INSTITUTE OF MATHEMATICS

THE CZECH ACADEMY OF SCIENCES

Approximately multiplicative maps between algebras of bounded operators on Banach spaces

Yemon Choi

Bence Horváth

Niels Jakob Laustsen

Preprint No. 72-2021

PRAHA 2021

Approximately multiplicative maps between algebras of bounded operators on Banach spaces

Yemon Choi, Bence Horváth and Niels Jakob Laustsen

7th October 2021

Abstract

We show that for any separable reflexive Banach space X and a large class of Banach spaces E, including those with a subsymmetric shrinking basis but also all spaces L_p for $1 \leq p \leq \infty$, every bounded linear map $\mathcal{B}(E) \to \mathcal{B}(X)$ which is approximately multiplicative is necessarily close in the operator norm to some bounded homomorphism $\mathcal{B}(E) \to \mathcal{B}(X)$. That is, the pair $(\mathcal{B}(E), \mathcal{B}(X))$ has the AMNM property in the sense of Johnson (*J. London Math. Soc.* 1988). Previously this was only known for $E = X = \ell_p$ with $1 < p < \infty$; even for those cases, we improve on the previous methods and obtain better constants in various estimates. A crucial role in our approach is played by a new result, motivated by cohomological techniques, which establishes AMNM properties relative to an amenable subalgebra; this generalizes a theorem of Johnson (*op cit.*).

2020 **Mathematics Subject Classification.** Primary 39B82, 47L10; Secondary 46B03, 46M18, 47B49

Key words and phrases. Algebra of bounded operators, amenable Banach algebra, approximately multiplicative, AMNM, Banach space, Hochschild cohomology, homomorphism, perturbation, Ulam stability.

Contents

1 Introduction

1.1 Background context, and the statement of our main theorem

The AMNM property referred to in the abstract was formulated by B. E. Johnson in [Jo88], and fits into the broader theme of "Ulam stability" for normed representations of groups or algebras: see [BOT13, Cho13, Ko21, MV19] for more recent work in a similar direction. The main purpose of the present paper is to extend our knowledge of the AMNM property to a class of Banach algebras where relatively little has been done, namely the algebras consisting of all bounded operators on E, for various Banach spaces E. (The more restricted setting of stability for **surjective** homomorphisms has recently been considered by the second author with Tarcsay; see [HT21].)

To state Johnson's original definition, and our own results, we need to set up some notation. For a Banach space X and $r \geq 0$, $\mathrm{ball}_r(X)$ denotes $\{x \in X : \|x\| \leq r\}$. Given Banach spaces E and F, and $n \in \mathbb{N}$, we write $\mathcal{L}^n(E,F)$ for the space of bounded n-multilinear maps $E \times \cdots \times E \to F$. If $n = 1$, then we shall usually modify this notation slightly and write $\mathcal{L}(E,F)$. One exception to this notational convention is that when $n = 1$ and $E = F$, we will denote the Banach algebra of all bounded linear operators $E \to E$ by $\mathcal{B}(E)$, to emphasise that this space is being equipped with extra algebraic structure. (We use the notation $\mathcal{L}^n(E,F)$ for the space of bounded, n-linear maps in place of $\mathcal{B}^n(E,F)$ to avoid confusion later in the paper; $\mathcal{B}^n$ usually stands for the space of continuous n-coboundaries in the context of Hochschild cohomology.)

For Banach algebras A and B we write $\mathrm{Mult}(\mathsf{A},\mathsf{B})$ for the set of bounded algebra homomorphisms $\mathsf{A}\to\mathsf{B}$ (the zero map is allowed). Then, given $\psi\in\mathcal{L}(\mathsf{A},\mathsf{B})$, we have a "global" measure of how far ψ is from being a homomorphism; namely, we can consider the distance of ψ from the set $\mathcal{L}(\mathsf{A},\mathsf{B})$ with respect to the operator norm. Explicitly,

$$\mathrm{dist}(\psi) := \inf\{\|\psi-\phi\| \,:\, \phi\in\mathrm{Mult}(\mathsf{A},\mathsf{B})\}.$$

(Note that since $\mathrm{Mult}(\mathsf{A},\mathsf{B})$ is closed, $\mathrm{dist}(\psi)=0$ if and only if $\psi\in\mathrm{Mult}(\mathsf{A},\mathsf{B})$.) On the other hand, since a linear map $\psi:\mathsf{A}\to\mathsf{B}$ is a homomorphism if and only if it satisfies the identity $\psi(a_1a_2)=\psi(a_1)\psi(a_2)$ for each a_1 and a_2 in the closed unit ball of A, we may consider the following "local" measure of how far ψ is from being a homomorphism.

Definition 1.1. Given a linear map $\psi:\mathsf{A}\to\mathsf{B}$, the **multiplicative defect** of ψ is

$$\mathrm{def}(\psi) := \sup\{\|\psi(a_1a_2)-\psi(a_1)\psi(a_2)\| \colon a_1,a_2\in\mathrm{ball}_1(\mathsf{A})\}\in[0,\infty].$$

If $\psi\in\mathcal{L}(\mathsf{A},\mathsf{B})$ and we have some a priori upper bound on $\|\psi\|$ (say $\|\psi\|\le 1000$), it is easily checked that $\mathrm{dist}(\psi)$ being small implies $\mathrm{def}(\psi)$ is small. That is: starting with a multiplicative and bounded linear map, adding a linear perturbation with small norm yields a bounded linear map that has small multiplicative defect. Ulam stability is then the phenomenon that, under certain conditions on our algebras A and B, we can go the other way. The following definition is due to B. E. Johnson, see [Jo88, Definition 1.2].

Definition 1.2. (AMNM pair) Let A and B be Banach algebras. The pair (A,B) is said to have the **AMNM property**, or be an **AMNM pair**, if the following holds:

> For any $\varepsilon>0$ and $L>0$ there exists $\delta>0$ such that for all $\phi\in\mathrm{ball}_L\,\mathcal{L}(\mathsf{A},\mathsf{B})$ with $\mathrm{def}(\phi)<\delta$, we have $\mathrm{dist}(\phi)<\varepsilon$.

Johnson investigated a diverse range of AMNM pairs (A,B), in addition to providing some explicit examples of A and B which do **not** form an AMNM pair. However, when it came to Banach algebras of the form $\mathcal{B}(E)$, only one infinite-dimensional example was considered in [Jo88]. Namely, Johnson showed (see [Jo88, Proposition 6.3]) that the pair $(\mathcal{B}(\ell_2),\mathcal{B}(\ell_2))$ has the AMNM property, which is striking since one is not making any assumptions about w*-w*continuity.

Johnson's result was extended from ℓ_2 to ℓ_p, for $1<p<\infty$, in the PhD thesis of Howey [How00, Theorem 5.2.1]; his proof is essentially identical to Johnson's. In both cases, the argument has a somewhat "monolithic" feel, and freely uses special features of ℓ_p, so that it is not obvious how one might adapt the proof to more general Banach spaces.

Our main theorem extends the Johnson–Howey results to a much wider range of Banach spaces, including the classical spaces L_p for $p\in(1,\infty)$, but also many of their complemented subspaces such as $\ell_p(\ell_2)$ or Rosenthal's X_p-spaces, and also any reflexive space with a subsymmetric basis. At the same time, we obtain results for pairs $(\mathcal{B}(E),\mathcal{B}(X))$ where $E\not\cong X$ and E need not be reflexive. To state our theorem, it will be convenient to make the following definition.

Definition 1.3. Let E be a Banach space. A **clone system** for E is a bounded family $(P_i)_{i\in\mathbb{I}}$ of idempotents in $\mathcal{B}(E)$, such that the operator P_iP_j has finite rank for all $i\neq j$, and $\sup_{i\in\mathbb{I}} d(E,\mathrm{Ran}(P_i))<\infty$ where d denotes the Banach–Mazur distance.

THEOREM 1.4. *Let X be any separable, reflexive Banach space. Let E be a Banach space such that both of the following conditions hold:*

(i) *$\mathcal{K}(E)$, the algebra of compact operators on E, is amenable as a Banach algebra;*

(ii) *E has an uncountable clone system.*

Then the pair $(\mathcal{B}(E), \mathcal{B}(X))$ has the AMNM property.

Although the hypotheses of Theorem 1.4 are rather technical, we will show in the next section that they hold for several classical examples of interest.

1.2 Examples covered by our main theorem

COROLLARY 1.5. *Let E be a Banach space with a subsymmetric shrinking basis. Then $(\mathcal{B}(E), \mathcal{B}(X))$ is an AMNM pair for every reflexive and separable X.*

Note that in this corollary, the hypothesis on E is satisfied by ℓ_p for all $p \in (1, \infty)$ and c_0 (see [AK, Section 9.2]), and also for several natural families of Orlicz sequence spaces (see [LT, Propositions 4.a.4 and 3.a.3]) and for Lorentz sequence spaces (see [LT, Propositions 4.e.3 and 1.c.12]).

Proof of Corollary 1.5. By [GJW94, Theorem 4.2] and [GJW94, Theorem 4.5], $\mathcal{K}(E)$ is amenable. The construction of an uncountable clone system for E is a straightforward consequence of the definition of "subsymmetric" and the existence of uncountable almost disjoint families of subsets of $\mathbb{N}$; given such a family $\mathcal{D} \subset \mathcal{P}(\mathbb{N})$ and a subsymmetric basis $(u_n)_{n \geq 1}$ for E, for each $S \in \mathcal{D}$ define P_S to be the projection $\sum_{n \geq 1} \lambda_n u_n \mapsto \sum_{n \in S} \lambda_n u_n$. For details, see e.g. the proof of [HT21, Proposition 3.5(1)] (although this technique was already well known to specialists in Banach space theory). $\square$

The construction of an uncountable clone system in Corollary 1.5 only used the fact that E possessed a subsymmetric basis; the shrinking condition was needed to invoke results from [GJW94] on amenability of $\mathcal{K}(E)$. On the other hand, it is well known that $\mathcal{K}(\ell_1)$ is amenable: this is a special case of [GJW94, Theorem 4.7]. We may therefore run the same argument as before to obtain an extra example.

COROLLARY 1.6. *$(\mathcal{B}(\ell_1), \mathcal{B}(X))$ is an AMNM pair for every reflexive and separable X.*

The spaces $L_p \equiv L_p[0, 1]$ do not have a subsymmetric basis unless $p = 2$; see *e.g.* [Si, Theorem 21.2, Chapter II, p. 568]. Thus, the next corollary shows that Corollary 1.5 is far from describing the full extent of the spaces covered by Theorem 1.4.

COROLLARY 1.7. *Let $p \in [1, \infty]$. Then $(\mathcal{B}(L_p), \mathcal{B}(X))$ is an AMNM pair for every reflexive and separable X.*

Proof. By [GJW94, Theorem 4.7] $\mathcal{K}(L_p)$ is amenable. For $1 \leq p < \infty$, an uncountable clone system for L_p is given by the construction in [HT21, Proposition 3.5]. While that construction does not work for $p = \infty$, we recall that by a celebrated application of Pełczyński's decomposition method $L_\infty \cong \ell_\infty$ as Banach spaces. Then it is simple to construct an uncountable clone system for ℓ_∞ using an uncountable family of almost disjoint subsets of $\mathbb{N}$, as in previous proofs. $\square$

For our final corollary, we rely on recent work of Johnson–Phillips–Schechtman [JPS21+], which we learned of after the initial work was done on this paper. For details we refer to [Ro70] and [JPS21+].

Corollary 1.8. *Let $p \in (1,2) \cup (2,\infty)$. Then $(\mathcal{B}(E), \mathcal{B}(X))$ is an AMNM pair for every reflexive and separable X, whenever E is any of the following Banach spaces:*

(i) $\ell_p \oplus \ell_2$;

(ii) $\ell_p(\ell_2) \equiv \ell_p(\mathbb{N}; \ell_2)$;

(iii) $\overbrace{X_p \otimes_p \cdots \otimes_p X_p}^{n}$ *for some $n \in \mathbb{N}$, where X_p denotes Rosenthal's X_p-space and $\otimes_p$ denotes the tensor product for closed subspaces of L_p.*

Proof. All of the listed choices for E are complemented subspaces of L_p, and hence are $\mathscr{L}_p$-spaces in the sense of Lindenstrauss–Pełczyński by [LP69, Theorem III]. Thus $\mathcal{K}(E)$ is amenable by [GJW94, Theorem 6.4], so it only remains to show that E has an uncountable clone system.

In [JPS21+, Definitions 1.2 and 2.1] the notion of an unconditional finite dimensional Schauder decomposition (UFDD) with a so-called *property* ($\sharp$) is introduced. We do not give the precise definition here, but it should be clear from the arguments below. It follows from Propositions 2.4 and 2.5 and the the paragraph after Definition 2.1 in [JPS21+] that all of the listed choices for E have a UFDD with ($\sharp$) with some constant $K > 0$, in the sense of [JPS21+, Definition 2.1].

We now show that whenever E is a Banach space with a UFDD that has property ($\sharp$) with some constant $K > 0$, then E has an uncountable clone system. Take a UFDD (E_n) with property ($\sharp$) with some constant $K > 0$. By taking an uncountable almost disjoint family $\mathcal{D}$ on $\mathbb{N}$, we obtain that $E_S := \overline{\operatorname{span}}(E_n \colon n \in S)$ is K-isomorphic to E for each $S \in \mathcal{D}$. Hence $\sup_{S \in \mathcal{D}} d(E, E_S) \leq K$. As outlined on page 2 in [JPS21+], for every $B \subseteq \mathbb{N}$ there is an idempotent $P_B \in \mathcal{B}(E)$ such that $\operatorname{Ran}(P_B) = \overline{\operatorname{span}}(E_n \colon n \in B)$. Moreover, there is a $C > 0$ (called the *suppression constant* in [JPS21+]) such that $\sup_{B \subseteq \mathbb{N}} \|P_B\| \leq C$. So $P_S \in \mathcal{B}(E)$ is an idempotent with $\operatorname{Ran}(P_S) = E_S$ and $\|P_S\| \leq C$ for each $S \in \mathcal{D}$. Also, $\operatorname{Ran}(P_S P_{S'}) = \overline{\operatorname{span}}(E_n \colon n \in S \cap S')$ is finite-dimensional, whenever $S, S' \in \mathcal{D}$ are distinct. Thus E has an uncountable clone system, as required. ☐

We hope that this selection of examples, while not exhaustive, shows that one can go far beyond the cases $E = X = \ell_p$ $(1 < p < \infty)$ studied by Johnson and Howey. Even for those special cases, our proof of Theorem 1.4 makes several technical improvements over their approach: we provide an argument with clearer structure, and we obtain better constants, which in principle could be made explicit.

Remark 1.9. One can show that the Tsirelson space T (as constructed by Figiel and Johnson [FJ74]) has an uncountable clone system. This may be folklore, but we include a proof in an appendix for sake of completeness (see Proposition A.1). On the other hand, Blanco and Grønbæk proved that $\mathcal{K}(T)$ is **not** amenable, see [BG09, Corollary 5.8], and so Theorem 1.4 cannot be applied to $\mathcal{B}(T)$. It is an open problem whether the pair $(\mathcal{B}(T), \mathcal{B}(T))$ has the AMNM property, and we believe this would be an interesting case to study further.

1.3 Comments on the proof of our main theorem, and other results of interest

Theorem 1.4 will follow by combining several other technical results. In this section we wish to highlight two of them, which correspond to the two conditions in the theorem. Proofs will be given in later sections.

The following definition will be used repeatedly throughout our arguments.

Definition 1.10 (Self-modular maps with respect to a subalgebra). Let A and B be Banach algebras and let D be a closed subalgebra of A. We denote by $\mathrm{SHom}_{\mathsf{D}}(\mathsf{A},\mathsf{B})$ the set of all bounded linear maps $\theta : \mathsf{A} \to \mathsf{B}$ which satisfy

$$\theta(ar) = \theta(a)\theta(r) \text{ and } \theta(ra) = \theta(r)\theta(a) \quad \text{for all } a \in \mathsf{A} \text{ and all } r \in \mathsf{D}.$$

Our main technical innovation is the following theorem, which provides a significant generalization of the main result in [Jo88].

