varepsilon > 0$, $x^* \in X^* \setminus \{0\}$ and $\sigma \in \omega^\omega$. By (3c), there is $i \in \omega$ corresponding to the triple $\varepsilon' := \frac{\varepsilon}{2\|x^*\|} > 0$, $x^* \in X^*$ and $\tau \in \omega^\omega$, where $\tau(j) = \sigma(j+1)$, for $j \in \omega$. We are going to verify that

$$|\{x \in B_{\sigma|i+1} : |x^*(x)| > \varepsilon\}| < \omega.$$

Assume by contradiction that this is not the case. Then there exists a stictly increasing sequence $(\alpha_k)_{k\in\omega}$ in κ such that $v_{\sigma(0)}^{\alpha_k} \in A_{\tau|i}$ and $|x^*(v_{\sigma(0)}^{\alpha_k})| > \varepsilon$, for every $k \in \omega$. Set $\alpha = \sup_k \alpha_k \leq \kappa$ and note that it follows from the fact that $(Q_\alpha)_{\alpha\leq\kappa}$ is $(A_{\tau|i}, \varepsilon')$-shrinking in x^* that there exists $k \in \omega$ such that $\rho_{A_{\tau|i}}(Q_{\alpha_k}^*(x^*), Q_\alpha^*(x^*)) < 2\varepsilon'\|x^*\| = \varepsilon$. In particular, it holds that $|Q_{\alpha_k}^*(x^*)(v_{\sigma(0)}^{\alpha_k}) - Q_\alpha^*(x^*)(v_{\sigma(0)}^{\alpha_k})| < \varepsilon$. Moreover using that $(Q_\alpha)_{\alpha\leq\kappa}$ is a SPRI in X and $v_{\sigma(0)}^{\alpha_k} \in (Q_{\alpha_k+1} - Q_{\alpha_k})[X]$, we conclude that $Q_{\alpha_k}(v_{\sigma(0)}^{\alpha_k}) = 0$ and $Q_\alpha(v_{\sigma(0)}^{\alpha_k}) = v_{\sigma(0)}^{\alpha_k}$ and thus we obtain that $|x^*(v_{\sigma(0)}^{\alpha_k})| < \varepsilon$, which contradicts the choice of $v_{\sigma(0)}^{\alpha_k}$. □

4.3. **Vašák spaces.** Following [13], we say that a Banach space X is *Vašák* if there exists a family $\{K_n : n \in \omega\}$ of weak*-compact subsets of X^{**} such that for every $x \in X$ and $x^{**} \in X^{**} \setminus X$, there exists $n \in \omega$ with $x \in K_n$ and $x^{**} \notin K_n$. The key tools to establish the characterization of Vašák spaces given in Theorem 27 are Theorem 24 and the characterization of Vašák spaces presented in [13, Theorem 3] that we recall below.

Lemma 26. *For a Banach space X, the following conditions are equivalent*

(i) X is Vašák;

(ii) there exist a set $\mathcal{A} \subset B_X$ that is linearly dense in X and a family $\{A_n : n \in \omega\}$ of subsets of $\mathcal{A}$ such that for every $\varepsilon > 0$, $x^* \in X^*$ and $x \in \mathcal{A}$, there exists $n \in \omega$ such that

$$x \in A_n \text{ and } |\{x' \in A_n : |x^*(x')| > \varepsilon\}| < \omega.$$

Theorem 27. *Let X be a Banach space with $\kappa := \mathrm{dens}(X)$. Then the following conditions are equivalent.*

(i) *X is Vašák.*

(ii) *There exist a projectional skeleton $\mathfrak{s} = (P_s)_{s\in\Gamma}$ on X and a family $\{A_n \subset B_X : n \in \omega\}$ of non-empty sets satisfying the following conditions*

  (a) *$\mathcal{A} := \bigcup_{n\in\omega} A_n$ is linearly dense in X and countably supports X^*;*