Theorem 1.11 (ANMM with respect to an amenable subalgebra). *Let A be a Banach algebra with a closed amenable subalgebra D_0, and let B be a unital dual Banach algebra with an isometric predual. Fix some $L \geq 1$. Then there exists a constant $C' \geq 1$ (possibly depending on L and D_0) such that the following holds: whenever $\psi \in \mathcal{L}(\mathsf{A},\mathsf{B})$ satisfies $\|\psi\| \leq L$ and $C' \operatorname{def}(\psi) \leq 1$, there exists $\theta \in \mathrm{SHom}_{\mathsf{D}_0}(\mathsf{A},\mathsf{B})$ with $\|\theta - \psi\| \leq C' \operatorname{def}(\psi)$.*

The case where A itself is amenable is [Jo88, Theorem 3.1], but in order to obtain our generalization, it does not suffice to bootstrap from the earlier result. Instead we rework the arguments in Johnson's proof, introducing a version of the multiplicative defect relative to a closed subalgebra, and putting certain calculations from that proof in the framework of "approximate cobounding" for a modified version of the Hochschild cochain complex. This will be treated in Sections 4 and 5.

We note that in the setting of Ulam stability for bounded representations of discrete groups on Hilbert space, a result analogous to Theorem 1.11 was given in [BOT13, Theorem 3.2]; the proof makes use of features particular to groups and to operators on Hilbert space.

Our other main ingredient in the proof of Theorem 1.4 is the following proposition, whose proof will be given in Section 3.2. It can be viewed as a "perturbed" version of [HT21, Proposition 3.8] (see also [BP69, Corollary 6.16]), and it generalizes an argument of Johnson (from the proof of [Jo88, Proposition 6.3]) in the case $X = E = \ell_2$. Moreover, we obtain better constants than those obtained by just repeating the steps in [Jo88]; see Remark 3.5 for further details.

Proposition 1.12. *Let E be a Banach space with an uncountable clone system. There exists a constant $c_E \in (0,1]$ such that the following holds: whenever X is a separable Banach space, and $\psi : \mathcal{B}(E)/\mathcal{K}(E) \to \mathcal{B}(X)$ is bounded linear with $\operatorname{def}(\psi) \leq c_E$, we have $\|\psi\| \leq \frac{3}{2} \operatorname{def}(\psi)$.*

The key point here is that the constant c_E does not depend on the chosen ψ, and so $\operatorname{def}(\psi)$ could be much smaller than c_E.

Note that in the conclusion of Proposition 1.12, we obtain the constant $3/2$ rather than some constant depending on the Banach algebras $\mathcal{B}(E)$ and $\mathcal{B}(X)$. Obtaining a universal constant (such as $3/2$) is not essential to the proof of Theorem 1.4 but it makes some of the epsilon-delta chasing significantly simpler.

2 Definitions and preliminary results

2.1 Basic properties of the multiplicative defect

First we have a general lemma. (A similar estimate is given without proof in [Jo88, Proposition 1.1].)

LEMMA 2.1. *Let* A *and* B *be Banach algebras and let* $\psi \in \mathcal{L}(\mathsf{A}, \mathsf{B})$. *Suppose that* $\theta \in \mathcal{L}(\mathsf{A}, \mathsf{B})$ *satisfies* $\|\theta - \psi\| \le 1$. *Then*

$$\operatorname{def}(\theta) \le \operatorname{def}(\psi) + 2\|\theta - \psi\|(1 + \|\psi\|).$$

Proof. Writing $\theta = \psi + \gamma$, for each a and b in A we have

$$\theta(ab) - \theta(a)\theta(b) = \psi(ab) + \gamma(ab) - \psi(a)\psi(b) - \psi(a)\gamma(b) - \gamma(a)\psi(b) - \gamma(a)\gamma(b).$$

Hence $\operatorname{def}(\theta) \le \operatorname{def}(\psi) + \|\gamma\| + 2\|\gamma\|\|\psi\| + \|\gamma\|^2$. Since we are assuming $\|\gamma\| \le 1$, the desired inequality follows. □

In the rest of this section we collect some general results concerning approximately multiplicative maps between Banach algebras, which do not seem to be spelled out in [Jo88]. These may be useful for future work on the AMNM property for other kinds of Banach algebras. It will be convenient to use the following terminology: given $\eta \in [0, \infty)$, we say that a linear map $\psi : \mathsf{A} \to \mathsf{B}$ is **η-multiplicative** if $\operatorname{def}(\psi) \le \eta$; equivalently, if

$$\|\psi(ab) - \psi(a)\psi(b)\| \le \eta\|a\|\|b\| \qquad \text{for all } a, b \in A.$$

The point is that often we are not concerned with the precise value of the multiplicative defect, but merely with whether it is controlled by some (small) constant or parameter.

LEMMA 2.2. *Let* A *and* B *be Banach algebras and let* $\eta \ge 0$. *Let* $\psi : \mathsf{A} \to \mathsf{B}$ *be linear and* η*-multiplicative.*

(i) *Suppose* $ab = b$ *with* $\|\psi(a)\| \le 1/3$. *Then* $\|\psi(b)\| \le \frac{3}{2}\eta\|a\|\|b\|$.

(ii) *Suppose* $bc = b$ *with* $\|\psi(c)\| \le 1/3$. *Then* $\|\psi(b)\| \le \frac{3}{2}\eta\|b\|\|c\|$.

Proof. We prove (i); the proof for (ii) is identical with left and right swapped.

Since $ab = b$, $\|\psi(b) - \psi(a)\psi(b)\| \le \eta\|a\|\|b\|$. Hence

$$\|\psi(b)\| \le \eta\|a\|\|b\| + \|\psi(a)\psi(b)\| \le \eta\|a\|\|b\| + \frac{1}{3}\|\psi(b)\|.$$

Rearranging we obtain the desired upper bound on $\|\psi(b)\|$. □

The following corollary is immediate.

COROLLARY 2.3. *Let* A *and* B *be Banach algebras with* A *unital. Let* $\psi : \mathsf{A} \to \mathsf{B}$ *be linear and* η*-multiplicative. If* $\|\psi(1_{\mathsf{A}})\| \le 1/3$ *then* ψ *is bounded with* $\|\psi\| \le 3\eta/2$.

REMARK 2.4. As observed in Section 1 of [Jo88], for a general linear $T : \mathsf{A} \to \mathsf{B}$ one can have $\operatorname{def}(T)$ small while T has large norm, even when $\mathsf{A} = \mathbb{C}$. But examination of Example 1.5 in that paper shows that $T(1_{\mathsf{A}})$ is large in that example. Corollary 2.3 shows that this is the only obstruction.

The next result will be applied to show that if p is an idempotent in a unital Banach algebra A and p is Murray–von Neumann equivalent to 1_{A}, then $\psi(p)$ being small implies $\psi(1_{\mathsf{A}})$ is small, provided that $\operatorname{def}(\psi)$ is small. Normally, in perturbing exact algebraic arguments, one has to impose an *a priori* upper bound on norms: informally, large times zero equals zero, but large times small might not be small. It is therefore somewhat surprising that in our result, we do not need to impose such a bound on $\|\psi\|$.

PROPOSITION 2.5. *Let* A *and* B *be Banach algebras. Let* $u, v \in \mathsf{A}$ *be such that* uv *and* vu *are idempotents. Let* $\psi : \mathsf{A} \to \mathsf{B}$ *be linear and* η*-multiplicative, for some* η *satisfying* $0 \leq \eta\|u\|^3\|v\|^3 \leq 2/9$*. If* $\|\psi(uv)\| \leq 1/3$ *then* $\|\psi(vu)\| \leq 1/3$*.*

Proof. If $vu = 0$ then $\psi(vu) = 0$ so there is nothing to prove. Hence we assume $vu \neq 0$; since vu is an idempotent $1 \leq \|vu\| \leq \|v\|\|u\|$.

Since uv is an idempotent, $uvu = uv \cdot uvu$ and $vuv = vuv \cdot uv$. Applying Lemma 2.2 gives

$$\|\psi(uvu)\| \leq \frac{3}{2}\eta\|uv\|\|uvu\| \quad \text{and} \quad \|\psi(vuv)\| \leq \frac{3}{2}\eta\|vuv\|\|uv\|$$

and so

$$\|\psi(uvu)\psi(vuv)\| \leq \left(\frac{3}{2}\eta\right)^2 \|u\|^5\|v\|^5 \leq \left(\frac{3}{2}\eta\right)^2 \|u\|^6\|v\|^6 \leq \left(\frac{3}{2}\right)^2\left(\frac{2}{9}\right)^2 = \frac{1}{9}.$$

But since vu is an idempotent, $vuv \cdot uvu = vu$. Hence

$$\|\psi(vu) - \psi(vuv)\psi(uvu)\| \leq \eta\|vuv\|\|uvu\| \leq \eta\|u\|^3\|v\|^3 \leq \frac{2}{9}$$

and so $\|\psi(vu)\| \leq \frac{2}{9} + \|\psi(vuv)\psi(uvu)\| \leq \frac{1}{3}$. ☐

REMARK 2.6. The choice of $\frac{1}{3}$ is somewhat arbitrary, and the reader may wonder why we did not attempt to prove sharper inequalities. In fact, it follows automatically from Corollary 2.11 below that if $\psi(uv)$ is "moderately small" then $\psi(vu)$ will be "very small". However, this refinement is not needed for the proofs of our main results.

2.2 Dual Banach algebras

There are various equivalent formulations in the literature of the notion of a dual Banach algebra. We follow the definition in [Da07, Section 1], although our terminology is slightly different and is influenced by [DPW09, Section 2].

DEFINITION 2.7. Let B be a Banach algebra and let V be a Banach space. We say that B is a **dual Banach algebra** with **isometric predual** V, if there is an isometric isomorphism of Banach spaces $j : \mathsf{B} \to \mathsf{V}^*$ such that multiplication $\mathsf{B} \times \mathsf{B} \to \mathsf{B}$ is separately $\sigma(\mathsf{B}, \mathsf{V})$-continuous.

Strictly speaking, in this definition, the choice of isometric isomorphism $j : \mathsf{B} \to \mathsf{V}^*$ should be part of the data. However, in most examples that occur in practice, it is clear from context which map j is being used. Moreover, as discussed in [DPW09, Section 2]:

- the "dual Banach algebra structure" induced on B only depends on the image of the isometry $j^*\kappa : \mathsf{V} \to \mathsf{B}^*$, where κ is the canonical embedding of V in its bidual;
- the condition that multiplication in B be separately $\sigma(\mathsf{B}, \mathsf{V})$-continuous is equivalent to requiring $j^*\kappa(\mathsf{V})$ to be a sub-B-bimodule of B^*.

This latter condition is often easier to check in practice.

If the choice of isometric predual for B is not important, or is clear from context, then we will usually just refer to the w*-topology on B without mentioning the particular predual.

EXAMPLE 2.8. The following Banach algebras are dual Banach algebras with an isometric predual.

– $M(G)$ where G is a locally compact group, with the isometric predual being $C_0(G)$;

– any von Neumann algebra N, with the isometric predual being the space of normal linear functionals on N;

– $\mathcal{B}(X)$ for any reflexive Banach space X, with the isometric predual being the projective tensor product $X^* \widehat{\otimes} X$.

REMARK 2.9. It was shown by Daws [Da07, Theorem 3.5 and Corollary 3.8] that the last of these examples is in some sense a universal one: given any dual Banach algebra B with an isometric predual, there exists a reflexive Banach space X and an isometric, w*-w*-continuous algebra homomorphism $\mathsf{B} \to \mathcal{B}(X)$.

2.3 A sharper dichotomy result

This section is not required for the proof of our main result, but it is included since the proofs are elementary and since it may be useful in future work. The following lemma is inspired by similar observations/calculations in [Cho13, Section 3.1], but we are able to give a simpler proof.

LEMMA 2.10. *Let $x \in [0,\infty)$ and suppose that $x \leq x^2 + c$ for some $c \in [0, 2/9]$. Then*

$$\min(x, 1-x) \leq \frac{3c}{2} \leq \frac{1}{3} \,.$$

Proof. By comparing the graphs of the functions $f(u) = u$ and $g(u) = u^2 + c$ for $u \geq 0$, which cross in exactly two points, we see that $x \in [0, u_1] \cup [u_2, \infty)$, where $0 \leq u_1 < u_2 \leq 1$ are the solutions of $u = u^2 + c$. Explicitly

$$u_1 = \frac{1}{2}(1 - \sqrt{1-4c}) \quad , \quad u_2 = \frac{1}{2}(1 + \sqrt{1-4c}) = 1 - u_1 \,.$$

It therefore suffices to prove that $u_1 \leq 3c/2$. This is equivalent to proving that $1 - 3c \leq \sqrt{1-4c}$, which (since both sides are non-negative) is equivalent to proving that $(1-3c)^2 \leq 1 - 4c$. Since $0 \leq c \leq 2/9$, we have $9c^2 \leq 2c$, and therefore $1 - 6c + 9c^2 \leq 1 - 4c$ as required. $\square$

COROLLARY 2.11 (A norm dichotomy). *Let A, B be Banach algebras and let p be an idempotent in A. Let δ satisfy $0 \leq \delta \|p\|^2 \leq \frac{2}{9}$, and suppose $\psi \in \mathcal{L}(\mathsf{A}, \mathsf{B})$ is δ-multiplicative. Then either $\|\psi(p)\| \leq \frac{3}{2}\|p\|^2\delta \leq \frac{1}{3}$, or $\|\psi(p)\| \geq 1 - \frac{3}{2}\|p\|^2\delta \geq \frac{2}{3}$.*

The point of this result is that we do not need *a priori* control on $\|\psi\|$ to choose how small δ must be; nor do we need any holomorphic functional calculus for the codomain B.

Proof. Since $p^2 = p$, we have $\|\psi(p)\| \leq \|\psi(p) - \psi(p)^2\| + \|\psi(p)\|^2 \leq \delta\|p\|^2 + \|\psi(p)\|^2$. Now applying Lemma 2.10 completes the proof. $\square$

3 Towards a proof of the main theorem

3.1 Self-modular maps relative to an ideal

Throughout this section, B is a dual Banach algebra with an isometric predual (Definition 2.7). We denote w*-limits in B by $\lim^{\sigma}$.

Proposition 3.1 (Decomposition relative to an ideal). *Let B be a dual Banach algebra with an isometric predual. Let A be a Banach algebra and J be a closed ideal in A with a b.a.i. Then each $\theta \in \mathrm{SHom}_{\mathsf{J}}(\mathsf{A},\mathsf{B})$ can be written as $\theta = \phi + \theta_s$, where $\phi : \mathsf{A} \to \mathsf{B}$ is a bounded homomorphism, $\theta_s|_{\mathsf{J}} = 0$, and $\mathrm{def}(\theta_s) = \mathrm{def}(\theta)$.*

Proof. Let B_0 denote the w*-closure of $\theta(\mathsf{J})$ inside B. Since J is an ideal and multiplication in B is separately w*-w*-continuous, the self-modular property of θ implies that

$$\theta(a)\mathsf{B}_0 \subseteq \mathsf{B}_0 \quad \text{and} \quad \mathsf{B}_0\theta(a) \subseteq \mathsf{B}_0 \quad \text{for all } a \in \mathsf{A}. \tag{3.1}$$

If $a_1, a_2 \in \mathsf{A}$ and $x \in \mathsf{J}$, then repeated use of the self-modularity property yields

$$\theta(x)\theta(a_1a_2) = \theta(xa_1a_2) = \theta(xa_1)\theta(a_2) = \theta(x)\theta(a_1)\theta(a_2); \tag{3.2}$$

hence, by taking w*-limits in (3.2), we have

$$b\theta(a_1a_2) = b\theta(a_1)\theta(a_2) \qquad \text{for all } a_1, a_2 \in \mathsf{A} \text{ and all } b \in \mathsf{B}_0. \tag{3.3}$$

Now let (e_i) be a b.a.i. in J. Passing to a subnet, we may assume that $\theta(e_i)$ w*-converges in B to some $p \in \mathsf{B}_0$. Then for any $x \in \mathsf{J}$,

$$\begin{aligned} \theta(x) = \lim_i \theta(e_i x) &= \lim_i \theta(e_i)\theta(x) \\ &= {\lim}^{\sigma}_i\, \theta(e_i)\theta(x) = \left({\lim}^{\sigma}_i\, \theta(e_i)\right)\theta(x) = p\theta(x), \end{aligned} \tag{3.4}$$

and similarly $\theta(x) = \theta(x)p$. Hence, by another application of w*-w*-continuity,

$$pb = b = bp \qquad \text{for all } b \in \mathsf{B}_0. \tag{3.5}$$

(In particular, p is idempotent, although we do not use this explicitly in what follows.)

For each $a \in \mathsf{A}$, (3.1) implies that $\theta(a)p \in \mathsf{B}_0$ and $p\theta(a) \in \mathsf{B}_0$. Hence by (3.5)

$$p\theta(a)p = \theta(a)p \quad \text{and} \quad p\theta(a) = p\theta(a)p \quad \text{for all } a \in \mathsf{A}. \tag{3.6}$$

Now define ϕ by putting $\phi(a) := p\theta(a)$. Combining (3.3) and (3.6), for all $a_1, a_2 \in \mathsf{A}$ we have

$$\phi(a_1a_2) = p\theta(a_1)\theta(a_2) = p\theta(a_1)p\theta(a_2) = \phi(a_1)\phi(a_2), \tag{3.7}$$

and thus ϕ is multiplicative.

Put $\theta_s(a) := \theta(a) - p\theta(a)$. Clearly $\phi + \theta_s = \theta$, and (3.4) implies that $\theta_s(x) = 0$ for all $x \in J$.

Finally: note that by (3.6), $\theta_s(a_1)p = 0$. Hence, for all $a_1, a_2 \in \mathsf{A}$,

$$\begin{aligned} \theta_s(a_1)\theta_s(a_2) = \theta_s(a_1)\theta(a_2) &= \theta(a_1)\theta(a_2) - p\theta(a_1)\theta(a_2) \\ &= \theta(a_1)\theta(a_2) - p\theta(a_1a_2), \end{aligned} \tag{3.8}$$

where the last equality follows from (3.3). Therefore

$$\theta_s(a_1a_2) - \theta_s(a_1)\theta_s(a_2) = \theta(a_1a_2) - \theta(a_1)\theta(a_2), \tag{3.9}$$

and we conclude that $\mathrm{def}(\theta_s) = \mathrm{def}(\theta)$. ☐

Remark 3.2. If the b.a.i. in J has norm $\le M$, then the functions ϕ and θ_s in this result can be taken to satisfy $\|\phi\| \le M\|\theta\|$ and $\|\theta_s\| \le (1+M)\|\theta\|$. However, we will not need these bounds in the applications of Proposition 3.1.

3.2 The proof of Proposition 1.12

In this section we prove Proposition 1.12. For convenience, we repeat the statement:

> *Let E be a Banach space with an uncountable clone system. There exists a constant $c_E \in (0,1]$ such that the following holds: whenever X is a separable Banach space, and $\psi : \mathcal{B}(E)/\mathcal{K}(E) \to \mathcal{B}(X)$ is bounded linear with $\operatorname{def}(\psi) \le c_E$, we have $\|\psi\| \le \frac{3}{2}\operatorname{def}(\psi)$.*

We start by shifting perspective slightly in the definition of a clone system. It is well known (see *e.g.* [La03, Lemma 1.4] for a proof) that an idempotent $P \in \mathcal{B}(E)$ satisfies $\operatorname{Ran}(P) \cong E$ if and only if P is Murray–von Neumann equivalent to I_E. We state a quantitative version in the following lemma, whose proof is left to the reader.

Lemma 3.3. *Let E be a Banach space and let $P \in \mathcal{B}(E)$ be an idempotent.*

(i) *If $\operatorname{Ran}(P) \cong E$, then for every $\varepsilon > 0$ there exist $U, V \in \mathcal{B}(E)$ such that $P = UV$, $I_E = VU$ and $\|U\|\|V\| \le (d(E, \operatorname{Ran}(P)) + \varepsilon)\|P\|$.*

(ii) *If $U, V \in \mathcal{B}(E)$ are such that $I_E = VU$ and $UV = P$, then $\operatorname{Ran}(U) = \operatorname{Ran}(P)$ and $V|_{\operatorname{Ran}(P)}$ is an isomorphism from $\operatorname{Ran}(P)$ onto E. Hence, $d(E, \operatorname{Ran}(P)) \le \|U\|\|V\|$ (and clearly $\|P\| \le \|U\|\|V\|$).*

We recall that idempotents p, q in a ring are said to be *orthogonal* if $pq = 0 = qp$.

Lemma 3.4. *Let Q be a Banach algebra containing an uncountable family Ω of pairwise orthogonal idempotents, and suppose $\sup_{p\in\Omega}\|p\| \le L$ for some $L \ge 1$. Let X be a separable Banach space, and suppose $\psi \in \mathcal{L}(\mathsf{Q}, \mathcal{B}(X))$ is η-multiplicative for some $\eta > 0$. Then $\|\psi(p)\| \le 2\eta L^2$ for uncountably many $p \in \Omega$.*

Proof. For $\varepsilon > 0$ let $\Omega_\varepsilon = \{p \in \Omega \colon \|\psi(p)\| > \varepsilon\}$. It suffices to show that $\Omega_{2\eta L^2}$ is countable; therefore, since $\Omega_{2\eta L^2} = \bigcup_{n=1}^\infty \Omega_{2\eta L^2 + 1/n}$, it suffices to show that Ω_c is countable for every $c > 2\eta L^2$.

Fix $c > 2\eta L^2$. We may assume that Ω_c is infinite (otherwise there is nothing to prove); in particular, this implies $\|\psi\| > 0$. For each $p \in \Omega_c$ pick a unit vector $x_p \in X$ such that $\|\psi(p)x_p\| \ge c$, and let $y_p = \psi(p)x_p$.

If $r \in \Omega_c$ and $r \ne p$, then

$$\|\psi(p)y_r\| = \|\psi(p)\psi(r)x_r\| \le \|\psi(p)\psi(r)\| = \|\psi(p)\psi(r) - \psi(pr)\| \le \eta L^2 \;;$$

on the other hand, since $\|\psi(p) - \psi(p)\psi(p)\| \le \eta L^2$,

$$\|\psi(p)y_p\| = \|\psi(p)\psi(p)x_p\| \ge \|\psi(p)x_p\| - \eta L^2 \ge c - \eta L^2 \;.$$

Combining these inequalities yields $\|\psi(p)y_p - \psi(p)y_r\| \ge c - 2\eta L^2$. Hence

$$\|y_p - y_r\| \ge \frac{c - 2\eta L^2}{\|\psi(p)\|} \ge \frac{c - 2\eta L^2}{\|\psi\| L} > 0 \quad \text{for all } p, r \in \Omega_c \text{ with } p \ne r.$$

Since X is separable this is only possible if Ω_c is countable. ∎

Proof of Proposition 1.12. Let Ω be an uncountable clone system for E. By Lemma 3.3(i), there is a constant $C \geq 1$ such that each $P \in \Omega$ can be factorized as $P = UV$, for some V and U in $\mathcal{B}(E)$ satisfying $\|U\|\|V\| \leq C$ and $VU = I_E$. We will show that the conclusion of Proposition 1.12 holds with $c_E := 6^{-1}C^{-3}$.

Let $\psi : \mathcal{B}(E)/\mathcal{K}(E) \to \mathcal{B}(X)$ be bounded linear. For convenience, let $\eta := \operatorname{def}(\psi)$, and suppose that $\eta \leq 6^{-1}C^{-3}$. Writing q for the quotient homomorphism $\mathcal{B}(E) \to \mathcal{B}(E)/\mathcal{K}(E)$, note that $q(\Omega)$ is an uncountable family of orthogonal idempotents in $\mathcal{B}(E)/\mathcal{K}(E)$ with $\|q(P)\| \leq \|P\| \leq C$ for every $P \in \Omega$. By Lemma 3.4 with $\mathsf{Q} = \mathcal{B}(E)/\mathcal{K}(E)$, there exists some $P \in \Omega$ such that

$$\|\psi q(P)\| \leq 2\eta C^2 \leq 2\eta C^3 \leq \frac{1}{3}\,.$$

(In fact there exist uncountably many, but we only need one!) Consider $\psi q : \mathcal{B}(E) \to \mathcal{B}(X)$, which satisfies $\operatorname{def}(\psi q) = \operatorname{def}(\psi) = \eta$. We have

$$\eta\|U\|^3\|V\|^3 \leq \eta C^3 \leq \frac{1}{6} < \frac{2}{9}\,.$$

Hence, applying Proposition 2.5 to the map $\psi q : \mathcal{B}(E) \to \mathcal{B}(X)$, we deduce that $\|\psi q(I_E)\| \leq 1/3$. Since $q(I_E)$ is the identity element of $\mathcal{B}(E)/\mathcal{K}(E)$, it follows from Corollary 2.3 that $\|\psi\| \leq 3\eta/2$ as required. ◻

Remark 3.5. Comparing our proof of Proposition 1.12 with Johnson's arguments in [Jo88]: he uses the fact that in any Banach algebra an element x for which $\|x^2 - x\|$ is "small" is "close in norm" to a genuine idempotent. The proof of this result relies on holomorphic functional calculus, and hence has implicit constants depending on the given algebra. Our approach bypasses this issue.

The proof works just as well if $\mathcal{B}(E)$ is replaced by an arbitrary unital Banach algebra A and $\mathcal{K}(E)$ by an arbitrary closed ideal $\mathsf{J} \trianglelefteq \mathsf{A}$. However, we do not know of natural examples that satisfy the **hypotheses** of Proposition 1.12 which are not of the form $\mathsf{A} = \mathcal{B}(E)$ and J being some closed operator ideal, so it seems more appropriate to restrict ourselves to this setting.

3.3 Deducing the main theorem from other results

We now show how Theorem 1.4 will follow from combining Theorem 1.11, Proposition 3.1 and Proposition 1.12. For convenience let us restate the theorem:

> *Let X be any separable, reflexive Banach space. Let E be a Banach space such that both of the following conditions hold:*
>
> (i) *$\mathcal{K}(E)$, the algebra of compact operators on E, is amenable as a Banach algebra;*
>
> (ii) *E has an uncountable clone system.*
>
> *Then the pair $(\mathcal{B}(E), \mathcal{B}(X))$ has the AMNM property.*

Proof of Theorem 1.4, assuming Theorem 1.11. $\mathcal{B}(X)$ is a dual Banach algebra with an isometric predual, since X is reflexive. Hence we may apply Theorem 1.11 with $\mathsf{A} = \mathcal{B}(E)$, $\mathsf{D}_0 = \mathcal{K}(E)$ and $\mathsf{B} = \mathcal{B}(X)$. Fix some $L \geq 1$, and let $C' \geq 1$ satisfy the conclusion of Theorem 1.11 (recall that C' may depend on the constant L and also on the Banach space E).

Given $\varepsilon > 0$, we fix some $\delta > 0$ to be determined later. Let $\psi : \mathcal{B}(E) \to \mathcal{B}(X)$ satisfy $\|\psi\| \le L$ and $\operatorname{def}(\psi) \le \delta$. It suffices to prove that there exists some bounded homomorphism $\phi : \mathcal{B}(E) \to \mathcal{B}(X)$ with $\|\phi - \psi\| \le \varepsilon$.

By Theorem 1.11, **provided that** $C'\delta \le 1$, there exists $\theta \in \mathrm{SHom}_{\mathcal{K}(E)}(\mathcal{B}(E), \mathcal{B}(X))$ such that $\|\theta - \psi\| \le C'\delta$. Note that by Lemma 2.1,

$$\operatorname{def}(\theta) \le \operatorname{def}(\psi) + 2(1 + \|\psi\|)\|\theta - \psi\| \le \delta + 2(1 + L)C'\delta \le 5LC'\delta.$$

By Proposition 3.1, applied with $\mathsf{A} = \mathcal{B}(E)$, $\mathsf{J} = \mathcal{K}(E)$ and $\mathsf{B} = \mathcal{B}(X)$, there exist

- a bounded homomorphism $\phi : \mathcal{B}(E) \to \mathcal{B}(X)$,
- a bounded linear map $\theta_s : \mathcal{B}(E) \to \mathcal{B}(X)$ which vanishes on $\mathcal{K}(E)$ and satisfies

$$\operatorname{def}(\theta_s) = \operatorname{def}(\theta) \le 5LC'\delta$$

such that $\theta = \phi + \theta_s$. Writing q for the quotient homomorphism $\mathcal{B}(E) \to \mathcal{B}(E)/\mathcal{K}(E)$, we may factorize θ_s as $\widetilde{\theta_s} q$ where $\|\widetilde{\theta_s}\| = \|\theta_s\|$.

Let c_E be the constant provided by Proposition 1.12 (recall that this depends only on the chosen clone system for E). By applying that proposition to $\widetilde{\theta_s}$: **provided that** $5LC'\delta \le c_E$, we have $\|\widetilde{\theta_s}\| \le 15LC'\delta/2$. Hence

$$\|\phi - \psi\| \le \|\theta - \psi\| + \|\theta_s\| \le C'\delta + \frac{15}{2}LC'\delta < 9LC'\delta.$$

Therefore, if we originally chose our δ to satisfy $0 < 5LC'\delta \le c_E$ and $9LC'\delta \le \varepsilon$, we have $\|\phi - \psi\| \le \varepsilon$ as required. ☐

At this point, the only piece missing from our proof of Theorem 1.4 is the proof of our main technical novelty, Theorem 1.11. This will take up the rest of the paper.

4 Towards a proof of Theorem 1.11

The process of proving Theorem 1.11 is quite long, and it may be helpful for the reader to know that the key implications are given by the following chain:

Theorem 1.11 $\Longleftarrow$ Theorem 4.2 $\Longleftarrow$ Proposition 4.3 $\Longleftarrow$ Section 5.3.

4.1 The projective tensor product and approximate diagonals

It turns out that we need to make **quantitative** (rather than merely **qualitative**) use of amenability. Thus, we shall briefly review the basic properties of the projective tensor norm for Banach spaces and the associated completed tensor product; a good source for background material is the monograph [Ry]. In what follows $\mathrm{Bil}(E, F; G)$ denotes the space of bounded, bilinear maps $E \times F \to G$ for Banach spaces E, F and G.

Rather than defining the projective tensor norm directly, we use the following characterization in terms of a universal property (see also [Ry, Theorem 2.9]).

> Given Banach spaces E and F, there exists a Banach space $E \widehat{\otimes} F$ and a norm 1 map $\iota_{E,F} \in \mathrm{Bil}(E, F; E \widehat{\otimes} F)$, such that for each Banach space X and each $\beta \in \mathrm{Bil}(E, F; X)$ there is a unique $T_\beta \in \mathcal{L}(E \widehat{\otimes} F, X)$ such that $\beta = T_\beta \circ \iota_{E,F}$. Moreover, $\|T_\beta\|_{\mathcal{L}(E \widehat{\otimes} F, X)} = \|\beta\|_{\mathrm{Bil}(E,F;X)}$.

As is standard, for $x \in E$ and $y \in F$ we write $x \otimes y$ for $\iota_{E,F}(x,y)$. It follows from the previous remarks that for each $T \in \mathcal{L}(E \widehat{\otimes} F, X)$,

$$\|T\|_{\mathcal{L}(E\widehat{\otimes}F,X)} = \|T \circ \iota_{E,F}\|_{\mathrm{Bil}(E,F;X)} = \sup\{\|T(x \otimes y)\| \colon x \in \mathrm{ball}_1(E), y \in \mathrm{ball}_1(F)\} \ . \tag{4.1}$$

That is: to determine the norm of $T \in \mathcal{L}(E \widehat{\otimes} F, X)$, it suffices to check how T acts on elementary tensors arising from the unit balls of E and F.

The theory of amenability for Banach algebras is now a vast topic (see *e.g.* [Ru] for a comprehensive modern study). We shall only need the following fragment. Let A be a Banach algebra. A bounded net $(\Delta_\alpha)_{\alpha\in\mathbb{I}}$ in $\mathsf{A} \widehat{\otimes} \mathsf{A}$ is called a **bounded approximate diagonal for** A if

$$\lim_\alpha (a \cdot \Delta_\alpha - \Delta_\alpha \cdot a) = 0, \quad \text{and} \quad \lim_\alpha a\pi_{\mathsf{A}}(\Delta_\alpha) = a \qquad \text{for all } a \in \mathsf{A}, \tag{4.2}$$

where the limits are taken in the norm topology, and $\pi_{\mathsf{A}} : \mathsf{A}\widehat{\otimes}\mathsf{A} \to \mathsf{A}$ is the unique bounded linear map satisfying $\pi_{\mathsf{A}}(a \otimes b) = ab$ for all $a, b \in \mathsf{A}$. We refer to $\sup_\alpha \|\Delta_\alpha\|$ as the norm of the bounded approximate diagonal.