  (b) *for every $\varepsilon > 0$ and $x^* \in X^*$, there exists $N \subset \omega$ such that $\mathcal{A} = \bigcup_{j\in N} A_j$ and $\mathfrak{s}$ is (A_j, ε)-shrinking in x^*, for every $j \in N$.*

(iii) *There exist a SPRI $(Q_\alpha)_{\alpha\leq\kappa}$ in X and a family $\{A_n \subset B_X : n \in \omega\}$ of non-empty sets satisfying the following conditions*

  (3a) *$\mathcal{A} := \bigcup_{n\in\omega} A_n$ is linearly dense in X and countably supports X^*;*

  (3b) *for every $x \in \mathcal{A}$ it holds that $\{Q_\alpha(x) \colon \alpha \leq \kappa\} \subset \{0, x\}$ and $\min\{\alpha \leq \kappa \colon Q_\alpha x = x\}$ is not a limit ordinal;*

  (3c) *for every $\varepsilon > 0$ and $x^* \in X^*$, there exists $N \subset \omega$ such that $\mathcal{A} = \bigcup_{n\in N} A_n$ and $(Q_\alpha)_{\alpha\leq\kappa}$ is (A_j, ε)-shrinking in x^*, for every $j \in N$.*

Proof. (i)$\Rightarrow$ (ii) Let $\mathcal{A} \subset B_X$ and $\{A_n \subset \mathcal{A} : n \in \omega\}$ be given by Lemma 26 and note that $\mathcal{A}$ countably supports X^*. Since X is Plichko, we have that X admits a projectional skeleton $\mathfrak{s} = (P_s)_{s\in\Gamma}$ and thus it follows from Corollary 20 that we may assume without loss of generality that $P_s(x) \in \{0, x\}$, for every $x \in \mathcal{A}$ and $s \in \Gamma$.

Clearly condition (a) is satisfied. Now let us show that condition (b) also holds. Fix $\varepsilon > 0$, $x^* \in X^* \setminus \{0\}$ and for each $x \in \mathcal{A}$, pick $n(x) \in \omega$ such that

$$x \in A_{n(x)} \text{ and } |\{x' \in A_{n(x)} : |x^*(x')| > \varepsilon\|x^*\|\}| < \omega.$$

Set $N := \{n(x) : x \in \mathcal{A}\}$. It is clear that $\mathcal{A} = \bigcup_{j\in N} A_j$ and fixed $j \in N$ the fact that $\mathfrak{s}$ is (A_j, ε)-shrinking in x^* is established with an argument identical to the one presented in the proof of (i)$\Rightarrow$ (ii) in Theorem A.

(ii) $\Rightarrow$ (iii) If X is separable, then the result is trivial and in the case when X is nonseparable, the result follows directly from Theorem 24.

(iii) $\Rightarrow$ (i) For every $\alpha \in \kappa$, set $R_\alpha := Q_{\alpha+1} - Q_\alpha$ and note that since $R_\alpha[X]$ is separable and $R_\alpha[\mathcal{A}]$ is linearly dense in $R_\alpha[X]$, there exists a countable set $\{v_n^\alpha : n \in \omega\} \subset R_\alpha[\mathcal{A}] \setminus \{0\}$ that is linearly dense in $R_\alpha[X]$. For each $n, m \in \omega$ define $B_{n,m} := A_m \cap \{v_n^\alpha : \alpha \in \kappa\}$ and set $\mathcal{B} := \bigcup_{n,m\in\omega} B_{n,m}$.