A Banach algebra A is **amenable** if there is a bounded approximate diagonal for A, and the **amenability constant of** A is the infimum of norms of all possible bounded approximate diagonals. It follows from compactness arguments in the bidual, together with Goldstine's lemma and a convexity argument, that we can always find a bounded approximate diagonal for A whose norm achieves this infimum.

4.2 Reduction to a unital version

Let us revisit the definition of the multiplicative defect. Given Banach algebras A and B and a linear map $\phi \colon \mathsf{A} \to \mathsf{B}$, we define $\phi^\vee : \mathsf{A} \times \mathsf{A} \to \mathsf{B}$ by

$$\phi^\vee(a,b) := \phi(ab) - \phi(a)\phi(b) \qquad \text{for all } a, b \in \mathsf{A}. \tag{4.3}$$

Our earlier definition merely says that

$$\mathrm{def}(\phi) = \sup\{\|\phi(a_1a_2) - \phi(a_1)\phi(a_2)\| \colon a_1, a_2 \in \mathrm{ball}_1(\mathsf{A})\} = \|\phi^\vee\|_{\mathrm{Bil}(\mathsf{A},\mathsf{A};\mathsf{B})} \tag{4.4}$$

Now let $\mathsf{D} \subseteq \mathsf{A}$ be a closed subalgebra. We will need to define quantities analogous to $\mathrm{def}(\phi)$ where the "multiplicative property" is only tested on pairs in $\mathsf{D} \times \mathsf{A}$ or $\mathsf{A} \times \mathsf{D}$. To be precise:

$$\begin{aligned} \mathrm{def}_{\mathsf{D}\times\mathsf{A}}(\phi) &= \|\phi^\vee\|_{\mathrm{Bil}(\mathsf{D},\mathsf{A};\mathsf{B})} \\ &= \sup\{\|\phi(a_1a_2) - \phi(a_1)\phi(a_2)\| \colon a_1 \in \mathrm{ball}_1(\mathsf{D}), a_2 \in \mathrm{ball}_1(\mathsf{A})\} \end{aligned} \tag{4.5}$$

with $\mathrm{def}_{\mathsf{A}\times\mathsf{D}}(\phi)$ defined similarly. The function $\mathrm{def}_{\mathsf{D}\times\mathsf{A}} \colon \mathcal{L}(\mathsf{A},\mathsf{B}) \to [0,\infty)$ is continuous.

The next lemma is a sharper version of Lemma 2.1.

Lemma 4.1. *Let* A, B *be Banach algebras and let* $\phi, \gamma \in \mathcal{L}(\mathsf{A},\mathsf{B})$. *Then for all* $a_1, a_2 \in \mathsf{A}$,

$$(\phi+\gamma)^\vee(a_1,a_2) = \phi^\vee(a_1,a_2) - \phi(a_1)\gamma(a_2) + \gamma(a_1a_2) - \gamma(a_1)\phi(a_2) - \gamma(a_1)\gamma(a_2) \ . \tag{4.6}$$

In particular, for any closed subalgebra $\mathsf{D} \subseteq \mathsf{A}$,

$$\mathrm{def}_{\mathsf{D}\times\mathsf{A}}(\phi+\gamma) \le \mathrm{def}_{\mathsf{D}\times\mathsf{A}}(\phi) + (2\|\phi\|+1)\|\gamma\| + \|\gamma\|^2 \,, \tag{4.7}$$

$$\mathrm{def}_{\mathsf{A}\times\mathsf{D}}(\phi+\gamma) \le \mathrm{def}_{\mathsf{A}\times\mathsf{D}}(\phi) + (2\|\phi\|+1)\|\gamma\| + \|\gamma\|^2 \,. \tag{4.8}$$

Proof. The first identity is a direct calculation, and we omit the details. The subsequent inequalities follow easily from the first identity and the definitions of $\operatorname{def}_{\mathsf{D}\times\mathsf{A}}$ *and* $\operatorname{def} \mathsf{A}\times\mathsf{D}$. $\square$

The following theorem, which extends [Jo88, Theorem 3.1], is the heart of Theorem 1.11. Note that unlike the earlier theorem, we impose the condition that the subalgebra is unital and restrict attention to unit-preserving maps, even though in the original application to Theorem 1.4 it was important to allow non-unital examples.

THEOREM 4.2 (AMNM with respect to a unital amenable subalgebra). *Let* A *be a Banach algebra, let* D *be a closed subalgebra of* A *which is unital and amenable with amenability constant* $\leq K$*, and let* B *be a unital dual Banach algebra with an isometric predual. Fix* $L \geq 1$ *and* $\delta > 0$ *satisfying* $K^2L^2\delta \leq 1/8$*.*

Let $\phi \in \mathcal{L}(\mathsf{A},\mathsf{B})$ *satisfy* $\|\phi\| \leq L$*,* $\phi(1_{\mathsf{D}}) = 1_{\mathsf{B}}$*,* $\operatorname{def}_{\mathsf{A}\times\mathsf{D}}(\phi) \leq \delta$*, and* $\operatorname{def}_{\mathsf{D}\times\mathsf{A}}(\phi) \leq \delta$*. Then there exists* $\psi \in \operatorname{SHom}_{\mathsf{D}}(\mathsf{A},\mathsf{B})$ *with* $\psi(1_{\mathsf{D}}) = 1_{\mathsf{B}}$ *and* $\|\phi - \psi\| \leq 12K^2L^3\delta$*.*

Recall the statement of Theorem 1.11:

> *Let* A *be a Banach algebra with a closed amenable subalgebra* D_0*, and let* B *be a unital dual Banach algebra with an isometric predual. Fix some* $L \geq 1$*. Then there exists a constant* $C' \geq 1$ *(possibly depending on* L *and* D_0*) such that the following holds: whenever* $\psi \in \mathcal{L}(\mathsf{A},\mathsf{B})$ *satisfies* $\|\psi\| \leq L$ *and* $C' \operatorname{def}(\psi) \leq 1$*, there exists* $\theta \in \operatorname{SHom}_{\mathsf{D}_0}(\mathsf{A},\mathsf{B})$ *with* $\|\theta - \psi\| \leq C' \operatorname{def}(\psi)$*.*

Deducing Theorem 1.11 from Theorem 4.2. We start by considering an arbitrary $\psi \in \mathcal{L}(\mathsf{A},\mathsf{B})$. Let $\mathsf{A}^\sharp = \mathbb{C}1 \oplus_1 \mathsf{A}$ denote the forced unitization of A (here $\oplus_1$ denotes the ℓ_1-sum of two Banach spaces). Then there is a natural extension of ψ to $\psi^\sharp : \mathsf{A}^\sharp \to \mathsf{B}$, given by

$$\psi^\sharp(\lambda, a) = \lambda 1_{\mathsf{B}} + \psi(a) \qquad \text{for all } \lambda \in \mathbb{C}, a \in \mathsf{A}.$$

It is easily checked that $\|\psi^\sharp\| = \|\psi\|$ (one direction is trivial since $\mathsf{A} \subset \mathsf{A}^\sharp$, and the other follows by our choice of norm on $\mathsf{A}^\sharp$). Moreover, a direct calculation shows that

$$\psi^\sharp((\lambda_1, a_1)(\lambda_2, a_2)) - \psi^\sharp(\lambda_1, a_1)\psi^\sharp(\lambda_2, a_2) = \psi(a_1a_2) - \psi(a_1)\psi(a_2); \tag{4.9}$$

and hence $\operatorname{def}(\psi^\sharp) = \operatorname{def}(\psi)$. (Once again, one direction is trivial since $\mathsf{A} \subset \mathsf{A}^\sharp$; the non-trivial direction follows from the identity (4.9).)

Let $\mathsf{D} = {\mathsf{D}_0}^\sharp$, which coincides with the closed subalgebra $\mathbb{C}1 \oplus_1 D_0$ of $A^\sharp$, where 1 is the adjoined unit. It is well known that the unitization of any amenable Banach algebra is itself amenable; let K be the amenability constant of D, which automatically satisfies $K \geq 1$.

Given $L \geq 1$, put $C' := 12K^2L^3$. Suppose $\psi \in \operatorname{ball}_L \mathcal{L}(\mathsf{A},\mathsf{B})$ satisfies $\operatorname{def}(\psi) = \delta$, for some $\delta \in [0, 1/C']$. By our previous remarks, the extended map $\psi^\sharp : \mathsf{A}^\sharp \to \mathsf{B}$ also has multiplicative defect δ and norm $\leq L$, and by construction it satisfies $\psi^\sharp(1) = 1_B$. Applying Theorem 4.2 to the triple $(\mathsf{D}, \mathsf{A}^\sharp, \mathsf{B})$ (note that $8K^2L^2\delta \leq 12K^2L^3\delta \leq 1$), we deduce that there exists $\phi \in \operatorname{SHom}_{\mathsf{D}}(\mathsf{A}^\sharp, \mathsf{B})$ with $\phi(1) = 1_{\mathsf{B}}$ and $\|\phi - \psi^\sharp\| \leq C'\delta$. Taking $\theta = \phi|_{\mathsf{A}} \in \operatorname{SHom}_{\mathsf{D}_0}(\mathsf{A},\mathsf{B})$, we see that the conclusions of Theorem 1.11 are satisfied. $\square$

4.3 Obtaining the unital version, using an improving operator

Guided by the case $\mathsf{D} = \mathsf{A}$ that is treated in [Jo88], we shall prove Theorem 4.2 by an iterative argument. Notably, the proof works by repeated application of a **nonlinear** operator $F : \mathcal{L}(A, B) \to \mathcal{L}(A, B)$ with certain "improving" properties. The operator F is designed in such a way that for each ϕ satisfying the assumptions of Theorem 4.2, the sequence of iterates $(F^n(\phi))_{n\in\mathbb{N}}$ is a fast Cauchy sequence in $\mathcal{L}(\mathsf{A}, \mathsf{B})$ and satisfies $\operatorname{def}_{\mathsf{D}\times\mathsf{A}}(F^n(\phi)) \to 0$. The map $\phi_\infty := \lim_{n\to\infty} F^n(\phi)$ then satisfies $\operatorname{def}_{\mathsf{D}\times\mathsf{A}}(\phi_\infty) = 0$; and $\|\phi - \phi_\infty\|$ can be bounded above in terms of $\operatorname{def}_{\mathsf{D}\times\mathsf{A}}(\phi)$, using the geometric decay from the fast Cauchy property. To get the final map ψ, one performs a "left–right switch" and exploits some *ad hoc* features of the operator F.

Before constructing the operator F, we isolate those of its properties which are needed for the argument in the previous paragraph.

Proposition 4.3 (A nonlinear improving operator). *Let A be a Banach algebra, let B be a unital dual Banach algebra with an isometric predual, and let D be a closed subalgebra of A which is unital and amenable with amenability constant $\le K$. Then there is a function $F : \mathcal{L}(\mathsf{A}, \mathsf{B}) \to \mathcal{L}(\mathsf{A}, \mathsf{B})$ with the following properties: for each $\phi \in \mathcal{L}(\mathsf{A}, \mathsf{B})$ satisfying $\phi(1_{\mathsf{D}}) = 1_{\mathsf{B}}$, we have*

(i) $F(\phi)(1_{\mathsf{D}}) = 1_{\mathsf{B}}$*;*

(ii) $\|F(\phi) - \phi\| \le K\|\phi\| \operatorname{def}_{\mathsf{D}\times\mathsf{A}}(\phi)$*;*

(iii) $\operatorname{def}_{\mathsf{D}\times\mathsf{A}}(F(\phi)) \le 3K^2\|\phi\|^2 \operatorname{def}_{\mathsf{D}\times\mathsf{D}}(\phi) \operatorname{def}_{\mathsf{D}\times\mathsf{A}}(\phi)$*.*

Moreover,

(iv) *if* $\operatorname{def}_{\mathsf{A}\times\mathsf{D}}(\phi) = 0$*, then* $\operatorname{def}_{\mathsf{A}\times\mathsf{D}}(F(\phi)) = 0$*.*

Proof of Theorem 4.2, given Proposition 4.3. We fix K, L and δ as in the statement of the theorem. Let $\phi \in \mathcal{L}(\mathsf{A}, \mathsf{B})$ with $\phi(1_{\mathsf{D}}) = 1_{\mathsf{B}}$, $\|\phi\| \le L$, $\operatorname{def}_{\mathsf{A}\times\mathsf{D}}(\phi) \le \delta$, and $\operatorname{def}_{\mathsf{D}\times\mathsf{A}}(\phi) \le \delta$.

The first step is to prove that $(F^n(\phi))_{n\ge 0}$ is a Cauchy sequence in $\mathcal{L}(\mathsf{A}, \mathsf{B})$. In fact, we prove a more precise technical statement, as follows.

Claim.
$\|F^n(\phi) - F^{n-1}(\phi)\| \le KL\delta 2^{-(n-1)}$ and $\operatorname{def}_{\mathsf{D}\times\mathsf{A}}(F^n(\phi)) \le 3\delta 2^{-2n-1}$, for each $n \ge 1$.

The claim is proved by strong induction on n. For the base case ($n = 1$): applying Proposition 4.3 to ϕ, we obtain $\|F(\phi) - \phi\| \le K\|\phi\| \operatorname{def}_{\mathsf{D}\times\mathsf{A}}(\phi) \le KL\delta$ and

$$\begin{aligned}\operatorname{def}_{\mathsf{D}\times\mathsf{A}}(F(\phi)) &\le 3K^2\|\phi\|^2 \operatorname{def}_{\mathsf{D}\times\mathsf{D}}(\phi) \operatorname{def}_{\mathsf{D}\times\mathsf{A}}(\phi) \\ &\le 3K^2\|\phi\|^2 \operatorname{def}_{\mathsf{D}\times\mathsf{A}}(\phi)^2 \\ &\le 3K^2L^2\delta^2 \\ &\le 3\delta/8\end{aligned}$$

as required. Now suppose the claim holds for all $1 \le j \le n$ for some $n \in \mathbb{N}$. Then

$$\begin{aligned}\|F^n(\phi)\| &\le \|\phi\| + \sum_{j=1}^{n} \|F^j(\phi) - F^{j-1}(\phi)\| \\ &\le L + KL\delta \sum_{j=1}^{n} 2^{-(j-1)} \le L + 2KL\delta \le 5L/4,\end{aligned} \tag{4.10}$$

using the fact that $K\delta \le KL\delta \le 1/8$. Combining (4.10) with the second part of the inductive hypothesis yields

$$\begin{aligned}\|F^n(\phi)\| \operatorname{def}_{\mathsf{D}\times\mathsf{A}}(F^n(\phi)) &\le (5L/4) \cdot 3\delta 2^{-2n-1} \\ &\le L\delta 2^{-2n+1} \le L\delta 2^{-n} \qquad (\text{since } n \ge 1).\end{aligned} \tag{4.11}$$

Applying Proposition 4.3 (ii) to $F^n(\phi)$ and using (4.11) yields

$$\|F^{n+1}(\phi) - F^n(\phi)\| \leq K\|F^n(\phi)\| \operatorname{def}_{\mathsf{D}\times\mathsf{A}}(F^n(\phi)) \leq KL\delta 2^{-n},$$

and applying Proposition 4.3 (iii) to $F^n(\phi)$ yields

$$\begin{aligned}
\operatorname{def}_{\mathsf{D}\times\mathsf{A}}(F^{n+1}(\phi)) &\leq 3K^2\|F^n(\phi)\|^2 \operatorname{def}_{\mathsf{D}\times\mathsf{D}}(F^n(\phi)) \operatorname{def}_{\mathsf{D}\times\mathsf{A}}(F^n(\phi)) \\
&\leq 3\,(K\|F^n(\phi)\| \operatorname{def}_{\mathsf{D}\times\mathsf{A}}(F^n(\phi)))^2 \\
&\leq 3(KL\delta 2^{-n})^2 \qquad \text{(using (4.11))} \\
&= 3K^2L^2\delta\cdot\delta 2^{-2n} \\
&\leq 3\delta 2^{-2n-3} \qquad \text{(since } K^2L^2\delta \leq 1/8).
\end{aligned}$$

This completes the inductive step, and hence proves the claim.

It follows from the claim that the sequence $(F^n(\phi))_{n\geq 0}$ is Cauchy in $\mathcal{L}(\mathsf{A},\mathsf{B})$. Let $\phi_\infty = \lim_{n\to\infty} F^n(\phi) \in \mathcal{L}(\mathsf{A},\mathsf{B})$. Since $F^n(\phi)(1_{\mathsf{D}}) = 1_{\mathsf{B}}$ for all $n \in \mathbb{N}$ and $\lim_n \operatorname{def}_{\mathsf{D}\times\mathsf{A}}(F^n(\phi)) = 0$, we have $\phi_\infty(1_{\mathsf{D}}) = 1_{\mathsf{B}}$ and $\operatorname{def}_{\mathsf{D}\times\mathsf{A}}(\phi_\infty) = 0$ by continuity. Also, $\|\phi - \phi_\infty\| \leq 2KL\delta$. This implies

$$\|\phi_\infty\| \leq \|\phi\| + \|\phi - \phi_\infty\| \leq L + 2KL\delta \leq L(1 + 2K^2L^2\delta) \leq 5L/4,$$

and, by the estimate given at the end of Lemma 4.1,

$$\begin{aligned}
\operatorname{def}_{\mathsf{A}\times\mathsf{D}}(\phi_\infty) &\leq \operatorname{def}_{\mathsf{A}\times\mathsf{D}}(\phi) + (2\|\phi\|+1)\|\phi_\infty - \phi\| + \|\phi_\infty - \phi\|^2 \\
&\leq \delta + (2L+1)2KL\delta + (2KL\delta)^2 \\
&\leq \delta(1 + 6KL^2 + 4K^2L^2\delta) \quad \leq \quad \delta(3/2 + 6KL^2) \quad \leq \quad 8KL^2\delta.
\end{aligned}$$

To obtain the final map ψ, let A^{op} and B^{op} be the Banach algebras whose underlying Banach spaces are the same as A and B respectively, but which have the opposite algebra structures, so that $a_1 \cdot_{(\mathsf{A}^{\mathsf{op}})} a_2 := a_2a_1$, etc. Note that D^{op} is a closed subalgebra of A^{op}. Moreover, D^{op} is unital and amenable with constant $\leq K$: for if $\sigma : \mathsf{D}\widehat{\otimes}\mathsf{D} \to \mathsf{D}\widehat{\otimes}\mathsf{D}$ is the flip map defined by $c_1 \otimes c_2 \mapsto c_2 \otimes c_1$, then σ maps bounded approximate diagonals for D to bounded approximate diagonals for D^{op}.

Let $\phi' \in \mathcal{L}(\mathsf{A}^{\mathsf{op}},\mathsf{B}^{\mathsf{op}})$ be the same function as $\phi_\infty \in \mathcal{L}(\mathsf{A},\mathsf{B})$ (we introduce new notation to emphasise that we are now working with different algebras as domain and codomain, which affects the definition of def). Then the following properties hold:

$$\phi'(1_{\mathsf{D}^{\mathsf{op}}}) = \phi_\infty(1_{\mathsf{D}}) = 1_{\mathsf{B}} = 1_{\mathsf{B}^{\mathsf{op}}}; \qquad \operatorname{def}_{\mathsf{A}^{\mathsf{op}}\times\mathsf{D}^{\mathsf{op}}}(\phi') = \operatorname{def}_{\mathsf{D}\times\mathsf{A}}(\phi_\infty) = 0.$$

Applying Proposition 4.3 to the triple $(\mathsf{A}^{\mathsf{op}},\mathsf{B}^{\mathsf{op}},\mathsf{D}^{\mathsf{op}})$, there is a function $F' : \mathcal{L}(\mathsf{A}^{\mathsf{op}},\mathsf{B}^{\mathsf{op}}) \to \mathcal{L}(\mathsf{A}^{\mathsf{op}},\mathsf{B}^{\mathsf{op}})$ such that

1. $F'(\phi')(1_{\mathsf{D}^{\mathsf{op}}}) = 1_{\mathsf{B}^{\mathsf{op}}}$;
2. $\|F'(\phi') - \phi'\| \leq K\|\phi'\| \operatorname{def}_{\mathsf{D}^{\mathsf{op}}\times\mathsf{A}^{\mathsf{op}}}(\phi') = K\|\phi_\infty\| \operatorname{def}_{\mathsf{A}\times\mathsf{D}}(\phi_\infty)$;
3. $\operatorname{def}_{\mathsf{D}^{\mathsf{op}}\times\mathsf{A}^{\mathsf{op}}}(F'(\phi')) \leq 3K^2\|\phi'\| \operatorname{def}_{\mathsf{D}^{\mathsf{op}}\times\mathsf{D}^{\mathsf{op}}}(\phi') \operatorname{def}_{\mathsf{D}^{\mathsf{op}}\times\mathsf{A}^{\mathsf{op}}}(\phi')$;
4. $\operatorname{def}_{\mathsf{A}^{\mathsf{op}}\times\mathsf{D}^{\mathsf{op}}}(F'(\phi')) = 0$.

Now observe that $\operatorname{def}_{\mathsf{D}^{\mathrm{op}}\times\mathsf{D}^{\mathrm{op}}}(\phi') \le \operatorname{def}_{\mathsf{A}^{\mathrm{op}}\times\mathsf{D}^{\mathrm{op}}}(\phi') = 0$. Hence we may improve the property 3 above to: $\operatorname{def}_{\mathsf{D}^{\mathrm{op}}\times\mathsf{A}^{\mathrm{op}}}(F'(\phi')) = 0$.

We define $\psi \in \mathcal{L}(\mathsf{A},\mathsf{B})$ to have the same underlying function as $F'(\phi')$. Then $\psi(1_{\mathsf{D}}) = F'(\phi')(1_{\mathsf{D}^{\mathrm{op}}}) = 1_{\mathsf{B}^{\mathrm{op}}} = 1_{\mathsf{B}}$, and

$$\begin{aligned}\|\psi-\phi\| &\le \|F'(\phi')-\phi'\| + \|\phi_\infty-\phi\|\\ &\le K(5L/4)8KL^2\delta + 2KL\delta \quad\ \le 12K^2L^3\delta\ .\end{aligned}$$

Finally, $\psi \in \operatorname{SHom}_{\mathsf{D}}(\mathsf{A},\mathsf{B})$ since $\operatorname{def}_{\mathsf{D}\times\mathsf{A}}(\psi) = \operatorname{def}_{\mathsf{A}^{\mathrm{op}}\times\mathsf{D}^{\mathrm{op}}}(F'(\phi')) = 0$ and $\operatorname{def}_{\mathsf{A}\times\mathsf{D}}(\psi) = \operatorname{def}_{\mathsf{D}^{\mathrm{op}}\times\mathsf{A}^{\mathrm{op}}}(F'(\phi'))) = 0$. ☐

4.4 Explanation for the improving operator

We have not given any definition of the operator F, let alone explained why amenability of D would allow us to find or construct F. In fact the definition of F is quite simple and explicit — see Equation (5.10) below — but attempting to prove directly that F has the required "improving properties" is far less straightforward. Subtle cancellations are required, and one has to pay attention to technical issues arising when carrying out repeated w*-averaging.

These issues are already present in the proof of [Jo88, Theorem 3.1], where an operator analogous to ours is constructed in the special case $\mathsf{D} = \mathsf{A}$. Although Johnson chooses in his proof to verify the necessary properties directly, he follows this with a brief sketch of how the construction of the operator and the proof that it has the required properties are motivated by a "vanishing H^2 argument" that is standard in the Hochschild cohomology theory of (amenable) Banach algebras.

In our setting, the algebra A is no longer amenable, but the unital subalgebra D is, and the corresponding notion in cohomology theory is that of **normalizing a 2-cocycle with respect to an amenable subalgebra**. It is this approach which guides our construction of the desired "improving operator" F. Rather than adapting the calculations in the proof of [Jo88, Theorem 3.1] in an *ad hoc* way to the setting of an amenable subalgebra $\mathsf{D} \subseteq \mathsf{A}$, it seems both more comprehensible and more robust to set up a general framework. This is our goal in the final section of the paper; the desired "improving operator" F will then emerge naturally as a special case of the general machinery.

5 Constructing the nonlinear improving operator

5.1 An approximate cochain complex

Throughout this subsection, we fix Banach algebras A, B and $\phi \in \mathcal{L}(\mathsf{A},\mathsf{B})$; we shall think of ϕ as defining an "approximate action" of A on B. As mentioned earlier, we are guided by a standard construction in the Hochschild cohomology theory of Banach algebras, which arises when normalizing cochains with respect to an amenable unital subalgebra. However, we require the actual techniques in the proofs and not just the results, and therefore we shall build the required machinery from scratch.

REMARK 5.1. After the original work was done for this section, it was brought to our attention that [Ka82] also adopts a similar setup with an approximate cochain complex; however, this is only done in the setting of (bounded) group cohomology for discrete groups. Moreover, [Ka82] does not explore the "relative" setting where one only has amenability for a subgroup rather than for the whole group.

DEFINITION 5.2. For each $n \in \mathbb{N}$, define the bounded linear map $\partial_\phi^n : \mathcal{L}^n(\mathsf{A},\mathsf{B}) \to \mathcal{L}^{n+1}(\mathsf{A},\mathsf{B})$ by

$$\partial_\phi^n \psi(a_1,\dots,a_{n+1}) = \begin{cases} \phi(a_1)\psi(a_2,\dots,a_{n+1}) \\ + \sum_{j=1}^{n}(-1)^j \psi(a_1,\dots,a_j a_{j+1},\dots,a_{n+1}) \\ +(-1)^{n+1}\psi(a_1,\dots,a_n)\phi(a_{n+1}). \end{cases}$$

In fact, to prove Proposition 4.3, we only need this definition for $n \in \{1,2\}$. We include the definitions for general n, to put the following arguments in their proper context.

REMARK 5.3. We make some remarks to provide context; they are not necessary for the proof of Proposition 4.3.

(i) If ϕ is multiplicative, then $(a,b) \mapsto \phi(a)b$ and $(b,a) \mapsto b\phi(a)$ give B the structure of an A-bimodule ${}_\phi\mathsf{B}_\phi$, and the operator ∂_ϕ^n is just the usual Hochschild coboundary operator for ${}_\phi\mathsf{B}_\phi$-valued cochains. If ϕ is not multiplicative, then we might have $\partial_\phi^{n+1} \circ \partial_\phi^n \neq 0$, but a direct calculation shows that $\|\partial_\phi^{n+1} \circ \partial_\phi^n\| \leq 4\,\mathrm{def}(\phi)$.

(ii) Recall that we have a nonlinear function $(\_\_)^\vee : \mathcal{L}(\mathsf{A},\mathsf{B}) \to \mathcal{L}^2(\mathsf{A},\mathsf{B})$; $\psi \mapsto \psi^\vee$ (where $\psi^\vee$ is defined as in Equation (4.3)), which satisfies $\mathrm{def}(\psi) = \|\psi^\vee\|$. If $\gamma \in \mathcal{L}(\mathsf{A},\mathsf{B})$, Equation (4.6) may be rewritten as

$$(\phi+\gamma)^\vee(a_1,a_2) = \phi^\vee - \partial_\phi^1(\gamma)(a_1,a_2) - \gamma(a_1,a_2) \quad \text{for all } a_1,a_2 \in \mathsf{A},$$

and it follows that the derivative of the function $(\_\_)^\vee$ at ϕ is just $-\partial_\phi^1$. (This observation is taken from remarks in [Jo88, Section 3].)

(iii) For now, we do not assume either A or B is unital; but when it comes to our analogue of "normalization of cocycles", some kind of unitality assumption is needed to obtain maps with the right properties.

Since ∂_ϕ^2 can be applied to arbitrary elements of $\mathcal{L}^2(\mathsf{A},\mathsf{B})$, we may apply it to the particular bilinear map $\phi^\vee$.

LEMMA 5.4 (A 2-cocycle for ∂_ϕ). $\partial_\phi^2(\phi^\vee) = 0$.

The proof is a straightforward calculation, which we omit.

DEFINITION 5.5 (Notation for restricting in first variable). Let E and V be Banach spaces, and let F be a closed subspace of E. Let $n \geq 2$. Given $\psi \in \mathcal{L}^n(E,V)$ we may regard it as an element of $\mathcal{L}(E,\mathcal{L}^{n-1}(E,V))$, which is defined by

$$x_1 \mapsto ((x_2,\dots,x_n) \mapsto \psi(x_1,\dots,x_n)).$$

Restricting this function to F yields a bounded linear map $F \to \mathcal{L}^{n-1}(E,V)$, which we denote by $\mathrm{Lres}_F(\psi) \in \mathcal{L}(F,\mathcal{L}^{n-1}(E,V))$. The function $\mathrm{Lres}_F : \mathcal{L}^n(E,V) \to \mathcal{L}(F,\mathcal{L}^{n-1}(E,V))$ is linear and contractive.

For the rest of this subsection, we fix a closed subalgebra $\mathsf{D} \subseteq \mathsf{B}$. Note that $\mathrm{def}_{\mathsf{D}\times\mathsf{A}}(\phi) = \|\mathrm{Lres}_{\mathsf{D}}(\phi^\vee)\|$.

Our goal is to define (linear) operators $\sigma_\phi^n : \mathcal{L}^{n+1}(\mathsf{A},\mathsf{B}) \to \mathcal{L}^n(\mathsf{A},\mathsf{B})$ such that for each $\psi \in \mathcal{L}^n(\mathsf{A},\mathsf{B})$, the map

$$\mathrm{Lres}_{\mathsf{D}}\left(\partial_\phi^{n-1}\,\sigma_\phi^{n-1}(\psi) + \sigma_\phi^n\,\partial_\phi^n(\psi) - \psi\right)$$

has norm controlled by $\operatorname{def}_{\mathsf{D}\times\mathsf{A}}(\psi)$ (we make this precise in Proposition 5.18 below). As a first step towards this, we set up a general construction by which elements of $\mathsf{D}\,\widehat{\otimes}\,\mathsf{D}$ define (linear) operators $\mathcal{L}^{n+1}(\mathsf{A},\mathsf{B}) \to \mathcal{L}^n(\mathsf{A},\mathsf{B})$.

Definition 5.6. Let $n \in \mathbb{N}$. Given $c, d \in \mathsf{D}$, we define the bounded linear map $[\![c\,;d]\!]_\phi^n : \mathcal{L}^{n+1}(\mathsf{A},\mathsf{B}) \to \mathcal{L}^n(\mathsf{A},\mathsf{B})$ by

$$[\![c\,;d]\!]_\phi^n(\psi)(a_1,\dots,a_n) = \phi(c)\ \psi(d,a_1,\dots,a_n) \qquad \text{for all } a_1,\dots,a_n \in A. \tag{5.1}$$

The function $\mathsf{D}\times\mathsf{D} \to \mathcal{L}(\mathcal{L}^{n+1}(\mathsf{A},\mathsf{B}),\mathcal{L}^n(\mathsf{A},\mathsf{B}))$ defined by $(c,d) \mapsto [\![c\,;d]\!]_\phi^n$ is a bounded bilinear map, with norm $\leq \|\phi\|\|\mathrm{Lres}_{\mathsf{D}}\|$. Therefore it extends uniquely to a bounded linear map $\mathsf{D}\,\widehat{\otimes}\,\mathsf{D} \to \mathcal{L}(\mathcal{L}^{n+1}(\mathsf{A},\mathsf{B}),\mathcal{L}^n(\mathsf{A},\mathsf{B}))$, which we denote by $w \mapsto [\![w]\!]_\phi^n$.

With this notation, $[\![c\otimes d]\!]_\phi^n$ is the same as $[\![c\,;d]\!]_\phi^n$. Note that for each $\psi \in \mathcal{L}^{n+1}(\mathsf{A},\mathsf{B})$

$$\|[\![w]\!]_\phi^n(\psi)\|_{\mathcal{L}^n(\mathsf{A},\mathsf{B})} \leq \|w\|_{\mathsf{D}\widehat{\otimes}\mathsf{D}}\|\phi\|\|\mathrm{Lres}_{\mathsf{D}}(\psi)\|_{\mathcal{L}(\mathsf{D},\mathcal{L}^n(\mathsf{A},\mathsf{B}))} \qquad \text{for all } w \in \mathsf{D}\,\widehat{\otimes}\,\mathsf{D}. \tag{5.2}$$

Remark 5.7. One could define $[\![w]\!]_\phi^n$ more directly by choosing a representation of w as an absolutely convergent sum of elementary tensors. We prefer to systematically use the universal property of $\widehat{\otimes}$, which makes it clearer that $[\![w]\!]_\phi^n$ depends only on w itself and not the choice of representation. Another benefit of our approach is that it generalises cleanly to other settings; for instance, if A and B are completely contractive Banach algebras in the sense of operator-space theory, then there are natural "completely bounded" versions of our results, with almost identical proofs.