In order to conclude that X is a Vašák space, let us show that the set $\mathcal{B}$ and the family $\{B_{n,m} : n, m \in \omega\}$ satisfy condition (ii) of Lemma 26. Using that (3b) ensures that $R_\alpha[\mathcal{A}] \setminus \{0\} \subset \mathcal{A}$, for every $\alpha \in \kappa$, it is easy to see that $\{v_n^\alpha : n \in \omega, \alpha \in \kappa\} \subset \mathcal{B}$, which implies that $\mathcal{B}$ is linearly dense in X. Now fix $\varepsilon > 0$, $x^* \in X^* \setminus \{0\}$ and $x \in \mathcal{B}$. Let $N \subset \omega$ be the set given by (3c) for $\varepsilon' := \frac{\varepsilon}{2\|x^*\|} > 0$ and x^*. Clearly, there exist $\alpha \in \kappa$ and $n \in \omega$ such that $x = v_n^\alpha$. Moreover, since $\mathcal{B} \subset \mathcal{A} = \bigcup_{j\in N} A_j$, there exists $j \in N$ such that $x \in A_j$ and thus we have that $x \in B_{n,j}$. In order to conclude the proof, it remains to show that the set $\{x' \in B_{n,j} : |x^*(x')| > \varepsilon\}$ is finite. Assume by contradiction that this set is infinite. In this case, there exits a strictly increasing sequence $(\alpha_k)_{k\in\omega}$ in κ such that $v_n^{\alpha_k} \in A_j$ and $|x^*(v_n^{\alpha_k})| > \varepsilon$, for every $k \in \omega$. Since $(Q_\alpha)_{\alpha\le\kappa}$ is (A_j, ε')-shrinking in x^*, arguing as in the proof of (iii) $\Rightarrow$ (i) in Theorem A, we achieve a contradiction and establish our result. □

4.4. **WCG spaces and their subspaces.** Recall that a Banach space is said to be *weakly compactly generated* (WCG) if it contains a linearly dense and weakly compact subset. The key tools to establish the characterization of WCG spaces given in Theorem 29 are Theorem 24 and the characterization of WCG spaces presented in [13, Theorem 1] that we recall below.

Lemma 28. *For a Banach space X, the following conditions are equivalent*

(i) X is WCG;
(ii) there exists a set $\mathcal{A} \subset B_X$ that is linearly dense in X and such that for every $\varepsilon > 0$ and $x^* \in X^*$ it holds that

$$|\{x \in \mathcal{A} : |x^*(x)| > \varepsilon\}| < \omega.$$

Theorem 29. *Let X be a Banach space with $\kappa := \mathrm{dens}(X)$. Then the following conditions are equivalent.*

(i) X is WCG.
(ii) There exist a projectional skeleton $\mathfrak{s} = (P_s)_{s\in\Gamma}$ on X and a non-empty set $\mathcal{A} \subset B_X$, satisfying the following conditions
  (a) $\mathcal{A}$ is linearly dense in X and countably supports X^*;
  (b) $\mathfrak{s}$ is $(\mathcal{A}, 0)$-shrinking in every element of X^*.
(iii) There exist a SPRI $(Q_\alpha)_{\alpha\le\kappa}$ in X and a non-empty set $\mathcal{A} \subset B_X$, satisfying the following conditions
  (3a) $\mathcal{A}$ is linearly dense in X and countably supports X^*;
  (3b) for every $x \in \mathcal{A}$ it holds that $\{Q_\alpha(x) : \alpha \le \kappa\} \subset \{0, x\}$ and $\min\{\alpha \le \kappa : Q_\alpha x = x\}$ is not a limit ordinal;
  (3c) $(Q_\alpha)_{\alpha\le\kappa}$ is $(\mathcal{A}, 0)$-shrinking in every element of X^*.