Lemma 5.8 (Approximate splitting, 1st version). *Let $n \geq 2$. Then*

$$\left.\begin{array}{r}\partial_\phi^{n-1}\,[\![w]\!]_\phi^{n-1}(\psi)(a_1,\dots,a_n)\\ +\,[\![w]\!]_\phi^n\,\partial_\phi^n(\psi)(a_1,\dots,a_n)\end{array}\right\} = \begin{cases}\ \ \pi_{\mathsf{B}}(\phi\,\widehat{\otimes}\,\phi)(w)\cdot\psi(a_1,\dots,a_n)\\ +\phi(a_1)\cdot[\![w]\!]_\phi^{n-1}(\psi)(a_2,\dots,a_n)\\ \ \ -[\![w\cdot a_1]\!]_\phi^{n-1}(\psi)(a_2,\dots,a_n)\end{cases} \tag{5.3}$$

for all $w \in \mathsf{D}\,\widehat{\otimes}\,\mathsf{D}$, $a_1,\dots,a_n \in \mathsf{A}$ and $\psi \in \mathcal{L}^n(\mathsf{A},\mathsf{B})$.

Proof. Fix $a_1,\dots,a_n \in \mathsf{A}$ and $\psi \in \mathcal{L}^n(\mathsf{A},\mathsf{B})$. We denote the left-hand side of (5.3) by $T_L(w)$ and denote the right-hand side by $T_R(w)$. Then T_L and T_R are bounded linear maps from $\mathsf{D}\,\widehat{\otimes}\,\mathsf{D}$ to B, so it suffices to prove that $T_L(c\otimes d) = T_R(c\otimes d)$ for all $c,d \in \mathsf{D}$.

Consider

$$T_L(c\otimes d) = \partial_\phi^{n-1}\,[\![c\otimes d]\!]_\phi^{n-1}\,\psi(a_1,\dots,a_n) + [\![c\otimes d]\!]_\phi^n\,\partial_\phi^n\,\psi(a_1,\dots,a_n).$$

Expanding these expressions, most of the terms cancel, leaving

$$\begin{aligned}\phi(a_1)\cdot\phi(c)\cdot\psi(d,a_2,\dots,a_n) &+ \phi(c)\cdot\phi(d)\cdot\psi(a_1,\dots,a_n) - \phi(c)\cdot\psi(da_1,a_2,\dots,a_n)\\ &= \phi(a_1)\cdot[\![c\otimes d]\!]_\phi^{n-1}\,\psi(a_2,\dots,a_n) + \pi_{\mathsf{B}}(\phi\,\widehat{\otimes}\,\phi)(c\otimes d)\cdot\psi(a_1,\dots,a_n)\\ &\qquad - [\![c\otimes da_1]\!]_\phi^{n-1}\,\psi(a_2,\dots,a_n)\end{aligned}$$

which equals $T_R(c\otimes d)$, as required. ∎

REMARK 5.9. The restriction to $n \geq 2$ is merely because we did not define $\mathcal{L}^0(\mathsf{A},\mathsf{B})$. If we put $\mathcal{L}^0(\mathsf{A},\mathsf{B}) := \mathsf{B}$, then the maps $\partial_\phi^0 : \mathsf{B} \to \mathcal{L}(\mathsf{A},\mathsf{B})$ and $[\![w]\!]_\phi^0 : \mathcal{L}(\mathsf{A},\mathsf{B}) \to \mathsf{B}$ can be defined in such a way that Lemma 5.8 remains valid for $n = 1$. However, this would require extra notation, and we do not need this case for the application to the proof of Proposition 4.3.

LEMMA 5.10. *Let $n \geq 2$. Let $w \in \mathsf{D}\widehat{\otimes}\mathsf{D}$ and let $a_1 \in \operatorname{ball}_1(\mathsf{D})$, $a_2, \dots, a_n \in \operatorname{ball}_1(\mathsf{A})$. Then for each $\psi \in \mathcal{L}^n(\mathsf{A},\mathsf{B})$,*

$$\begin{aligned} &\left\| \phi(a_1) \cdot [\![w]\!]_\phi^{n-1}(\psi)(a_2,\dots,a_n) - [\![a_1 \cdot w]\!]_\phi^{n-1}(\psi)(a_2,\dots,a_n) \right\| \\ &\qquad \leq \operatorname{def}_{\mathsf{D}\times\mathsf{D}}(\phi) \, \|w\|_{\mathsf{D}\widehat{\otimes}\mathsf{D}} \, \|\operatorname{Lres}_{\mathsf{D}}(\psi)\|_{\mathcal{L}(\mathsf{D},\mathcal{L}^{n-1}(\mathsf{A},\mathsf{B}))}. \end{aligned} \tag{5.4}$$

Proof. Fixing $a_1 \in \operatorname{ball}_1(\mathsf{D})$ and $a_2, \dots, a_n \in \operatorname{ball}_1(\mathsf{A})$, let

$$T(w) := \phi(a_1) \cdot [\![w]\!]_\phi^{n-1}(\psi)(a_2,\dots,a_n) - [\![a_1 \cdot w]\!]_\phi^{n-1}(\psi)(a_2,\dots,a_n) \quad \text{for all } w \in \mathsf{D}\widehat{\otimes}\mathsf{D}.$$

Then $T\colon \mathsf{D}\widehat{\otimes}\mathsf{D} \to \mathsf{B}$ is a bounded linear map and it suffices to prove that $\|T\| \leq \operatorname{def}_{\mathsf{D}\times\mathsf{D}}(\phi) \, \|\operatorname{Lres}_{\mathsf{D}}(\psi)\|_{\mathcal{L}(\mathsf{D},\mathcal{L}^{n-1}(\mathsf{A},\mathsf{B}))}$. By (4.1) it suffices to prove that

$$\|T(c \otimes d)\| \leq \operatorname{def}_{\mathsf{D}\times\mathsf{D}}(\phi) \, \|\operatorname{Lres}_{\mathsf{D}}(\psi)\|_{\mathcal{L}(\mathsf{D},\mathcal{L}^{n-1}(\mathsf{A},\mathsf{B}))} \quad \text{for all } c, d \in \operatorname{ball}_1(\mathsf{D}).$$

This is now a straightforward calculation:

$$\begin{aligned} \|T(c \otimes d)\| &= \left\| \phi(a_1) \cdot [\![c \otimes d]\!]_\phi^{n-1}(\psi)(a_2,\dots,a_n) - [\![a_1 c \otimes d]\!]_\phi^{n-1}(\psi)(a_2,\dots,a_n) \right\| \\ &= \|\phi(a_1)\phi(c)\psi(d,a_2,\dots,a_n) - \phi(a_1 c)\psi(d,a_2,\dots,a_n)\| \\ &\leq \|\phi(a_1)\phi(c) - \phi(a_1 c)\| \, \|\psi(d,a_2,\dots,a_n)\| \\ &= \|\phi(a_1)\phi(c) - \phi(a_1 c)\| \, \|[\operatorname{Lres}_{\mathsf{D}}(\psi)(d)](a_2,\dots,a_n)\| \\ &\leq \operatorname{def}_{\mathsf{D}\times\mathsf{D}}(\phi) \, \|\operatorname{Lres}_{\mathsf{D}}(\psi)\|_{\mathcal{L}(\mathsf{D},\mathcal{L}^{n-1}(\mathsf{A},\mathsf{B}))}, \end{aligned}$$

as required. ◻

5.2 Defining the approximate homotopy

To construct our approximate homotopy, we have to place further restrictions on B and D. Thus throughout this subsection:

- A is a Banach algebra, B is a unital dual Banach algebra with an isometric predual, and $\phi \in \mathcal{L}(\mathsf{A},\mathsf{B})$;
- D is a closed subalgebra of A, which is unital and amenable with constant $\leq K$;

We also fix a net $(\Delta_\alpha)_{\alpha\in I}$ which is a bounded approximate diagonal for D and has the following properties: $\sup_\alpha \|\Delta_\alpha\|_{\mathsf{D}\widehat{\otimes}\mathsf{D}} \leq K$; and there exists $\boldsymbol{\Delta} \in (\mathsf{D}\widehat{\otimes}\mathsf{D})^{**}$ such that $\Delta_\alpha \xrightarrow{\mathrm{w}^*} \boldsymbol{\Delta}$. The desired operators $\sigma_\phi^n : \mathcal{L}^{n+1}(\mathsf{A},\mathsf{B}) \to \mathcal{L}^n(\mathsf{A},\mathsf{B})$ will be constructed as limits of the operators $[\![\Delta_\alpha]\!]_\phi^n$, with respect to an appropriate topology which we now describe.

Let E and F be Banach spaces and let $n \in \mathbb{N}$ be fixed. For the sake of readability, elements of E^n will be written as $\underline{x} := (x_1, x_2, \dots, x_n)$. For every $\underline{x} \in E^n$ and $y \in F$, we introduce the bounded linear maps

$$\begin{aligned} \mathsf{ev}_{\underline{x}}\colon \mathcal{L}^n(E,F^*) \to F^*; &\quad \psi \mapsto \psi(\underline{x}), \\ \varepsilon_y\colon F^* \to \mathbb{C}; &\quad f \mapsto f(y). \end{aligned}$$

Definition 5.11. The projective topology on $\mathcal{L}^n(E,F^*)$ with respect to the family $(F^*,\sigma(F^*,F),\mathsf{ev}_{\underline{x}})_{\underline{x}\in E^n}$ is called the **topology of point-to-w* convergence** and will be denoted by τ. That is, τ is the smallest topology on $\mathcal{L}^n(E,F^*)$ such that $\mathsf{ev}_{\underline{x}}$ is τ-to-$\sigma(F^*,F)$ continuous for each $\underline{x}\in E^n$.

Proposition 5.12. *The topology τ is a linear, locally convex, Hausdorff topology on the vector space $\mathcal{L}^n(E,F^*)$. A neighbourhood basis of zero in τ is given by*

$$\mathfrak{B} := \left\{ \bigcap_{i=1}^{N} \mathsf{ev}_{\underline{x}^{(i)}}^{-1}(U_i) :\ N\in\mathbb{N},\ \underline{x}^{(i)}\in E^n,\ U_i\in\mathfrak{N} \text{ for each } 1\le i\le N \right\},$$

where $\mathfrak{N}$ is a neighbourhood basis of zero in $\sigma(F^*,F)$.

Proof. A projective topology is automatically linear, in other words, it is compatible with the vector space operations (see *e.g.* Chapter 1, Section 7, Proposition 7 in [Bo]). That $\mathfrak{B}$ is a neighbourhood basis of zero in τ follows from Chapter 1, Section 7, the paragraph before Corollary 1 in [Bo] or from Chapter 5, Section 5.1 in [SW]. From this it is immediate that τ is locally convex. Indeed, $\mathfrak{N}$ consists of convex sets and therefore $\mathfrak{B}$ consists of convex sets too.

It remains to show that τ is Hausdorff, which is equivalent to $\bigcap_{V\in\mathfrak{B}} V=\{0\}$. Let $\psi\in\bigcap_{V\in\mathfrak{B}} V$. Fix an arbitrary $\underline{x}\in E^n$ and $y\in F$. We define

$$V_r := \mathsf{ev}_{\underline{x}}^{-1}\big(\varepsilon_y^{-1}(\mathrm{ball}_r(\mathbb{C}))\big) \qquad \text{for all } r>0,$$

clearly $V_r\in\mathfrak{B}$. Hence $\psi\in V_r$, or equivalently, $|\langle y,\psi(\underline{x})\rangle|\le r$ for each $r>0$. Thus $\langle y,\psi(\underline{x})\rangle=0$, and as $\underline{x}\in E^n$ and $y\in F$ were arbitrary, we conclude $\psi=0$. ☐

Lemma 5.13. *A net $(\psi_\gamma)_{\gamma\in\Gamma}$ in $\mathcal{L}^n(E,F^*)$ converges to zero with respect to τ (in notation, $\lim_\gamma^\tau \psi_\gamma=0$) if and only if*

$$\lim\nolimits_\gamma^\sigma \psi_\gamma(\underline{x})=0 \qquad \textit{for all } \underline{x}\in E^n.$$

Proof. Assume (ψ_γ) is a net in $\mathcal{L}^n(E,F^*)$ which converges to zero with respect to τ. Let $U\in\mathfrak{N}$ be arbitrary. Fix an $\underline{x}\in E^n$, clearly $\Omega:=\mathsf{ev}_{\underline{x}}^{-1}(U)\in\mathfrak{B}$. Hence there is $\delta\in\Gamma$ such that $\psi_\gamma\in\Omega$ for each $\gamma\ge\delta$. Thus $\psi_\gamma(\underline{x})\in U$ for each $\gamma\ge\delta$, showing $\lim_\gamma^\sigma\psi_\gamma(\underline{x})=0$.

Assume $\lim_\gamma^\sigma\psi_\gamma(\underline{x})=0$ *for each* $\underline{x}\in E^n$. Let $\Omega\in\mathfrak{B}$ be be arbitrary. Hence there is an $N\in\mathbb{N}$ and there are $\underline{x}^{(i)}\in E^n$ and $U_i\in\mathfrak{N}$ for each $i\in\{1,\dots,N\}$ such that

$$0\in\bigcap_{i=1}^{N}\mathsf{ev}_{\underline{x}^{(i)}}^{-1}(U_i)=\Omega.$$

Fix $i\in\{1,\dots,N\}$. By $\lim_\gamma^\sigma\psi_\gamma(\underline{x}^{(i)})=0$, we can pick $\delta_i\in\Gamma$ such that $\psi_\gamma(\underline{x}^{(i)})\in U_i$ for each $\gamma\ge\delta_i$. Take $\delta\in\Gamma$ such that $\delta\ge\delta_i$ for each $i\in\{1,\dots,N\}$. Fix $\gamma\in\Gamma$ with $\gamma\ge\delta$, then

$$\psi_\gamma\in\bigcap_{i=1}^{N}\mathsf{ev}_{\underline{x}^{(i)}}^{-1}(U_i)=\Omega.$$

Hence (ψ_γ) converges to zero with respect to τ. ☐

Lemma 5.14. *Suppose B is a dual Banach algebra with an isometric predual. Then for every $n\in\mathbb{N}$, the operator $\partial_\phi^n:\mathcal{L}^n(\mathsf{A},\mathsf{B})\to\mathcal{L}^{n+1}(\mathsf{A},\mathsf{B})$ is τ-to-τ continuous.*

Proof. Let (ψ_i) be a τ-convergent net in $\mathcal{L}^n(\mathsf{A},\mathsf{B})$, with limit ψ. Let $a_1,\dots,a_{n+1}\in\mathsf{A}$. By Lemma 5.13, for each $j=1,\dots,n$ we have

$$\psi_i(a_1,\dots,a_ja_{j+1},\dots,a_{n+1}) \xrightarrow{\mathrm{w}^*} \psi(a_1,\dots,a_ja_{j+1},\dots,a_{n+1}).$$

Also, since B is a dual Banach algebra, multiplication in B is separately w*-continuous. Hence

$$\begin{aligned}\lim_i^\sigma \big(\phi(a_1)\psi_i(a_2,\dots,a_{n+1})\big) &= \phi(a_1)\ \lim_i^\sigma \psi_i(a_2,\dots,a_{n+1})\\ &= \phi(a_1)\psi(a_2,\dots,a_{n+1})\ ,\end{aligned}$$

and similarly

$$\lim_i^\sigma \big(\psi_i(a_1,\dots,a_n)\phi(a_{n+1})\big) = \psi(a_1,\dots,a_n)\phi(a_{n+1}).$$

Thus $(\partial_\phi^n\,\psi_i)(a_1,\dots,a_{n+1}) \xrightarrow{\mathrm{w}^*} (\partial_\phi^n\,\psi)(a_1,\dots,a_{n+1})$, as required. ☐

LEMMA 5.15. *Given $n\in\mathbb{N}$ and $\psi\in\mathcal{L}^{n+1}(\mathsf{A},\mathsf{B})$, the net $([\![\Delta_\alpha]\!]_\phi^n\,\psi)$ τ-converges in $\mathcal{L}^n(\mathsf{A},\mathsf{B})$.*

Proof. Fix $\psi\in\mathcal{L}^{n+1}(\mathsf{A},\mathsf{B})$. Given $a_1,\dots,a_n\in\mathsf{A}$, define $T\in\mathcal{L}(\mathsf{D}\mathbin{\widehat{\otimes}}\mathsf{D},\mathsf{B})$ by $T(w):=[\![w]\!]_\phi^n(\psi)(a_1,\dots,a_n)$. Then $T:\mathsf{D}\mathbin{\widehat{\otimes}}\mathsf{D}\to\mathsf{B}$ is a bounded linear map with values in a dual Banach space, and hence has a unique w*-w*-continuous extension $\widetilde{T}:(\mathsf{D}\mathbin{\widehat{\otimes}}\mathsf{D})^{**}\to\mathsf{B}$, which satisfies $\|\widetilde{T}\|=\|T\|$. In particular,

$$[\![\Delta_\alpha]\!]_\phi^n(\psi)(a_1,\dots,a_n) = T(\Delta_\alpha) \xrightarrow{\mathrm{w}^*} \widetilde{T}(\boldsymbol{\Delta}). \tag{5.5}$$

Denote the right-hand side of (5.5) by $\Psi(a_1,\dots a_n)$.

Routine calculations show that the map $\Psi\colon\mathsf{A}^n\to\mathsf{B}$ is n-multilinear. Using (5.5) and the bound in (5.2), we obtain

$$\begin{aligned}\|\Psi(a_1,\dots,a_n)\| = \|\widetilde{T}(\boldsymbol{\Delta})\| &\le \liminf_\alpha \|T(\Delta_\alpha)\| \le K\|T\|\\ &\le K\|\phi\|\|\mathrm{Lres}_{\mathsf{D}}(\psi)\|\|a_1\|\dots\|a_n\|\ .\end{aligned}$$

Thus $\Psi\in\mathcal{L}^n(\mathsf{A},\mathsf{B})$, and $[\![\Delta_\alpha]\!]_\phi^n(\psi)\xrightarrow{\tau}\Psi$ by (5.5) and Lemma 5.13. ☐

DEFINITION 5.16 (Approximate homotopy). Define $\sigma_\phi^n:\mathcal{L}^{n+1}(\mathsf{A},\mathsf{B})\to\mathcal{L}^n(\mathsf{A},\mathsf{B})$ by

$$\sigma_\phi^n(\psi) = \lim_\alpha^\tau [\![\Delta_\alpha]\!]_\phi^n(\psi) \qquad \text{for all } \psi\in\mathcal{L}^n(\mathsf{A},\mathsf{B}). \tag{5.6}$$

This is well-defined by Lemma 5.15.

The following lemma is basic, and is included just for sake of convenient reference. We leave the proof to the reader.

LEMMA 5.17. *Let F be a Banach space, and let (f_i) be a net in F^* which converges* w* *to some $f\in F^*$. Suppose also that there is a convergent net (c_i) in $[0,\infty)$ such that $\|f_i\|\le c_i$. Then $\|f\|\le\lim_i c_i$.*

PROPOSITION 5.18 (Approximate splitting, 2nd version). *Suppose $\phi(1_{\mathsf{D}})=1_{\mathsf{B}}$. Then for all $n\ge 2$ and all $\psi\in\mathcal{L}^n(\mathsf{A},\mathsf{B})$,*

$$\begin{aligned}&\left\|\mathrm{Lres}_{\mathsf{D}}\left(\partial_\phi^{n-1}\,\sigma_\phi^{n-1}(\psi)+\sigma_\phi^n\,\partial_\phi^n(\psi)-\psi\right)\right\|_{\mathcal{L}(\mathsf{D},\mathcal{L}^{n-1}(\mathsf{A},\mathsf{B}))}\\ &\qquad\le 2K\,\mathrm{def}_{\mathsf{D}\times\mathsf{D}}(\phi)\|\mathrm{Lres}_{\mathsf{D}}(\psi)\|_{\mathcal{L}(\mathsf{D},\mathcal{L}^{n-1}(\mathsf{A},\mathsf{B}))}\ .\end{aligned}$$

Proof. To ease notational congestion, throughout this proof we let

$$M := \|\mathrm{Lres}_{\mathsf{D}}(\psi)\|_{\mathcal{L}(\mathsf{D},\mathcal{L}^{n-1}(\mathsf{A},\mathsf{B}))}\ .$$

Let $a_1 \in \mathrm{ball}_1(\mathsf{D})$ and let $a_2,\dots,a_n \in \mathrm{ball}_1(\mathsf{A})$; it suffices to prove that

$$\left\| \begin{matrix} \partial_\phi^{n-1}\,\sigma_\phi^{n-1}(\psi)(a_1,\dots,a_n) + \sigma_\phi^n\,\partial_\phi^n(\psi)(a_1,\dots,a_n) \\ -\psi(a_1,\dots,a_n) \end{matrix} \right\| \le 2K\,\mathrm{def}_{\mathsf{D}\times\mathsf{D}}(\phi)M\ . \tag{5.7}$$

Since $\partial_\phi^{n-1} : \mathcal{L}^{n-1}(\mathsf{A},\mathsf{B}) \to \mathcal{L}^n(\mathsf{A},\mathsf{B})$ is τ-to-τ continuous by Lemma 5.14,

$$\partial_\phi^{n-1}\,\sigma_\phi^{n-1}(\psi) + \sigma_\phi^n\,\partial_\phi^n(\psi) - \psi \quad = \lim_\alpha^\tau \left(\partial_\phi^{n-1}\,[\![\Delta_\alpha]\!]_\phi^{n-1}(\psi) + [\![\Delta_\alpha]\!]_\phi^n\,\partial_\phi^n(\psi) - \psi\right).$$

Thus the left-hand side of the desired inequality (5.7) is equal to

$$\left\| \lim_\alpha^\sigma \begin{pmatrix} \partial_\phi^{n-1}\,[\![\Delta_\alpha]\!]_\phi^{n-1}(\psi)(a_1,\dots,a_n) + [\![\Delta_\alpha]\!]_\phi^n\,\partial_\phi^n(\psi)(a_1,\dots,a_n) \\ -\psi(a_1,\dots,a_n) \end{pmatrix} \right\|. \tag{5.8}$$

Combining Lemma 5.8, Lemma 5.10, and the bound in (5.2) yields

$$\begin{aligned}
&\left\| \begin{matrix} \partial_\phi^{n-1}\,[\![\Delta_\alpha]\!]_\phi^{n-1}(\psi)(a_1,\dots,a_n) + [\![\Delta_\alpha]\!]_\phi^n\,\partial_\phi^n(\psi)(a_1,\dots,a_n) \\ -\pi_{\mathsf{B}}(\phi\,\widehat{\otimes}\,\phi)(\Delta_\alpha)\cdot\psi(a_1,\dots,a_n) \end{matrix} \right\| \\
&= \left\| \phi(a_1)\cdot[\![\Delta_\alpha]\!]_\phi^{n-1}(\psi)(a_2,\dots,a_n) - [\![\Delta_\alpha\cdot a_1]\!]_\phi^{n-1}(\psi)(a_2,\dots,a_n) \right\| \\
&\le \begin{cases} \left\| \phi(a_1)\cdot[\![\Delta_\alpha]\!]_\phi^{n-1}(\psi)(a_2,\dots,a_n) - [\![a_1\cdot\Delta_\alpha]\!]_\phi^{n-1}(\psi)(a_2,\dots,a_n) \right\| \\ \qquad + \left\| [\![a_1\cdot\Delta_\alpha - \Delta_\alpha\cdot a_1]\!]_\phi^{n-1}(\psi)(a_2,\dots,a_n) \right\| \end{cases} \\
&\le \mathrm{def}_{\mathsf{D}\times\mathsf{D}}(\phi)\|\Delta_\alpha\|M + \|\phi\|\|a_1\cdot\Delta_\alpha - \Delta_\alpha\cdot a_1\|M.
\end{aligned}$$

Also, since $\phi(1_{\mathsf{D}}) = 1_{\mathsf{B}}$, using $\mathrm{def}_{\mathsf{D}\times\mathsf{D}}(\phi) = \|\phi\pi_{\mathsf{D}} - \pi_{\mathsf{B}}(\phi\,\widehat{\otimes}\,\phi)\|$, we obtain

$$\begin{aligned}
&\left\|\pi_{\mathsf{B}}(\phi\,\widehat{\otimes}\,\phi)(\Delta_\alpha)\cdot\psi(a_1,\dots,a_n) - \psi(a_1,\dots,a_n)\right\| \\
&\le \|\pi_{\mathsf{B}}(\phi\,\widehat{\otimes}\,\phi)(\Delta_\alpha) - \phi(1_{\mathsf{D}})\|\|\psi(a_1,\dots,a_n)\| \\
&\le \|\pi_{\mathsf{B}}(\phi\,\widehat{\otimes}\,\phi)(\Delta_\alpha) - \phi(\pi_{\mathsf{D}}(\Delta_\alpha))\|M + \|\phi(\pi_{\mathsf{D}}(\Delta_\alpha) - 1_{\mathsf{D}})\|M \\
&\le \mathrm{def}_{\mathsf{D}\times\mathsf{D}}(\phi)\|\Delta_\alpha\|M + \|\phi\|\|\pi_{\mathsf{D}}(\Delta_\alpha) - 1_{\mathsf{D}}\|M\ .
\end{aligned}$$

Putting things together, and recalling that $K \ge \sup_\alpha \|\Delta_\alpha\|$, we have:

$$\left\| \begin{matrix} \partial_\phi^{n-1}\,[\![\Delta_\alpha]\!]_\phi^{n-1}(\psi)(a_1,\dots,a_n) + [\![\Delta_\alpha]\!]_\phi^n\,\partial_\phi^n(\psi)(a_1,\dots,a_n) \\ -\psi(a_1,\dots,a_n) \end{matrix} \right\| \le 2\,\mathrm{def}_{\mathsf{D}\times\mathsf{D}}(\phi)KM + \varepsilon_\alpha\ , \tag{5.9}$$

where $\varepsilon_\alpha := \|\phi\|\|a_1\cdot\Delta_\alpha - \Delta_\alpha\cdot a_1\|M + \|\phi\|\|\pi_{\mathsf{D}}(\Delta_\alpha) - 1_{\mathsf{D}}\|M$, which tends to 0 by (4.2). Comparing (5.8) and (5.9), and appealing to Lemma 5.17, the desired inequality (5.7) follows. ∎

5.3 Defining the “improving operator”

As in the previous subsection:

- A is a Banach algebra, B is a unital dual Banach algebra with an isometric predual;
- D is a closed subalgebra of A, which is unital and amenable with constant $\leq K$.

Then, for any given $\phi \in \mathcal{L}(\mathsf{A},\mathsf{B})$, we may still form the splitting maps σ_ϕ^n, as in Definition 5.16. However, rather than fixing a single $\phi \in \mathcal{L}(\mathsf{A},\mathsf{B})$ and working with it throughout, we will now allow ϕ to vary.

Definition 5.19. The **improving operator** $F : \mathcal{L}(\mathsf{A},\mathsf{B}) \to \mathcal{L}(\mathsf{A},\mathsf{B})$ is defined by the formula

$$F(\phi) := \phi + \sigma_\phi^1(\phi^\vee) = \phi + \lim_\alpha^\tau [\![\Delta_\alpha]\!]_\phi^1(\phi^\vee). \tag{5.10}$$

The desired properties of F follow from applying the machinery of Section 5.2 to the bilinear map $\phi^\vee \in \mathcal{L}^2(\mathsf{A},\mathsf{B})$, viewed as a “2-cocycle” with respect to the operator ∂_ϕ^2 (see Lemma 5.4). We first deal with some technical details that do not depend on amenability of D.

Lemma 5.20. *Let $\phi,\psi \in \mathcal{L}(\mathsf{A},\mathsf{B})$ and let $w \in \mathsf{D}\widehat{\otimes}\mathsf{D}$.*

(i) *If $\psi(1_\mathsf{D}) = 1_\mathsf{B}$, then $[\![w]\!]_\phi^1(\psi^\vee)(1_\mathsf{D}) = 0$.*

(ii) *If $\operatorname{def}_{\mathsf{A}\times\mathsf{D}}(\psi) = 0$, then $[\![w]\!]_\phi^1(\psi^\vee)(x) = 0$ for all $x \in \mathsf{D}$, and*

$$[\![w]\!]_\phi^1(\psi^\vee)(ax) = [\![w]\!]_\phi^1(\psi^\vee)(a)\cdot\psi(x) \qquad \textit{for all } a \in \mathsf{A},\ x \in \mathsf{D}.$$

Proof. For fixed ϕ and $\psi \in \mathcal{L}(\mathsf{A},\mathsf{B})$, the map $w \mapsto [\![w]\!]_\phi^1(\psi^\vee)$ is bounded linear from $\mathsf{D}\widehat{\otimes}\mathsf{D}$ to $\mathcal{L}(\mathsf{A},\mathsf{B})$. Hence, for both (i) and (ii), it suffices to prove the desired identity in the special case $w = c\otimes d$, where $c,d \in \mathsf{D}$.

(i) $[\![c\otimes d]\!]_\phi^1(\psi^\vee)(1_\mathsf{D}) = \phi(c)\psi^\vee(d,1_\mathsf{D}) = \phi(c)\psi(d1_\mathsf{D}) - \phi(c)\psi(d)\psi(1_\mathsf{D}) = 0$.

(ii) Let $a \in \mathsf{A}$ and $x \in \mathsf{D}$. Then

$$[\![c\otimes d]\!]_\phi^1(\psi^\vee)(x) = \phi(c)\psi^\vee(d,x) = \phi(c)\psi(dx) - \phi(c)\psi(d)\psi(x) = 0$$

and

$$\begin{aligned}[\![c\otimes d]\!]_\phi^1(\psi^\vee)(ax) &= \phi(c)\psi^\vee(d,ax)\\ &= \phi(c)\psi(dax) - \phi(c)\psi(d)\psi(ax)\\ &= \phi(c)\psi(da)\psi(x) - \phi(c)\psi(d)\psi(a)\psi(x)\\ &= [\![c\otimes d]\!]_\phi^1(\psi^\vee)(a)\cdot\psi(x).\end{aligned}$$

□

Proof of Proposition 4.3. Let $\phi \in \mathcal{L}(\mathsf{A},\mathsf{B})$ with $\phi(1_\mathsf{D}) = 1_\mathsf{B}$. Let F be as in Definition 5.19.

Part (i): show that $F(\phi)(1_\mathsf{D}) = 1_\mathsf{B}$.
By the definition of F, this is equivalent to showing that $\lim_\alpha^\sigma [\![\Delta_\alpha]\!]_\phi^1(\phi^\vee)(1_\mathsf{D}) = 0$, which in turn follows from Lemma 5.20(i).

Part (ii): show that $\|F(\phi)-\phi\| \leq K\|\phi\|\operatorname{def}_{\mathsf{D}\times\mathsf{A}}(\phi)$.
Applying the bound in (5.2) with $\psi = \phi^\vee$ and $w = \Delta_\alpha$ yields

$$\|[\![\Delta_\alpha]\!]_\phi^1(\phi^\vee)\| \leq K\|\phi\|\|\operatorname{Lres}_\mathsf{D}(\phi^\vee)\| = K\|\phi\|\operatorname{def}_{\mathsf{D}\times\mathsf{A}}(\phi)\ .$$

Taking the limit on the left-hand side gives the desired bound on $\|F(\phi)-\phi\|$.