Proof. (i)$\Rightarrow$(ii) Let $\mathcal{A}$ be the set given by Lemma 28 and note that $\mathcal{A}$ countably supports X^*. Since X is Plichko, we have that X admits a projectional skeleton $\mathfrak{s} = (P_s)_{s\in\Gamma}$ and thus it follows from Corollary 20 that we may assume without loss of generality that $P_s(x) \in \{0, x\}$, for every $x \in \mathcal{A}$ and $s \in \Gamma$. Finally fixed $x^* \in X^*$, the fact that $\mathfrak{s}$ is $(\mathcal{A}, 0)$-shrinking in x^* can be easily checked using that for every $\varepsilon > 0$, the set $\{x \in \mathcal{A} : |x^*(x)| > \varepsilon\}$ is finite, similarly as argued in the proof of (i)$\Rightarrow$ (ii) in Theorem A.
(ii)$\Rightarrow$ (iii) If X is separable, then the result is trivial and in the case when X is nonseparable, the result follows directly from Theorem 24.
(iii)$\Rightarrow$(i) For every $\alpha \in \kappa$, set $R_\alpha := Q_{\alpha+1} - Q_\alpha$ and note that since $R_\alpha[X]$ is separable and $R_\alpha[\mathcal{A}]$ is linearly dense in $R_\alpha[X]$, there exists a countable set $\{v_n^\alpha : n \ge 1\} \subset R_\alpha[\mathcal{A}] \setminus \{0\}$ that is linearly dense in $R_\alpha[X]$. In order to conclude that X is WCG, we will show that the set $\mathcal{B} := \bigcup_{\alpha<\kappa}\{\frac{v_n^\alpha}{n} : n \ge 1\}$ satisfies condition (ii) of Lemma 28. Note that $\mathcal{B} \subset B_X$, since it follows

from condition (3b) that $R_\alpha[\mathcal{A}] \setminus \{0\} \subset \mathcal{A}$. Moreover the linear density of $\mathcal{B}$ in X follows from the fact that $\mathcal{B}$ is linearly dense in $\bigcup_{\alpha\in\kappa} R_\alpha[X]$ and $\bigcup_{\alpha\in\kappa} R_\alpha[X]$ is linearly dense in X. Now fix $\varepsilon > 0$, $x^* \in X^*$ and let us show that the set $\{x \in \mathcal{B} : |x^*(x)| > \varepsilon\}$ is finite. Assume by contradiction that this is not the case. Note that if there are only finitely many $\alpha \in \kappa$ such that $\{x \in \mathcal{B} : |x^*(x)| > \varepsilon\} \cap R_\alpha[X] \neq \emptyset$, we easily arrive to a contradiction, since $\lim_{n\to\infty} \frac{v_n^\alpha}{n} = 0$, for every $\alpha \in \kappa$. Otherwise, there exist a strictly increasing sequence $(\alpha_k)_{k\in\omega}$ in κ and a sequence of nonzero natural numbers $(n_k)_{k\in\omega}$ such that

$$\forall k \in \omega : \quad \left| x^*\left(\frac{v_{n_k}^{\alpha_k}}{n_k}\right) \right| > \varepsilon.$$

Since $(Q_\alpha)_{\alpha\leq\kappa}$ is $(\mathcal{A}, 0)$-shrinking in x^*, arguing as in the proof of (iii) $\Rightarrow$ (i) in Theorem A, we obtain that there exists $k \in \omega$ such that $|x^*(v_{n_k}^{\alpha_k})| < \varepsilon$, since $v_{n_k}^{\alpha_k} \in \mathcal{A}$. But this contradicts the choice of $v_{n_k}^{\alpha_k}$ and thus establishes the result. $\square$

It is worth mentioning that a subspace of a WCG space is not necessarily WCG (see [24]). The key tools to establish the characterization of subspaces of WCG spaces given in Theorem 31 are Theorem 24 and the characterization of subspaces of WCG spaces presented in [13, Theorem 2] that we recall below.

Lemma 30. *For a Banach space X, the following conditions are equivalent*

- *(i) X is a subspace of a WCG space;*
- *(ii) there exists a set $\mathcal{A} \subset B_X$ that is linearly dense in X and such that for every $\varepsilon > 0$ there exists a decomposition $\mathcal{A} = \bigcup_{n\in\omega} A_n^\varepsilon$ satisfying the following condition*

$$\forall n \in \omega,\ \forall x^* \in X^* \Rightarrow |\{x \in A_n^\varepsilon : |x^*(x)| > \varepsilon\}| < \omega.$$

Theorem 31. *Let X be a Banach space with $\kappa := \operatorname{dens}(X)$. Then the following conditions are equivalent.*