Part (iii): show that $\operatorname{def}_{\mathsf{D}\times\mathsf{A}}(F(\phi)) \le 3K^2\|\phi\|^2 \operatorname{def}_{\mathsf{D}\times\mathsf{D}}(\phi)\operatorname{def}_{\mathsf{D}\times\mathsf{A}}(\phi)$.
We put $\gamma := F(\phi)-\phi = \sigma^1_\phi(\phi^\vee)$ in order to simplify some formulas. Rewriting the identity (4.6) in terms of the operator ∂^1_ϕ, we have

$$(\phi+\gamma)^\vee = \phi^\vee - \partial^1_\phi(\gamma) - \pi_{\mathsf{B}}\circ(\gamma\mathbin{\widehat{\otimes}}\gamma)\circ\iota_{\mathsf{A},\mathsf{A}}\ ,$$

where $\iota_{\mathsf{A},\mathsf{A}} \in \operatorname{Bil}(\mathsf{A},\mathsf{A};\mathsf{A}\mathbin{\widehat{\otimes}}\mathsf{A})$ is the canonical map. Hence

$$\begin{aligned}\operatorname{def}_{\mathsf{D}\times\mathsf{A}}(F(\phi)) &= \|\operatorname{Lres}_{\mathsf{D}}(\phi+\gamma)^\vee\| \\ &\le \|\operatorname{Lres}_{\mathsf{D}}\left(\phi^\vee-\partial^1_\phi(\gamma)\right)\| + \|\operatorname{Lres}_{\mathsf{D}}(\pi_{\mathsf{B}}\circ(\gamma\mathbin{\widehat{\otimes}}\gamma)\circ\iota_{\mathsf{A},\mathsf{A}})\| \\ &\le \|\operatorname{Lres}_{\mathsf{D}}\left(\phi^\vee-\partial^1_\phi(\gamma)\right)\| + \|\gamma|_{\mathsf{D}}\|_{\mathcal{L}(\mathsf{D},\mathsf{B})}\,\|\gamma\|\ . \end{aligned} \tag{5.11}$$

To bound the first term on the right-hand side of (5.11), we take $n=2$ and $\psi=\phi^\vee$ in Proposition 5.18. This yields

$$\begin{aligned}\|\operatorname{Lres}_{\mathsf{D}}\left(\partial^1_\phi\,\sigma^1_\phi(\phi^\vee)+\sigma^2_\phi\,\partial^2_\phi(\phi^\vee)-\phi^\vee\right)\| &\le 2K\operatorname{def}_{\mathsf{D}\times\mathsf{D}}(\phi)\|\operatorname{Lres}_{\mathsf{D}}(\phi^\vee)\| \\ &= 2K\operatorname{def}_{\mathsf{D}\times\mathsf{D}}(\phi)\operatorname{def}_{\mathsf{D}\times\mathsf{A}}(\phi).\end{aligned} \tag{5.12}$$

Recall that $\partial^2_\phi(\phi^\vee)=0$ (by Lemma 5.4) and $\gamma=\sigma^1_\phi(\phi^\vee)$. Hence (5.12) may be rewritten as

$$\|\operatorname{Lres}_{\mathsf{D}}\left(\partial^1_\phi(\gamma)-\phi^\vee\right)\| \le 2K\operatorname{def}_{\mathsf{D}\times\mathsf{D}}(\phi)\operatorname{def}_{\mathsf{D}\times\mathsf{A}}(\phi). \tag{5.13}$$

The second term is easier to deal with. We already know from part (ii) of this proposition that $\|\gamma\|\le K\|\phi\|\operatorname{def}_{\mathsf{D}\times\mathsf{A}}(\phi)$. By the same argument, using (5.2), we obtain

$$\|\gamma|_{\mathsf{D}}\|_{\mathcal{L}(\mathsf{D},\mathsf{B})} \le K\|\phi\|\|\phi^\vee|_{\mathsf{D}\times\mathsf{D}}\|_{\mathcal{L}^2(\mathsf{D},\mathsf{B})} = K\|\phi\|\operatorname{def}_{\mathsf{D}\times\mathsf{D}}(\phi).$$

Hence

$$\|\gamma|_{\mathsf{D}}\|_{\mathcal{L}(\mathsf{D},\mathsf{B})}\|\gamma\| \le K^2\|\phi\|^2\operatorname{def}_{\mathsf{D}\times\mathsf{D}}(\phi)\operatorname{def}_{\mathsf{D}\times\mathsf{A}}(\phi). \tag{5.14}$$

Combining (5.11) with (5.13) and (5.14) yields

$$\operatorname{def}_{\mathsf{D}\times\mathsf{A}}(F(\phi)) \le (2K+K^2\|\phi\|^2)\operatorname{def}_{\mathsf{D}\times\mathsf{D}}(\phi)\operatorname{def}_{\mathsf{D}\times\mathsf{A}}(\phi).$$

To finish off the proof of part (iii) it suffices to observe that $K\ge 1$ (because $\pi_{\mathsf{D}}\colon \mathsf{D}\widehat{\otimes}\mathsf{D}\to\mathsf{D}$ is contractive and $(\pi_{\mathsf{D}}(\Delta_\alpha))_{\alpha\in I}$ is a b.a.i. for D) and $\|\phi\|\ge 1$ (since $\phi(1_{\mathsf{D}})=1_{\mathsf{B}}$ and both A and B are unital).

Part (iv): show that if $\operatorname{def}_{\mathsf{A}\times\mathsf{D}}(\phi)=0$*, then* $\operatorname{def}_{\mathsf{A}\times\mathsf{D}}(F(\phi))=0$.
Applying Lemma 5.20 (ii) with $w=\Delta_\alpha$ and $\psi=\phi$, and then taking the limit, we have

$$\gamma(x) = \lim^\sigma_\alpha [\![\Delta_\alpha]\!]^1_\phi(\phi^\vee)(x) = 0 \qquad \text{for all } x\in\mathsf{D}$$

and

$$\gamma(ax)-\gamma(a)\phi(x) = \lim^\sigma_\alpha\left([\![\Delta_\alpha]\!]^1_\phi(\phi^\vee)(ax)-[\![\Delta_\alpha]\!]^1_\phi(\phi^\vee)(a)\cdot\phi(x)\right)=0 \quad \text{for all } a\in\mathsf{A}, x\in\mathsf{D}.$$

Hence, whenever $a\in\mathsf{A}$ and $x\in\mathsf{D}$, we have

$$\begin{aligned}F(\phi)^\vee(a,x) &= \phi(ax)+\gamma(ax)-(\phi(a)+\gamma(a))(\phi(x)+\gamma(x)) \\ &= \phi(ax)+\gamma(a)\phi(x)-(\phi(a)+\gamma(a))\phi(x) \\ &= 0\end{aligned}$$

as required. ∎

This completes the proof of Proposition 4.3, and hence — via Theorem 4.2 — the proof of Theorem 1.11.

Acknowledgements. The main results of this paper were obtained during the second-named author's Ph.D. studies at the Department of Mathematics and Statistics of Lancaster University, UK, supported by a studentship from the Faculty of Science and Technology at that institution. He is indebted to his examiners Dr. Matthew Daws and Prof. J. Martin Lindsay for their careful reading of his thesis [Hor19], which contains an earlier version of the work presented here. He also acknowledges with thanks the support received from the Czech Science Foundation (GAČR project 19-07129Y; RVO 67985840) while this paper was completed and various technical improvements were made, such as the additional examples (Corollary 1.8, Remark 1.9) listed in Section 1.2 and the machinery in Section 5.

A Constructing an uncountable clone system for the Tsirelson space

Let T denote the Tsirelson space. In this appendix we prove the following result.

Proposition A.1. *There is an uncountable clone system for T.*

Proof. We use the notation and terminology of [CS] and [BKL20, Section 3]. Let (t_n) denote the unit vector basis for T. For a subset M of $\mathbb{N}$, P_M is the norm one basis projection onto the closed linear span of $\{t_m : m \in M\}$, denoted by T_M. We first recall a few definitions. We say that $J \subseteq \mathbb{N}$ is a nonempty **Schreier set** if J is a finite set with $|J| \leq \min J$. Let $M \subseteq \mathbb{N}$. We say that J is an **interval in** $\mathbb{N} \setminus M$ if J is of the form $J = [a, b] \cap \mathbb{N}$ for some real numbers $b > a \geq 1$, such that $J \cap M = \emptyset$. Lastly, if $M \subseteq \mathbb{N}$ and J is an interval in $\mathbb{N} \setminus M$, we define

$$\sigma(\mathbb{N}, J) = \sup\Big\{\sum_{j\in J} s_j : s_j \in [0,1]\ (j \in J),\ \Big\|\sum_{j\in J} s_j t_j\Big\|_T \leq 1\Big\}.$$

We rely on the following two results:

- Let $J \subseteq \mathbb{N}$ be a nonempty Schreier set. Then
$$\|x\|_T \geq \frac{1}{2}\sum_{j\in J}|x_j| \qquad \text{for all } x = (x_j) \in T.$$
This is an immediate consequence of how the Tsirelson norm is defined.
- For an infinite $M \subseteq \mathbb{N}$, we have $T_M \cong T$ if and only if there is a constant $C \geq 1$ such that $\sigma(\mathbb{N}, J) \leq C$ for every interval J in $\mathbb{N} \setminus M$. This is a special case of a result of Casazza–Johnson–Tzafriri [CJT84], stated in [BKL20, Corollary 3.2], and applied here only in the particular case where $N = \mathbb{N}$.

Combining these two results, we obtain the following conclusion: Suppose that $M = \{m_1 < m_2 < \cdots\} \subseteq \mathbb{N}$ is an infinite set with

$$m_1 = 1 \qquad \text{and} \qquad m_{j+1} \leq 2m_j + 2 \qquad \text{for all } j \in \mathbb{N}. \tag{A.1}$$

For every nonempty interval J in $\mathbb{N} \setminus M$, there is a unique $j \in \mathbb{N}$ such that $J \subseteq [m_j + 1, m_{j+1} - 1]$. This implies that

$$|J| \leq (m_{j+1} - 1) - (m_j + 1) + 1 \leq 2m_j + 1 - m_j = m_j + 1 \leq \min J,$$

so J is a Schreier set, and therefore

$$\Big\| \sum_{j \in J} s_j t_j \Big\|_T \geq \frac{1}{2} \sum_{j \in J} s_j \qquad \text{for all } s_j \in [0,1],\ j \in J$$

by the first bullet point, so $\sigma(\mathbb{N}, J) \leq 2$. Hence $T_M \cong T$ by the second bullet point. In fact, it follows from the second part of the proof of Theorem 10 and the paragraph before Proposition 3 in [CJT84] that T_M and T are 4-isomorphic.

We can therefore establish the result by constructing an uncountable, almost disjoint family $\mathcal{D}$ of sets whose elements satisfy (A.1). For then, the uncountable family of norm one idempotents $(P_M)_{M \in \mathcal{D}}$ will be the desired clone system. We construct $\mathcal{D}$ as follows.

Given a function $f \in \{0,1\}^{\mathbb{N}}$, define

$$m_n(f) = 2^{n-1} + \sum_{j=1}^{n-1} f(j) 2^{n-1-j} \qquad \text{for all } n \in \mathbb{N}.$$

Alternatively, we can state this definition recursively as follows:

$$m_1(f) = 1 \qquad \text{and} \qquad m_{n+1}(f) = 2m_n(f) + f(n) \qquad \text{for all } n \in \mathbb{N}. \tag{A.2}$$

Set

$$M(f) = \{m_n(f) : n \in \mathbb{N}\} \qquad \text{and} \qquad \mathcal{D} = \{M(f) : f \in \{0,1\}^{\mathbb{N}}\}.$$

Clearly $M(f)$ is an infinite subset of $\mathbb{N}$ for each $f \in \{0,1\}^{\mathbb{N}}$. Since $f(n) \in \{0,1\}$, the recursive definition (A.2) shows that the elements of $M(f)$ satisfy (A.1).

It remains to verify that the family $\mathcal{D}$ is almost disjoint. More precisely, for distinct functions $f, g \in \{0,1\}^{\mathbb{N}}$, we claim that $|M(f) \cap M(g)| = k$, where $k \in \mathbb{N}$ is the smallest number such that $f(k) \neq g(k)$. This however follows from an easy induction argument and (A.2). $\square$

References

[AK] F. Albiac and N. J. Kalton, *Topics in Banach Space Theory*, Springer-Verlag, USA, 2006.

[BKL20] K. Beanland, T. Kania and N. J. Laustsen, The lattices of closed operator ideals on the Tsirelson and Schreier spaces, *J. Funct. Anal.* **279** (2020), 108668.

[BP69] E. Berkson and H. Porta, Representations of $\mathcal{B}(X)$, *J. Funct. Anal.* **3**, No. 1 (1969), 1–34.

[BG09] A. Blanco and N. Grønbæk, Amenability of algebras of approximable operators, *Israel J. Math.* **171** (2009), 127–156.

[Bo] N. Bourbaki, *Topological Vector Spaces (Chapters 1–5)*, Springer-Verlag, Berlin, 2003.

[BOT13] M. Burger, N. Ozawa and A. Thom, On Ulam stability, *Israel J. Math.* **193**, No. 1 (2013), 109—129.

[CJT84] P. G. Casazza, W. B. Johnson and L. Tzafriri, On Tsirelson's space, *Israel J. Math.* **47** (1984), 81–98.

[CS] P. G. Casazza and T. J. Shura, *Tsirelson's space.* Lecture Notes in Mathematics 1363. Springer-Verlag, Berlin, 1989.

[Cho13] Y. Choi, Approximately multiplicative maps from weighted semilattice algebras, *J. Aust. Math. Soc.* **95**, No. 1 (2013), 36–67.

[Da07] M. Daws, Dual Banach algebras: representations and injectivity, *Studia Math.* **178**, No. 3 (2007), 231–275.

[DPW09] M. Daws, H. L. Pham and S. White, Conditions implying the uniqueness of the weak*-topology on certain group algebras, *Houston J. Math.* **35**, No. 1 (2009), 253–272.

[FJ74] T. Figiel and W. B. Johnson, A uniformly convex Banach space which contains no ℓ_p, *Compos. Math.* **29** (1974), 179–190.

[GJW94] N. Grønbæk, B. E. Johnson and G. A. Willis, Amenability of Banach algebras of compact operators, *Israel J. Math.* **87**, No. 1 (1994), 289–324.

[Hor19] B. Horváth, *Algebras of operators on Banach spaces, and homomorphisms thereof*, (PhD thesis), Lancaster University (2019), https://doi.org/10.17635/lancaster/thesis/614

[HT21] B. Horváth and Zs. Tarcsay, Perturbations of surjective homomorphisms between algebras of operators on Banach spaces, to appear in *Proc. Amer. Math. Soc.* https://doi.org/10.1090/proc/15666. See also arXiv 2003.04572v2

[How00] R. Howey, *Approximately multiplicative maps between some Banach algebras*, (PhD thesis), University of Newcastle upon Tyne (2000).

[Jo88] B. E. Johnson, Approximately multiplicative maps between Banach algebras, *J. Lond. Math. Soc.* **37**, No. 2 (1988), 294–316.

[JPS21+] W. B. Johnson, N. C. Phillips and G. Schechtman, The SHAI property for the operators on L^p. Preprint, available at arXiv 2102.03966

[Ka82] D. Kazhdan, On ε-representations, *Israel J. Math.* **43**, No. 4 (1982), 315—323.

[Ko21] T. Kochanek, Approximately order zero maps between C^* algebras, *J. Funct. Anal.* **281**, No. 2 (2021), 109025.

[La03] N. J. Laustsen, On ring-theoretic (in)finiteness of Banach algebras of operators on Banach spaces, *Glasgow Math. J.* **45**, No. 1 (2003), 1–19.

[LP69] J. Lindenstrauss and A. Pełczyński, The $\mathscr{L}_p$ spaces, *Israel J. Math.* **7** (1969), 325–349.

[LT] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces I*, Springer-Verlag, Berlin, 1996.

[MV19] P. McKenney, A. Vignati, Ulam stability for some classes of C^*-algebras, *Proc. Roy. Soc. Edinburgh Sect. A* **149**, No. 1, (2019), 45–59.

[Ro70] H. P. Rosenthal, On the subspaces of L^p $(p > 2)$ spanned by sequences of independent random variables, *Israel J. Math.* **8** (1970), 273–303.

[Ru] V. Runde, *Amenable Banach Algebras — A Panorama*, Springer-Verlag, Berlin, 2020.

[Ry] R. A. Ryan, *Introduction to Tensor Products of Banach Spaces*, Springer, London, 2002.

[Si] I. Singer, *Bases in Banach Spaces, Vol. I*, Springer-Verlag, Germany, 1970.

[SW] H. H. Schaefer and M. P. Wolff, *Topological Vector Spaces* (Second Edition), Springer-Verlag, New York, 1999.

(Yemon Choi) Department of Mathematics and Statistics, Fylde College, Lancaster University, Lancaster, LA1 4YF, United Kingdom
E-mail address: y.choi1@lancaster.ac.uk

(Bence Horváth) Institute of Mathematics, Czech Academy of Sciences, Žitná 25, 115 67 Prague 1, Czech Republic
E-mail address: horvath@math.cas.cz, hotvath@gmail.com

(Niels Jakob Laustsen) Department of Mathematics and Statistics, Fylde College, Lancaster University, Lancaster, LA1 4YF, United Kingdom
E-mail address: n.laustsen@lancaster.ac.uk