- *(i) X is isometric to a subspace of a WCG space.*
- *(ii) There exist a projectional skeleton $\mathfrak{s} = (P_s)_{s\in\Gamma}$ on X and a non-empty set $\mathcal{A} \subset B_X$, satisfying the following conditions*
  - *(a) $\mathcal{A}$ is linearly dense in X and countably supports X^*;*
  - *(b) for every $\varepsilon > 0$, there exists a decomposition $\mathcal{A} = \bigcup_{n\in\omega} A_n^\varepsilon$ such that $\mathfrak{s}$ is $(A_n^\varepsilon, \frac{\varepsilon}{\|x^*\|})$-shrinking in x^*, for every $n \in \omega$ and $x^* \in X^* \setminus \{0\}$.*
- *(iii) There exist a SPRI $(Q_\alpha)_{\alpha\leq\kappa}$ in X and a non-empty set $\mathcal{A} \subset B_X$, satisfying the following conditions*
  - *(3a) $\mathcal{A}$ is linearly dense in X and countably supports X^*;*
  - *(3b) for every $x \in \mathcal{A}$ it holds that $\{Q_\alpha(x) \colon \alpha \leq \kappa\} \subset \{0, x\}$ and $\min\{\alpha \leq \kappa \colon Q_\alpha x = x\}$ is not a limit ordinal;*
  - *(3c) for every $\varepsilon > 0$, there exists a decomposition $\mathcal{A} = \bigcup_{n\in\omega} A_n^\varepsilon$ such that $(Q_\alpha)_{\alpha\leq\kappa}$ is $(A_n^\varepsilon, \frac{\varepsilon}{\|x^*\|})$-shrinking in x^*, for every $n \in \omega$ and $x^* \in X^* \setminus \{0\}$.*

Proof. (i)⇒(ii) Let $\mathcal{A}$ be the set given by Lemma 30 and note that $\mathcal{A}$ countably supports X^*. Since X is Plichko, we have that X admits a projectional skeleton $\mathfrak{s} = (P_s)_{s\in\Gamma}$ and thus it follows from Corollary 20 that we may assume without loss of generality that $P_s(x) \in \{0, x\}$, for every $x \in \mathcal{A}$ and $s \in \Gamma$. Now fix $\varepsilon > 0$, and let $\mathcal{A} = \bigcup_{n\in\omega} A_n^\varepsilon$ be the decomposition given by Lemma 30. The fact that $\mathfrak{s}$ is $(A_n^\varepsilon, \frac{\varepsilon}{\|x^*\|})$-shrinking in x^*, for every $n \in \omega$ and $x^* \in X^* \setminus \{0\}$ can be easily checked using that the set $\{x \in A_n^\varepsilon : |x^*(x)| > \varepsilon\}$ is finite, similarly as argued in the proof of (i)⇒ (ii) in Theorem A.
(ii)⇒ (iii) If X is separable, then the result is trivial and in the case when X is nonseparable, the result follows directly from Theorem 24.
(iii)⇒(i) For every $\alpha \in \kappa$, set $R_\alpha := Q_{\alpha+1} - Q_\alpha$ and note that since $R_\alpha[X]$ is separable and $R_\alpha[\mathcal{A}]$ is linearly dense in $R_\alpha[X]$, there exists a countable set $\{v_n^\alpha : n \geq 1\} \subset R_\alpha[\mathcal{A}] \setminus \{0\}$ that is linearly dense in $R_\alpha[X]$. In order to conclude that X is a subspace of a WCG space, we will show that the set $\mathcal{B} := \bigcup_{\alpha<\kappa}\{v_n^\alpha : n \in \omega\}$ satisfies condition (ii) of Lemma 30. Note that $\mathcal{B} \subset B_X$, since it follows from condition (3b) that $R_\alpha[\mathcal{A}] \setminus \{0\} \subset \mathcal{A}$. Moreover the linear density of $\mathcal{B}$ in X follows from the fact that $\mathcal{B}$ is linearly dense in $\bigcup_{\alpha\in\kappa} R_\alpha[X]$ and $\bigcup_{\alpha\in\kappa} R_\alpha[X]$ is linearly dense in X.

Now fix $\varepsilon > 0$, set $\varepsilon' := \varepsilon/2$ and let $\mathcal{A} = \bigcup_{n\in\omega} A_n^{\varepsilon'}$ be the decomposition given by (3c). For each $n, m \in \omega$, define $B_{n,m}^\varepsilon := A_m^{\varepsilon'} \cap \{v_n^\alpha : \alpha \in \kappa\}$ and note that $\mathcal{B} = \bigcup_{n,m\in\omega} B_{n,m}^\varepsilon$, since $\mathcal{B} \subset \mathcal{A}$. In order to conclude the proof, it remains to show that fixed $n, m \in \omega$ and $x^* \in X^* \setminus \{0\}$, the set $\{x \in B_{n,m}^\varepsilon : |x^*(x)| > \varepsilon\}$ is finite. Assume by contradiction that this set is infinite. In this case, there exits a strictly increasing sequence $(\alpha_k)_{k\in\omega}$ in κ such that $v_n^{\alpha_k} \in A_m^{\varepsilon'}$ and $|x^*(v_n^{\alpha_k})| > \varepsilon$, for every $k \in \omega$. Since $(Q_\alpha)_{\alpha\leq\kappa}$ is $(A_m^{\varepsilon'}, \frac{\varepsilon'}{\|x^*\|})$-shrinking in x^*, arguing as in the proof of (iii) ⇒ (i) in Theorem A, we achieve a contradiction and establish our result. ☐

5. Open problems and remarks

As mentioned previously, we consider the relationship between SPRI and projectional skeletons quite interesting. Observe that combining an inductive argument with Proposition 22 and a result similar to [16, Proposition 6.2.7] (just replacing the assumption that the sequence of projections is a PRI by the assumption that it satisfies conditions (P1), (P2), (P3), (P4) and (P6)), one can easily construct a SPRI in any Banach space with a projectional skeleton. We would like to understand better the properties of the SPRI built from a given projectional skeleton. In particular, this could be useful when dealing with questions proposed in [21, Section 6], where the author already obtained many deep results concerning the connection between projectional skeletons and PRI. As an example of a problem we suggest the following.

Problem 32. *Find a property* (P) *of SPRI such that a Banach space* X *has a projectional skeleton if and only if it admits a SPRI satisfying* (P).

It is important to note that the methods used in the present paper depend heavily on Corollary 20, that is, on the fact that we work with subclasses of Plichko spaces. We observe that given a Plichko space X, it follows from [21, Theorem 3.1] that X admits a projectional skeleton $\mathfrak{s} = (P_s)_{s\in\Gamma}$ and a a Markushevich basis $(x_i, x_i^*)_{i\in I}$ such that $\{x_i : i \in I\}$ countably supports $D(\mathfrak{s})$ and therefore Theorem 24 ensures that X admits a SPRI $(Q_\alpha)_{\alpha\leq \operatorname{dens} X}$ such that $Q_\alpha[\{x_i : i \in I\}] \subset \{x_i : i \in I\} \cup \{0\}$, for every $\alpha \leq \operatorname{dens} X$. In this context, it is natural to propose Question 33, that seems to be related to [21, Question 6.3].

Question 33. *Let X be a Banach space with a projectional skeleton. Does X admit a Markushevich basis $(x_i, x_i^*)_{i\in I}$ and a SPRI $(Q_\alpha)_{\alpha\leq \operatorname{dens} X}$ such that $Q_\alpha[\{x_i\colon i \in I\}] \subset \{x_i\colon i \in I\} \cup \{0\}$, for every $\alpha \leq \operatorname{dens} X$?*

A final suggestion for further research is to characterize other subclasses of Plichko spaces that were considered in [13]. For instance, we suggest the following.

Problem 34. *Characterize the class of Hilbert generated spaces using projectional skeletons and SPRI.*

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